MULTIDIMENSIONAL VISCOUS SHOCKS II: THE SMALL VISCOSITY LIMIT

OLIVIER GUES, GUY MÉTIVIER, MARK WILLIAMS, KEVIN ZUMBRUN

Abstract. In this paper we prove the existence of curved multiD viscous shocks and also justify the small viscosity limit.

Starting with a curved, multidimensional (inviscid) shock solution to a system of hyperbolic conservation laws, we show that the shock can be obtained as a small viscosity limit of solutions to an associated parabolic problem (viscous shocks). The two main hypotheses are a natural Evans function assumption on the viscous profile, together with a restriction on how much the shock can deviate from flatness. The main tools are a conjugation lemma which removes $\frac{x_N}{\epsilon}$ dependence from the linearization of the parabolic problem about the viscous profile, new degenerate Kreiss-type symmetrizers used to prove an L^2 estimate for the linearized problem, and a finite regularity calculus of semiclassical and mixed type (classical-semiclassical) pseudodifferential operators.

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Part 1. Introduction

1. The problem

This paper presents a rigorous study of the zero viscosity limit for multiD curved shocks, and at the same time proves the existence of curved viscous shocks for systems of conservation laws. Starting with a curved shock (a piecewise smooth solution of a system of hyperbolic conservation laws), we show that this shock can be obtained as the limit as viscosity goes to zero of solutions to an associated parabolic problem (hyperbolic + viscosity). In [GW] an arbitrarily high order asymptotic expansion was constructed for the viscous boundary layer on each side of the shock, but the expansion was rigorously justified there only for sufficiently weak shocks in dimension one. Here we are able to prove stability of the layer and thereby justify the expansion in all dimensions for shocks of arbitrary strength satisfying: (a) an appropriate Evans function condition and (b) a hypothesis that limits how much the curved shock we start with can deviate from flatness ((H7) and (H6) in section 2). Recent work by Freistühler and Szmolyan [FS] and independently by Plaza and Zumbrun [PZ] shows that the Evans condition holds for sufficiently weak Lax shocks. We recall that the existence of multiD curved shocks in the inviscid case was proved by Majda [M2, M3].

Consider the $m \times m$ system of conservation laws on $\mathbb{R}^{N+1}$

$$\sum_{j=0}^{N} A_j(u)\partial_{x_j} u = 0. \tag{1.1}$$

where $A_j(u) = f_j'(u)$ and $f_j : \mathbb{R}^m \to \mathbb{R}^m$ are C^∞ functions with $f_0(u) = u$.

Set $x = (x_0, x'', x_N) = (x', x_N)$, where x_0 denotes time, and suppose that $(U^0_\pm(x),\, x_N = \psi_0(x'))$ is a given shock solution of (1.1), not necessarily planar, which exists for $x_0 \in [-T_0, T_0]$. For convenience, we assume $(U^0_\pm, d\psi_0)$ is constant $(\underline{U}_\pm, \underline{\sigma})$ outside some ball centered at the origin in $[-T_0, T_0] \times \mathbb{R}^N$. It is also convenient to suppose that $(U^0_\pm, \psi_0) \in C^\infty([-T_0, T_0] \times \mathbb{R}^N$. (Even though $U^0_\pm$ is initially defined just in $\pm(x_N - \psi_0(x')) \geq 0$, we can extend each of these functions smoothly to all of $[-T_0, T_0] \times \mathbb{R}^N$.) The case of sufficiently high but finite regularity can be handled as below, but much more bookkeeping is needed.

To say that $(U^0_\pm, d\psi_0)$ is a shock solution of (1.1) means that both of the following conditions hold. Let S be the surface $x_N = \psi_0(x')$.

1. U^0_+ (resp., U^0_-) satisfies (1.1) in $x_N \geq \psi_0(x')$ (resp., $x_N \leq \psi_0(x')$).

2. $\sum_0^{N-1}[f_j(U^0)]\partial_{x_j}\psi_0 - [f_N(U^0)] = 0$ on S (the Rankine-Hugoniot condition). Here $[u]$ means the jump at S, $u_+ - u_-$.

Together these conditions imply that U^0 defined as U^0_+ (resp., U^0_-) in $x_N > \psi_0(x')$ (resp., $x_N < \psi_0(x')$) is a distribution solution of (1.1) in the whole space.

The problem we study is the following one:

Under suitable hypotheses show that on the time interval $[0, T_0]$, U^0 is the limit as $\epsilon \to 0$ in some appropriate norm (e.g., L^2) of solutions to the associated parabolic problem on $\mathbb{R}^{N+1}$:

$$\sum_{j=0}^{N} A_j(u^\epsilon)\partial_{x_j} u^\epsilon - \epsilon \triangle u^\epsilon = 0. \tag{1.2}$$

The asymptotic expansion constructed in [GW] provides an arbitrarily high order approximate solution to the parabolic problem (1.2) with the given shock as the "outer part" of the leading term. Our main result is that these approximate solutions are close in L^∞ for ϵ small to exact solutions of the parabolic problem. This yields the small viscosity limit as a simple corollary.

As in [GW] the first step is to reformulate the parabolic problem as a doubled boundary problem involving an unknown "front". We fix once and for all a high order approximation to that front, constructed as part of the expansion in [GW], and use it to define a change of variables leading to a flat boundary ($x_N = 0$). From this point on the main tools are the conjugation argument of [MZ] and degenerate symmetrizers.

We look for an exact solution to the doubled problem as the sum of the approximate solution and an error term which satisfies a nonlinear "error equation". The [MZ] conjugation argument allows us to reduce the problem of proving L^2 estimates for the linearized error equation to the study of a similar problem with $\frac{x_N}{\epsilon}$ dependence removed from the coefficients. The doubled boundary problem that remains fails to satisfy the uniform Lopatinski condition (since the Evans function vanishes for zero frequency). Indeed, that failure is the main point that distinguishes our problem from the question of stability of viscous Dirichlet boundary layers studied in [MZ]. There the uniform Lopatinski condition was satisfied. Here as in [GMWZ1] we construct degenerate Kreiss-type symmetrizers to cope with the degeneracy in the Lopatinski determinant. These symmetrizers yield a degenerate L^2 estimate - see Theorem 9.1. The singularity in the estimate (which really occurs only in the low frequency regime) makes it harder to absorb the various error terms that arise in using the pseudodifferential calculus, and also complicates the proof of nonlinear stability. We are nevertheless able to use the estimate to prove nonlinear stability because we have a high order approximate solution.

Assumptions on Evans functions (Definition 7.3) have been shown to give necessary and sufficient conditions for nonlinear stability in the small viscosity limit in the cases of 1D Dirichlet boundary layers [GR] and 1D curved shocks [R]. More recently, the same was shown for multiD Dirichlet boundary layers in the paper [MZ]. It is to be expected then, that assumptions on the Evans function (see (H7)) should be the correct approach for studying the stability of the boundary layers that arise in the small viscosity limit for multiD shocks.

The goal of [GMWZ1] was to find a way to use energy estimates to recover and extend some of the results proved in [Z] by constructive techniques based on estimation of Green's functions. In both papers the problem of long time stability for multiD *planar* viscous shocks is studied under the Evans assumption (H7) on the viscous profile. In [GMWZ1] we had no high order approximate solution, but again the basic L^2 estimate obtained with degenerate symmetrizers was singular. However, the planar hypothesis meant that after conjugation the linearized error equation had constant coefficients, so we were able to prove additional mixed norm and $L^1 - L^2$ estimates that led to nonlinear long time stability. It is not clear to us yet whether such estimates can be proved in a variable coefficient situation such as the case of curved shocks. In any case such estimates are not needed to study the short time small viscosity problem considered here.

We'll refer to the small viscosity problem studied here as the *small viscosity problem with prepared data*, since we use the approximate solution to define initial data for the associated parabolic problem. This prepared data problem was solved in [GX] for sufficiently weak 1D shocks, and in [R] for 1D shocks of arbitrary strength satisfying an Evans function hypothesis like the one we make here (H7). A more difficult problem is the *small viscosity problem with unprepared data* in which one takes the discontinuous initial data given by the hyperbolic shock as initial data for the associated parabolic problem. In this problem one has additional phenomena such as the formation of an initial layer (see [Y] for an analysis in 1D).

In their work on the small viscosity problem in 1D the authors mentioned above [GX, R, Y] all found it helpful to put the error equation in conservative form and then integrate. The Evans function for the resulting problem does not vanish for zero frequency. Conservative form also proved useful in the study of long time stability for 1D [KK] and multiD ([GMWZ1], part 2) planar shocks under zero mass perturbations. While it is not hard to use the approximate solution to write the error equation for the problem considered here in conservative form, we see no way to take advantage of that fact. On the one hand there appears to be no useful way to integrate the equation in the multiD small viscosity problem for systems (in the *scalar* multiD case integration helps [Go2]). In addition, the various error terms introduced by use of the pseudodifferential calculi during the process of conjugating the problem to simpler forms wreck the conservative structure anyway. Instead, here we apply symmetrizer arguments directly to the unintegrated equation.

We conclude the paper with some observations about the difference between the long time stability and small viscosity problems. One might at first think that the problems are equivalent after rescaling, but in fact this is not so. Although both problems can be formulated (as we do here and in [GMWZ1]) as doubled boundary value problems, the small viscosity problem involves a boundary layer, while the long time problem does not (see section 12).

A more detailed overview and guide to the proof is given in section 3 after the statement of the main result. As far as we know, this is the first existence proof and justification of the small viscosity limit for multiD curved shocks. In a subsequent paper we hope to remove the restriction (assumption (H6)) on how much the inviscid shock can deviate from flatness. In addition, it is clear that the methods of this paper yield similar results under weaker hyperbolicity hypotheses than (H2), and for more general, even nonlinear, viscosities. We plan to discuss these generalizations in a future work.

This work builds on the classic stability analysis for multi-D inviscid shocks in [M2, M3]. An analogous stability problem for highly oscillatory, multi-D, inviscid shocks is studied in [W]. The nonlinear stability of curved multi-D weak inviscid shocks is studied in [FM].

2. Assumptions

2.1. Assumptions on the equations.

(H1) $f_j \in C^\infty(\mathbb{R}^m, \mathbb{R}^m)$.

(H2) There are neighborhoods $\mathcal{O}_\pm \subset \mathcal{R}^m$ of $U^0_\pm(0)$, respectively, such that for $u_\pm \in \mathcal{O}_\pm$, $\sum_{j=1}^N A_j(u_\pm)\xi_j$ has simple real eigenvalues for $\xi \in \mathbb{R}^N \setminus 0$ (strict hyperbolicity).

2.2. Assumptions on the inviscid shock.

(H3) $(U^0_\pm(x), \psi_0(x'))$ exists for $x_0 \in [-T_0, T_0]$. ψ_0 is C^∞ and $U^0_\pm$ are C^∞ up to the shock surface $\mathcal{S} = \{x_N = \psi_0(x')\}$. We take $\psi_0(0) = 0$.

(H4) $U^0_\pm(x) \in \mathcal{O}_\pm$ for all $x \in [-T_0, T_0] \times \mathbb{R}^N$, and $(U^0_\pm, \psi_0)$ is constant $(\underline{U}_\pm, \underline{\sigma})$ for (x'', x_N) outside some ball centered at the origin in $\mathbb{R}^N$.

(H5) $(U^0_\pm, d\psi_0)$ is uniformly stable in the sense of Majda [M2] at each point of $\mathcal{S}$.

(H6) The set $\{(U^0_\pm(x), d\psi_0(x')) : x \in [-T_0, T_0] \times \mathbb{R}^N\}$ is a subset (necessarily compact) of the neighborhood ω_1 of $(U^0_+(0), U^0_-(0), d\psi^0(0))$ chosen in Remark 7.9.

