Remarks on Bronštein's root theorem

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1 Introduction

In [Br1], M.D.Bronštein proved that the roots of hyperbolic polynomials

$$p(t,\tau) = \tau^m + \sum_{k=1}^{m} p_k(t)\tau^{m-k}. \tag{1.1}$$

which depend on t varying in some interval $I \subset \mathbb{R}$, are locally Lipschitz functions of t, if the coefficients p_k are C^r functions of t and the multiplicity of the roots does not exceed r; more importantly, he also proved that the roots are locally uniformly Lipschitz if p depends also continuously of parameters varying in a compact set.

His proof is based on delicate inequalities between the coefficients p_k implied by hyperbolicity. The goal of this note is to give a different and simpler proof, based on an induction on the degree. We will also drop compactness, to keep only uniform bounds of coefficients in $W^{r,\infty}$.

This result is crucial in the proof of other estimates that M.D.Bronštein proved in his other paper [Br2], on the well posedness in Gevrey spaces of hyperbolic equations. We will also quickly discuss these estimates in the last section of this note, as they are indeed equivalent to the Lispchitz smoothness of the roots.

To fix the notations, we make the following assumption.

Assumption 1.1. *For $t \in I := [-T, T]$, $p(t, \cdot)$ is a monic hyperbolic polynomial of degree m, which means that it has only real roots which denote by*

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$\lambda_1(t) \leq \ldots \leq \lambda_m(t)$. *We assume that their multiplicity is less than or equal to* $r \in \{1, \ldots, m\}$. *We further assume that all the coefficients* p_k *belong to the space* $W^{r,\infty}(I)$ *and that*

$$\forall k \in \{1, \ldots, m\}, \quad \|p_k\|_{W^{r,\infty}(I)} \leq R, \tag{1.2}$$

$$\forall t \in I, \ \forall \tau, , \quad \sum_{j=0}^{r} |\partial_\tau^j p(t, \tau)| \geq \delta > 0. \tag{1.3}$$

Here $W^{r,\infty}(I)$ denotes the space of functions in L^∞ with derivatives up to order r in L^∞. This is the space of C^{r-1} functions such that the $(r-1)$-th derivative satisfies a Lipschitz condition. The condition (1.3) is a quantitive way to express that the multiplicities of the roots do not exceed r.

Theorem 1.2. *For all* $T_1 < T$, *there is a constant* K *which depends on* T,T_1, R, δ *and* m, *such that for all family of polynomials* $p(t, \cdot)$ *which satisfies the Assumption* 1.1 *one has for all* $t \in [-T_1, T_1]$*:*

$$p(t, \tau) = 0 \quad \Rightarrow \quad |\partial_t p(t, \tau)| \leq K |\partial_\tau p(t, \tau)|. \tag{1.4}$$

Moreover, for all $j = 1, \ldots, m$ *and* t *and* t' *in* $[-T_1, T_1]$*:*

$$|\lambda_j(t) - \lambda_j(t')| \leq K |t - t'|. \tag{1.5}$$

The property (1.4) is stated in Lemma 4 of [Br1], as an intermediate step in the proof of the Lipschitz estimate. The equivalence of (1.4) and (1.5) for strictly hyperbolic polynomials is immediate, since the implicit function theorem implies that the roots roots λ_j and that

$$\partial_t \lambda_j(t) = -\partial_t p(t, \lambda_j) / \partial_\tau p(t, \lambda_j).$$

Approximating p by strictly hyperbolic polynomials, immediately implies the equivalence of (1.4) and (1.5) (see Lemma 2.5 below an the discussion which follows).

To prove (1.4) at t_0, we will make several reductions. When $r < m$, we will factorize p, on an interval of length which depends only on R and δ, into products of polynomials of degree at most r, and thus the general case follows from the case where we take $r = m$. Note that in this case, the condition (1.3) is trivial since $\partial_\tau^m p(t, \tau) = m!$. Moreover, by translation and dilatation in t, the key estimate is the following:

Proposition 1.3. *For all R, there is a constant K which depends on R and m, such that for all family of polynomials $p(t,\cdot)$ on $[-1,1]$ which satisfies*

$$\forall k \in \{1,\dots,m\}, \quad \|p_k\|_{W^{m,\infty}([-1,1])} \le R, \tag{1.6}$$

then

$$p(0,\tau) = 0 \quad \Rightarrow \quad |\partial_t p(0,\tau)| \le K|\partial_\tau p(0,\tau)|. \tag{1.7}$$

This proposition is proved by induction on m in Section 3 and the proof of Theorem 1.2 is completed in Section 4. Additional remarks are given in the last Section 5. Before starting the proof we collect in Section 2 several preliminary lemmas which will be needed later on.

Remark 1.4. The Lipschitz regularity of the root stated in Theorem 1.2 is local in $]-T,T[$ and in general, does not extend to the boundary, as shown by the example $p(t,\tau) = \tau^2 - t$ on $[0,1]$. The estimates are not valid at $t = 0$.

