TIME-DEPENDENT LOSS OF DERIVATIVES FOR HYPERBOLIC OPERATORS WITH NON-REGULAR COEFFICIENTS

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Abstract

In this paper we will study the Cauchy problem for strictly hyperbolic operators with low regularity coefficients in any space dimension $N \geq 1$. We will suppose the coefficients to be log-Zygmund continuous in time and log-Lipschitz continuous in space. Paradifferential calculus with parameters will be the main tool to get energy estimates in Sobolev spaces and these estimates will present a time-dependent loss of derivatives.

1 Introduction

This paper is devoted to the study of the Cauchy problem for a second order strictly hyperbolic operator defined in a strip $[0, T] \times \mathbb{R}^N$, for some $T > 0$ and $N \geq 1$. Consider a second order operator of the form

def:op

$$Lu \;:=\; \partial_t^2 u \,-\, \sum_{j,k=1}^{N} \partial_j \left(a_{jk}(t,x)\, \partial_k u\right) \tag{1}$$

(with $a_{jk} = a_{kj}$ for all j, k) and assume that L is strictly hyperbolic with bounded coefficients, i.e. there exist two constants $0 < \lambda_0 \leq \Lambda_0$ such that

$$\lambda_0\, |\xi|^2 \;\leq\; \sum_{j,k=1}^{N} a_{jk}(t,x)\, \xi_j\, \xi_k \;\leq\; \Lambda_0\, |\xi|^2$$

for all $(t,x) \in [0,T] \times \mathbb{R}^N$ and all $\xi \in \mathbb{R}^N$.

It is well-known (see e.g. [11] or [15]) that, if the coefficients a_{jk} are Lipschitz continuous with respect to t and only measurable in x, then the Cauchy problem for L is well-posed in H^1–L^2. If the a_{jk}'s are Lipschitz continuous with respect to t and C_b^∞ (i.e. $\mathcal{C}^\infty$ and bounded with all their derivatives) with respect to the space variables, one can recover the well-posedness in H^{s+1}–H^s for all $s \in \mathbb{R}$. Moreover, in the latter case, one gets, for all $s \in \mathbb{R}$ and for a constant C_s depending only on it, the following energy estimate:

est:no-loss (2)
$$\sup_{0\leq t\leq T}\left(\|u(t,\cdot)\|_{H^{s+1}}\,+\,\|\partial_t u(t,\cdot)\|_{H^s}\right)\,\leq$$
$$\leq\,C_s\left(\|u(0,\cdot)\|_{H^{s+1}}+\|\partial_t u(0,\cdot)\|_{H^s}+\int_0^T\|Lu(t,\cdot)\|_{H^s}\,dt\right)$$

for all $u\in\mathcal{C}([0,T];H^{s+1}(\mathbb{R}^N))\,\cap\,\mathcal{C}^1([0,T];H^s(\mathbb{R}^N))$ such that $Lu\in L^1([0,T];H^s(\mathbb{R}^N))$. Let us explicitly remark that previous inequality involves no loss of regularity for the function u: estimate (2) holds for every $u\in\mathcal{C}^2([0,T];H^\infty(\mathbb{R}^N))$ and the Cauchy problem for L is well-posed in H^∞ *with no loss of derivatives*.

If the Lipschitz continuity (in time) hypothesis is not fulfilled, then (2) is no more true. Nevertheless, one can still try to recover H^∞-well-posedness, with a *loss of derivatives* in the energy estimate.

The first case to consider is the case of the coefficients a_{jk} depending only on t:

$$Lu\;=\;\partial_t^2u\;-\;\sum_{j,k=1}^N a_{jk}(t)\,\partial_j\partial_k u\,.$$

In [6], Colombini, De Giorgi and Spagnolo assumed the coefficients to satisfy an integral log-Lipschitz condition:

hyp:int-LL (3)
$$\int_0^{T-\varepsilon}|a_{jk}(t+\varepsilon)\;-\;a_{jk}(t)|\,dt\;\leq\;C\,\varepsilon\,\log\left(1\,+\,\frac{1}{\varepsilon}\right)\,,$$

for some constant $C>0$ and all $\varepsilon\in]0,T]$. To get the energy estimate, they first smoothed coefficients using a mollifier kernel (ρ_ε). Then, by Fourier transform, they defined an approximated energy $E_\varepsilon(\xi,t)$ in phase space, where the problem becomes a family of ordinary differential equations. At that point, the key idea was to perform a different approximation of the coefficients in different zones of the phase space: in particular, they set $\varepsilon\,=\,|\xi|^{-1}$. Finally, they obtained an energy estimate *with a fixed loss of derivatives*: there exists a constant $\delta>0$ such that, for all $s\in\mathbb{R}$, the inequality

est:c-loss (4)
$$\sup_{0\leq t\leq T}\left(\|u(t,\cdot)\|_{H^{s+1-\delta}}\,+\,\|\partial_t u(t,\cdot)\|_{H^{s-\delta}}\right)\,\leq$$
$$\leq\,C_s\left(\|u(0,\cdot)\|_{H^{s+1}}+\|\partial_t u(0,\cdot)\|_{H^s}+\int_0^T\|Lu(t,\cdot)\|_{H^s}\,dt\right)$$

holds true for all $u\in\mathcal{C}^2([0,T];H^\infty(\mathbb{R}^N))$, for some constant C_s depending only on s. Let us remark that if the coefficients a_{jk} are not Lipschitz continuous then a loss of regularity cannot be avoided, as shown by Cicognani and Colombini in [4]. Besides, in this paper the authors prove that, if the regularity of the coefficients a_{jk} is measured by a modulus of continuity, any intermediate modulus of continuity between the Lipschitz and the log-Lipschitz ones entails necessarily a loss of regularity, which, however, can be made arbitrarily small.

Recently Tarama (see paper [16]) analysed the problem when coefficients satisfy an integral log-Zygmund condition: there exists a constant $C>0$ such that, for all j, k and all $\varepsilon\in]0,T/2[$, one has

hyp:int-LZ (5)
$$\int_\varepsilon^{T-\varepsilon}|a_{jk}(t+\varepsilon)\,+\,a_{jk}(t-\varepsilon)\,-\,2\,a_{jk}(t)|\,dt\;\leq\;C\,\varepsilon\,\log\left(1\,+\,\frac{1}{\varepsilon}\right)\,.$$

On the one hand, this condition is somehow related to the punctual condition (for a function $a\in\mathcal{C}^2([0,T])$) $|a(t)|+|t\,a'(t)|+|t^2\,a''(t)|\;\leq\;C$, considered by Yamazaki in [17]. On the other hand, it's obvious that, if the a_{jk}'s satisfy (3), then they satisfy also (5): so, a more general class of functions is considered. Again, Fourier transform, smoothing the cofficients and linking the approximation parameter with the dual variable were fundamental tools in the analysis of Tarama. The improvement with respect to paper [6], however, was obtained defining a new energy, which involved (by derivation in time) second derivatives of the approximated coefficients. Finally, he got an estimate analogous to (4), which implies, in particular, well-posedness in the space H^∞.

In paper [8], Colombini and Lerner considered instead the case in which coefficients a_{jk} depend both in time and in space variables. In particular, they assumed an isotropic punctual log-Lipschitz condition, i.e. there exists a constant $C>0$ such that, for all $\zeta=(\tau,\xi)\in\mathbb{R}\times\mathbb{R}^N$, one has

$$\sup_{z=(t,x)\in\mathbb{R}\times\mathbb{R}^N} |a_{jk}(z+\zeta)\,-\,a_{jk}(z)|\;\leq\;C\,|\zeta|\,\log\left(1\,+\,\frac{1}{|\zeta|}\right)\,.$$

Again, smoothing coefficients with respect to the time variable is required; on the contrary, one cannot use the Fourier transform, due to the dependence of a_{jk} on x. The authors bypassed this problem taking advantage of the Littlewood-Paley decomposition and paradifferential calculus. Hence, they considered the energy concerning each localized part $\Delta_\nu u$ of the solution u, and then they performed a weighed summation to put all these pieces together. Also in this case, they had to consider a different approximation of the coefficients in different zones of the phase space, which was obtained setting $\varepsilon=2^{-\nu}$ (recall that 2^ν is the "size" of the frequencies in the ν-th ring, see subsection 3.1 below). In the end, they got the following statement: for all $s\in\,]0,1/4]$, there exist positive constants β and C_s and a time $T^*\in\,]0,T]$ such that

est:t-loss

$$(6)\qquad \sup_{0\leq t\leq T^*}\left(\|u(t,\cdot)\|_{H^{-s+1-\beta t}}\,+\|\partial_t u(t,\cdot)\|_{H^{-s-\beta t}}\right)\,\leq$$
$$\leq\,C_s\left(\|u(0,\cdot)\|_{H^{-s+1}}+\|\partial_t u(0,\cdot)\|_{H^{-s}}+\int_0^{T^*}\|Lu(t,\cdot)\|_{H^{-s-\beta t}}\,dt\right)$$

for all $u\in\mathcal{C}^2([0,T];H^\infty(\mathbb{R}^N))$. Let us point out that the bound on s was due to this reasons: the product by a log-Lipschitz function is well-defined in H^s if and only if $|s|<1$. Note also that this fact gives us a bound on the lifespan of the solution: the regularity index $-s+1-\beta T^*$ has to be strictly positive, so one can expect only local-in-time existence of a solution. Moreover in the case the coefficients a_{jk} are $\mathcal{C}^\infty_b$ in space, the authors proved inequality (6) for all s: so, they still got well-posedness in H^∞, but *with a loss of derivatives increasing in time.*

The case of a complete strictly hyperbolic second order operator,

$$Lu\;=\;\sum_{j,k=0}^{N}\partial_{y_j}\left(a_{jk}\,\partial_{y_k}u\right)\;+\;\sum_{j=0}^{N}\big(b_j\,\partial_{y_j}u\;+\;\partial_{y_j}(c_j\,u)\big)\;+\;d\,u$$

(here we set $y=(t,x)\in\mathbb{R}_t\times\mathbb{R}^N_x$), was considered by Colombini and Métivier in [9]. They assumed the same isotropic log-Lipschitz condition of [8] on the coefficients of the second order part of L, while b_j and c_j were supposed to be α-Hölder continuous (for some $\alpha\in\,]1/2,1[$) and d to be only bounded. The authors headed towards questions such as local existence and uniqueness, and also finite propagation speed for local solutions.