Observe that (H2) implies that for $u_\pm \in \mathcal{O}_\pm$

$$\text{the eigenvalues } \lambda \text{ of } -i\sum_{j=1}^N A_j(u_\pm)\xi_j - |\xi|^2 \text{ satisfy } \Re\lambda = -|\xi|^2. \tag{2.1}$$

Remark 2.1. 1. (H5) implies that $(U^0_\pm(x', \psi_0(x')), \psi_0(x'))$ satisfies the *Lax shock inequalities* for all x'. Let

$$A_N(u, d\phi) \equiv A_N(u) - \sum_{j=0}^{N-1} A_j(u)\partial_{x_j}\phi.$$

The Lax shock inequalities in turn imply that if we let k (resp., l) be the number of positive (resp., negative) eigenvalues of $A_N(U^0_+(x',\psi_0(x')),d\psi_0(x'))$ (resp., $A_N(U^0_-(x',\psi_0(x')),d\psi_0(x'))$, then

$$k+l=m-1. \tag{2.2}$$

2. For Lax shocks (H5) is a consequence of (H7) below (see Remark 7.6). Thus, we could as well replace (H5) by the assumption that $(U^0_\pm, d\psi_0)$ is a Lax shock.

The final hypothesis is an assumption on the viscous profile $\mathcal{U}^0(x',z)$ which is stated in terms of the corresponding Evans function $\mathcal{D}(x',\widehat{\beta},\rho)$. Our definitions of these terms are the standard ones and they are recalled in (7.23)-(7.24) and Definition 7.3 respectively. Let $S^N_+=\{\beta=(\beta',\gamma')\in\mathbb{R}^{N+1}:\gamma'\geq 0 \text{ and } |\beta|=1\}$, and introduce polar coordinates $\beta=\rho\widehat{\beta}$, $\widehat{\beta}\in S^N_+$.

2.3. Assumption on the viscous profile.

(H7) For each $x'\in[-T_0,T_0]\times\mathbb{R}^{N-1}_{x''}$, $\mathcal{D}(x',\widehat{\beta},\rho)$ vanishes to precisely first order at $\rho=0$ (where it must vanish) for all $\widehat{\beta}\in S^N_+$, and has no other zeros in $S^N_+\times\overline{\mathbb{R}}_+$.

Remark 2.2. 1. Recent work by Freistühler and Szmolyan [FS] and independently by Plaza and Zumbrun [PZ] shows that (H7) holds for sufficiently weak Lax shocks.

2. When the Evans function vanishes in $\gamma'>0$, the linearized problem is strongly unstable. The analogue of (H7) in one space dimension has been shown by Rousset [R] to imply nonlinear stability in that case.

3. The neighborhood ω_1 specifies how much the shock can deviate from flatness.

4. The choice of ω_1 and (H6) imply that $A_N(U^0_\pm(x),d\psi_0(x'))$ has a uniformly bounded inverse for all $x\in\overline{\mathbb{R}}^{N+1}_+$.

2.4. Choice of extension of the shock and profile. It will be convenient to smoothly extend $(U^0_\pm(x),d\psi_0(x'))$ and the corresponding viscous profile $\mathcal{U}^0(x',z)$ to all time ($x_0\in\mathbb{R}$) so that (H3)-(H7) continue to hold.

In addition we can choose the extension so that $\{(U^0_\pm(x',\psi_0(x')),d\psi_0(x')):x'\in\mathbb{R}^N\}$ is a compact subset of ω_1 (see Remark 7.9), and so that $\{(V^0_+,\partial_zV^0_+,V^0_-,\partial_zV^0_-)(x',0):x'\in\mathbb{R}^N\}$ is a compact subset of $\mathbb{R}^{4m}$ (see (7.18)).

It is not necessary (nor is it possible in general) to extend $(U^0_\pm(x),d\psi_0(x'))$ as a *solution* of the system of conservation laws.

Henceforth we assume that such extensions have been chosen.

3. Main result and guide to the proof

Definition 3.1. Let $x=(x',x_N)$ be the original variables on $\mathbb{R}^{N+1}$ in which the problems (1.1) and (1.2) are stated. For a fixed choice of Ψ^ϵ define *flat variables* $\tilde{x}$ (globally on $[-T_0,T_0]\times\mathbb{R}^N$) by

$$\begin{aligned}\tilde{x}'&=x'\\ \tilde{x}_N&=x_N-\Psi^\epsilon(x').\end{aligned} \tag{3.1}$$

When the use of flat variables is clear from the context, we drop the tilde.

Notation 3.1. 1. When working in flat variables, if we are given functions $f_\pm(x)$ defined on $x_N\geq 0$, we define $f(x)$ for $x_N\in\mathbb{R}$ by

$$f(x)=\begin{cases}f_+(x',x_N) \text{ for } x_N\geq 0\\ f_-(x',-x_N) \text{ for } x_N\leq 0\end{cases}.$$

2. Similarly, given $f(x)$ defined for $x_N\in\mathbb{R}$, define $f_\pm(x',x_N)=f(x',\pm x_N)$ for $x_N\geq 0$.

In the statement of the following theorem $\tilde{u}^\epsilon$ is obtained from $\tilde{u}^\epsilon_\pm$ as in the above Notation, where $(\tilde{u}^\epsilon_\pm, \Psi^\epsilon)$ is the smooth high order approximate solution to the doubled parabolic problem (4.4) on $\tilde{x}_N \geq 0$ constructed in [GW] and recalled in section 4 (4.6), (4.7). The viscous front is approximated by $x_N = \Psi^\epsilon(x')$ where

$$\Psi^\epsilon(x') = \psi_0(x') + \epsilon\psi_1(x') + \cdots + \epsilon^M\psi_M(x') \tag{3.2}$$

and ψ_0 defines the inviscid shock.

The functions $\tilde{u}^\epsilon_\pm$ have expansions (in flat variables) $\tilde{u}^\epsilon_\pm(x) =$

$$\left(\mathcal{U}^0_\pm(x,z) + \epsilon\mathcal{U}^1_\pm(x,z) + \cdots + \epsilon^M\mathcal{U}^M_\pm(x,z)\right)|_{z=\frac{x_N}{\epsilon}} + \epsilon^M r(x). \tag{3.3}$$

Here

$$\mathcal{U}^j_\pm(x,z) = U^j_\pm(x) + V^j_\pm(x',z), \tag{3.4}$$

$U^0_\pm(x)$ is the original shock and $V^j_\pm(x',z)$ are boundary layer profiles exponentially decreasing in z.

It is shown in [GW] that these expansions can be constructed to arbitrarily high order under the assumptions of section 2.

Theorem 3.1. *Under assumptions H1-H7 of section 2 there exists an ϵ_0 such that for $0 < \epsilon \leq \epsilon_0$ the parabolic problem (1.2) has an exact solution on $\Omega_{T_0} \equiv [0,T_0] \times \mathbb{R}^N$ of the form (in the original x variables)*

$$u^\epsilon(x) = \tilde{u}^\epsilon(x', x_N - \Psi^\epsilon(x')) + \epsilon^L w, \tag{3.5}$$

where $\tilde{u}^\epsilon$ and Ψ^ϵ have the above expansions. Note that the original inviscid shock $(U^0_\pm, \psi_0)$ appears in the leading terms.

The exponent L can be chosen as large as desired provided the approximate solution is constructed with sufficiently many terms ($M(L)$) and in that case we have the estimates (in flat variables),

$$\begin{aligned} &|\partial^\alpha(w_\pm, \epsilon\partial_N w_\pm)|_{L^\infty} \leq 1 \\ &|\partial^\alpha(w_\pm, \epsilon\partial_N w_\pm)|_{L^2} \leq C(T_0)\epsilon^L \end{aligned} \tag{3.6}$$

for $|\alpha| \leq L$, $0 < \epsilon \leq \epsilon_0$. Here $\partial = (\partial_0, \ldots, \partial_{N-1})$.

Remark 3.1. 1. This theorem is an immediate consequence of Theorem 11.1. For a given L as in Theorem 3.1 one can use Theorem 11.1 to see how many terms in the expansions of $(\tilde{u}^\epsilon_\pm, \Psi^\epsilon)$ are needed to yield the estimates (3.6).

2. Denote the original variables by (y', y_N), and write the right and left sides of the inviscid shock as $U_R(y', y_N)$ and $U_L(y', y_N)$. The shock surface is $y_N = \psi_0(y')$. Set $x' = y'$ and $x_N = y_N - \psi_0(y')$. Then $U^0_+(x)$ in (3.4) is $U_R(x', x_N + \psi_0(x'))$ and $U^0_-(x) = U_L(x', -x_N + \psi_0(x'))$.

Given the properties of the profiles as described in Proposition 4.1, Theorem 3.1 has the following immediate corollary.

Corollary 3.1. *Let $U^0(x)$ be the function on Ω_{T_0} defined as U^0_+ (resp., U^0_-) in $x_N > \psi_0(x')$ (resp., $x_N < \psi_0(x')$), where $(U^0_\pm, \psi_0)$ is the given inviscid shock. Then for u^ϵ as in (3.5) and any compact $K \subset \Omega_{T_0}$ we have*

$$|u^\epsilon - U^0|_{L^2(K)} \leq C(K)\sqrt{\epsilon}.$$

Of course, Theorem 3.1 contains much more information than this, since it rigorously justifies the explicit high order asymptotic description given by the expansions (3.2), (3.3) of the viscous boundary layer on each side of the inviscid shock.

Remark 3.2. Henceforth, we'll work exclusively in flat variables. In those variables the viscous front is given by $x_N = 0$, while the inviscid shock is $x_N = -(\Psi^\epsilon(x') - \psi_0(x'))$.

The main steps in the proof are:

I. Approximate solution. Construct an arbitrarily high order approximate solution to the $m \times m$ parabolic problem (1.2) in which the inviscid shock appears in the leading term. This is done in [GW] by introducing an unknown "front"

$$x_N = \Psi^\epsilon(x'),$$

and reformulating the original problem on the whole space as a doubled boundary problem with transmission boundary conditions on the half-space $\tilde{x}_N \geq 0$, where $\tilde{x}_N = x_N - \Psi^\epsilon(x')$. The expansion of Ψ^ϵ is constructed along with that of $\tilde{u}$. This approximate solution is recalled in section 4.

II. Reduce to a forward error problem. Henceforth, we work with the doubled parabolic boundary problem (4.4). Advantages are that we are now in a position to apply Kreiss-type symmetrizer techniques developed for boundary problems. In addition, we have a single limiting problem (7.10) as $z = \frac{x_N}{\epsilon} \to +\infty$, instead of two distinct limiting problems at $\pm\infty$.

We fix a high order approximate solution $(\tilde{u}, \Psi^\epsilon)$ to the doubled problem (4.4), and look for an exact solution of the form

$$u^\epsilon_\pm = \tilde{u}^\epsilon_\pm + w^\epsilon_\pm, \tag{3.7}$$

where $w_\pm$ (drop epsilons) satisfies the "error equation" (really an initial boundary value problem) (6.4).

Note that the "viscous profile" $\mathcal{U}^0(x', z)$ (7.23) is essentially the leading term in the expansion of $\tilde{u}$.

Initial data for the error problem (6.4) satisfying high order corner compatibility conditions (at the corner $x_0 = 0$, $x_N = 0$) is chosen in section 5, and that allows us in section 6 to reformulate the error problem as a "forward problem" (i.e., one where the forcing and the solution are zero in the past, $x_0 < 0$) with homogeneous boundary data.