Remark 1.5. The condition on the $W^{r,\infty}$ norm of the coefficients is sharp, in the sense that it cannot be replaced by $W^{s,\infty}$ with $s < r$. For $r = m = 2$, this is seen on the example $p(t,\tau) = \tau^2 - a(t)$ where one can choose $a \ge 0$ in $W^{s,\infty}$ with $s < 2$ and $\sqrt{a}$ not Lipschitz. This can be extended in the following way.

Proposition 1.6. *For all $m \ge 2$ and $\epsilon > 0$, there are hyperbolic polynomials (1.1) with coefficients p_k in $C^{m-\epsilon}([-1,1])$ such that there is a root $\lambda_j(t)$ which is not Lipschitz continuous near the origin.*

The proof is given in Section 4 below.

2 Preliminaries

In this section, we gather several elementary lemmas which will be useful during the proof.

2.1 Sets of hyperbolic polynomials

Denote by $\mathcal{P}^m$ the set of monic polynomials

$$p(\tau) = \tau^m + \sum_{k=0}^{m-1} p_{m-k}\tau^k = \prod_{j=1}^{m}(\tau - \lambda_j) \tag{2.1}$$

and we identify p with $(p_1, \dots, p_m) \in \mathbb{C}^m$. In particular we use the norm $|p| = \max |p_k|$ on $\mathcal{P}^m$. It is also natural to introduce the quantity

$$\langle p \rangle = \max |p_k|^{1/k}, \tag{2.2}$$

since for all $\rho > 0$,

$$p_\rho(\tau) = \rho^m p(\tau/\rho) \quad \Rightarrow \quad \langle p_\rho \rangle = \rho \langle p \rangle. \tag{2.3}$$

Moreover, the roots satisfy

$$\forall j, \qquad |\lambda_j| \leq 2\langle p \rangle \tag{2.4}$$

since

$$0 = |p(\lambda_j)| \geq |\lambda_j|^m \Big(1 - \sum_{k=0}^{m-1} \big(\langle p \rangle / |\lambda_j|\big)^{m-k}\Big)$$

and the parenthesis is positive when $\langle p \rangle / |\lambda_j| < 1/2$.

We denote by $\mathcal{H}^m$ the set of hyperbolic polynomials, that is polynomials with only reals roots λ_j. We label them in nondecreasing order. For $p \in \mathcal{H}^m$, the coefficients p_k are real. Let $\mathcal{H}_0^m$ denote the subset of polynomials $p \in \mathcal{H}^m$ such that the coefficient p_1 vanishes. In this case, the roots λ_j satisfy

$$p_1 = \sum \lambda_j = 0, \qquad p_2 = -\frac{1}{2} \sum \lambda_j^2 \leq 0. \tag{2.5}$$

This implies that forall $k > 2$, $|p_k| \leq m^k |2p_2|^{k/2}$, or in short

$$p \in \mathcal{H}_0^m \quad \Rightarrow \quad \langle p \rangle \leq m |2p_2|^{\frac{1}{2}}. \tag{2.6}$$

2.2 Smooth factorisation

Our proof relies on the factorization of polynomials and a good control of the factors. It is convenient to introduce the following terminology.

Definition 2.1. *We say that there is an holomorphic factorization on a a neighborhood $\mathcal{O}$ of $\underline{p} \in \mathcal{P}^m$ if there are holomorphic mappings from ϕ_j from $\mathcal{O}$ to $\mathcal{P}^{m_j}$, for $j = 1, \dots, l$, with $1 \leq m_j < m$, such that for all $p \in \mathcal{O}$,*

$$p(\tau) = p^1(\tau) \dots p^l(\tau), \qquad p^j = \phi_j(p). \tag{2.7}$$

Recall first the following local result.

Lemma 2.2. *Suppose that $\underline{p} \in \mathcal{P}^m$ can be factorized as $\underline{p}^1 \dots \underline{p}^l$ with $\underline{p}^j \in \mathcal{P}^{m_j}$ with $1 \le m_j < m$ with non common roots. Then, there is a neighborhood $\mathcal{O}$ and a holomorphic factorization on $\mathcal{O}$ with $\phi_j(\underline{p}) = \underline{p}^j$. Moreover, if p is hyperbolic, all the factors $\phi_j(p)$ are hyperbolic.*

Proof. Consider sets D_j which contains of all the roots of $\underline{a}_j$ and such that $\overline{D}_j \cap \overline{D}_k = \emptyset$ for $j \neq k$. For p in a neighborhood $\mathcal{O}$ of $\underline{p}$, p does not vanish on $\cup \partial D_j$ and by Rouché's theorem p has m_j roots in D_j, which yields the decomposition $p = p^1 \dots p^l$. Moreover, the symmetric functions of the roots of p^j are given by

$$\sigma_k(p^j) = \frac{1}{2i\pi} \int_{\partial D_j} \frac{\partial_\tau p(\tau)}{p(\tau)} \, \tau^k d\tau.$$

Thus they are holomorphic functions of the coefficients of p. Thus the coefficients of p^j, which are polynomials of the $\sigma_k(p^j)$, are also holomorphic functions of the coefficients of p, proving the first part of the lemma.