Recently, Colombini and Del Santo, in [7] (for a first approach to the problem see also [10], where smoothness in space were required), came back to the Cauchy problem for the operator (1), mixing up a Tarama-like hypothesis (concerning the dependence on the time variable) with the one of Colombini and Lerner (with respect to x). More precisely, they assumed a punctual log-Zygmund condition in time and a punctual log-Lipschitz condition in space, uniformly with respect to the other variable (see conditions (9) and (10) below). However, they had to restrict themselves to the case of space dimension $N=1$: as a matter of fact, a Tarama-kind energy was somehow necessary to compensate the bad behaviour of the coefficients with respect to t, but it was not clear how to define it in higher space dimensions. Again, localizing energy by Littlewood-Paley decomposition and linking approximation parameter and dual variable lead to an estimate analogous to (6).

The aim of the present paper is to extend the result of Colombini and Del Santo to any dimension $N \geq 1$. As just pointed out, the main difficulty was to define a suitable energy related to the solution. So, the first step is to pass from functions $a(t,x)$ with low regularity modulus of continuity, to more general symbols $\sigma_a(t,x,\xi)$ (obviously related to the initial function a) satisfying the same hypothesis in t and x, and then to consider paradifferential operators associated to these symbols. Nevertheless, positivity hypothesis on a (required for defining a strictly hyperbolic problem) does not translate, in general, to positivity of the corresponding operator, which is fundamental in obtaining energy estimates. At this point, paradifferential calculus depending on a parameter $\gamma \geq 1$, defined and developed by Métivier in [12] (see also [14]), comes into play and allows us to recover positivity of the (new) paradifferential operator associated to a. Defining a localized energy and an approximation of the coefficients depending on the dual variable are, once again, basic ingredients in closing estimates. Hence, in the end we will get an inequality similar to (6), for any $s \in]0,1[$.

The paper is organized as follows.

First of all, we will introduce the work hypothesis for our strictly hyperbolic problem, and we will state our main results.

Then, we will present the tools we need, all from Fourier Analysis. In particular, we will recall Littlewood-Paley decomposition and some results about (classical) paradifferential calculus, as introduced first by J.-M. Bony in the famous paper [2]. We will need also to define a different class of Sobolev spaces, of logarithmic type, as done in [9]: they will come into play in our computations. Moreover, we will present also paradifferential calculus depending on a parameter (which is basic in our analysis, as already pointed out), as introduced in [12] and [14]. A complete treatement about functions with low regularity modulus of continuity will end this section. In particular, we will focus on log-Zymgund and log-Lipschitz conditions: taking advantage of paradifferential calculus, we will state properties of functions satisfying such hypothesis and of the relative smoothed-in-time (by a convolution kernel) ones. Hence, we will pass to consider more general symbols and the associated paradifferential operators, for which we will develop also a symbolic calculus and we will state a fundamental positivity estimate.

This having been done, we will be then ready to tackle the proof of our main result: we will go back to the main ideas of paper [7]. First of all, taking advantage of a convolution kernel, we will smooth the coefficients, but with respect to the time variable only. As a matter of facts, low regularity in x will be compensated by considering paradifferential operators associated to our coefficients. Then, we will decompose the solution u to the Cauchy problem for (1) into dyadic blocks $\Delta_\nu u$, for which we will define an approximate localized energy e_ν: the dependence on the approximation parameter ε will be linked to the phase space localization, setting $\varepsilon = 2^{-\nu}$. The piece of energy e_ν will be of Tarama type, but this time multiplication by functions will be replaced by action of paradifferential operators associated to them. A weighed summation of these pieces will define the total energy $E(t)$ associated to u. The rest of the proof is classical: we will derive E with respect to time and, using Gronwall Lemma, we will get a control for it in terms of initial energy $E(0)$ and external force Lu only.

2 Basic definitions and main result

s:results

This section is devoted to the presentation of our work setting and of our main results.

Let us consider the operator over $[0,T_0]\times\mathbb{R}^N$ (for some $T_0>0$ and $N\geq 1$)

eq:op (7)
$$Lu\,=\,\partial_t^2 u\,-\,\sum_{i,j=1}^{N}\partial_i\left(a_{ij}(t,x)\,\partial_j u\right)\,,$$

and let us suppose L to be strictly hyperbolic with bounded coefficients, i.e. there exist two positive constants $0<\lambda_0\leq\Lambda_0$ such that, for all $(t,x)\in\mathbb{R}_t\times\mathbb{R}^N_x$ and all $\xi\in\mathbb{R}^N$, one has

h:hyp (8)
$$\lambda_0\,|\xi|^2\,\leq\,\sum_{i,j=1}^{N}a_{ij}(t,x)\,\xi_i\,\xi_j\,\leq\,\Lambda_0\,|\xi|^2\,.$$

Moreover, let us suppose the coefficients to be log-Zygmund-continuous in the time variable t, uniformly with respect to x, and log-Lipschitz-continuous in the space variables, uniformly with respect to t. This hypothesis reads as follow: there exists a constant K_0 such that, for all $\tau>0$ and all $y\in\mathbb{R}^N\setminus\{0\}$, one has

h:LZ_t (9)
$$\sup_{(t,x)}\left|a_{ij}(t+\tau,x)+a_{ij}(t-\tau,x)-2a_{ij}(t,x)\right|\;\leq\;K_0\,\tau\,\log\left(1\,+\,\frac{1}{\tau}\right)$$

h:LL_x (10)
$$\sup_{(t,x)}\left|a_{ij}(t,x+y)-a_{ij}(t,x)\right|\;\leq\;K_0\,|y|\,\log\left(1\,+\,\frac{1}{|y|}\right)\,.$$

Now, let us state our main result, i.e. an energy estimate for the operator (7).

t:en-est **Theorem 2.1.** *Let us consider the operator L defined in* (7)*, and let us suppose L to be strictly hyperbolic, i.e. relation* (8) *holds true. Moreover, let us suppose that coefficients a_{ij} satisfy also conditions* (9) *and* (10)*.*

Then, for all fixed $\theta\in\,]0,1[$, there exist a $\beta^*>0$, a time $T>0$ and a constant $C>0$ such that the following estimate,

est:thesis (11)
$$\sup_{0\leq t\leq T}\left(\|u(t,\cdot)\|_{H^{-\theta+1-\beta^* t}}\,+\|\partial_t u(t,\cdot)\|_{H^{-\theta-\beta^* t}}\right)\,\leq$$
$$\leq\,C\left(\|u(0,\cdot)\|_{H^{-\theta+1}}+\|\partial_t u(0,\cdot)\|_{H^{-\theta}}+\int_0^T\|Lu(t,\cdot)\|_{H^{-\theta-\beta^* t}}\,dt\right)\,,$$

holds true for all $u\in\mathcal{C}^2([0,T];H^\infty(\mathbb{R}^N))$.

So, it's possible to control the Sobolev norms of solutions to (7) in terms of those of initial data and of the external force only: the price to pay is a loss of derivatives, increasing (linearly) in time.

3 Tools

In this section we will introduce the main tools, from Fourier Analysis, we will need to prove our statement.

First of all, we will recall classical Littlewood-Paley decomposition and some basic results on dyadic analysis. We will also define a different class of Sobolev spaces, of logarithmic type.
Then, we will need to introduce a paradifferential calculus depending on some parameter $\gamma\geq 1$: the main ideas are the same of the classic version, but the introduction of the parameter allows us to perform a more refined analysis. This will be a basic tool to get our result.
After this, we will consider functions with low regularity modulus of continuity. In particular, we will focus on log-Zygmund and log-Lipschitz functions: dyadic decomposition allows us to get some of their properties. Moreover, we will analyse also the convolution of a log-Zygmund

function by a smoothing kernel.
Finally, taking advantage of paradifferential calculus with parameters, we will consider general symbols having such a low regularity in time and space variables. Under suitable hypothesis on such a symbol, we will also get positivity estimates for the associated operator.

3.1 Littlewood-Paley decomposition

`ss:L-P`

Let us first define the so called “Littlewood-Paley decomposition”, based on a non-homogeneous dyadic partition of unity with respect to the Fourier variable. We refer to [1], [2] and [13] for the details.

So, fix a smooth radial function χ supported in the ball $B(0,2)$, equal to 1 in a neighborhood of $B(0,1)$ and such that $r \mapsto \chi(r\,e)$ is nonincreasing over $\mathbb{R}_+$ for all unitary vectors $e \in \mathbb{R}^N$. Set also $\varphi(\xi) = \chi(\xi) - \chi(2\xi)$.

The dyadic blocks $(\Delta_j)_{j\in\mathbb{Z}}$ are defined by[1]

$$\Delta_j := 0 \quad \text{if} \quad j \leq -1, \qquad \Delta_0 := \chi(D) \quad \text{and} \quad \Delta_j := \varphi(2^{-j}D) \quad \text{if} \quad j \geq 1.$$

We also introduce the following low frequency cut-off:

$$S_j u := \chi(2^{-j}D) = \sum_{k\leq j} \Delta_k \quad \text{for} \quad j \geq 0.$$

The following classical properties will be used freely throughout the paper:

- for any $u \in \mathcal{S}'$, the equality $u = \sum_j \Delta_j u$ holds true in $\mathcal{S}'$;
- for all u and v in $\mathcal{S}'$, the sequence $(S_{j-3}u\;\Delta_j v)_{j\in\mathbb{N}}$ is spectrally supported in dyadic annuli.

Let us also mention a fundamental result, which explains, by the so-called *Bernstein's inequalities*, the way derivatives act on spectrally localized functions.

`l:bern` **Lemma 3.1.** *Let $0 < r < R$. A constant C exists so that, for any nonnegative integer k, any couple (p,q) in $[1,\infty]^2$ with $1 \leq p \leq q$ and any function $u \in L^p$, we have, for all $\lambda > 0$,*

$$\begin{aligned} \operatorname{supp}\widehat{u} \subset B(0,\lambda R) \quad &\Longrightarrow \quad \|\nabla^k u\|_{L^q} \,\leq\, C^{k+1}\,\lambda^{k+N\left(\frac{1}{p}-\frac{1}{q}\right)}\,\|u\|_{L^p}\,; \\ \operatorname{supp}\widehat{u} \subset \{\xi\in\mathbb{R}^N \,|\, r\lambda \leq |\xi| \leq R\lambda\} \quad &\Longrightarrow \quad C^{-k-1}\,\lambda^k\|u\|_{L^p} \,\leq\, \|\nabla^k u\|_{L^p} \,\leq\, C^{k+1}\,\lambda^k\|u\|_{L^p}\,. \end{aligned}$$

Let us recall the characterization of (classical) Sobolev spaces via dyadic decomposition: for all $s \in \mathbb{R}$ there exists a constant $C_s > 0$ such that

`est:dyad-Sob`

$$\frac{1}{C_s}\sum_{\nu=0}^{+\infty} 2^{2\nu s}\,\|u_\nu\|^2_{L^2} \,\leq\, \|u\|^2_{H^s} \,\leq\, C_s \sum_{\nu=0}^{+\infty} 2^{2\nu s}\,\|u_\nu\|^2_{L^2}\,, \tag{12}$$

where we have set $u_\nu := \Delta_\nu u$.