The problem has been reduced to solving the $2m \times 2m$ forward error problem (6.14). The most difficult remaining step will be to prove an L^2 estimate for the corresponding linearized problem (6.16) (the linearization is about $\tilde{u}_\pm$).

III. Symbolic preparation. All the work in section 7 is done at the symbol level. The arguments are quantized in section 9 (that is, operators are associated to symbols) after the needed pseudodifferential calculi are developed in section 8.

The discussion in section 7 applies almost entirely to behavior in the small ($|\beta| \leq \delta$) and medium-sized ($\delta \leq |\beta| \leq R$) frequency regions, with the main difficulties centered in the small frequency region. Here $\beta = (\beta', \gamma') = \epsilon\zeta$, where $\zeta = (\zeta', \gamma)$, ζ' is dual to x', and $\gamma \geq 1$. Sometimes we need to use polar coordinates $\beta = \rho\hat{\beta}$, where $|\hat{\beta}| = 1$.

IV. Conjugation to remove $\frac{x_N}{\epsilon}$ dependence. In section 7 to prepare the way for the use of symmetrizers, we first rewrite (6.16) as the $4m \times 4m$ first order system (7.6) (the system is first order in ∂_N, but second order tangential derivatives do appear).

It has been known for a long time (see, e.g., [Go, GX] for the 1D case) that the main obstacle to proving an L^2 estimate for the linearized error problem is the $z = \frac{x_N}{\epsilon}$ dependence of the coefficients. Here we use a key idea from [MZ], which is to replace the original linearized problem (7.6) by a "limiting" problem in which the $\frac{x_N}{\epsilon}$ dependence (but not the x_N dependence!) has been removed. This is achieved by conjugating the original problem with a semiclassical pseudodifferential operator

$\mathcal{W}_D$ (9.14) whose symbol

$$\mathcal{W} = \mathcal{W}_0(x', \frac{x_N}{\epsilon}, p^\epsilon(x), \beta) + \epsilon \mathcal{W}_1(x', \frac{x_N}{\epsilon}, p^\epsilon(x), \beta) \tag{3.8}$$

is constructed in section 7.

The limiting problem (7.10) is obtained from (7.6) simply by letting $z \to +\infty$ in the coefficients of that problem. The construction of the symbols $\mathcal{W}_0$, $\mathcal{W}_1$ combines the Gap Lemma of [GZ] with our semiclassical pseudodifferential calculus (section 8 and the Appendix).

The construction of the conjugator $\mathcal{W}$ and the later construction of symmetrizers depend on a knowledge of the spectral properties of the symbol $\mathcal{G}_\infty(p, \beta)$ of the limiting operator. These properties are recalled in section 7.

V. Degenerate Evans implies degenerate Lopatinski. In section 7 we also define the Evans function, first giving the classical definition $\mathcal{D}(x', \hat{\beta}, \rho)$ (7.29) for problems on the whole space (as in [ZS], e.g.), and then relating that to the Evans function $\mathbb{D}(x', \hat{\beta}, \rho)$ (7.35) for the corresponding doubled boundary problem.

The Evans function (a Wronskian of solutions to the ODE (7.34)(a)) encodes information about the linearized stability of the viscous profile and also, less obviously, of the original inviscid shock. The existence of the profile itself implies $\mathbb{D}(x', \hat{\beta}, 0) = 0$, and a key hypothesis of this paper (H7) is the assumption that $\mathbb{D}$ vanishes to precisely first order at $\rho = 0$ and has no other zeros in the unstable (closed) half plane ($\gamma' \geq 0$). We recall from [ZS] how the first order vanishing of $\mathbb{D}$ at $\rho = 0$ is equivalent to the simultaneous validity of: (a) transversality at the connection $\mathcal{U}^0$ (of the stable/unstable manifolds for $U^0_+(x')/U^0_-(x')$ of the travelling wave ODE (7.24)), and (b) the uniform stability in the sense of Majda [M2] of the original inviscid shock . The properties (a) and (b) are necessary for the construction of high order approximate solutions as in [GW].

In Proposition 7.2 (recalled from [GMWZ1]) we describe how the small frequency behavior of the Evans function translates into failure at $\rho = 0$ of the uniform Lopatinski condition for the boundary problem (6.16). It is important for the construction of degenerate symmetrizers to know precisely how the boundary operator Γ behaves on the decaying generalized eigenspace $\mathcal{E}_-(x', \hat{\beta}, \rho)$ (7.4), and in particular to identify the one dimensional subspace $\mathcal{E}_{-,\phi}$ on which it vanishes. That subspace is essentially the span of the doubled differentiated profile $\partial_z \mathcal{U}^0$ (7.36).

VI. Conjugation to block structure. The last element of symbolic preparation carried out in section 7 is the conjugation of $\mathcal{G}_\infty$, the symbol of the limiting problem (7.10), to block structure. The (main) stages of the conjugation are

$$\mathcal{G}_\infty \to \mathcal{G}_{1,\infty} \text{ (7.52) } \to \mathcal{G}_{2,\infty} \text{ (7.58) } \to \mathcal{G}_{HP} \text{ (7.65) } \to \mathcal{G}_{B,\infty} \text{ (7.59)}. \tag{3.9}$$

The H block of $\mathcal{G}_{HP}$ is associated to the generalized eigenspace of $\mathcal{G}_\infty$ corresponding to small eigenvalues - that is, eigenvalues that approach zero as $\rho \to 0$. The P block corresponds to eigenvalues whose real parts remain strictly bounded away from zero as $\rho \to 0$.

The conjugation from the H block of $\mathcal{G}_{HP}$ to the H_B block of $\mathcal{G}_{B,\infty}$ is done as in [MZ]. Again, the main difficulty is associated with "glancing modes", that is, points $(p', \hat{\beta}', \hat{\gamma}', \rho) = (p', \hat{\beta}', 0, 0)$ such that $\hat{H}_B(p', \hat{\beta}', 0, 0)$ has multiple pure imaginary eigenvalues (here $\hat{H}_B$ (7.61) is defined by $H_B(p', \hat{\beta}, \rho) = \rho \hat{H}_B(p', \hat{\beta}, \rho)$). The argument is a modification of the classic perturbation argument of Kreiss [K], the difference being that now the perturbation is performed with respect to the parameters $\hat{\gamma}'$ and ρ instead of just $\hat{\gamma}'$.

The conjugations associated to the P block pose no significant difficulty.

In fact only the conjugations represented by the first three arrows in (3.9) will be quantized in section 9. The final arrow is the only one that requires localization on the unit sphere S^N_+ in β space, and is needed because the piece of the symmetrizer corresponding to the H block of $\mathcal{G}_{HP}$ has to be constructed microlocally (that is, using spatial cutoffs and cutoffs on S^N_+ simultaneously).

Microlocal pieces $S_{\hat{H}_B}$ symmetrizing $\hat{H}_B$ are constructed first. They are then conjugated back a step and assembled by a microlocal partition of unity to produce the symbol S_H.

VII. Symmetrizer construction at the symbol level.

The degenerate symmetrizer $S = \begin{pmatrix} S_H & 0 \\ 0 & S_P \end{pmatrix}$ is constructed so that $\Re S\mathcal{G}_{HP}$ has certain positivity properties in the interior (7.82), (7.83), (7.90), and so that $S+\Gamma_1^*\Gamma_1$ has certain positivity properties on the boundary (7.91) (here $\Gamma_1 = \Gamma\mathcal{T}^0$, where the symbol T^0 is a composition of conjugator symbols (7.67)).

The S_H block is constructed as in [MZ] by modifying the ansatz used in [K]; an extra term is added to the kth subblock of $S_{\hat{H}_B}$ corresponding to the extra ρ parameter. The S_H block of S is not the "degenerate" one.

Assumption (H7) together with the fact that the viscous profile approaches its endstate with fast exponential decay as $z \to \infty$ implies that the boundary operator fails to satisfy the uniform Lopatinski condition on a one dimensional subspace $E_{P_{1,-}}$ of the eigenspace associated to the P block of $\mathcal{G}_{HP}$ (see Proposition 7.2, 2(b)). To deal with this we construct the S_P block with a degeneracy as $\rho \to 0$:

$$S_P = \begin{pmatrix} CI & 0 \\ 0 & -\rho^2 I \end{pmatrix}. \tag{3.10}$$

In Proposition 7.4 we show that S_P and S_H can be chosen so that

$$\begin{aligned} &(a)\ c_1\rho^2|U|^2 \le ((S+\Gamma_1^*\Gamma_1)(x',\widehat{\beta},\rho)U,U) \le c_2\rho^2|U|^2 \text{ for } U \in E_{P_{1,-}} \text{ and} \\ &(b)\ ((S+\Gamma_1^*\Gamma_1)(x',\widehat{\beta},\rho)U,U) \ge c_1|U|^2 \text{ for } U \in E^c_{P_{1,-}}, \end{aligned} \tag{3.11}$$

where $E^c_{P_{1,-}}$ is the subspace of $\mathbb{C}^{4m}$ orthogonal to $E_{P_{1,-}}$.

Note that when the uniform Lopatinski condition (Definition 7.6) holds, S can be constructed so that an estimate like (3.11)(b) holds for all $U \in \mathbb{C}^{4m}$.

VIII. Pseudodifferential calculi and the mixed Garding inequality. In section 8 and the Appendix we present the semiclassical, classical, and mixed pseudodifferential calculi we need to quantize the symbolic portion of the argument. The calculi are rather simple in the sense that the proofs are based just on Taylor's formula and standard properties of the Fourier transform. Even though our inviscid shock and approximate solution are piecewise C^∞, we construct the calculi under weaker regularity hypotheses in order to allow the arguments of this paper to be applied when C^∞ is replaced by C^M for M large enough.

To an element $a(x',\beta,\zeta)$ of the mixed symbol class $\mathcal{M}^m_M$ (8.11) we associate the operator

$$a(x',\epsilon D, D)u = \int e^{ix'\zeta'} a(x',\epsilon\zeta,\zeta)\hat{u}(\zeta')d\zeta'. \tag{3.12}$$

Operations like composition and taking adjoints with pseudodifferential calculi produce error terms, and a quick glance at our main L^2 estimate, Theorem 9.1, shows that this estimate cannot absorb $O(|U|_{L^2})$ errors. Partly for this reason (in contrast to [MZ], where such errors can be absorbed), in section 9 we often need to keep track of terms beyond the leading term in applications of the calculus and estimate the associated higher order errors.

One of the main applications of the mixed calculus is the proof of the Garding inequality for mixed pseudodifferential operators stated at the end of section 8. In particular that proof requires both composition and adjoint formulas for mixed type operators.

IX. Localization, assumption (H6), and limiting the deviation from flatness of the inviscid shock. Spatial localization is accomplished with smooth cutoffs $\kappa(x)$, while frequency localization is performed with pseudodifferential operators associated to semiclassical symbols like

$\chi_1(\epsilon\zeta)$ (for localization by frequency size) and classical symbols like $\chi_2(\frac{\zeta}{|\zeta|})$ (for localization by frequency direction).

Note that if one tries to commute a cutoff like $\kappa(x)$ through

$$\partial_N U - \frac{1}{\epsilon}\mathcal{G}U = F, \tag{3.13}$$

the commutator is an unacceptable $O(|U|_{L^2})$ error. The error is unacceptable because a degenerate L^2 estimate proved for the localized problem cannot be used to conclude *anything* about the solution to the original problem (3.13).

Localization by frequency direction leads to a similar problem, so these two types of localization have to be avoided in the early conjugations. They can in fact be tolerated once the block structure $\mathcal{G}_{HP}$ has been achieved (see Remark 9.5).