Because the roots of p^j are roots of p, it is clear that if p is hyperbolic then all the p^j are hyperbolic. □

Corollary 2.3. *Suppose that $\mathcal{K}$ is a compact subset of $\mathcal{P}^m$ of polynomials with roots of multiplicity at mots $r < m$. Then there is $\rho > 0$ such that for all $\underline{p} \in \mathcal{K}$, there is an analytic factorization on the ball of center $\underline{p}$ and radius ρ in polynomials p^j of degree $m_j \le r$.*

Proof. By assumption and the lemma, there is an analytic factorization on a neighborhood of all $\underline{p} \in \mathcal{K}$. By compactness, there is $\rho > 0$ such that one can cover $\mathcal{K}$ by balls $\mathcal{B}_j$ of radius ρ such that there is analytic factorization on the ball $\tilde{\mathcal{B}}_j$ of radius 2ρ with the same center. All $\underline{p}$ is contained in one of the $\mathcal{B}_j$ and the ball of center $\underline{\rho}$ and radius ρ is contained in $\tilde{\mathcal{B}}_j$. □

In particular, we will use the following corollary.

Corollary 2.4. *For $R > 0$, there is $\rho > 0$ such that for all $\underline{p} \in \mathcal{H}_0^m$ such that*

$$|\underline{p}| \le R, \quad |\underline{p}_2| \ge 1. \tag{2.8}$$

there is an analytic factorization $p = p^1 p^2$ on the ball of center $\underline{p}$ and radius ρ.

Proof. Denote by $\mathcal{K}$ the set of polynomials in $\mathcal{H}_0^m$ which satisfy (2.8). The roots of $p \in \mathcal{K}$ satisfy $\sum \lambda_j = 0$ and $\sum \lambda_j^2 \ge 2$. Therefore there is at least one positive root and at least one negative root, which shows that the multiplicities of the roots is strictly less than m. □

2.3 Approximation by strictly hyperbolic polynomials

Recall the following result proved by W.Nuij [Nuij] .

Lemma 2.5. *If $p \in \mathcal{H}^m$ and $\varepsilon \neq 0$, then $p_\varepsilon = (1 + \varepsilon\partial_\tau)^m p$ has m distinct real roots.*

For the sake of completeness we sketch the proof. Let $p = \prod(\tau - \mu_j)^{r_j}$ where the $\mu_1 < \ldots < \mu_l$ are distinct. The roots of $p + \varepsilon\partial_\tau p$ are the multiple roots μ_j of p, with multiplicity decreased by one, and the zeros of $f(\tau) = 1 + \varepsilon \sum r_j/(\tau - \mu_j)$, which vanishes in each interval $]\mu_j, \mu_{j+1}[$ and in $-]-\infty, \mu_1[$ or in $]\mu_l, +\infty[$ depending on the sign of ε. Hence $p + \varepsilon\partial_\tau p$ has m real roots counted with their multiplicity and the maximal multiplicity is decreased by one. Iterating m times, all the roots are real and simple.

2.4 Estimates for functions of one real variable

On intervals $I \subset \mathbb{R}$ and for integers $r \geq 1$, we consider the spaces $W^{r,\infty}(I)$ of functions of class C^{r-1} such that the $(r-1)$-th derivative is Lipschitz continuous on I. We give below two elementary lemmas.

Lemma 2.6. *There is a consant C which depends only on m such that for all $a \in W^{m,\infty}([-1,1]])$ and $j \leq m$:*

$$\big\|\partial_t^j a\big\|_{L^\infty} \leq C\big(\big\|a\big\|_{L^\infty} + \big\|\partial_t^m a\big\|_{L^\infty}\big). \tag{2.9}$$

Proof. Intregrating m times from 0 to t we see that the function $b \in W^{m,\infty}([0,1])$ such that $\partial_t^m b = \partial_t^m a$, $\partial_t^j b(0) = 0$ for $j < m$, is bounded and

$$\|\partial_t^j b\|_{L^\infty} \leq \|\partial_t^m a\|_{L^\infty}.$$

Moreover, the function $a - b$ is a polynomial of degree less than m and therefore its coefficients are bounded by its L^∞ norm, and there is a constant C which depends only on m such that

$$\|\partial_t^j (a-b)\|_{L^\infty} \leq C\|a-b\|_{L^\infty} \leq C\big(\|a\|_{L^\infty} + \|\partial_t^m a\|_{L^\infty}\big).$$

The lemma follows. □

Lemma 2.7. *For all nonnegative $a \in W^{2,\infty}([-1,1])$ one has*

$$|t| \leq \theta\sqrt{a(0)/M} \quad \Rightarrow \quad \frac{1}{2}a(0) \leq a(t) \leq 2a(0), \tag{2.10}$$

where $M = \|a\|_{W^{2,\infty}([-1,1])}$ and $\theta = (\sqrt{5} - 2)$.