So, the H^s norm of a tempered distribution is the same as the l^2 norm of the sequence $(2^{s\nu}\,\|\Delta_\nu u\|_{L^2})_{\nu\in\mathbb{N}}$. Now, one may ask what we get if, in the sequence, we put weights different to the exponential term $2^{s\nu}$. Before answering this question, we introduce some definitions. For the details of the presentiation, we refer also to [9].

Let us set $\Pi(D) := \log(2+|D|)$, i.e. its symbol is $\pi(\xi) := \log(2+|\xi|)$.

`d:log-H^s` **Definition 3.2.** For all $\alpha \in \mathbb{R}$, we define the space $H^{s+\alpha\log}$ as the space $\Pi^{-\alpha}H^s$, i.e.

$$f \,\in\, H^{s+\alpha\log} \quad\Leftrightarrow\quad \Pi^\alpha f \,\in\, H^s \quad\Leftrightarrow\quad \pi^\alpha(\xi)\,\left(1+|\xi|^2\right)^{s/2}\widehat{f}(\xi) \,\in\, L^2\,.$$

[1]Throughout we agree that $f(D)$ stands for the pseudo-differential operator $u \mapsto \mathcal{F}^{-1}(f\,\mathcal{F}u)$.

From the definition, it's obvious that the following inclusions hold for $s_1 > s_2$, $\alpha_1 > \alpha_2 > 0$:

$$H^{s_1+\alpha_1 \log} \hookrightarrow H^{s_1+\alpha_2 \log} \hookrightarrow H^{s_1} \hookrightarrow H^{s_1-\alpha_2 \log} \hookrightarrow H^{s_1-\alpha_1 \log} \hookrightarrow H^{s_2}.$$

We have the following dyadic characterization of these spaces (see [13, Prop. 4.1.11]).

p:log-H **Proposition 3.3.** *Let s, $\alpha \in \mathbb{R}$. A tempered distribution u belongs to the space $H^{s+\alpha \log}$ if and only if:*

(i) for all $k \in \mathbb{N}$, $\Delta_k u \in L^2(\mathbb{R}^N)$;

(ii) set $\delta_k := 2^{ks}(1+k)^\alpha \|\Delta_k u\|_{L^2}$ for all $k \in \mathbb{N}$, the sequence $(\delta_k)_k$ belongs to $l^2(\mathbb{N})$.

Moreover, $\|u\|_{H^{s+\alpha \log}} \sim \|(\delta_k)_k\|_{l^2}$.

Hence, this proposition generalizes property (12).

This new class of Sobolev spaces, which are in a certain sense of logarithmic type, will come into play in our analysis. As a matter of fact, operators associated to log-Zygmund or log-Lipschitz symbols give a logarithmic loss of derivatives. We will clarify in a while what we have just said; first of all, we need to introduce a new version of paradifferential calculus, depending on a parameter $\gamma \geq 1$.

3.2 Paradifferential calculus with parameters

Let us here present the paradifferential calculus depending on some parameter γ. One can find a complete and detailed treatement in [14].

Fix $\gamma \geq 1$ and take a cut-off function $\psi \in \mathcal{C}^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ which verifies the following properties:

- there exist $0 < \varepsilon_1 < \varepsilon_2 < 1$ such that

$$\psi(\eta, \xi) = \begin{cases} 1 & \text{for} \quad |\eta| \leq \varepsilon_1 (\gamma + |\xi|) \\ 0 & \text{for} \quad |\eta| \geq \varepsilon_2 (\gamma + |\xi|) \, ; \end{cases}$$

- for all $(\beta, \alpha) \in \mathbb{N}^N \times \mathbb{N}^N$, there exists a constant $C_{\beta,\alpha}$ such that

$$\left| \partial_\eta^\beta \partial_\xi^\alpha \psi(\eta, \xi) \right| \leq C_{\beta,\alpha} (\gamma + |\xi|)^{-|\alpha|-|\beta|}.$$

For example, if $\gamma = 1$, one can take

$$\psi(\eta, \xi) \equiv \psi_{-3}(\eta, \xi) := \sum_{k=0}^{+\infty} \chi_{k-3}(\eta)\, \varphi_k(\xi),$$

where χ and φ are the localization (in phase space) functions associated to a Littlewood-Paley decomposition, see [13, Ex. 5.1.5]. Similarly, if $\gamma > 1$ it is possible to find a suitable integer $\mu \geq 0$ such that

od_eq:pp_symb

$$\psi(\eta, \xi) \equiv \psi_\mu(\eta, \xi) := \chi_\mu(\eta)\, \chi_{\mu+2}(\xi) + \sum_{k=\mu+3}^{+\infty} \chi_{k-3}(\eta)\, \varphi_k(\xi) \tag{13}$$

is a function with the properties just described.

Define

$$G^\psi(x, \xi) := \left(\mathcal{F}_\eta^{-1} \psi \right)(x, \xi),$$

the inverse Fourier transform of ψ with respect to the variable η.

1:G **Lemma 3.4.** *For all* $(\beta,\alpha)\in\mathbb{N}^N\times\mathbb{N}^N$,

pd_est:G_1 (14)
$$\left\|\partial_x^\beta\partial_\xi^\alpha G^\psi(\cdot,\xi)\right\|_{L^1(\mathbb{R}^N_x)} \;\leq\; C_{\beta,\alpha}\,(\gamma+|\xi|)^{-|\alpha|+|\beta|}\,,$$

pd_est:G_2 (15)
$$\left\||\cdot|\,\log\left(2+\frac{1}{|\cdot|}\right)\,\partial_x^\beta\partial_\xi^\alpha G^\psi(\cdot,\xi)\right\|_{L^1(\mathbb{R}^N_x)} \;\leq\; C_{\beta,\alpha}\,(\gamma+|\xi|)^{-|\alpha|+|\beta|-1}\,\log(1+\gamma+|\xi|)\,.$$

Proof. See [13, Lemma 5.1.7]. □

Thanks to G, we can smooth a symbol a in the x variable and we can define the paradifferential operator associated to a as the pseudodifferential operator associated to this smooth function. We set the classical symbol associated to a to be

$$\sigma_a(x,\xi)\;:=\;(\psi(D_x,\xi)a)\,(x,\xi)\;=\;\Big(G^\psi(\cdot,\xi)\,*_x\,a(\cdot,\xi)\Big)\,(x)\,,$$

and then the paradifferential operator associated to a:

$$T_a\;:=\;\sigma_a(x,D_x)\,,$$

where we have omitted ψ because the definition is independent of it, up to lower order terms.

r:p-prod **Remark 3.5.** Let us note that if $a=a(x)\in L^\infty$ and if we take the cut-off function ψ_{-3}, we get that T_a is actually the usual paraproduct operator. If we take ψ_μ as defined in (13), instead, we get a paraproduct operator which starts from high enough frequencies, which will be indicated with T_a^μ(see [9, Par. 3.3]).

Let us point out that we can also define a γ-dyadic decomposition. First of all, we set

$$\Lambda(\xi,\gamma)\;:=\;\left(\gamma^2\,+\,|\xi|^2\right)^{1/2}\,.$$

Then, taken the usual smooth function χ associated to a Littlewood-Paley decomposition, we define

$$\chi_\nu(\xi,\gamma)\;:=\;\chi\left(2^{-\nu}\Lambda(\xi,\gamma)\right)\,,\quad S_\nu^\gamma\;:=\;\chi_\nu(D_x,\gamma)\,,\quad \Delta_\nu^\gamma\;:=\;S_{\nu+1}^\gamma-S_\nu^\gamma\,.$$

The usual properties of the support of the localization functions still hold, and for all fixed $\gamma\geq1$ and all tempered distributions u, we have

$$u\;=\;\sum_{\nu=0}^{+\infty}\Delta_\nu^\gamma\,u\qquad\text{in}\quad\mathcal{S}'\,.$$

Moreover, with natural modifications in definitions, we can introduce the space $H_\gamma^{s+\alpha\log}$ as the set of tempered distributions for which

$$\|u\|^2_{H_\gamma^{s+\alpha\log}}\;:=\;\int_{\mathbb{R}^N_\xi}\Lambda^{2s}(\xi,\gamma)\,\log^{2\alpha}(1+\gamma+|\xi|)\,|\widehat{u}(\xi)|^2\,d\xi\;<\;+\infty\,.$$

For the details see [14, Par. 2.1.1]. What is important to retain is that, once we fix $\gamma\geq1$ (for example, to obtain positivity of paradifferential operators involved in our computations), all the previous construction is equivalent to the classical one; in particular, the space $H_\gamma^{s+\alpha\log}$ coincides with $H^{s+\alpha\log}$, the respective norms are equivalent and the characterization given by Proposition 3.3 still holds true.

3.3 On log-Lipschitz and log-Zygmund functions

Let us now give the rigorous definitions of the modulus of continuity of funtions we are dealing with, and state some of their properties.

d:LL **Definition 3.6.** A function $f \in L^\infty(\mathbb{R}^N)$ is said to be log-Lipschitz, and we write $f \in LL(\mathbb{R}^N)$, if the quantity

$$|f|_{LL} := \sup_{x,y\in\mathbb{R}^N,\,|y|<1} \left(\frac{|f(x+y) - f(x)|}{|y| \log\left(1 + \frac{1}{|y|}\right)} \right) < +\infty .$$

We define $\|f\|_{LL} := \|f\|_{L^\infty} + |f|_{LL}$.

Let us define also the space of log-Zygmund functions. We will give the general definition in $\mathbb{R}^N$, even if one dimensional case will be the only relevant one for our purposes.

d:LZ **Definition 3.7.** A function $g \in L^\infty(\mathbb{R}^N)$ is said to be log-Zygmund, and we write $g \in LZ(\mathbb{R}^N)$, if the quantity

$$|g|_{LZ} := \sup_{x,y\in\mathbb{R}^N,\,|y|<1} \left(\frac{|g(x+y) + g(x-y) - 2\,g(x)|}{|y| \log\left(1 + \frac{1}{|y|}\right)} \right) < +\infty .$$

We define $\|g\|_{LZ} := \|g\|_{L^\infty} + |g|_{LZ}$.

r:log-g **Remark 3.8.** Let us immediately point out that, by monotonicity of logarithmic function, we can replace the factor $\log(1 + 1/|y|)$ in previous definitions with $\log(1 + \gamma + 1/|y|)$, for all parameters $\gamma \geq 1$. As paradifferential calculus with parameters will play a fundamental role in our computations, it's convenient to perform such a change, and so does also in hypothesis (9) and (10) of section 2.