On the other hand localization by frequency size ($\chi_1(\epsilon\zeta)$) can be tolerated in (3.13) (see step **1** in the proof of Proposition 9.1).

There are two points in the argument where spatial localization is needed. One is at the stage of the very first conjugation $\mathcal{G} \to \mathcal{G}_\infty$, where in order to apply the Gap Lemma we need to limit how much the coefficients of $\mathcal{G}_\infty$ (which depend on the inviscid shock $(U^0_\pm, d\psi_0)$) can vary. We can't introduce a spatial cutoff at this point, so we introduce a hypothesis (H6) instead . That is, instead of using a spatial cutoff to restrict to a small neighborhood on which the inviscid shock varies only slightly, we assume that the global deviation of that shock from flatness (a piecewise constant shock) is not too large. The neighborhood ω_1 in the statement of (H6) specifies how much deviation is allowed.

The viscous profile satisfies

$$|\partial_z \mathcal{U}^0(x', z)| \leq Ce^{-\delta z}, \tag{3.14}$$

for some $\delta > 0$. The discussion in Remark 7.9 shows, for example, that the larger δ is, the more one can allow the inviscid shock to deviate from flatness.

The second point where spatial localization is needed is in the construction of the S_H block of the symmetrizer symbol and (therefore) also of the corresponding operator $s^\epsilon_{h,D}$ (9.50), (9.71). As indicated earlier the resulting $O(|U|_{L^2})$ errors can be absorbed now since the $\mathcal{G}_{HP}$ form has already been achieved (Remark 9.5).

X. L^2 estimate - error control. The L^2 estimate is proved in section 9. The main technical challenge here is to control the size of errors arising from use of the calculi.

We've already discussed the cutoff errors. Another source of $O(|U|_{L^2})$ errors is the conjugation process. For example, if one attempts to conjugate $\mathcal{G}$ to $\mathcal{G}_\infty$ using a first order conjugator whose symbol is given by just the first term $\mathcal{W}_0$ in (3.8) (as is done in [MZ]), this produces an $O(|U|_{L^2})$ error. The operator associated to $\mathcal{W}_1$ removes that error (step **3** in the proof of Proposition 9.1). The semiclassical calculus tells us what equation the symbol $\mathcal{W}_1$ must satisfy, and the Gap Lemma enables us to solve that equation.

The quantized version of the conjugation represented by the first arrow in (3.9) also produces $O(|U|_{L^2})$ errors that cannot simply be thrown on the right as new forcing terms. Instead, we "incorporate these errors back into the system" (they are the r_0 terms in the matrix (9.38)). By a careful choice of the conjugating operator T_D (7.54) and its left and right (approximate) inverses, we can arrange so that these incorporated errors occupy relatively harmless positions in (9.38). The positions are harmless because the $H_{R,L}$ blocks are unaffected, and a subsequent conjugation (by the operator $T_{a,D}$ (9.40)) removes the off-diagonal terms while leaving behind acceptable $O(\epsilon|U|_{L^2})$ errors (step **4** in the proof of Proposition 9.1).

XI. L^2 estimate - use of Garding inequalities. We note first that the estimates on $\chi_M U$ and $\chi_L U$ in (9.6), corresponding to the medium and large frequency regimes, are taken from [MZ].

Indeed, estimates (9.6)(b),(c) are essentially estimates (4.37),(4.28), respectively, in [MZ]. We say "essentially" because, although [MZ] considers Dirichlet boundary conditions, the same argument in the medium and large frequency regions yields estimates for any boundary condition satisfying the uniform Lopatinski condition in those regions. We refer the reader to [GMWZ1], section 3, for more detail on how the standard symmetrizer argument works in those regions for such boundary conditions. Thus, the focus in this paper is almost entirely on the small frequency estimate for $\chi_S U$.

Having quantized the symmetrizer symbol in step **5** (9.71) and introduced spatial and frequency ($\chi_1(\epsilon\zeta)$) cutoffs in step **6**, we proceed in step **7** to obtain the desired bounds on the solution $U_{5,S}$ to (9.73).

We start from the simple identities (9.75) (obtained by integrating $\partial_N(s^\epsilon_D U_{5,S}, U_{5,S})$ on $x_N \geq 0$ and using the equation to rewrite $\partial_N U_{5,S}$). The interior estimates (9.77), (9.79), and (9.80) are done by blocks. The main new point here is the estimate corresponding to the degenerate $s^\epsilon_{p-,D}$ block. Here we rewrite the symbol

$$s^\epsilon_{p-}\mathcal{P}^\epsilon_- \text{ as } \epsilon^2(\frac{1}{\epsilon^2}s^\epsilon_{p-}\mathcal{P}^\epsilon_-) \tag{3.15}$$

and observe that $\frac{1}{\epsilon^2}s^\epsilon_{p-}\mathcal{P}^\epsilon_-$ is a smooth symbol of order two in the mixed calculus satisfying the positivity property (9.57), (9.58). Thus, the mixed Garding inequality gives the estimate (9.79).

The most delicate part of the estimate is the treatment of boundary terms. Here again we have a degeneracy (the one described above in (3.11), but the analysis cannot be done by blocks.

Thus, we introduce pseudodifferential projections, that is, mixed operators whose matrix symbols $\pi_1(x', \epsilon\zeta, \zeta)$ and $\pi_2(x', \epsilon\zeta, \zeta)$ project onto orthogonal invariant subspaces for the operator $S + \Gamma_1^*\Gamma_1$. We note that $\pi_1 = e_{4m}e^*_{4m}$, where $e_{4m} = f_{4m} + |\epsilon\zeta|\mathcal{F}$ and $f_{4m}(x', \epsilon\zeta)$ is obtained by doubling the differentiated profile $\partial_z\mathcal{U}^0$, extending to $\rho > 0$, and transporting by $(\mathcal{T}^0)^{-1}$.

The projectors allow us to quantize the symbolic positivity estimates in (3.11) using Garding inequalities. The classical Garding inequality can be used to estimate $((s^\epsilon_D + \Gamma^*_{1,D}\Gamma_{1,D})\pi_{2,D}U_5, \pi_{2,D}U_5)$ (Prop. 9.4), while the mixed Garding inequality is used for $((s^\epsilon_D + \Gamma^*_{1,D}\Gamma_{1,D})\pi_{1,D}U_5, \pi_{1,D}U_5)$ (Prop. 9.3). Mixed Garding applies since

$$((s^\epsilon + \Gamma_1^*\Gamma_1)\pi_1 U, \pi_1 U) = (B_1^\epsilon v_{4m}, v_{4m}) \tag{3.16}$$

where $B_1^\epsilon = \epsilon^2 b_1(x', \epsilon\zeta, \zeta)$ is 1×1 and b_1 is a smooth mixed symbol of order two satisfying $b_1 \geq c\langle\zeta\rangle^2$ (9.70).

The mixed terms involving both $\pi_{1,D}U_5$ and $\pi_{2,D}U_5$ are shown to give acceptable errors (Proposition 9.2).

XII. Higher derivative estimates . If one simply differentiates the equation (10.3) and throws commutators on the right as new forcing terms, those commutators are unacceptably large errors. Instead, we consider an enlarged system for the new unknown $U^{*,k} = ((\frac{\gamma}{\epsilon^2})^k U, (\frac{\gamma}{\epsilon^2})^{k-1}\partial U, \dots, \partial^k U)$. The system can be put in a simple block diagonal form (10.4). The choice of the power ϵ^2 in the definition of $U^{*,k}$ makes the commutator error appearing on the right in (10.4) an acceptable error.

We can now simply repeat the entire argument of section 9 on this block diagonal system to prove the higher derivative estimates of Proposition 10.1. These estimates involve only the tangential derivatives $(\partial_0, \dots, \partial_{N-1})$.

XIII. Nonlinear stability. Here we take advantage of the large powers of ϵ appearing in the two terms on the right in the nonlinear forward error problem (6.14) to prove convergence of the obvious iteration scheme (11.4), (11.5). To control L^∞ norms we use the Sobolev inequalities (11.9) - it suffices to control just one ∂_N derivative provided one has control of sufficiently many tangential derivatives. The ∂_N control comes from the equation.

The nonlinear term on the right in (11.6) depends on $\partial'' U$ as well as U, so we need to take advantage of the extra gain in the high frequency estimates in order to be able to estimate k derivatives of U_{n+1} with only k derivatives of U_n (as we must do to make the iteration scheme work). This is the point of (11.13) and (11.21). The rest is routine.

Part 2. **Reductions**

4. Reduction to a doubled boundary problem

Following [GW] we make the change of coordinates

$$\tilde{x}' = x',\ \tilde{x}_N = x_N - \Psi^\epsilon(x'), \tag{4.1}$$

where the smooth function Ψ^ϵ remains to be determined. Set $\tilde{u}^\epsilon(\tilde{x}) = u^\epsilon(x)$, and drop the tildes to rewrite (1.2) (suppressing some epsilons) as

$$\sum_{j=0}^{N-1} A_j(u)\partial_{x_j} u + A_N(u, d\Psi)\partial_{x_N} u - \epsilon \sum_1^N (\partial_{x_j} - \partial_{x_j}\Psi\partial_{x_N})^2 u = 0, \tag{4.2}$$

where

$$A_N(u, d\Psi) = A_N(u) - \sum_0^{N-1} A_j(u)\partial_{x_j}\Psi. \tag{4.3}$$

On $\overline{\mathbb{R}}^{N+1}_+ = \{x_N \geq 0\}$ define

$$u^\epsilon_\pm(x) = u^\epsilon(x', \pm x_N),$$

and note that u^ϵ satisfies the free problem (4.2) if and only if $u^\epsilon_\pm$ satisfies the doubled parabolic boundary problem on $\overline{\mathbb{R}}^{N+1}_+$:

$$\begin{aligned} &(a) \sum_0^{N-1} A_j(u_\pm)\partial_{x_j} u_\pm \pm A_N(u_\pm, d\Psi)\partial_{x_N} u_\pm \mp \epsilon \left(\sum_1^{N-1} \partial^2_{x_j}\Psi\right)\partial_{x_N} u_\pm \\ &\quad - \epsilon\left(\sum_1^{N-1} \partial^2_{x_j} + C^\epsilon(x')\partial^2_{x_N} \mp 2\sum_1^{N-1} \partial_{x_j}\Psi\partial_{x_j}\partial_{x_N}\right) u_\pm = 0 \\ &(b) u_+ - u_- = 0 \text{ on } x_N = 0 \\ &(c) \partial_{x_N} u_+ + \partial_{x_N} u_- = 0 \text{ on } x_N = 0, \end{aligned} \tag{4.4}$$

where

$$C^\epsilon(x') = 1 + |\nabla_{x''}\Psi^\epsilon|^2\ . \tag{4.5}$$

At this stage, we decide to look for a function Ψ^ϵ which is polynomial with respect to ϵ, that is:

$$\Psi^\epsilon(x') = \psi_0(x') + \epsilon\psi_1(x') + \cdots + \epsilon^M \psi_M(x'), \tag{4.6}$$

where $\psi_1, \ldots, \psi_M$ remain to be determined.

[GW] constructs an approximate solution to (4.4) of the form $(\tilde{u}^\epsilon_\pm, \Psi^\epsilon)$ where Ψ^ϵ is given by (4.6), and $\tilde{u}^\epsilon_\pm(x) =$

$$\left(\mathcal{U}^0_\pm(x,z) + \epsilon\mathcal{U}^1_\pm(x,z) + \cdots + \epsilon^M \mathcal{U}^M_\pm(x,z)\right)|_{z=\frac{x_N}{\epsilon}} + \epsilon^M r(x). \tag{4.7}$$

Here

$$\mathcal{U}^j_\pm(x,z) = U^j_\pm(x) + V^j_\pm(x', z),$$

$U^0_\pm(x)$ is the original shock (see Remark 3.4), and $V^j_\pm(x',z)$ are boundary layer profiles exponentially decreasing in z. The $\epsilon-$dependence is suppressed in the notation.