Proof. Recall first that for nonnegative functions $a \in W^{2,\infty}([-1,1])$:

$$|\partial_t a(0)| \le \max\left\{ (2a(0)\|\partial_t^2 a\|_{L^\infty})^{\frac12}\,,\, 2a(0)\right\} \le 2M^{\frac12} a(0)^{\frac12}. \tag{2.11}$$

The first estimate follows from the inequality which is valid for all $s \in [-1,1]$:

$$0 \le a(s) \le a(0) + s\partial_t a(0) + \frac{1}{2} s^2 M_2$$

with $M_2 = \|\partial_t^2 a\|_{L^\infty}$. If $|\partial_t a(0)| \le M_2$ we can choose $s = -\partial_t a(0)/M_2$ and we find that $|\partial_t a(0)|^2 \le 2M_2|a(0)|$. If $|\partial_t a(0)| \ge M_2$, we take $s = \pm 1$ to obtain that

$$0 \le a(0) - |\partial_t a(0)| + \frac{1}{2} M_2 \le 0 \le a(0) - \frac{1}{2}|\partial_t a(0)|$$

and thus $|\partial_t a(0)| \le 2a(0)$.

Hence, with $\delta = \sqrt{a(0)}$, one has

$$|a(t) - \delta^2| \le 2|t|\sqrt{M}\delta + \frac{1}{2} t^2 M \le \frac{1}{2}\delta^2$$

if $|t| \le (\sqrt{5} - 2)\delta/\sqrt{M}$ proving the lemma. □

3 The main induction

In this section we prove Proposition 1.3 by induction on m, the case $m = 1$ being trivial. So we assume that $m \ge 2$ and that the proposition is proved for polynomials of degree less than m. More precisely, we assume that for all R_1, there is K_1 such that for all family of monic hyperbolic polynomials $\{q(t,\cdot), t \in [-1,1]\}$ of degree $\le m-1$ with $W^{m-1,\infty}$ coefficients and such that

$$\forall k, \quad \|q_k\|_{W^{m-1,\infty}([-1,1])} \le R_1, \tag{3.1}$$

one has

$$q(0,\tau) = 0 \quad \Rightarrow \quad |\partial_t q(0,\tau)| \le K_1 |\partial_\tau q(0,\tau)|. \tag{3.2}$$

We fix $R > 0$ and we look for K such that (1.7) is satisfied for all family of polynomials $p(t,\cdot) \in \mathcal{H}^m$ which satisfy (1.6).

We first consider families of hyperbolic polynomials in $\mathcal{H}_0^m$.

Proposition 3.1. *Given R, there is K such that for all family of hyperbolic polynomials $\{p(t,\cdot) \in \mathcal{H}_0^m, t \in [-1,1]\}$, which satisfies the bound* (1.6)*, the property* (1.7) *is satisfied.*

Proof. **a)** Because $p(t,\cdot) \in \mathcal{H}_0^m$, we know that $p_2(t) \leq 0$. If $p_2(0) = 0$ then, because $m \geq 2$, $\partial_t p_2(0) = 0$ and $|p_2(t)| \leq \frac{1}{2}R|t|^2$. Hence by (2.6) all the coefficients satisfy $p_k(t) = O(|t|^k)$. In particular, $\tau = 0$ is the unique root of $p(0,\cdot)$ and therefore the estimate is trivially true since $\partial_\tau p(0,0) = \partial_t p(0,0) = 0$.

From now on we assume that $p_2(0) < 0$ and we set $\delta = \sqrt{-p_2(0)/2} > 0$.

b) Let $\rho = (\sqrt{5}-2)\sqrt{(2/R)}$. By Lemma (2.7), for $|t| \leq \rho\delta$ one has $\delta^2 \leq -p_2(t) \leq 4\delta^2$. Note that the interval $[-\rho\delta, \rho\delta]$ is contained in $[-1,1]$ since $\delta \leq \sqrt{R/2}$. By (2.6), the coefficients of p satisfy

$$\forall t \in [-\rho\delta, \rho\delta], \quad |p_k| \leq C\delta^k \tag{3.3}$$

where C depends only on m. For $s \in [-1,1]$, introduce the polynomial

$$q(s,\tau) = \delta^{-m} p(s\rho\delta, \delta\tau) = \tau^m + \sum_{k=2}^{m} q_k(s)\tau^{m-k} \tag{3.4}$$

the coefficients of which are

$$q_k(s) = \delta^{-k} p_k(\rho\delta s). \tag{3.5}$$

The estimate

$$\|q_k\|_{L^\infty([-1,1]} \leq C \tag{3.6}$$

follow from (3.3). Moreover,

$$|\partial_s^m q_k(s)| \leq \rho^m \delta^{m-k} \|\partial_t^m p_k\|_{L^\infty} \leq R', \tag{3.7}$$

where R' depends only on R. Here we crucially use that the order of derivation is larger than or equal to k, so that the exponent of δ is nonnegative.