Let us give a characterization of the space LZ. Recall that the space of Zygmund functions is actually $B^1_{\infty,\infty}$: following the same proof of this case (see e.g. [3]) one can prove next proposition.

p:LZ **Proposition 3.9.** *The space $LZ(\mathbb{R}^N)$ coincides with the logarithmic Besov space $B^{1-\log}_{\infty,\infty}$, i.e. the space of tempered distributions u such that*

est:LZ (16)
$$\sup_{\nu\geq 0} \left(2^\nu\,(\nu+1)^{-1}\,\|\Delta_\nu u\|_{L^\infty}\right) < +\infty .$$

Proof. (i) Let us first consider a $u \in B^{1-\log}_{\infty,\infty}$ and take x and $y \in \mathbb{R}^N$, with $|y| < 1$. For all fixed $n \in \mathbb{N}$ we can write:

$$\begin{aligned} u(x+y) + u(x-y) - 2u(x) &= \sum_{k<n} \left(\Delta_k u(x+y) + \Delta_k u(x-y) - 2\Delta_k u(x)\right) + \\ &\quad + \sum_{k\geq n} \left(\Delta_k u(x+y) + \Delta_k u(x-y) - 2\Delta_k u(x)\right) . \end{aligned}$$

First, we take advantage of the Taylor's formula up to second order to handle the former terms; then, we use property (16). Hence we get

$$\begin{aligned} |u(x+y) + u(x-y) - 2u(x)| &\leq C\,|y|^2 \sum_{k<n} \left\|\nabla^2 \Delta_k u\right\|_{L^\infty} + 4 \sum_{k\geq n} \|\Delta_k u\|_{L^\infty} \\ &\leq C \left(|y|^2 \sum_{k<n} 2^k\,(k+1) + \sum_{k\geq n} 2^{-k}(k+1) \right) \\ &\leq C\,(n+1)\left(|y|^2\,2^n + 2^{-n}\right) . \end{aligned}$$

Now, as $|y| < 1$, the choice $n = 1 + [\log_2(1/|y|)]$ (where with $[\sigma]$ we mean the greatest positive integer less than or equal to σ) completes the proof of the first part.

(ii) Now, given a log-Zygmund function u, we want to estimate the L^∞ norm of its localized part $\Delta_k u$.

Let us recall that applying the operator Δ_k is the same of the convolution with the inverse Fourier transform of the function $\varphi(2^{-k}\cdot)$, which we call $h_k(x) = 2^{kN} h(2^k\cdot)$, where we set $h = \mathcal{F}_\xi^{-1}(\varphi)$. As φ is an even function, so does h; moreover we have

$$\int h(z)\,dz \;=\; \int \mathcal{F}_\xi^{-1}(\varphi)(z)\,dz \;=\; \varphi(\xi)_{|\xi=0} \;=\; 0\,.$$

Therefore, we can write:

$$\Delta_k u(x) \;=\; 2^{kN-1}\int h(2^k y)\,\big(u(x+y)+u(x-y)-2u(x)\big)\,dy\,,$$

and noting that $\sigma \;\mapsto\; \sigma\,\log\left(1+\gamma+1/\sigma\right)$ is increasing completes the proof of the second part.

$\square$

From definitions 3.6 and 3.7, it's obvious that $LL(\mathbb{R}^N) \;\hookrightarrow\; LZ(\mathbb{R}^N)$: Proposition 3.3 of [8] explains this property in terms of dyadic decomposition.

p:dyadic-LL **Proposition 3.10.** *There exists a constant C such that, for all $a \in LL(\mathbb{R}^N)$ and all integers $k>0$, we have*

est:a_k (17)
$$\left\|\Delta_k a\right\|_{L^\infty} \;\leq\; C\,(k+1)\,2^{-k}\,\|a\|_{LL}\,.$$

Moreover, for all $k\in\mathbb{N}$ we have

est:a-S_ka (18)
est:S_ka (19)
$$\begin{aligned} \left\|a \;-\; S_k a\right\|_{L^\infty} \;&\leq\; C\,(k+1)\,2^{-k}\,\|a\|_{LL} \\ \left\|S_k a\right\|_{\mathcal{C}^{0,1}} \;&\leq\; C\,(k+1)\,\|a\|_{LL}\,. \end{aligned}$$

r:LL **Remark 3.11.** Note that, again from Proposition 3.3 of [8], property (19) is a characterization of the space LL.

Using dyadic characterization of the space LZ and following the same ideas of the proof of Proposition 3.9, we can prove the following property. For our purposes, it's enough to consider a log-Zygmund function a depending only on the time variable t, but the same reasonning holds also in higher dimensions.

l:LZ **Lemma 3.12.** *For all $a\in LZ(\mathbb{R})$, there exists a constant C, depending only on the LZ norm of a, such that, for all $\gamma\geq 1$ and all $0<|\tau|<1$ one has*

est:lip-LZ (20)
$$\sup_{t\in\mathbb{R}}\left|a(t+\tau)-a(t)\right| \;\leq\; C\,|\tau|\,\log^2\!\left(1+\gamma+\frac{1}{|\tau|}\right)\,.$$

Proof. As done in proving Proposition 3.9, for all $n\in\mathbb{N}$ we can write

$$a(t+\tau)-a(t) \;=\; \sum_{k<n}\left(\Delta_k a(t+\tau)-\Delta_k a(t)\right) \;+\; \sum_{k\geq n}\left(\Delta_k a(t+\tau)-\Delta_k a(t)\right)\,,$$

where, obviously, the localization in frequencies is done with respect to the time variable. For the former terms we use the mean value theorem, while for the latter ones we use characterization (17); hence, we get

$$\begin{aligned} \left|a(t+\tau)-a(t)\right| \;&\leq\; \sum_{k<n}\left\|\frac{d}{dt}\Delta_k a\right\|_{L^\infty}|\tau| \;+\; 2\sum_{k\geq n}\left\|\Delta_k a\right\|_{L^\infty} \\ &\leq\; C\left(n^2\,|\tau| \;+\; \sum_{k\geq n}2^{-k}k\right)\,. \end{aligned}$$

The series in the right-hand side of the previous inequality can be bounded, up to a multiplicative constant, by $2^{-n}n$; therefore

$$|a(t+\tau) - a(t)| \;\leq\; C\, n\, \left(n\,|\tau| \,+\, 2^{-n}\right)\,,$$

and the choice $n = 1 + [\log_2(1/|\tau|)]$ completes the proof. □

Now, given a log-Zygmund function $a(t)$, we can regularize it by convolution. So, take an even function $\rho \in \mathcal{C}_0^\infty(\mathbb{R}_t)$, $0 \leq \rho \leq 1$, whose support is contained in the interval $[-1,1]$ and such that $\int \rho(t)dt = 1$, and define the mollifier kernel

$$\rho_\varepsilon(t) \;:=\; \frac{1}{\varepsilon}\,\rho\!\left(\frac{t}{\varepsilon}\right) \qquad\qquad \forall\,\varepsilon \in \,]0,1]\,.$$

We smooth the function a setting, for all $\varepsilon \in \,]0,1]$,

eq:a^e

$$a_\varepsilon(t) \;:=\; \left(\rho_\varepsilon \,*\, a\right)(t) \;=\; \int_{\mathbb{R}_s} \rho_\varepsilon(t-s)\,a(s)\,ds\,. \tag{21}$$

The following proposition holds.

p:LZ-reg

Proposition 3.13. *Let a be a log-Zygmund function. For all $\gamma \geq 1$, there exist constants C_γ such that*

est:a^e-a

$$|a_\varepsilon(t) - a(t)| \;\leq\; C_\gamma\,\|a\|_{LZ}\;\varepsilon\,\log\left(1+\gamma+\frac{1}{\varepsilon}\right) \tag{22}$$

est:d_t-a^e

$$|\partial_t a_\varepsilon(t)| \;\leq\; C_\gamma\,\|a\|_{LZ}\,\log^2\left(1+\gamma+\frac{1}{\varepsilon}\right) \tag{23}$$

est:d_tt-a^e

$$\left|\partial_t^2 a_\varepsilon(t)\right| \;\leq\; C_\gamma\,\|a\|_{LZ}\;\frac{1}{\varepsilon}\,\log\left(1+\gamma+\frac{1}{\varepsilon}\right)\,. \tag{24}$$

Proof. For first and third inequalities, the proof is the same as in [7]. We have to pay attention only to (23). As ρ' has null integral, the relation

$$\partial_t a_\varepsilon(t) \;=\; \frac{1}{\varepsilon^2}\int_{|s|\leq\varepsilon} \rho'\left(\frac{s}{\varepsilon}\right)\,\left(a(t-s) - a(t)\right)ds$$

holds, and hence, taking advantage of (20), it implies

$$|\partial_t a_\varepsilon(t)| \;\leq\; \frac{C}{\varepsilon^2}\int_{|s|\leq\varepsilon}\left|\rho'\left(\frac{s}{\varepsilon}\right)\right|\,|s|\,\log^2\left(1+\gamma+\frac{1}{|s|}\right)ds\,.$$

Observing that the function $\sigma \mapsto \sigma\log^2(1+\gamma+1/\sigma)$ is increasing in the interval $[0,1]$, and so does in $[0,\varepsilon]$, allows us to complete the proof. □

3.4 Low regularity symbols and calculus

For the analysis of our strictly hyperbolic problem, it's important to pass from $LZ_t - LL_x$ functions to more general symbols in variables (t,x,ξ) which have this same regularity in t and x.

We want to investigate properties of such these symbols and of the associated operators. For reasons which will appear clear in the sequel, we will have to take advantage not of the classical paradifferential calculus, but of the calculus with parameters.