Let us write (4.4)(a) as

$$\mathcal{P}^\epsilon_\pm(u^\epsilon_\pm, d\Psi^\epsilon)\partial u^\epsilon_\pm = 0. \tag{4.8}$$

Plugging (4.7) and (4.6) into (4.4) and setting coefficients of distinct powers of ϵ equal to zero yields a sequence of profile equations for the $\psi_j(x')$, $U^j_\pm(x)$, and $V^j_\pm(x',z)$. Under the assumptions of section 2 these equations can be solved to yield $(\tilde{u}^\epsilon_\pm, \Psi^\epsilon)$ satisfying (suppress some epsilons):

$$\begin{aligned} &\mathcal{P}_\pm(\tilde{u}_\pm, d\Psi)\partial\tilde{u}_\pm = \epsilon^M R^{\epsilon,M}_\pm(x)\\ &\tilde{u}_+ - \tilde{u}_- = 0 \text{ on } x_N = 0\\ &\partial_{x_N}\tilde{u}_+ + \partial_{x_N}\tilde{u}_- = 0 \text{ on } x_N = 0, \end{aligned} \tag{4.9}$$

on $[\frac{-T_0}{2}, T_0]$.

Notation 4.1. 1. Set $\Omega_{T_0} = [\frac{-T_0}{2}, T_0] \times \overline{\mathbb{R}}^N_+$, where $\overline{\mathbb{R}}^N_+ = \{(x'', x_N) : x_N \geq 0\}$. Let $H^M_{T_0}(x) = H^M(\Omega_{T_0})$ and $H^M_{T_0}(x') = H^M(b\Omega_{T_0})$.

2. Set $\Omega = \overline{\mathbb{R}}^{N+1}_+ = \{(x_0, x'', x_N) : x_N \geq 0\}$. Let $H^M(x) = H^M(\Omega)$ and $H^M(x') = H^M(b\Omega)$.

3. Set $H^M(\{x_0 = 0, x_N \geq 0\}) = H^M(x'', x_N)$.

4. Many of the functions in this paper have an ϵ-dependence that is usually suppressed in the notation (when it is harmless). For functions with $\pm$ dependence, we set $u = (u_+, u_-)$.

5. z is a placeholder for $\frac{x_N}{\epsilon}$.

6. For $j = 0, \ldots, N$ set $\partial_j = \partial_{x_j}$ and $D_j = \frac{1}{i}\partial_j$.

The remainder $R^{\epsilon,M}_\pm$ is C^∞ and satisfies

$$\begin{aligned} &|\partial^\alpha R^{\epsilon,M}_\pm|_{L^\infty} \leq C_\alpha \epsilon^{-\alpha_N}\\ &|\partial^\alpha R^{\epsilon,M}_\pm|_{L^2} \leq C_\alpha \epsilon^{\frac{1}{2}-\alpha_N}. \end{aligned} \tag{4.10}$$

on Ω_{T_0} for all multi-indices α.

For later reference we record here some properties of the profiles:

Proposition 4.1 ([GW]). *1. $U^0_\pm(x)$, $V^0_\pm(x',z)$, and $d\psi_0(x')$ are independent of (x'', x_N) for $|x'', x_N|$ large. Each is a smooth function of its arguments with derivatives of every order uniformly bounded with respect to (x,z). Moreover, there exist $\delta > 0$, $C > 0$ such that for all x'*

$$|V^0_\pm(x',z)| \leq Ce^{-\delta z}. \tag{4.11}$$

2. For $j \geq 1$ $U^j_\pm(x)$, $V^j_\pm(x',z)$, and $d\psi_j(x')$ vanish for $|x'', x_N|$ large. Each is a smooth function of its arguments, $V^j_\pm$ is exponentially decreasing in z, and

$$U^j_\pm \in H^\infty_{T_0}(x). \tag{4.12}$$

3. The function $r(x)$ in (4.7) lies in $H^\infty_{T_0}(x)$.

4. For $j \geq 1$ the functions $U^j_\pm(x)$, $V^j_\pm(x',z)$, $d\psi_j(x')$, and r can be extended from Ω_{T_0} to Ω so that statements 1-3 continue to hold on Ω. (This gives an extension of $\tilde{u}$ that we'll use later.)

Remark 4.1. If $\psi_1, \ldots, \psi_M$ are not included in (4.6), the profile equations turn out to be overdetermined and consequently unsolvable. See ([GW],4.4).

We seek an exact solution to (4.4) of the form

$$u^\epsilon_\pm = \tilde{u}^\epsilon_\pm + w^\epsilon_\pm, \tag{4.13}$$

where $w^\epsilon_\pm$ satisfies for $x_0 \in [0, T_0]$

$$\begin{aligned} &\mathcal{P}_\pm(\tilde{u}_\pm + w_\pm, d\Psi)\partial w_\pm = \\ &\qquad [\mathcal{P}_\pm(\tilde{u}_\pm, d\Psi) - \mathcal{P}_\pm(\tilde{u}_\pm + w_\pm, d\Psi)]\partial \tilde{u}_\pm - \epsilon^M R^{\epsilon,M}_\pm, \\ &w_+ - w_- = 0 \text{ on } x_N = 0, \\ &\partial_{x_N} w_+ + \partial_{x_N} w_- = 0 \text{ on } x_N = 0. \end{aligned} \tag{4.14}$$

Clearly, we also need some initial condition. If we simply try

$$w_\pm = 0 \text{ on } x_0 = 0,$$

then corner compatibility fails and we can't expect regular solutions $w^\epsilon_\pm$. So we should try to choose initial data of the form

$$w^\epsilon_\pm = \epsilon^{M'} \omega^\epsilon_{0,\pm}(x'', x_N) \text{ on } x_0 = 0, \tag{4.15}$$

which is corner-compatible to sufficiently high order with (4.14). This is done in the next section. Standard parabolic theory (e.g., as in [KL, E]) then gives existence on some $[0, T_\epsilon]$. The task remains of showing existence on $[0, T_0]$ for small enough ϵ.

5. Corner compatibility

Let $w_{0,\pm}(x'', x_N) = w_\pm|_{x_0=0}$. Corner compatibility to a given high enough order is arranged by correctly specifying $\partial^k_N w_{0,\pm}$ for $k = 1, \dots, k_0$, for k_0 large enough, at the corner $x_0 = 0, x_N = 0$.

For $k = 0, 1$ choose $\partial^k_N w_{0,\pm}(0, x'', 0)$ to be any functions satisfying the boundary conditions, say, the constant function zero in both cases (the choice of compatible data is far from unique).

For $k = 2$ use the interior equation (4.14) and the differentiated boundary condition

$$\partial_0(w_+ - w_-) = 0$$

to determine

$$\partial^2_N w_\pm(0, x'', 0) = \epsilon^{M-1} a_{2,\pm}(x'').$$

Then differentiate the interior equation with ∂_N and use the boundary condition

$$\partial_0(\partial_N w_+ + \partial_N w_-) = 0$$

to get

$$\partial^3_N w_\pm(0, x'', 0) = \epsilon^{M-2} a_{3,\pm}(x'').$$

Here $a_k \in H^\infty$. Continue in this way. Then, for a smooth cutoff $\rho(x_N)$ identically equal to 1 near $x_N = 0$, take (with slightly modified a_k)

$$w_{0,\pm}(x'', x_N) = \rho(x_N)[x^2_N a_{2,\pm}(x'')\epsilon^{M-1} + \cdots + x^{k_0}_N a_{k_0,\pm}\epsilon^{M-k_0+1}]. \tag{5.1}$$

If k_0 is odd, then for $j = 1, \dots, \frac{k_0-1}{2}$ this choice of initial data is compatible with

$$\begin{aligned} &\partial^j_0(w_+ - w_-) = 0 \\ &\partial^j_0(\partial_N w_+ + \partial_N w_-) = 0 \end{aligned} \tag{5.2}$$

at the corner. In this case we say the initial data is *corner compatible to order* $\frac{k_0-1}{2}$.

Observe that for a given choice of k_0

$$w_\pm|_{x_0=0} = \epsilon^{M-k_0+1}\omega^\epsilon_{0,\pm}(x'', x_N), \tag{5.3}$$

where $\omega_{0,\pm} \in H^\infty(x'', x_N)$ uniformly with respect to ϵ..

Remark 5.1. Thus, we have now a *carefully constructed* choice of initial data for (4.4), namely

$$u^\epsilon_\pm = \tilde{u}^\epsilon_\pm(0, x'', x_N) + \epsilon^{M-k_0+1}\omega^\epsilon_{0,\pm}(x'', x_N) \text{ on } x_0 = 0. \tag{5.4}$$

6. Reduction to a forward problem

We will look for an exact solution $u^\epsilon_\pm$ to (4.4) as a perturbation $\tilde{u}^\epsilon_\pm + w_\pm$ of $\tilde{u}^\epsilon_\pm$. Our next task is to put the "error equation" for w in convenient form.

Notation 6.1. 1. Set $\mathcal{A}_N(u, d\Psi) = A_N(u, d\Psi) - \epsilon\left(\sum_1^{N-1} \partial^2_{x_j}\Psi\right)$ where, we recall,

$$A_N(u, d\Psi) = A_N(u) - \sum_0^{N-1} A_j(u)\partial_{x_j}\Psi, \tag{6.1}$$

and $A_j = df_j$, $j = 0, \dots, N$, $f_0(u) = u$.

2. Let $\mathcal{F}_N(u, d\Psi) \equiv f_N(u) - \sum_0^{N-1} f_j(u)\partial_{x_j}\Psi - \epsilon(\sum_1^{N-1} \partial^2_{x_j}\Psi)u$.
3. Let $\mathcal{F}_\pm(u, d\Psi) = (f_1(u), \dots, f_{N-1}(u), \pm\mathcal{F}_N(u, d\Psi))$.
4. Let $\mathbb{E}_\pm(d\Psi, \partial^2) = \sum_1^{N-1}\partial^2_{x_j} + C(x')\partial^2_{x_N} \mp 2\sum_1^{N-1}\partial_{x_j}\Psi(x')\partial_{x_j}\partial_{x_N}$.
5. Let $\mathcal{H}_\pm(v, d\Psi)w \equiv (A_1(v)w, \dots, A_{N-1}(v)w, \pm\mathcal{A}_N(v, d\Psi)w)$.
6. For $j = 1, \dots, N-1$ let $Q_j(v, w) = f_j(v + w) - f_j(v) - A_j(v)w$ and set

$$\begin{aligned} &Q_N(v, w) \equiv \mathcal{F}_N(v + w, d\Psi) - \mathcal{F}_N(v, d\Psi) - \mathcal{A}_N(v, d\Psi)w, \\ &\mathcal{Q}_\pm(v, w) = (Q_1(v, w), \dots, \pm Q_N(v, w)). \end{aligned} \tag{6.2}$$

$\mathcal{Q}_\pm$ is at least quadratic in w.

7. Set $B(u\pm) = \begin{pmatrix} u_+ - u_- \\ \partial_N u_+ + \partial_N u_- \end{pmatrix}|_{x_N=0}$.
8. Let $\partial_j = \partial_{x_j}$ and $\nabla = (\partial_1, \dots, \partial_N)$, so $\nabla \cdot U = \operatorname{div} U$.