By Lemma 2.6 one can interpolate between (3.6) and (3.7) and there is R' which depends only on R and m such that

$$\forall j \leq m, \forall s \in [-1,1], \quad |\partial_t^j q_k(s)| \leq R'. \tag{3.8}$$

Moreover, by construction

$$-q_2(s) = -\delta^{-2} p_2(s\rho\delta) \geq 1. \tag{3.9}$$

Summing up, we have proved that the $q(s,\cdot)$ are hyperbolic polynomials in $\mathcal{H}_0^m$ for $s \in [-1,1]$, which satisfy uniform bounds (3.8), and the additional condition $-q_2(s) \geq 1$.

c) By Corollary 2.4 there is $\rho > 0$, which depends only on R', and an analytic factorization $q = q^1 q^2$ on the ball centered at $q(0, \cdot)$ of radius ρ. The condition $|q(s, \cdot) - q(0, \cdot)| = \max_k |q_k(s) - q_k(0)| \leq \rho$ is satisfied on the interval $|s| \leq \rho / R'$, and hence, there is $T > 0$, $T \leq 1$, depending only on R' and m, such that for $s \in [-T, T]$

$$q(s, \tau) = q^1(s, \tau) q^2(s, \tau). \tag{3.10}$$

Moreover, the coefficients of q^1 and q^2 are analytic functions of the coefficients of q and there is R'' such that their coefficients satisfy

$$\forall j \leq m, \forall s \in [-T, T],, \quad |\partial_t^j q_k^l(s)| \leq R''.$$

Consider then $\tilde{q}^l(s, \tau) = q(sT, \tau)$ defined for $|s| \leq 1$. Then, there is R_1, depending only on R and m, such that the $\tilde{q}^l$ are of degree less than m and satisfy the estimates (3.1) of the induction hypothesis. Therefore, there is K_1 which depends only on R_1 and m such that the polynomials $\tilde{q}^1$ and $\tilde{q}^2$ satisfy (3.2). Therefore, the q^l satisfy (3.2) with the constant K_1/T. At the roots of q^1 one has $\partial_t q = \partial_t q^1 q^2$ and $\partial_\tau q = \partial_\tau q^1 q^2$. There is a similar and symmetric result at the roots of q^2. Because q^1 and q^2 have no root in common, this implies that

$$q(0, \tau) = 0 \quad \Rightarrow \quad |\partial_t q(0, \tau)| \leq K_1/T |\partial_\tau q(0, \tau)|. \tag{3.11}$$

d) From the definition (3.4) of q, we see that $q(0, \tau) = 0$ is equivalent to $p(0, \delta\tau) = 0$ and that

$$\partial_s q(0, \tau) = \delta^{1-m} \rho \partial_t p(0, \delta\tau), \qquad \partial_\tau q(0, \tau) = \delta^{1-m} p(0, \delta\tau).$$

Hence (3.11) implies that at roots of $p(0, \tau) = 0$

$$|\partial_t p(t_0, \tau)| \leq K_1/T\rho \; |\partial_\tau p(t_0, \tau)|. \tag{3.12}$$

This shows that (1.7) is satisfied with the constant $K = K_1/T\rho$ which depends only on R and m. $\square$

We now relax the condition $p_1 = 0$ and finish the proof of Proposition 1.3.

Proposition 3.2. *Given R, there is K such that for all family of hyperbolic polynomials $\{p(t, \cdot) \in \mathcal{H}^m, t \in [-1, 1]\}$, which satisfies the bound* (1.6)*, the property* (1.7) *is satisfied.*

Proof. Consider

$$\tilde{p}(t,\tau) = p(\tau + p_1(t)/m). \tag{3.13}$$

The polynomials $\tilde{p}(t,\cdot)$ all belong to $\mathcal{H}_0^m$: one has $\tilde{p}_1 = 0$ and the other coefficients are

$$\tilde{p}_k = \sum_{l=0}^{k} \binom{m-l}{m-k} (-p_1/m)^{k-l} p_l.$$

Thus, by (1.2),

$$|\partial_t^j \tilde{p}_k(t)| \le R_1$$

where R_1 depends only on R and m. Thus, by Proposition 3.1, there is K_1, depending only on R_1, such that

$$\tilde{p}(0,\tau) = 0 \quad \Rightarrow \quad |\partial_t \tilde{p} 0, \tau)| \le K_1 |\partial_\tau \tilde{p}(0,\tau)|.$$

Because

$$\partial_t \tilde{p}(t,\tau) = \partial_t p(t, \tau + p_1/m) + \frac{1}{m} \partial_t p_1(t)\, \partial_\tau p(t, \tau + p_1/m)$$

the estimate (1.7) follows with $K = K_1 + R/m$. □

The proof of Proposition 1.3 is now complete.