So, let us take a symbol $a(t,x,\xi)$ of order $m \geq 0$, such that a is log-Zygmund in t and log-Lipschitz in x, uniformly with respect to the other variables. Now we smooth a with respect to time, as done in (21). Next lemma provides us some estimates on classical symbols associated to a_ε and its time derivatives.

l:lzll-symb **Lemma 3.14.** *The classical symbols associated to a_ε and its time derivatives satisfy:*

$$
\begin{aligned}
\left|\partial_\xi^\alpha \sigma_{a_\varepsilon}\right| &\leq C_\alpha\,(\gamma+|\xi|)^{m-|\alpha|} \\
\left|\partial_x^\beta \partial_\xi^\alpha \sigma_{a_\varepsilon}\right| &\leq C_{\beta,\alpha}\,(\gamma+|\xi|)^{m-|\alpha|+|\beta|-1}\,\log\left(1+\gamma+|\xi|\right) \\
\left|\partial_\xi^\alpha \sigma_{\partial_t a_\varepsilon}\right| &\leq C_\alpha\,(\gamma+|\xi|)^{m-|\alpha|}\,\log^2\!\left(1+\gamma+\frac{1}{\varepsilon}\right) \\
\left|\partial_x^\beta \partial_\xi^\alpha \sigma_{\partial_t a_\varepsilon}\right| &\leq C_{\beta,\alpha}\,\,(\gamma+|\xi|)^{m-|\alpha|+|\beta|-1}\,\log\left(1+\gamma+|\xi|\right)\,\frac{1}{\varepsilon} \\
\left|\partial_\xi^\alpha \sigma_{\partial_t^2 a_\varepsilon}\right| &\leq C_\alpha\,(\gamma+|\xi|)^{m-|\alpha|}\,\log\!\left(1+\gamma+\frac{1}{\varepsilon}\right)\,\frac{1}{\varepsilon} \\
\left|\partial_x^\beta \partial_\xi^\alpha \sigma_{\partial_t^2 a_\varepsilon}\right| &\leq C_{\beta,\alpha}\,\,(\gamma+|\xi|)^{m-|\alpha|+|\beta|-1}\,\log\left(1+\gamma+|\xi|\right)\,\frac{1}{\varepsilon^2}\,.
\end{aligned}
$$

Proof. The first inequality is a quite easy computation.

For the second one, we have to observe that

$$
\int \partial_i G(x-y,\xi)dx \;=\; \int \partial_i G(z,\xi)dz \;=\; \int \mathcal{F}_\eta^{-1}\left(\eta_i\,\psi(\eta,\xi)\right)dz \;=\; \left(\eta_i\,\psi(\eta,\xi)\right)_{|\eta=0} \;=\; 0\,.
$$

So, we have

$$
\partial_i \sigma_{a_\varepsilon} \;=\; \int \partial_i G(y,\xi)\left(a_\varepsilon(t,x-y,\xi)-a_\varepsilon(t,x,\xi)\right)dy\,,
$$

and from this, remembering lemma 3.4, we get the final control.

The third estimate immediately follows from the hypothesis on a and from (23).

Moreover, in the case of space derivatives, we can take advantage once again of the fact that $\partial_i G$ has null integral:

$$
\begin{aligned}
\partial_i \sigma_{\partial_t a_\varepsilon} &= \int \partial_i G(x-y,\xi)\,\partial_t a_\varepsilon(t,y,\xi)\,dy \\
&= \int_{\mathbb{R}_s} \frac{1}{\varepsilon^2}\,\rho'\left(\frac{t-s}{\varepsilon}\right)\left(\int_{\mathbb{R}_y^N} \partial_i G(y,\xi)\left(a(s,x-y,\xi)-a(s,x,\xi)\right)dy\right)ds\,.
\end{aligned}
$$

Hence, the estimate follows from the log-Lipschitz continuity hypothesis and from inequality (15).

About the $\partial_t^2 a_\varepsilon$ term, the first estimate comes from (24), while for the second one we argue as before:

$$
\begin{aligned}
\partial_i \sigma_{\partial_t^2 a_\varepsilon} &= \int \partial_i G(x-y,\xi)\,\partial_t^2 a_\varepsilon(t,y,\xi)\,dy \\
&= \int_{\mathbb{R}_y^N} \partial_i G(x-y,\xi)\,\frac{1}{\varepsilon^3}\left(\int_{\mathbb{R}_s} \rho''\left(\frac{t-s}{\varepsilon}\right)\left(a(s,y,\xi)-a(s,x,\xi)\right)ds\right)dy \\
&= \frac{1}{\varepsilon^3}\int_{\mathbb{R}_s} \rho''\left(\frac{t-s}{\varepsilon}\right)\left(\int_{\mathbb{R}_y^N} \partial_i G(y,\xi)\left(a(s,x-y,\xi)-a(s,x,\xi)\right)dy\right)ds\,,
\end{aligned}
$$

and the thesis follows again from log-Lipschitz continuity and from (15). □

Note that first and second inequalities are satisfied also by the symbol a (not smoothed in time).

Now let us recall some basic facts on symbolic calculus, which follow from previous lemma.

p:symb-calc **Proposition 3.15.** *(i) Let a be a symbol of order m which is LL in the x variable. Then T_a maps $H_\gamma^{s+\alpha\log}$ into $H_\gamma^{s-m+\alpha\log}$.*

(ii) *Let us take two symbols a, b of order m and m' respectively. Suppose that a, b are LL in the x variable. The composition of the associated operators can be approximated by the symbol associated to the product of these symbols, up to a remainder term:*

$$T_a \circ T_b \,=\, T_{ab} \,+\, R\,.$$

The remainder operator R maps $H^{s+\alpha\log}_{\gamma}$ into $H^{s-m-m'+1+(\alpha+1)\log}_{\gamma}$.

(iii) *Let a be a symbol of order m which is LL in the x variable. The adjoint (over L^2) operator of T_a is, up to a remainder operator, $T_{\overline{a}}$. The remainder operator maps $H^{s+\alpha\log}_{\gamma}$ into $H^{s-m+1+(\alpha+1)\log}_{\gamma}$.*

Let us end this subsection stating a basic positivity estimate. In this situation, paradifferential calculus with parameters comes into play.

p:pos **Proposition 3.16.** *Let $a(t,x,\xi)$ be a symbol of order m, which is LL in the x variable and such that*

$$\mathrm{Re}\,(a(t,x,\xi)) \,\geq\, \lambda_0\,\left(\gamma+|\xi|\right)^m\,.$$

Then, there exists a constant λ_1, depending only on the LL norm of a and on λ_0, such that, for γ large enough, one has

$$\mathrm{Re}\,(T_a u,u)_{L^2} \,\geq\, \lambda_1\,\|u\|^2_{H^{m/2}_{\gamma}}\,.$$

Proof. The result is an immediate consequence of Theorem 2.19 of [14]. □

r:pos **Remark 3.17.** Let us note the following fact, which comes again from Theorem 2.19 of [14]. If the positive symbol a has low regularity in time and we smooth it by convolution with respect to this variable, we obtain a family $\left(a_\varepsilon\right)_\varepsilon$ of positive symbols, with same constant λ_0. Now, all the paradifferential operators associated to these symbols will be positive operators, uniformly in ε: i.e. the constant λ_1 of previous inequality can be choosen independently of ε.

Let us observe that previous proposition generalizes Corollary 3.12 of [9] (stated for the paraproduct by a positive LL function) to the more general case of a paradifferential operator with a strictly positive symbol of order m.

Finally, thanks to Theorem 2.18 of [14] about the remainder operator for the adjoint, we have the following corollary, which turns out to be fundamental in our energy estimates.

c:pos **Corollary 3.18.** *Let a be a positive symbol of order 1 and suppose that a is LL in the x variable. Then there exists $\gamma\geq 1$, depending only on the symbol a, such that*

$$\|T_a u\|_{L^2} \,\sim\, \|\nabla u\|_{L^2}$$

for all $u\in H^1(\mathbb{R}^N)$.

4 Proof of the energy estimate for L

Finally, we are able to tackle the proof of Theorem 2.1. We argue in a stadard way: first of all, we define an energy associated to a solution of equation (7), and then we prove estimates on its time derivative in terms of the energy itself. In the end, we will close the estimates thanks to Gronwall Lemma.

The key idea to the proof is to split the total energy into localized components e_ν, each one of them associated to the dyadic block $\Delta_\nu u$, and then to put all these pieces together (see also [8] and [7]). Let us see the proof into details.

4.1 Approximate and total energy

Let us first regularize coefficients a_{ij} in the time variable by convolution, as done in (21), and let us define the 0-th order symbol

$$\alpha_\varepsilon(t,x,\xi) := \left(\gamma^2+|\xi|^2\right)^{-1/2}\left(\gamma^2+\sum_{i,j}a_{ij,\varepsilon}(t,x)\,\xi_i\,\xi_j\right)^{1/2}.$$

We take $\varepsilon = 2^{-\nu}$ (see also [8] and [7]), and (for notation convenience) we will miss out the ε.

Before going on, let us fix a real number $\gamma \geq 1$, which will depend only on λ_0 and on the $\sup_{i,j}\|a_{ij}\|_{LL_x}$, such that (see Corollary 3.18)

est:param

$$\|T_{\alpha^{-1/2}}\,w\|_{L^2} \geq \frac{\lambda_0}{2}\,\|w\|_{L^2} \qquad \text{and} \qquad \left\|T_{\alpha^{1/2}(\gamma^2+|\xi|^2)^{1/2}}\,w\right\|_{L^2} \geq \frac{\lambda_0}{2}\,\|\nabla w\|_{L^2} \tag{25}$$

for all $w \in H^\infty$. Let us remark that the choice of γ is equivalent to the choice of the parameter μ in (13) (see Remark 3.5) and from now on, we will consider paraproducts starting from this μ, according to definition (13), even if we will omit it in the notations.

Consider in (7) a function $u \in \mathcal{C}^2([0,T_0];H^\infty)$. We want to get energy estimate for u. We rewrite the equation using paraproduct operators by the coefficients a_{ij}:

$$\partial_t^2 u = \sum_{i,j}\partial_i\left(a_{ij}\,\partial_j u\right) + Lu = \sum_{i,j}\partial_i\left(T_{a_{ij}}\partial_j u\right) + \widetilde{L}u,$$

where $\widetilde{L}u = Lu + \sum_{i,j}\partial_i\big((a_{ij}-T_{a_{ij}})\partial_j u\big)$. Let us apply operator Δ_ν: we get

eq:loc

$$\partial_t^2 u_\nu = \sum_{i,j}\partial_i\left(T_{a_{ij}}\,\partial_j u_\nu\right) + \sum_{i,j}\partial_i\left(\left[\Delta_\nu, T_{a_{ij}}\right]\,\partial_j u\right) + (\widetilde{L}u)_\nu, \tag{26}$$

where $u_\nu = \Delta_\nu u$, $(\widetilde{L}u)_\nu = \Delta_\nu(\widetilde{L}u)$ and $\left[\Delta_\nu, T_{a_{ij}}\right]$ is the commutator between Δ_ν and the para-multiplication by a_{ij}.