In this notation the doubled parabolic problem satisfied by $\tilde{u}_\pm$, (4.9), is

$$\begin{aligned} &\partial_0\tilde{u}_\pm + \nabla\cdot(\mathcal{F}_\pm(\tilde{u}_\pm, d\Psi)) - \epsilon\mathbb{E}_\pm(d\Psi, \partial^2)\tilde{u}_\pm = \epsilon^M R^{\epsilon,M}_\pm(x) \\ &B(\tilde{u}_\pm) = 0. \end{aligned} \tag{6.3}$$

Now $u = \tilde{u} + w$ will be an exact solution of (4.4) provided $w_\pm$ is a solution to the *error problem*

$$\begin{aligned} &(a)\ \partial_0 w_\pm + \nabla\cdot(\mathcal{H}_\pm(\tilde{u}_\pm, d\Psi)w_\pm) - \epsilon\mathbb{E}_\pm(d\Psi, \partial^2)w_\pm = \\ &\qquad -\nabla\cdot(\mathcal{Q}_\pm(\tilde{u}_\pm, w_\pm)) - \epsilon^M R^{\epsilon,M}_\pm(x) \\ &(b)\ B(w_\pm) = 0 \\ &(c)\ w_\pm|_{x_0=0} = w_{0,\pm}(x'', x_N) = \epsilon^{M_0}\zeta^\epsilon_\pm(x'', x_N), \end{aligned} \tag{6.4}$$

where $\zeta_\pm \in H^\infty(x'', x_N)$ is chosen so that $w_{0,\pm}$ is corner compatible to order k_0. Here k_0 and $M_0 = M_0(M, k_0) < M$ can be taken large provided $M >> k_0$.

Remark 6.1. The "bad term" with coefficient of order $\frac{1}{\epsilon}$ in (6.4) arises when $\tilde{u}_\pm$ in $\nabla\cdot\mathcal{H}_\pm$ is differentiated with ∂_N.

6.1. **Reduction to a forward problem with nonhomogeneous boundary conditions.** We proceed to replace (6.4) by a problem with nonhomogeneous boundary and interior forcing both supported in $x_0 \geq 0$. The reduction is carried out in the following steps:

1. First extend $\tilde{u}$ and Ψ as indicated in Proposition 4.1 from Ω_{T_0} to Ω. Extend $R^{\epsilon,M}_{\pm}$ to Ω to have support in $|x_0| < T_0 + 1$ and so that the estimates (4.10) still hold.

Next, ignore the boundary and extend the function ζ^ϵ in (6.4)(c) without any loss of regularity into $x_N < 0$. Similarly extend $\tilde{u}_\pm$ and $R^{\epsilon,M}_{\pm}$ so that (6.4)(a) is now an equation on the full space $\mathbb{R}^{N+1}$.

2. Consider the initial value problem for the new unknown $w_{1,\pm}$ given by the extended (6.4)(a) and the extended initial data. Let $\mathcal{G}_\pm(w_{1,\pm})$ denote the expression obtained by replacing $w_\pm$ by $w_{1,\pm}$ in (6.4)(a) and subtracting the right side from the left. Construct a k_0-th order solution at $x_0 = 0$ that is, a function $w_{1,\pm}$ satisfying

$$\partial_0^j(\mathcal{G}_\pm(w_{1,\pm})) = 0 \text{ at } x_0 = 0 \text{ for } j = 1, \ldots, k_0. \tag{6.5}$$

$w_{1,\pm}$ should be defined for all x_0, but supported in $[-\delta, \delta]$ for some $\delta > 0$ (easily arranged by multiplying by a cutoff $\chi(x_0)$ identically one near 0). The high power of ϵ in the initial data is useful here, since each time the equation is used to solve for some $\partial_0^j w_{1,\pm}|_{x_0=0}$, a factor of $\frac{1}{\epsilon}$ is introduced.

3. Define

$$\tilde{\mathcal{G}}_\pm \equiv \begin{cases} \mathcal{G}_\pm(w_{1,\pm}), & x_0 \geq 0 \\ 0, & x_0 < 0 \end{cases}, \tag{6.6}$$

which lies in $H^{k_0+1}(x)$ by (6.5).

4. Corner compatibility conditions on the original initial data $w_{0,\pm}$ imply the function

$$g \equiv \begin{cases} B(w_{1,\pm}), & x_0 \geq 0 \\ 0, & x_0 < 0 \end{cases} \tag{6.7}$$

belongs to $H^{k_0+1}(x')$.

5. Looking for a solution to (6.4) of the form $w_\pm = w_{1,\pm} + w_{2,\pm}$, we have reduced to solving the forward boundary problem on $x_N \geq 0$:

$$\begin{aligned} &(a)\ \partial_0 w_{2,\pm} + \nabla \cdot (\mathcal{H}_\pm(\tilde{u}_\pm, d\Psi) w_{2,\pm}) - \epsilon \mathbb{E}_\pm(d\Psi, \partial^2) w_{2,\pm} = \\ &-\nabla \cdot [\mathcal{Q}_\pm(\tilde{u}_\pm, w_{1,\pm} + w_{2,\pm}) - \mathcal{Q}_\pm(\tilde{u}_\pm, w_{1,\pm})] - \tilde{\mathcal{G}}_\pm \\ &(b)\ B(w_{2,\pm}) = -g \\ &(c)\ w_{2,\pm} = 0 \text{ in } x_0 < 0. \end{aligned} \tag{6.8}$$

Remark 6.2. Fix M_1 large. Provided k_0 and M were taken large enough in the construction above, the functions $w_{1,\pm}$ and $\tilde{\mathcal{G}}_\pm$ (resp. g) can be taken to have the form

$$\begin{aligned} &\epsilon^{M_1} f^\epsilon(x), \text{ for } f^\epsilon \in H^{M_1}(x), \\ &(\text{resp.}, \epsilon^{M_1} h^\epsilon(x'), \text{ for } h^\epsilon \in H^{M_1}(x')) \end{aligned} \tag{6.9}$$

uniformly with respect to ϵ.

6.2. **Reduction to a forward problem with homogeneous boundary conditions.** Next we transfer the nonzero boundary data of (6.8) to interior forcing:

1. Look for a solution to (6.8) of the form $w_{2,\pm} = u'_\pm + v'_\pm$, where the first two terms of the Taylor series of u' at $x_N = 0$ are chosen so that u' satisfies the boundary condition (6.8)(b). Then extend u' without loss of regularity into $x_N \geq 0$ so that its support lies in $x_0 \geq 0$.

2. Let $\mathcal{K}_\pm(x)$ denote the expression obtained by replacing $w_{2,\pm}$ by $u'_\pm$ in (6.8)(a) and subtracting the right side from the left. We have now reduced to solving the forward problem with homogeneous boundary conditions

$$\begin{aligned}&(a)\ \partial_0 v'_\pm + \nabla\cdot(\mathcal{H}_\pm(\tilde u_\pm, d\Psi)v'_\pm) - \epsilon\mathbb{E}_\pm(d\Psi,\partial^2)v'_\pm = \\ &\qquad -\nabla\cdot[\mathcal{Q}_\pm(\tilde u_\pm, w_{1,\pm}+u'_\pm+v'_\pm) - \mathcal{Q}_\pm(\tilde u_\pm, w_{1,\pm}+u'_\pm)] - \mathcal{K}_\pm. \\ &(b)\ B(v'_\pm) = 0 \\ &(c)\ v'_\pm = 0 \text{ in } x_0 < 0.\end{aligned} \tag{6.10}$$

Remark 6.3. Fix M_2 large. Provided k_0 and M were taken large enough in the construction above, the functions

$$u'_\pm, w_{1,\pm}, \text{ and } \mathcal{K}$$

can be chosen with the form

$$\epsilon^{M_2} f^\epsilon(x), \text{ for } f^\epsilon \in H^{M_2}(x) \tag{6.11}$$

uniformly with respect to ϵ. In addition,

$$\begin{aligned}&\text{supp } u' \cup \text{supp } \mathcal{K} \subset \{0 \leq x_0 \leq T_0 + 1\}, \\ &\text{supp } w_{1,\pm} \subset \{-\delta \leq x_0 \leq \delta\}.\end{aligned} \tag{6.12}$$

Next relabel $v' = w$, $b = w_1 + u'$ and write (6.10) in a simpler form

$$\begin{aligned}&(a)\ \partial_0 w_\pm + \nabla\cdot(\mathcal{H}_\pm(\tilde u_\pm, d\Psi)w_\pm) - \epsilon\mathbb{E}_\pm(d\Psi,\partial^2)w_\pm = \\ &\qquad -\nabla\cdot[\mathcal{Q}_\pm(\tilde u_\pm, b_\pm + w_\pm) - \mathcal{Q}_\pm(\tilde u_\pm, b_\pm)] - \mathcal{K}_\pm \\ &(b)\ B(w_\pm) = 0 \\ &(c)\ w_\pm = 0 \text{ in } x_0 < 0.\end{aligned} \tag{6.13}$$

Let us write $b_\pm = \epsilon^{M_2}\tilde b_\pm$, $\mathcal{K} = \epsilon^{M_2}F$, and $w = \epsilon^L \tilde w$, for $L \leq M_2$. Drop tildes, relabel $M_2 = M$, and cancel ϵ^L to obtain our final form for the error problem on $x_N \geq 0$:

$$\begin{aligned}&(a)\ \partial_0 w_\pm + \nabla\cdot(\mathcal{H}_\pm(\tilde u_\pm, d\Psi)w_\pm) - \epsilon\mathbb{E}_\pm(d\Psi,\partial^2)w_\pm = \\ &\qquad -\nabla\cdot\big((\epsilon^M b_\pm, \epsilon^L w_\pm)\mathcal{N}(\tilde u_\pm, d\Psi, \epsilon^M b_\pm, \epsilon^L w_\pm)w_\pm\big) - \epsilon^{M-L}F_\pm \\ &(b)\ B(w_\pm) = 0 \\ &(c)\ w_\pm = 0 \text{ in } x_0 < 0.\end{aligned} \tag{6.14}$$

Here, we've used that $\mathcal{Q}(\tilde u, p) = O(|p|^2)$ and introduced an obvious notation in defining $\mathcal{N}$. Moreover, $b_\pm$ and $F_\pm$ are in $H^M(x)$ uniformly with respect to ϵ, and

$$\text{supp } F \subset \{0 \leq x_0 \leq T_0 + 1\}. \tag{6.15}$$

Recall that M in (6.14) can be taken arbitrarily large as long as the approximate solution $\tilde u$ is constructed with sufficiently many terms.

To complete the study of the small viscosity limit, it is enough to show that for some $\epsilon_0 > 0$, this problem has a solution on Ω for $0 < \epsilon < \epsilon_0$.

The first step is to obtain an L^2 estimate for the forward linearized problem on Ω:

$$\begin{aligned}&(a)\ \partial_0 w_\pm + \nabla\cdot(\mathcal{H}_\pm(\tilde u_\pm, d\Psi)w_\pm) - \epsilon\mathbb{E}_\pm(d\Psi,\partial^2)w_\pm = F_\pm\\ &(b)\ B(w_\pm)=0 \text{ on } x_N=0\\ &(c)\ w_\pm = 0 \text{ in } x_0<0,\end{aligned}\tag{6.16}$$

where

$$\text{supp } F\subset\{0\le x_0\le T_0+1\}.\tag{6.17}$$

Part 3. Symbolic preparation

7. Evans functions, conjugators, block structure, symmetrizers

7.1. $4m\times 4m$ **first order system.** We rewrite the problem yet again, this time putting it in a form needed for the symmetrizer argument to follow.