4 Proof of Theorem 1.2

4.1 Proof of (1.4)

Recall the notations. The polynomials are defined for $|t| \le T$ and we want to prove (1.4) for $|t_0| \le T_1$, where $T_1 < T$ are given. By translation, we can always assume that $t_0 = 0$ and that the polynomials are defined on the fixed interval $|t| \le T - T_1$. By dilation, we can further assume that $T - T_1 \ge 1$, and therefore, the estimate (1.4) immediately follows from Proposition 3.2 when $r = m$.

We now consider the case $r < m$. In addition to the uniform bound (1.2) of the $W^{r,\infty}$ norm of the coefficients, we assume (1.3). Again it is sufficient to prove the estimate for $t_0 = 0$ and $T > 0$, R and δ are fixed. Consider the compact set $\mathcal{K} \subset \mathcal{H}^m$ of polynomials with coefficients bounded by R and satisfying

$$\min_{\tau \in \mathbb{R}} \sum_{j=0}^{r} |\partial_\tau^j p(\tau)| \ge \delta. \tag{4.1}$$

This is a compact set of polynomials with roots of mulitplicity $\leq r$. By Corollary 2.3, there is ρ which depends only on $\mathcal{K}$, that is on R and δ, such that there is an holomorphic factorisation $p = p^1 \dots p^l$ for $|p - p(0, \cdot)| \leq \rho$. Hence, for $|t| \leq \rho/R$, there is a factorisation

$$p(t, \tau) = p^1(t, \tau) \dots p^l(t, \tau)$$

where the coefficients of p^j are holomorphic functions of the coefficients of p. Therefore, there is R_1, which depends only on R and δ, such that the $W^{r,\infty}$ norms of the coefficients of the p^j are bounded by R_1. Moreover, the p^j are hyperbolic of degree at most r. Hence, by the first step, there is K_1 such that

$$p^j(0, \tau) = 0 \quad \Rightarrow \quad |\partial_t p^j(0, \tau)| \leq K_1 |\partial_\tau p^j(0, \tau)|. \tag{4.2}$$

This implies (1.4) with $K = K_1 R_1^{l-1}$.

4.2 Proof of (1.5)

As already said, (1.4) and (1.5) are equivalent when p is strictly hyperbolic. By Lemma 2.5, $p_\varepsilon = (1 + \varepsilon \partial_\tau)^m \in \mathcal{H}^m$ and the $W^{r,\infty}$ norm of its coefficients is uniformly bounded if $\varepsilon \in [0, 1[$. Moreover, decreasing δ if necessary, the property (1.3) remains satisfied uniformly in ε for ε small. Hence, there is a constant K such that the property (1.4) is satisfied for all p_ε. Hence, for $\varepsilon > 0$ small, the roots of p_ε are simple and satisfy on $[-T_1, T_1]$

$$|\lambda_{\varepsilon,j}(t) - \lambda_{\varepsilon,j}(t')| \leq K |t - t'|.$$

The roots are continuous functions of ε and hence (1.5) follows.

4.3 Proof of Proposition 1.6

Consider a strictly hyperbolic polynomial $q(\tau)$ with 0 as a root:

$$q(\tau) = \tau^m + \sum_{k=1}^{m-1} q_{m-k} \tau^k. \tag{4.3}$$

In particular, $q_{m-1} = \partial_\tau q(0) \neq 0$. There is $\delta > 0$ such that for all real $a \in [-\delta, \delta]$, $q(\tau) + a$ is strictly hyperbolic, with roots $\mu_1(a) < \ldots < \mu_m(a)$ which are analytic in a. One of them, say μ_j, vanishes at $a = 0$ and $\partial_a \mu_j(0) = -1/q_{m-1} \neq 0$. Therefore, decreasing δ if necessary, there is $c > 0$ such that

$$|a| \leq \delta \quad \Rightarrow \quad |\partial_a \mu_j(a)| \geq c. \tag{4.4}$$

For $t \in [-1, 1]$, consider the polynomial

$$p(t, a, \tau) = t^m (q(\tau/t) + a) = \tau^m + \sum_{k=1}^{m-1} t^{m-k} q_{m-k} \tau^k + t^m a.$$

It is hyperbolic and its roots are $t\mu_l(a)$. In particular, for $t > 0$, one has $\lambda_j(t, a) = t\mu_j(a)$. Now we chose $a = a(t) = \delta \cos(t^{-\alpha})$ with $0 < \alpha < 1/m$. The coefficients $p_1, \ldots, p_{m-1}$ of $p(t, a(t), \cdot)$ are smooth, and the last one $p_m(t) = \delta t^m \cos(t^{-1-\alpha})$ belongs to the Hölder class $C^{m-m\alpha}$. On the other hand, the t derivative of its j-th root is

$$\mu_j(a(t)) + \alpha\delta t^{-\alpha} \sin(t^{-\alpha})\ \partial_a \mu_j(a(t)),$$

which by (4.4) is not bounded as $t \to 0$.