Now, we set

$$\begin{aligned} v_\nu(t,x) &:= T_{\alpha^{-1/2}}\,\partial_t u_\nu - T_{\partial_t(\alpha^{-1/2})}\,u_\nu \\ w_\nu(t,x) &:= T_{\alpha^{1/2}(\gamma^2+|\xi|^2)^{1/2}}\,u_\nu \\ z_\nu(t,x) &:= u_\nu \end{aligned}$$

and we define the approximate energy associated to the ν-th component of u (as done in [7]):

eq:appr_en

$$e_\nu(t) := \|v_\nu(t)\|_{L^2}^2 + \|w_\nu(t)\|_{L^2}^2 + \|z_\nu(t)\|_{L^2}^2\,. \tag{27}$$

r:w_nu

Remark 4.1. Let us note that, thanks to hypothesis (8) and our choice of the frequence μ from which defining the paraproduct, we have that $\|w_\nu(t)\|_{L^2}^2 \sim \|\nabla u_\nu\|_{L^2}^2 \sim 2^{2\nu}\,\|u_\nu\|_{L^2}^2$.

Now, we fix a $\theta \in]0,1[$, as required in hypothesis, and we take a $\beta > 0$ to be chosen later; we can define the total energy associated to the solution u to be the quantity

eq:tot_E

$$E(t) := \sum_{\nu\geq 0} e^{-2\beta(\nu+1)t}\,2^{-2\nu\theta}\,e_\nu(t)\,. \tag{28}$$

It's not difficult to prove (see also inequality (32) below) that there exist constants C_θ and C'_θ, depending only on the fixed θ, for which one has:

est:E(0)

$$(E(0))^{1/2} \leq C_\theta\left(\|\partial_t u(0)\|_{H^{-\theta}} + \|u(0)\|_{H^{-\theta+1}}\right) \tag{29}$$

est:E(t)

$$(E(t))^{1/2} \geq C'_\theta\left(\|\partial_t u(t)\|_{H^{-\theta-\beta^* t}} + \|u(t)\|_{H^{-\theta+1-\beta^* t}}\right), \tag{30}$$

where we have set $\beta^* = \beta\,(\log 2)^{-1}$.

4.2 Time derivative of the approximate energy

We want to find an estimate on time derivative of the energy in order to get a control on it by Gronwall Lemma. Let us start analysing each term of (27).

4.2.1 z_ν term

For the third term we have:

eq:d_t-z
$$\frac{d}{dt}\|z_\nu(t)\|^2_{L^2} \;=\; 2\,\mathrm{Re}\,(u_\nu\,,\,\partial_t u_\nu)_{L^2}\;. \tag{31}$$

Now, we have to control the term $\partial_t u_\nu$: using positivity of operator $T_{\alpha^{-1/2}}$, we have

est:d_t-u_nu
$$\|\partial_t u_\nu\|_{L^2} \;\leq\; C\,\|T_{\alpha^{-1/2}}\partial_t u_\nu\|_{L^2} \;\leq\; C\left(\|v_\nu\|_{L^2} + \left\|T_{\partial_t\left(\alpha^{-1/2}\right)}u_\nu\right\|_{L^2}\right) \;\leq\; C\,(e_\nu)^{1/2}\;. \tag{32}$$

So, we get the estimate:

est:d_t-z
$$\frac{d}{dt}\|z_\nu(t)\|^2_{L^2} \;\leq\; C\,e_\nu(t)\,. \tag{33}$$

4.2.2 v_ν term

Straightforward computations show that

$$\partial_t v_\nu(t,x) \;=\; T_{\alpha^{-1/2}}\partial_t^2 u_\nu \;-\; T_{\partial_t^2(\alpha^{-1/2})}u_\nu\,.$$

Therefore, keeping in mind relation (26), we get:

eq:d_t-v
$$\begin{aligned}\frac{d}{dt}\|v_\nu(t)\|^2_{L^2} \;=\;& -2\,\mathrm{Re}\left(v_\nu\,,\,T_{\partial_t^2(\alpha^{-1/2})}u_\nu\right)_{L^2} + 2\sum_{i,j}\mathrm{Re}\,\left(v_\nu\,,\,T_{\alpha^{-1/2}}\partial_i\left(T_{a_{ij}}\,\partial_j u_\nu\right)\right)_{L^2} + \\ & +2\sum_{i,j}\mathrm{Re}\,\left(v_\nu\,,\,T_{\alpha^{-1/2}}\partial_i\left[\Delta_\nu, T_{a_{ij}}\right]\partial_j u\right)_{L^2} + \\ & +2\,\mathrm{Re}\left(v_\nu\,,\,T_{\alpha^{-1/2}}\left(\widetilde{L}u\right)_\nu\right)_{L^2}.\end{aligned} \tag{34}$$

Obviously, we have

est:Lu_nu
$$\left|2\,\mathrm{Re}\left(v_\nu\,,\,T_{\alpha^{-1/2}}\left(\widetilde{L}u\right)_\nu\right)_{L^2}\right| \leq C\,(e_\nu)^{1/2}\left\|\left(\widetilde{L}u\right)_\nu\right\|_{L^2}\,, \tag{35}$$

while from Lemma 3.14 we immediately get

est:zeta
$$\begin{aligned}\left|2\,\mathrm{Re}\left(v_\nu\,,\,T_{\partial_t^2(\alpha^{-1/2})}u_\nu\right)_{L^2}\right| \;&\leq\; C\,\|v_\nu\|_{L^2}\,\log\!\left(1+\gamma+\frac{1}{\varepsilon}\right)\frac{1}{\varepsilon}\,\|u_\nu\|_{L^2} \\ &\leq\; C\,(\nu+1)\,e_\nu\,,\end{aligned} \tag{36}$$

where we have used the fact that $\varepsilon = 2^{-\nu}$. The other two terms of (34) will be treated later.

4.2.3 w_ν term

We now derive w_ν with respect to the time variable: thanks to a broad use of symbolic calculus, we get the following sequence of equalities:

eq:d_t-w

$$
\begin{aligned}
(37)\frac{d}{dt}\|w_\nu\|^2_{L^2} &= 2\,\mathrm{Re}\left(T_{\partial_t(\alpha^{1/2})(\gamma^2+|\xi|^2)^{1/2}}u_\nu\,,\,w_\nu\right)_{L^2} + 2\,\mathrm{Re}\left(T_{\alpha^{1/2}(\gamma^2+|\xi|^2)^{1/2}}\partial_t u_\nu\,,\,w_\nu\right)_{L^2}\\
&= 2\,\mathrm{Re}\left(T_{\alpha(\gamma^2+|\xi|^2)^{1/2}}T_{-\partial_t(\alpha^{-1/2})}u_\nu\,,\,w_\nu\right)_{L^2} + 2\,\mathrm{Re}\left(R_1u_\nu\,,\,w_\nu\right)_{L^2} +\\
&\quad + 2\,\mathrm{Re}\left(T_{\alpha(\gamma^2+|\xi|^2)^{1/2}}T_{\alpha^{-1/2}}\partial_t u_\nu\,,\,w_\nu\right)_{L^2} + 2\,\mathrm{Re}\left(R_2\partial_t u_\nu\,,\,w_\nu\right)_{L^2}\\
&= 2\,\mathrm{Re}\left(v_\nu\,,\,T_{\alpha(\gamma^2+|\xi|^2)^{1/2}}w_\nu\right)_{L^2} + 2\,\mathrm{Re}\left(v_\nu\,,\,R_3w_\nu\right)_{L^2} +\\
&\quad + 2\,\mathrm{Re}\left(R_1u_\nu\,,\,w_\nu\right)_{L^2} + 2\,\mathrm{Re}\left(R_2\partial_t u_\nu\,,\,w_\nu\right)_{L^2}\\
&= 2\,\mathrm{Re}\left(v_\nu\,,\,T_{\alpha^{-1/2}}T_{\alpha^{3/2}(\gamma^2+|\xi|^2)^{1/2}}w_\nu\right)_{L^2} + 2\,\mathrm{Re}\left(v_\nu\,,\,R_4w_\nu\right)_{L^2} +\\
&\quad + 2\,\mathrm{Re}\left(v_\nu\,,\,R_3w_\nu\right)_{L^2} + 2\,\mathrm{Re}\left(R_1u_\nu\,,\,w_\nu\right)_{L^2} + 2\,\mathrm{Re}\left(R_2\partial_t u_\nu\,,\,w_\nu\right)_{L^2}\\
&= 2\,\mathrm{Re}\left(v_\nu\,,\,T_{\alpha^{-1/2}}T_{\alpha^2(\gamma^2+|\xi|^2)}u_\nu\right)_{L^2} +\\
&\quad + 2\,\mathrm{Re}\left(v_\nu\,,\,T_{\alpha^{-1/2}}R_5u_\nu\right)_{L^2} + 2\,\mathrm{Re}\left(v_\nu\,,\,R_4w_\nu\right)_{L^2} +\\
&\quad + 2\,\mathrm{Re}\left(v_\nu\,,\,R_3w_\nu\right)_{L^2} + 2\,\mathrm{Re}\left(R_1u_\nu\,,\,w_\nu\right)_{L^2} + 2\,\mathrm{Re}\left(R_2\partial_t u_\nu\,,\,w_\nu\right)_{L^2}\,.
\end{aligned}
$$

The important fact is that remainder terms are not bad and can be controlled in terms of approximate energy. As a matter of facts, taking advantage of Proposition 3.15 and Lemma 3.14, we get the following estimates.