Perform the differentiations on the $\mathcal{H}$ term in (6.16), set $U_\pm=(w_\pm,\epsilon\partial_N w_\pm)$ and observe that (6.16)(a) can be rewritten as

$$\partial_N U_\pm - \frac{1}{\epsilon}G_\pm U_\pm = C^\epsilon(x')^{-1}F_\pm,\tag{7.1}$$

where we have relabelled $\begin{pmatrix}0\\F_\pm\end{pmatrix}$ as $F_\pm$. Here

$$G_\pm = \begin{pmatrix}0 & I\\ M_\pm & A_\pm\end{pmatrix}\tag{7.2}$$

with

$$\begin{aligned}&M_\pm = C^\epsilon(x')^{-1}\left[\epsilon\partial_0+\sum_1^{N-1}A_j(\tilde u_\pm)\epsilon\partial_j-\sum_1^{N-1}\epsilon^2\partial_j^2+E_\pm\right]\\ &E_\pm w_\pm = \pm(\partial_u\mathcal{A}_N(\tilde u_\pm,d\Psi)w_\pm)\partial_z\tilde u_\pm+\sum_1^{N-1}(\partial_u A_j(\tilde u_\pm)w_\pm)\epsilon\partial_j\tilde u_\pm\\ &A_\pm = C^\epsilon(x')^{-1}\left[\pm\mathcal{A}_N(\tilde u_\pm,d\Psi)\pm 2\sum_1^{N-1}\partial_j\Psi\epsilon\partial_j\right].\end{aligned}\tag{7.3}$$

To prove weighted estimates we introduce $\tilde U_\pm = e^{-\gamma x_0}U_\pm$, $\tilde F_\pm = e^{-\gamma x_0}F_\pm$ and observe that (7.1) is equivalent to

$$\partial_N\tilde U_\pm - \frac{1}{\epsilon}G^\gamma_\pm\tilde U_\pm = C^\epsilon(x')^{-1}\tilde F_\pm,\tag{7.4}$$

where $G^\gamma_\pm$ is the same as $G_\pm$ except that the ∂_0 in $M_\pm$ is replaced by $\partial_0+\gamma$.

Drop tildes, define the $2m\times 4m$ matrix Γ and the $4m\times 4m$ matrix $\mathcal{G}$ by

$$\begin{aligned}&\Gamma = \begin{pmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix}\\ &\mathcal{G} = \begin{pmatrix}G^\gamma_+ & 0\\ 0 & G^\gamma_-\end{pmatrix},\end{aligned}\tag{7.5}$$

and let $U = \begin{pmatrix} U_+ \\ U_- \end{pmatrix}$ to obtain the following equivalent form ($\mathcal{G}$ form) for the doubled boundary problem (6.16):

$$\begin{aligned} &\partial_N U - \frac{1}{\epsilon}\mathcal{G}U = F \\ &\Gamma U = 0 \text{ on } x_N = 0 \\ &U = 0 \text{ in } x_0 < 0. \end{aligned} \tag{7.6}$$

Here we have relabeled $C(x')^{-1}F$ as F.

7.2. **Limiting and frozen problems.** Let us write the approximate solution $\tilde{u}^\epsilon(x)$ in (4.7) as

$$\tilde{u}^\epsilon_\pm(x) = u^\epsilon_{1\pm}(x) + u^\epsilon_{2\pm}(x', z)|_{z=\frac{x_N}{\epsilon}}, \tag{7.7}$$

where for some $\delta > 0$

$$\begin{aligned} &|u^\epsilon_2(x', z)| + |\nabla_{x'} u^\epsilon_2(x', z)| + |\partial_z u^\epsilon_2(x', z)| \le Ce^{-\delta z}, \\ &u^\epsilon_{1\pm}(x) = U^0_\pm(x) + O(\epsilon), \end{aligned} \tag{7.8}$$

and $(U^0_\pm(x), \psi_0(x'))$ is the original (ideal) shock. We also have

$$d\Psi^\epsilon(x') = d\psi_0(x') + O(\epsilon). \tag{7.9}$$

Definition 7.1. The limiting problem corresponding to (7.6) is

$$\begin{aligned} &\partial_N U - \frac{1}{\epsilon}\mathcal{G}_\infty U = F \\ &\Gamma \mathcal{W}_D U = 0 \text{ on } x_N = 0 \\ &U = 0 \text{ in } x_0 < 0, \end{aligned} \tag{7.10}$$

where $\mathcal{G}_\infty$ is the matrix with blocks

$$\begin{pmatrix} 0 & I \\ M_{\infty,\pm} & A_{\infty,\pm} \end{pmatrix}$$

obtained from those of $\mathcal{G}$ by letting $z = \frac{x_N}{\epsilon} \to +\infty$, and $\mathcal{W}_D$ is a pseudodifferential operator whose symbol is constructed later in this section. We have (dropping epsilons)

$$\begin{aligned} &M_{\infty,\pm} = C(x')^{-1}\left[\epsilon(\partial_0 + \gamma) + \sum_1^{N-1} A_j(u_{1,\pm})\epsilon\partial_j - \sum_1^{N-1} \epsilon^2\partial_j^2 + E_{\infty,\pm}\right] \\ &E_{\infty,\pm}w_\pm = \sum_1^{N-1}(\partial_u A_j(u_{1,\pm})w_\pm)\epsilon\partial_j u_{1,\pm} \\ &A_{\infty,\pm} = C(x')^{-1}\left[\pm\mathcal{A}_N(u_{1,\pm}, d\Psi) \pm 2\sum_1^{N-1}\partial_j\Psi\epsilon\partial_j\right]. \end{aligned} \tag{7.11}$$

Notation 7.1. 1. Recall that the x'-dependence in $C^\epsilon(x')$ enters through $d\Psi^\epsilon$ (4.5). Thus, the $x-$dependence in the coefficients of $\mathcal{G}$ enters through

$$\begin{aligned} &p^\epsilon_{1,\pm} = u_{1,\pm},\ p^\epsilon_2 = d_{x'}\Psi, \\ &p^\epsilon_3 = (\epsilon\nabla_{x''}u_{1,+}, \epsilon\nabla_{x''}u_{1,-}, \epsilon\sum_1^{N-1}\partial^2_{x_j}\Psi), \\ &u_{2,\pm},\ \epsilon\nabla_{x''}u_{2,\pm},\ \text{ and } \partial_z u_{2,\pm}. \end{aligned} \tag{7.12}$$

In view of (7.8) the x-dependence of $\mathcal{G}_\infty$ enters just through

$$p^\epsilon(x) = (p^\epsilon_{1,+}, p^\epsilon_{1,-}, p^\epsilon_2, p^\epsilon_3), \tag{7.13}$$

where p^ϵ varies in some neighborhood of

$$\underline{p} = (U^0_+(0), U^0_-(0), d\psi_0(0), 0) \in \mathbb{R}^{2m} \times \mathbb{R}^N \times \mathbb{R}^{2m(N-1)+1}. \tag{7.14}$$

For later reference we note, setting $\epsilon = 0$, that

$$p^0(x) = (U^0_+(x), U^0_-(x), d\psi_0(x'), 0), \tag{7.15}$$

the first three components giving the original inviscid shock.

2. Let us set $\beta = \epsilon\zeta$

$$\begin{aligned} &\mathcal{V}_{\epsilon,\pm}(x', z) = (u_{2,\pm}, \epsilon\nabla_{x''} u_{2,\pm}, \partial_z u_{2,\pm})(x', z), \\ &\mathcal{V}_\epsilon(x', z) = (\mathcal{V}_{\epsilon,+}(x', z), \mathcal{V}_{\epsilon,-}(x', z)) \end{aligned} \tag{7.16}$$

and write the symbols associated to the operators $\mathcal{G}$, $\mathcal{G}_\infty$ as

$$\begin{aligned} &\mathcal{G} = \mathcal{G}(\mathcal{V}_\epsilon(x', z), p^\epsilon(x), \beta) \\ &\mathcal{G}_\infty = \mathcal{G}_\infty(p^\epsilon(x), \beta). \end{aligned} \tag{7.17}$$

3. Observe that

$$\mathcal{V}_{0,\pm}(x', z) = (V^0_\pm, 0, \partial_z V^0_\pm)(x', z) \tag{7.18}$$

where $V^0_\pm$ are the leading profiles from (4.7).

Remark 7.1. For the purposes of proving an L^2 estimate we would normally (e.g., when the uniform Lopatinski condition is satisfied) drop the second term in the expression for $E_\pm w_\pm$, since doing so would result just in an absorbable $O(|U|_{L^2})$ error. Similarly, if we were free to localize in x' we could treat it as another nearly constant parameter, say p^ϵ_4. Such localization would introduce another error of the same order. However, since our symmetrizer is degenerate, our estimate (9.1) cannot absorb such errors. So instead we proceed as above.

In preparation for the construction of the conjugator that will allow us to replace the original $\mathcal{G}$ problem with the limiting $\mathcal{G}_\infty$ problem, we define the associated frozen coefficient problems (only p is frozen):

$$\begin{aligned} &(a) U_z - \mathcal{G}(\mathcal{V}_\epsilon(x', z), p, \beta) U = F \\ &(b) U_z - \mathcal{G}_\infty(p, \beta) = F \end{aligned} \tag{7.19}$$

Remark 7.2. The estimates (7.8) show that when one passes to the limit as $z \to \infty$ in (7.19)(a), the (x', z) dependence in $\mathcal{G}$ disappears.

7.3. Spectral properties of $\mathcal{G}_\infty(p, \beta)$.

Notation 7.2. 1. Recall $\beta = (\beta', \gamma')$ is a placeholder for $\epsilon\zeta$), and introduce polar coordinates

$$\beta = \rho\widehat{\beta}, \text{ where } \widehat{\beta} = (\widehat{\beta'}, \widehat{\gamma'}) \text{ and } \widehat{\beta} \in S^N. \tag{7.20}$$

We'll always take $\gamma' \geq 0$, so define $S^N_+ = S^N \cap \{\widehat{\gamma'} \geq 0\}$.

Notation 7.3. Given a function $q(x)$ shall sometimes denote $q(x', 0)$ by $q(x')$. For example, $U^0_\pm(x', 0) = U^0_\pm(x')$.

Remark 7.3. Observe that smooth functions $f(\beta)$ of $\beta \in \mathbb{R}^{N+1}$ can be rewritten as smooth functions $f(\widehat{\beta},\rho)$ with $(\widehat{\beta},\rho) \in S^N \times \overline{\mathbb{R}}_+$. For such functions we'll use both notations interchangeably. However, when $f(\widehat{\beta},0)$ is not constant on S^N, the function $f(\beta)$ corresponding to $f(\widehat{\beta},\rho)$ is not continuous at $\beta = 0$.

Proposition 7.1 ([Z, ZS])**.**
1. Assume $p_{1,\pm} \in \mathcal{O}_\pm$, $p_2 \in \mathbb{R}^N$, and $p_3 = 0$. When $\rho > 0$ and $\gamma' \geq 0$, $\mathcal{G}_\infty(p,\beta)$ has $2m$ eigenvalues counted with multiplicities in $\Re\mu > 0$ and $2m$ eigenvalues in $\Re\mu < 0$.