5 Bronštein estimates for hyperbolic polynomials

In [Br2], M.D Bronštein proved the well posed-ness of the Cauchy problem in Gevrey spaces for a hyperbolic operator $P(t, x, D_t, D_x)$, using the construction of a parametrix for $e^{-t\gamma(D_x)}\ Pe^{t\gamma(D_x)}$, with principal symbol $q = 1/P(t, x, \tau - i\gamma(\xi), \xi)$. The key step is to prove that q is a good symbol, and this relies on the following estimates.

Theorem 5.1. *Suppose that $\{p(t, \cdot), t \in [-T, T]\}$ is a family of hyperbolic polynomials which satisfies Assumption* 1.1. *Then, for all $T_1 < T$ there is a constant C such that for $k \leq m$ and $l \leq r$, one has for $t \in [T_1, T_1]$ and $\tau \in \mathbb{C}$ with $|\operatorname{Im} \tau| \leq 1$*

$$|\operatorname{Im} \tau|^{k+l} \left|\partial_\tau^k \partial_t^l p(t, \tau)\right| \leq C\, |p(\tau, a)|. \tag{5.1}$$

This theorem essentially rephrases the Proposition 2 of [Br2]. What we want to emphasize here is that these estimates are equivalent to (1.4) (1.5) so that Theorems 1.2 and 5.1 are somehow equivalent.

The estimates of the τ derivatives are classical, using that

$$|p(t, \tau)| \geq c|\operatorname{Im} \tau|^r \tag{5.2}$$

and that $\frac{m!}{k!}\partial_\tau^k p$ is monic and hyperbolic with maximal multiplicity at most $r - 1$.

The estimates of the derivatives with respect to the parameter t are proved by induction on the degree m using the following lemma which follows from (1.4) (1.5). This lemma is implicit in [Br2] at the bottom of page 92, but it is sufficiently important to be stated on its own.

Lemma 5.2. *Under Assumption* 1.1*, given $T_1 < T$, there is $\varepsilon_1 > 0$ such that for $|\varepsilon| \le \varepsilon_1$ $q = \partial_\tau p + \varepsilon \partial_t p$ is hyperbolic with roots of multiplicity $\le r - 1$.*

Proof. For the convenience of the reader we recall the essence of Bronštein's proof. Write

$$p(\tau, a) = \tau^m + \sum_{k=1}^{m} p_k(a) \tau^{m-k} = \prod_{j=1}^{m} (\tau - \lambda_j(a)). \tag{5.3}$$

Because p is monic, q is of degree $\le m - 1$ and the coefficient of τ^{m-1} is

$$a(t) = m + \varepsilon \partial_a p_1(t) \tag{5.4}$$

Thus there is $\varepsilon_1 > 0$ depending only on R, such that for $\varepsilon \le \varepsilon_1$, $a \ge \frac{1}{2} m$ and q is of degree $m - 1$.

Suppose that $\mu_1 < \ldots < \mu_n$ are the distinct roots of $p(t_0, \cdot)$ with multiplicity r_j. For $k \le r_j - 1$, μ_j is a root of $\partial_\tau^{k-1} p(t_0, \cdot)$ and thus (1.4) implies that there is K, which depends only on T_1 and on the bounds R and δ such that

$$|\partial_t \partial_\tau^{k-1} p(t_0, \mu_j)| \le K |\partial_\tau^k p(t_0, \mu_j)|. \tag{5.5}$$

In particular, if $r_j > 1$, μ_j is a root of $\partial_\tau p(t_0, \cdot)$, thus of $\partial_t p(t_0, \cdot)$ and $q(t_0, \cdot)$, of multiplicity $\ge r_j - 1$. Moreover (5.5) implies that when $\tau \to \mu_j$,

$$|\partial_t p(t_0, \tau)| \le \big(K + O\big(|\tau - \mu_j|\big)\big) |\partial_\tau p(t_0, \tau)|.$$

Therefore, if $|\varepsilon| K < 1$, $q(t_0, \tau)/\partial_\tau p(t_0, \tau) > 0$ for τ close to μ_j.