- R_1 has principal symbol equal to $\partial_\xi\left(\alpha(\gamma^2+|\xi|^2)^{1/2}\right)\partial_x\partial_t(\alpha^{-1/2})$, so

est:R_1

$$
(38)\qquad \left|2\,\mathrm{Re}\left(R_1u_\nu\,,\,w_\nu\right)_{L^2}\right| \;\leq\; C\,(\nu+1)\,e_\nu\,.
$$

- The principal symbol of R_2 is instead $\partial_\xi\left(\alpha(\gamma^2+|\xi|^2)^{1/2}\right)\partial_x(\alpha^{-1/2})$, so, remembering also the control on $\|\partial_t u_\nu\|_{L^2}$, we have:

est:R_2

$$
(39)\qquad \left|2\,\mathrm{Re}\left(R_2\partial_t u_\nu\,,\,w_\nu\right)_{L^2}\right| \leq C\,\nu\,(e_\nu)^{1/2}\,\|w_\nu\|_{L^2} \leq C\,(\nu+1)\,e_\nu\,.
$$

- Symbolic calculus tells us that the principal part of R_3 is given by $\partial_\xi\partial_x\left(\alpha(\gamma^2+|\xi|^2)^{1/2}\right)$, therefore

est:R_3

$$
(40)\qquad \left|2\,\mathrm{Re}\left(v_\nu\,,\,R_3w_\nu\right)_{L^2}\right| \leq C\,\|v_\nu\|_{L^2}\,\nu\,\|w_\nu\|_{L^2} \leq C\,(\nu+1)\,e_\nu\,.
$$

- Now, R_4 has $\partial_\xi\left(\alpha^{-1/2}\right)\partial_x\left(\alpha^{3/2}(\gamma^2+|\xi|^2)^{1/2}\right)$ as principal symbol, so

est:R_4

$$
(41)\qquad \left|2\,\mathrm{Re}\left(v_\nu\,,\,R_4w_\nu\right)_{L^2}\right| \leq C\,\|v_\nu\|_{L^2}\,\nu\,\|w_\nu\|_{L^2} \leq C\,(\nu+1)\,e_\nu\,.
$$

- Finally, R_5 is given, at the higher order, by the product of symbols $\partial_\xi\left(\alpha^{3/2}(\gamma^2+|\xi|^2)^{1/2}\right)$ and $\partial_x\left(\alpha^{1/2}(\gamma^2+|\xi|^2)^{1/2}\right)$, and so we get

est:R_5

$$
(42)\qquad \left|2\,\mathrm{Re}\left(v_\nu\,,\,T_{\alpha^{-1/2}}R_5u_\nu\right)_{L^2}\right| \leq C\,\|v_\nu\|_{L^2}\,2^\nu\,\nu\,\|u_\nu\|_{L^2} \leq C\,(\nu+1)\,e_\nu\,.
$$

4.2.4 Principal part of the operator L

Now, thanks to previous computations, it's natural to pair up the second term of (34) with the last one of the last equality of (37). As α is a symbol of order 0, we have

$$
\left|2\,\mathrm{Re}\left(v_\nu\,,\,T_{\alpha^{-1/2}}\sum_{i,j}\partial_i\left(T_{a_{ij}}\partial_j u_\nu\right)\right)_{L^2} + 2\,\mathrm{Re}\left(v_\nu\,,\,T_{\alpha^{-1/2}}T_{\alpha^2(\gamma^2+|\xi|^2)}u_\nu\right)_{L^2}\right| \leq C\,\|v_\nu\|_{L^2}\,\|\zeta_\nu\|_{L^2}\,,
$$

where we have set

eq:zeta_nu

$$\zeta_\nu \;:=\; T_{\alpha^2(\gamma^2+|\xi|^2)}u_\nu \,+\, \sum_{i,j}\partial_i\left(T_{a_{ij}}\,\partial_j u_\nu\right) \;=\; \sum_{ij} T_{a_{ij,\varepsilon}\xi_i\xi_j+\gamma^2}u_\nu \,+\, \partial_i\left(T_{a_{ij}}\,\partial_j u_\nu\right)\,. \tag{43}$$

We remark that

$$\partial_i\left(T_{a_{ij}}\,\partial_j u_\nu\right) = T_{\partial_i a_{ij}}\partial_j u_\nu \,-\, T_{a_{ij}\xi_i\xi_j}u_\nu,$$

where, with a little abuse of notations, we have written the derivative $\partial_i a_{ij}$ meaning that we are taking the derivative of the classical symbol associated to a_{ij}.

First of all, we have that

est:T_1

$$\begin{aligned}\left\|T_{\partial_i a_{ij}}\partial_j u_\nu\right\|_{L^2} &\le \|S_\mu\,\partial_i a_{ij}\|_{L^\infty}\,\|S_{\mu+2}\partial_j u_\nu\|_{L^2} \,+\, \sum_{k\ge\mu+3}\|\nabla S_{k-3}a_{ij}\|_{L^\infty}\,\|\Delta_k\nabla u_\nu\|_{L^2} \\ &\le C\,(\mu+1)\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)\|\nabla u_\nu\|_{L^2} \,+ \\ &\qquad + \sum_{k\ge\mu+3\,,\,k\sim\nu}(k+1)\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)\|\nabla\Delta_k u_\nu\|_{L^2} \\ &\le C_\mu\,(\nu+1)\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)(e_\nu)^{1/2}\,,\end{aligned} \tag{44}$$

where μ is the parameter fixed in (13) and we have also used (19). Next, we have to control the term

$$T_{a_{ij,\varepsilon}\xi_i\xi_j+\gamma^2}u_\nu \,-\, T_{a_{ij}\xi_i\xi_j}u_\nu \;=\; T_{(a_{ij,\varepsilon}-a_{ij})\xi_i\xi_j}u_\nu \,+\, T_{\gamma^2}u_\nu\,.$$

It's easy to see that

$$\left\|T_{(a_{ij,\varepsilon}-a_{ij})\xi_i\xi_j}u_\nu\right\|_{L^2} \le C\,\varepsilon\,\log\left(1+\frac{1}{\varepsilon}\right)2^\nu\,\|\nabla u_\nu\|_{L^2}\,,$$

and so, keeping in mind that $\varepsilon = 2^{-\nu}$,

est:delta-T

$$\left\|T_{(a_{ij,\varepsilon}-a_{ij})\xi_i\xi_j+\gamma^2}u_\nu\right\|_{L^2} \le C_\gamma\,(\nu+1)\,(e_\nu)^{1/2}\,. \tag{45}$$

Therefore, from (44) and (45) we finally get

est:sec-ord

$$\left|2\,\mathrm{Re}\left(v_\nu, T_{\alpha^{-1/2}}\sum_{i,j}\partial_i\left(T_{a_{ij}}\partial_j u_\nu\right)\right)_{L^2} + 2\,\mathrm{Re}\,\left(v_\nu, T_{\alpha^{-1/2}}T_{\alpha^2(\gamma^2+|\xi|^2)}u_\nu\right)_{L^2}\right| \le C\,(\nu+1)\,e_\nu\,, \tag{46}$$

where the constant C depends on the log-Lipschitz norm of coefficients a_{ij} of the operator L and on the fixed parameters μ and γ.

To sum up, from inequalities (33), (35), (36) and (46) and from estimates of remainder terms (38)-(42), we can conclude that

est:d_t-e

$$\begin{aligned}\frac{d}{dt}e_\nu(t) \;&\le\; C_1\,(\nu+1)\,e_\nu(t) \,+\, C_2\,(e_\nu(t))^{1/2}\left\|\left(\widetilde{L}u\right)_\nu(t)\right\|_{L^2} \,+ \\ &\qquad + \left|2\sum_{i,j}\mathrm{Re}\,\left(v_\nu\,,\,T_{\alpha^{-1/2}}\partial_i\left[\Delta_\nu, T_{a_{ij}}\right]\partial_j u\right)_{L^2}\right|\,.\end{aligned} \tag{47}$$

ss:comm

4.3 Commutator term

We want to estimate the quantity

$$\left|\sum_{i,j} \mathrm{Re}\,\left(v_\nu\,,\, T_{\alpha^{-1/2}}\partial_i\left[\Delta_\nu, T_{a_{ij}}\right]\partial_j u\right)_{L^2}\right|.$$

We start remarking that

$$[\Delta_\nu, T_{a_{ij}}]w = [\Delta_\nu, S_\mu a]S_{\mu+2}w + \sum_{k=\mu+3}^{+\infty}[\Delta_\nu, S_{k-3}a_{ij}]\,\Delta_k w,$$

where μ is fixed, as usual (see Remark 3.5). In fact Δ_ν and Δ_k commute so that

$$\Delta_\nu(S_\mu a_{ij}S_{\mu+2}w) - S_\mu a_{ij}(S_{\mu+2}\Delta_\nu w) = \Delta_\nu(S_\mu a_{ij}S_{\mu+2}w) - S_\mu a_{ij}\Delta_\nu(S_{\mu+2}w),$$

and similarly

$$\Delta_\nu(S_{k-3}a_{ij}\Delta_k w) - S_{k-3}a_{ij}\Delta_k(\Delta_\nu w) = \Delta_\nu(S_{k-3}a_{ij}\Delta_k w) - S_{k-3}a_{ij}\Delta_\nu(\Delta_k w).$$

Consequently, taking into account also that $S_{\mu+2}$ and Δ_k commute with ∂_j, we have

$$\partial_i\left([\Delta_\nu, T_{a_{ij}}]\,\partial_j u\right) \;=\; \partial_i\left([\Delta_\nu, S_\mu a_{ij}]\,\partial_j(S_{\mu+2}u)\right) \;+\; \partial_i\left(\sum_{k=\mu+3}^{+\infty}[\Delta_\nu, S_{k-3}a_{ij}]\,\partial_j(\Delta_k u)\right).$$

Let's consider first the term

$$\partial_i\left([\Delta_\nu, S_\mu a_{ij}]\,\partial_j(S_{\mu+2}u)\right).$$

Looking at the support of the Fourier transform of $[\Delta_\nu, S_\mu a_{ij}]\,\partial_j(S_{\mu+2}u)$, we have that it is contained in $\{|\xi| \le 2^{\mu+4}\}$ and moreover $[\Delta_\nu, S_\mu a_{ij}]\,\partial_j(S_{\mu+2}u)$ is identically 0 if $\nu \ge \mu+5$. From Bernstein's inequality and [5, Th. 35] we have that

$$\left\|\partial_i\left([\Delta_\nu, S_\mu a_{ij}]\,\partial_j(S_{\mu+2}u)\right)\right\|_{L^2} \;\le\; C_\mu\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)\,\|S_{\mu+2}u\|_{L^2},$$

hence, putting all these facts together, we have

est:comm0

$$\begin{aligned}(48)\qquad &\left|\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\,2^{-2\nu\theta}\sum_{ij}2\,\mathrm{Re}\left(v_\nu\,,\,T_{\alpha^{-1/2}}\,\partial_i\left([\Delta_\nu, S_\mu a_{ij}]\partial_j(S_{\mu+2}u)\right)\right)_{L^2}\right| \le \\ &\le C_\mu\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)\sum_{\nu=0}^{\mu+4} e^{-2\beta(\nu+1)t}\,2^{-2\nu\theta}\|v_\nu\|_{L^2}\left(\sum_{h=0}^{\mu+2}\|u_h\|_{L^2}\right) \\ &\le C_\mu\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)\,e^{\beta(\mu+5)T}\,2^{(\mu+4)\theta}\sum_{\nu=0}^{\mu+4} e^{-\beta(\nu+1)t}\,2^{-\nu\theta}\|v_\nu\|_{L^2}\,\cdot \\ &\qquad\cdot\sum_{h=0}^{\mu+4} e^{-\beta(h+1)t}\,2^{-h\theta}\|u_h\|_{L^2} \\ &\le C_\mu\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)\,e^{\beta(\mu+5)T}\,2^{(\mu+4)\theta}\sum_{\nu=0}^{\mu+4}e^{-2\beta(\nu+1)t}\,2^{-2\nu\theta}e_\nu(t).\end{aligned}$$

Next, let's consider

$$\partial_i\left(\sum_{k=\mu+3}^{+\infty}[\Delta_\nu, S_{k-3}a_{ij}]\,\partial_j(\Delta_k u)\right).$$

Looking at the support of the Fourier transform, it is possible to see that

$$[\Delta_\nu, S_{k-3}a_{ij}]\,\partial_j(\Delta_k u)$$

is identically 0 if $|k-\nu|\geq 3$. Consequently the sum over k is reduced to at most 5 terms: $\partial_i([\Delta_\nu, S_{\nu-5}a_{ij}]\,\partial_j(\Delta_{\nu-2}u))+\cdots+\partial_i([\Delta_\nu, S_{\nu-1}a_{ij}]\,\partial_j(\Delta_{\nu+2}u))$, each of them having the support of the Fourier transform contained in $\{|\xi|\leq 2^{\nu+1}\}$. Let's consider one of these terms, e.g. $\partial_i([\Delta_\nu, S_{\nu-3}a_{ij}]\,\partial_j(\Delta_\nu u))$, the computation for the other ones being similar. We have, from Bernstein's inequality,

$$\|\partial_i\left([\Delta_\nu, S_{\nu-3}a_{ij}]\,\partial_j(\Delta_\nu u)\right)\|_{L^2}\;\leq\; C\,2^\nu\,\|[\Delta_\nu, S_{\nu-3}a_{ij}]\,\partial_j(\Delta_\nu u)\|_{L^2}\,.$$

On the other hand, using [5, Th. 35] again, we have:

$$\|[\Delta_\nu, S_{\nu-3}a_{ij}]\partial_j(\Delta_\nu u)\|_{L^2}\;\leq\; C\,\|\nabla S_{\nu-3}a_{ij}\|_{L^\infty}\,\|\Delta_\nu u\|_{L^2}\,,$$

where C does not depend on ν. Consequently, using also (19), we deduce

$$\|\partial_i\left([\Delta_\nu, S_{\nu-3}a_{ij}]\,\partial_j(\Delta_\nu u)\right)\|_{L^2}\;\leq\; C\,2^\nu\,(\nu+1)\,\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)\,\|\Delta_\nu u\|_{L^2}\,.$$

From this last inequality and similar ones for the other terms, it is easy to obtain that

$$\left|\sum_{i,j}\mathrm{Re}\left(v_\nu\,,\,T_{\alpha^{-1/2}}\partial_i\left(\sum_{k=\mu+3}^{+\infty}[\Delta_\nu, S_{k-3}a_{ij}]\,\partial_j(\Delta_k u)\right)\right)_{L^2}\right|\;\leq\; C\,\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)\,(\nu+1)\,e_\nu(t)$$

and then

est:comm1

$$\left|\sum_{\nu=0}^{+\infty}e^{-2\beta(\nu+1)t}\,2^{-2\nu\theta}\,\sum_{ij}2\,\mathrm{Re}\left(v_\nu\,,\,T_{\alpha^{-1/2}}\partial_i\left(\sum_{k=\mu+3}^{+\infty}[\Delta_\nu, S_{k-3}a_{ij}]\,\partial_j(\Delta_k u)\right)\right)_{L^2}\right|\;\leq$$
$$\leq\; C\,\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)\,\sum_{\nu=0}^{+\infty}(\nu+1)\,e^{-2\beta(\nu+1)t}\,2^{-2\nu\theta}\,e_\nu(t)\,. \tag{49}$$

Collecting the informations from (48) and (49), we obtain

est:comm

$$\left|\sum_{\nu=0}^{+\infty}e^{-2\beta(\nu+1)t}\,2^{-2\nu\theta}\,\sum_{ij}2\,\mathrm{Re}\left(v_\nu\,,\,T_{\alpha^{-1/2}}\partial_i\left[\Delta_\nu, T_{a_{ij}}\right]\partial_j u\right)_{L^2}\right|\;\leq$$
$$\leq\; C_3\,\sum_{\nu=0}^{+\infty}(\nu+1)\,e^{-2\beta(\nu+1)t}\,2^{-2\nu\theta}\,e_\nu(t)\,, \tag{50}$$

where C_3 depends on μ, $\sup_{i,j}\|a_{ij}\|_{LL_x}$, on θ and on the product $\beta\,T$.

ss:f-est

4.4 Final estimate

From (47) and (50) we get

$$\begin{aligned}
\frac{d}{dt}E(t) \; &\leq \; (C_1 \,+\, C_3 \,-\, 2\beta)\sum_{\nu=0}^{+\infty}(\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_\nu(t) \,+ \\
&\qquad + C_2 \sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, (e_\nu(t))^{1/2}\, \left\| \left(\widetilde{L}u(t)\right)_\nu \right\|_{L^2} \\
&\leq \; (C_1 \,+\, C_3 \,-\, 2\beta)\sum_{\nu=0}^{+\infty}(\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_\nu(t) \,+ \\
&\qquad + C_2 \sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, (e_\nu(t))^{1/2}\, \left\| \left(\sum_{i,j} \partial_i \left((a_{ij} - T_{a_{ij}})\partial_j u \right) \right)_\nu \right\|_{L^2} \,+ \\
&\qquad + C_2 \sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, (e_\nu(t))^{1/2}\, \left\| (Lu(t))_\nu \right\|_{L^2} \,.
\end{aligned}$$

Now, applying Hölder inequality for series implies

$$\begin{aligned}
&\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, (e_\nu(t))^{1/2}\, \left\| \left(\sum_{i,j} \partial_i \left((a_{ij} - T_{a_{ij}})\partial_j u \right) \right)_\nu \right\|_{L^2} \;\leq \\
&\qquad \leq \left(\sum_{\nu=0}^{+\infty}(\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_\nu(t) \right)^{1/2} \cdot \\
&\qquad\qquad \cdot \left(\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, (\nu+1)^{-1} \left\| \left(\sum_{i,j} \partial_i \left((a_{ij} - T_{a_{ij}})\partial_j u \right) \right)_\nu \right\|_{L^2}^2 \right)^{1/2} ,
\end{aligned}$$

and, by definition, one has

$$\begin{aligned}
&\left(\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, (\nu+1)^{-1} \left\| \left(\sum_{i,j} \partial_i \left((a_{ij} - T_{a_{ij}})\partial_j u \right) \right)_\nu \right\|_{L^2}^2 \right)^{1/2} \;= \\
&\qquad\qquad = \left\| \sum_{i,j} \partial_i \left((a_{ij} - T_{a_{ij}})\partial_j u \right) \right\|_{H^{-\theta-\beta^* t - \frac{1}{2}\log}} .
\end{aligned}$$

From [9, Prop. 3.4] we have that

a-T_a_est

$$\left\| \sum_{i,j} \partial_i \left((a_{ij} - T_{a_{ij}})\partial_j u \right) \right\|_{H^{-s-\frac{1}{2}\log}} \;\leq\; C \left(\sup_{i,j} \|a_{ij}\|_{LL_x} \right) \|u\|_{H^{1-s+\frac{1}{2}\log}} , \tag{51}$$

with C uniformly bounded for s in a compact set of $]0,1[$. Consequently,

$$\begin{aligned}
&\left(\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, (\nu+1)^{-1} \left\| \left(\sum_{i,j} \partial_i \left((a_{ij} - T_{a_{ij}})\partial_j u \right) \right)_\nu \right\|_{L^2}^2 \right)^{1/2} \;\leq \\
&\qquad\qquad \leq\; C \left(\sup_{i,j} \|a_{ij}\|_{LL_x} \right) \|u\|_{H^{1-\theta-\beta^* t+\frac{1}{2}\log}} \\
&\qquad\qquad \leq\; C \left(\sum_{\nu=0}^{+\infty}(\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_\nu(t) \right)^{1/2} ,
\end{aligned}$$

and finally

$$\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, (e_\nu(t))^{1/2} \left\| \left(\sum_{i,j} \partial_i \left((a_{ij} - T_{a_{ij}}) \partial_j u \right) \right)_\nu \right\|_{L^2} \leq$$

$$\leq C_4 \sum_{\nu=0}^{+\infty} (\nu+1) e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_\nu(t)\,,$$

with C_4 uniformly bounded for $\beta^* t + \theta$ in a compact set of $]0,1[$. So, if we take $\beta > 0$ and $T \in]0, T_0]$ such that (recall that $\beta^* = \beta (\log 2)^{-1}$)

eq:T (52)

$$\beta^* T \;=\; K \;<\; 1 - \theta$$

we have $0 < \theta \leq \theta + \beta^* t \leq \theta + K < 1$. Therefore we obtain

$$\begin{aligned} \frac{d}{dt} E(t) \;\;\leq\;\; & (C_1 + C_4 C_2 + C_3 - 2\beta) \sum_{\nu=0}^{+\infty} (\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_\nu(t) \;+ \\ & + C_2 \sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, (e_\nu(t))^{1/2}\, \|(Lu(t))_\nu\|_{L^2}\,. \end{aligned}$$

Now, taking β large enough such that $C_1 + C_4 C_2 + C_3 - 2\beta \leq 0$, which corresponds to take $T > 0$ small enough, we finally arrive to the estimate

$$\frac{d}{dt} E(t) \;\leq\; C_2\, (E(t))^{1/2}\, \|Lu(t)\|_{H^{-\theta-\beta^* t}}\;;$$

applying Gronwall's Lemma and keeping in mind (29) and (30) give us estimate (11). □

r:T **Remark 4.2.** Let us point out that condition (52) gives us a condition on the lifespan T of a solution to the Cauchy problem for (7). It depends on $\theta \in]0,1[$ and on $\beta^* > 0$, hence on constants $C_1 \ldots C_4$. Going after the guideline of the proof, one can see that, in the end, the time T depends only on the index θ, on the parameter μ defined by conditions (25), on constants λ_0 and Λ_0 defined by (8) and on the quantities $\sup_{i,j} \|a_{ij}\|_{LZ_t}$ and $\sup_{i,j} \|a_{ij}\|_{LL_x}$.

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