On $]\mu_j, \mu_{j+1}[$, $p(t_0, \cdot)$ has a constant sign, and $\partial_\tau p(t_0, \cdot)$ has opposite sign near the ends μ_j and μ_{j+1}. By the remark above, near the ends, $q(t_0, \cdot)$ has the same sign as $\partial_\tau p(t_0, \cdot)$, provided that $|\varepsilon| K < 1$. Thus $q(t_0, \cdot)$ has opposite sign near the ends and q has also a root in the interval $]\mu_j, \mu_{j+1}[$. Because q is of degree $m - 1$, this implies that $q(t_0, \cdot)$ has only real roots: the μ_j with multiplicity $r_j - 1$ and one simple root in each of the interval $]\mu_j, \mu_{j+1}[$. □

Proof of Theorem 5.1. We proceed by induction on l.

a) If $l = 0$, denoting by λ_j the roots of p, one has

$$\partial_\tau^k p / p = \sum_{|I| = k} \prod_{j \in I} (\tau - \lambda_j)^{-1},$$

where I denotes subsets of $\{1, \ldots, m\}$ and $|I|$ is the number of elements in it. The estimate follows since $|\tau - \lambda_j| \ge |\operatorname{Im} \tau|$.

b) Similarly, if $l = 1$, $(\partial_\tau^k \partial_t p)/p$ is a sum of terms

$$\nabla \lambda_{j'} \prod_{j \in I} (\tau - \lambda_j)^{-1}$$

with $|I| = k + 1$. The estimate follows, knowing the estimate of $\nabla \lambda_{j'}$ from Theorem 1.2.

c) Suppose now that the estimate has been proved up to derivatives of order $l - 1$. Consider the polynomial

$$q = \frac{1}{a}\big(\partial_\tau p + \varepsilon \partial_a p\big), \qquad a = m + \varepsilon \partial_t p_1.$$

By Lemma 5.2, q is monic and hyperbolic, and for $T_2 \in]T_1, T[$ and ε small, depending only on R and δ, l the family $\{q(t, \cdot), t \in [-T_2, T_2]\}$ satisfies the Assumption 1.1 for some constants R_1 and δ_1 which depend only on R and δ. Hence, one can apply the induction hypothesis to q and conclude that

$$\big|\partial_\tau^k \partial_t^{l-1} q(\tau, a)\big| \leq C |\mathrm{Im}\,\tau|^{-k-l+1} |q(\tau, a)| \leq C |\mathrm{Im}\,\tau|^{-k-|\alpha|} |p(\tau, a)|,$$

since, by step b),

$$|q| \leq C |\mathrm{Im}\,\tau|^{-1} |p|.$$

One has similar estimates for aq, since a is smooth, and for $\partial_\tau p$ by the induction hypothesis, thus

$$\big|\varepsilon \partial_\tau^k \partial_t^l p(\tau, a)\big| \lesssim |\mathrm{Im}\,\tau|^{-k-|\alpha|} |p(\tau, a)|$$

and the estimate is proved at the order l. □

Remark 5.3. Theorems 1.2 and (5.1) are closely related. We have used the former to prove the latter, but the converse is also true : if the estimates (5.1) are satisfied, then the main estimate of (1.4) is satisfied. Indeed, it is sufficient to assume that

$$\gamma\, |\partial_t p(t_0, \tau - i\gamma)| \leq K_1 |p(t_0, \tau - i\gamma)|.$$

At a root τ_0 of $p(t_0, \tau_0) = 0$, the right hand side is $\gamma\, |\partial_\tau p(t_0, \tau_0| + O(\gamma^2)$. Thus dividing out by γ and letting γ tend to 0 implies that

$$|\partial_t p(t_0, \tau_0)| \leq K_1 |\partial_\tau p(t_0, \tau_0)|.$$

Remark 5.4. When $p(\tau, \xi)$ is polynomial in (τ, ξ) the estimates (5.1) are proved in a completely different way in [Hör] Theorem 12.4.6, using real algebraic geometry. Bronštein's theorem is much more general since it applies when the coefficient are only C^r functions of the parameters.

Remark 5.5. The estimates (5.1) have another formulation. Introduce

$$\tilde{p}(t,s,\tau) = \sum_{l=0}^{r} \frac{1}{l!} s^l \partial_t^l p(t,\tau). \tag{5.6}$$

Then, (5.1) implies that for $|s| \lesssim |\mathrm{Im}\,\tau|$,

$$|\tilde{p}(t,s,\tau)| \lesssim |p(t,\tau)| \tag{5.7}$$

and that there is $\delta > 0$ such that for $|s| \leq \delta |\mathrm{Im}\,\tau|$,

$$\frac{1}{2}|p(t,\tau)| \leq |\tilde{p}(t,s,\tau)| \leq 2|p(t,\tau)|. \tag{5.8}$$

Note that (5.7) is indeed equivalent to (5.1) for $k = 0$, since there is a constant C such that for all polynomial and all $\gamma > 0$, $q(s) = \sum_{l \leq r} q_l s^l$ one has

$$C^{-1} \sup \gamma^l |q_l| \leq \sup_{|s| \leq \gamma} |q(s)| \leq C \sup \gamma^l |q_l|$$

A direct proof of (5.7) and (5.8) is given in [CNR].

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