*2. $\mathcal{G}_\infty(p^0(x'),0)$ has 0 as a semisimple eigenvalue of multiplicity $2m$. The nonvanishing eigenvalues (*fast modes*) are those of*

$$(7.21)\qquad \begin{aligned} &A_{\infty,+}(p^0(x'),0)\ (k \text{ positive, } m-k \text{ negative) and}\\ &A_{\infty,-}(p^0(x'),0)\ (l \text{ positive, } m-l \text{ negative)} \end{aligned}$$

3. Consider the multiple zero eigenvalue of $\mathcal{G}_\infty(p^0(x'),\widehat{\beta},0)$ (polar coordinates). For $\widehat{\gamma}' > \delta > 0$, this eigenvalue splits for $\rho > 0$ small into $k + l = m - 1$ slow decaying modes

$$(7.22)\qquad \mu = c_\delta \rho + O(\rho^2) \text{ where } \Re c_\delta < 0$$

and $(m - k) + (m - l) = m + 1$ slow growing modes ($\Re c_\delta > 0$).

Here "decaying" and "growing" refer to the corresponding exponential solutions $e^{\mu z}v$ of (7.19)(b). A proof of Proposition 7.1 is also given in [GMWZ1], Proposition 2.1, where a slightly different reduction of the original problem to a first order system is used.

7.4. **Evans function on the whole line.** In order to make a clear connection with the earlier work [Z, ZS] on planar shocks, we first define the Evans function for the curved shock problem on the whole space, and then relate this to the Evans function for the doubled boundary problem.

Note first that the profiles (recall Notation 7.3)

$$\mathcal{U}^0_\pm(x',z) = U^0_\pm(x') + V^0_\pm(x',z)$$

as in (4.7), defined for $z \geq 0$, patch together to give a smooth profile on $\mathbb{R}_z$:

$$(7.23)\qquad \mathcal{U}^0(x',z) = \begin{cases} \mathcal{U}^0_+(x',z), & z \geq 0 \\ \mathcal{U}^0_-(x',-z), & z \leq 0 \end{cases}.$$

Setting

$$\mathbb{G}(u,d\phi) \equiv f_N(u) - \sum_0^{N-1} f_j(u)\partial_j\phi,$$

we recall that the profile $\mathcal{U}^0(x',z)$ is constructed in [GW] as a solution of the "travelling wave" equation

$$(7.24)\qquad \begin{aligned} &C^0(x')\partial_z\mathcal{U}^0 = \mathbb{G}(\mathcal{U}^0,d\psi_0) - \mathbb{G}(U^0_-,d\psi_0)\\ &\lim_{z\to\pm\infty}\mathcal{U}(x',z) = U^0_\pm(x'). \end{aligned}$$

Definition 7.2. Define the $2m \times 2m$ matrix for $z \in \mathbb{R}$

$$G_0(x', z, \beta) = \begin{pmatrix} 0 & I \\ M^0 & A^0 \end{pmatrix}, \quad \text{where}$$

$$\begin{aligned}
&M^0(x', z, \beta) = \\
&C^0(x')^{-1}\left[(i\beta_0 + \gamma') + \sum_1^{N-1} A_j(\mathcal{U}^0(x', z))i\beta_j + \sum_1^{N-1} \beta_j^2 + E^0(x', z)\right], \\
&E^0(x', z)w = (\partial_u A_N(\mathcal{U}^0(x', z), d\psi_0(x'))w)\, \partial_z \mathcal{U}^0(x', z), \\
&A^0(x', z, \beta) = C^0(x')^{-1}\left[A_N(\mathcal{U}^0(x', z), d\psi_0(x')) + 2\sum_1^{N-1} \partial_j \psi_0(x') i\beta_j\right].
\end{aligned} \tag{7.25}$$

We shall also work with the $4m \times 4m$ matrices on $z \geq 0$ given by

$$\begin{aligned}
&\mathcal{G}_0(x', z, \beta) \equiv \mathcal{G}(\mathcal{V}_0(x', z), p^0(x'), \beta), \\
&\mathcal{G}_{\infty,0}(x', \beta) \equiv \mathcal{G}_\infty(p^0(x'), \beta),
\end{aligned} \tag{7.26}$$

where the matrices on the right are obtained from those in (7.17) by setting $\epsilon = 0$ and evaluating on the inviscid shock.

Remark 7.4. The matrix G_0 is the same as the upper left block of $\mathcal{G}_0$, except that the latter matrix is restricted to $z \geq 0$.

The Evans function is a Wronskian of solutions to the following $2m \times 2m$ system on $\mathbb{R}_z$ in which x' is a smoothly varying parameter:

$$\mathbb{U}_z - G_0(x', z, \beta)\mathbb{U} = 0. \tag{7.27}$$

Lemma 7.1. *For $\beta = (\beta', \gamma')$ with $\gamma' > 0$, there exist bases of solutions*

$$\{\mathbb{U}_1^R(x', z, \beta), \cdots, \mathbb{U}_m^R\},\ \{\mathbb{U}_1^L, \cdots, \mathbb{U}_m^L\} \tag{7.28}$$

of (7.27) spanning the stable/unstable manifolds at $z = +\infty/-\infty$, respectively, such that

$$\mathcal{D}(x', \beta) \equiv \det(\mathbb{U}_1^R, \cdots, \mathbb{U}_m^R, \mathbb{U}_1^L, \cdots, \mathbb{U}_m^L)|_{z=0} \tag{7.29}$$

is C^∞ in x', analytic in β, and continuously extendible to $\gamma' = 0$.

Proof. The proof in [ZS], based on the Gap Lemma of [GZ] and Proposition 7.1, works as well in the presence of the parameter x'. □

Definition 7.3. The function $\mathcal{D}(x', \beta)$ in (7.29) is called the Evans-Lopatinski determinant (or *Evans function* for short) for the problem (7.27). We always take the solutions defining the columns in (7.29) to be of size ~ 1.

Remark 7.5. In $\rho > 0$ we may write $\mathcal{D}(x', \beta) = \mathcal{D}(x', \widehat{\beta}, \rho)$. The argument of [ZS] Lemma 5.1 shows that $\mathcal{D}(x', \widehat{\beta}, \rho)$ and $\partial_\rho \mathcal{D}(x', \widehat{\beta}, \rho)$ are analytic in $(\widehat{\beta}, \rho)$ on $\{\widehat{\gamma'} > 0, \rho > 0\}$ and continuously extendible to $\{\widehat{\gamma'} \geq 0, \rho \geq 0\}$.

7.5. **Assumption on the viscous profile.**

Recall from the Introduction the assumption (H7):

(H7) For each $x' \in \mathbb{R}^N$, $\mathcal{D}(x', \widehat{\beta}, \rho)$ vanishes to precisely first order at $\rho = 0$ (where it must vanish) for all $\widehat{\beta} \in S_+^N$, and has no other zeros in $S_+^N \times \overline{\mathbb{R}}_+$.

Remark 7.6. 1. Nonvanishing of $\mathcal{D}(x',\beta)$ for $\gamma'>0$ is necessary even for linearized stability of the viscous boundary layer. See Remark 1.3 of [GMWZ1].

2. For Lax shocks the argument of Proposition 5.3 of [ZS] implies

$$\mathcal{D}(x',\widehat{\beta},\rho)=C\kappa(x')\Delta(x',\widehat{\beta})\rho+o(\rho) \tag{7.30}$$

as $\rho\to 0$, for some $C\neq 0$. Here $\kappa(x')$ is nonvanishing if and only if the stable/unstable manifolds for $U^0_+(x')/U^0_-(x')$ of the travelling wave ODE (7.24) are transverse at the connection $\mathcal{U}^0(x',z)$. $\Delta(x',\widehat{\beta})$ is the Lopatinski-Kreiss-Majda determinant for the ideal shock problem linearized at $(U^0_\pm,d\psi_0)$. Since uniform stability of the inviscid shock is equivalent to nonvanishing of $\Delta(x',\widehat{\beta})$ for $\widehat{\beta}\in S^N_+$, this explains why Assumption (H6) implies (H5)$_r$ for Lax shocks.

The computation giving (7.30) shows that

$$\mathcal{D}(x',\widehat{\beta},\rho)\in C^\infty(x',C(S^N_+,C^1(\overline{\mathbb{R}}_+)). \tag{7.31}$$

3. The nonvanishing of both κ and Δ is needed to carry out the construction of the high order approximate solution in [GW]. Thus, the assumptions in section 2 imply the hypotheses of [GW] are satisfied.

4. The vanishing of $\mathcal{D}(x',\hat{\beta},0)$ reflects the fact that at $\rho=0$ equation (7.27) has the solution $\mathbb{U}(x',z)=(\phi,\phi_z)$, where $\phi=\partial_z\mathcal{U}^0(x',z)$ (differentiate (7.24) twice). This solution is fast-decaying at both $\pm\infty$. It will be convenient later to normalize

$$\mathbb{U}^R_1(x',z,\widehat{\beta},0)=\mathbb{U}^L_m(x',z,\widehat{\beta},0)=(\phi(x',z),\phi_z(x',z)). \tag{7.32}$$

7.6. **Evans function for the doubled boundary problem.**

Notation 7.4. 1. Given a function $\mathbb{U}(z)=\begin{pmatrix}u(z)\\v(z)\end{pmatrix}$ defined for $z\in\mathbb{R}$, we set for $z\geq 0$

$$\begin{aligned}&\mathbb{U}_+(z)=\begin{pmatrix}u(z)\\v(z)\end{pmatrix}\\&\mathbb{U}_-(z)=\begin{pmatrix}u(-z)\\-v(-z)\end{pmatrix}.\end{aligned} \tag{7.33}$$

2. Similarly, given a vector $e=\begin{pmatrix}a\\b\end{pmatrix}\in\mathbb{C}^{2m}$, set $e_-=\begin{pmatrix}a\\-b\end{pmatrix}$.

Observe that $\mathbb{U}(x',z,\widehat{\beta},\rho)$ is a solution of the $2m\times 2m$ system (7.27) if and only if $U=\begin{pmatrix}\mathbb{U}_+(z)\\\mathbb{U}_-(z)\end{pmatrix}$ solves the $4m\times 4m$ boundary problem

$$\begin{aligned}&(a)\ U_z-\mathcal{G}_0(x',z,\beta)U=0\\&(b)\ \Gamma U=0 \text{ on } z=0.\end{aligned} \tag{7.34}$$

Definition 7.4. For $\widehat{\gamma'}>0$, $\rho>0$ define $\mathcal{E}_-(x',\widehat{\beta},\rho)$ as the space of boundary values at $z=0$ of decaying solutions to (7.34)(a). In view of Proposition 7.1 $\mathcal{E}_-(x',\widehat{\beta},\rho)$ has dimension $2m$. Moreover, it has a continuous extension to $\widehat{\gamma'}\geq 0$, $\rho\geq 0$.

Remark 7.7. The individual functions $\mathbb{U}^{R,L}_j(x',z,\widehat{\beta},\rho)$ appearing in the definition of $\mathcal{D}(x',\beta)$ are locally analytic in $(\widehat{\beta},\rho)$ on $\{\widehat{\gamma'}>0,\ \rho>0\}$. This is a consequence of a standard contraction mapping argument [Co] together with the corresponding fact for solutions to the systems obtained from (7.27) by taking limits as $z\to\pm\infty$. This argument also shows that the individual solutions corresponding to fast decaying modes extend analytically to $\{\widehat{\gamma'}\geq 0,\ \rho\geq 0\}$. The fast decaying solutions are independent of $\widehat{\beta}$ at $\rho=0$, and so extend smoothly as functions of β as well.

Since the subspace $\mathcal{E}_-(x',\widehat{\beta},\rho)$ has a continuous extension to $\{\widehat{\gamma}'\geq 0,\ \rho\geq 0\}$ (this can be seen by arguing as in [CP], Chapter 7), we can if necessary redefine the individual solutions $\mathbb{U}_j^{R,L}$ corresponding to slowly decaying modes so that they have continuous extensions to $\widehat{\gamma}'\geq 0$, $\rho\geq 0$. Henceforth, we assume this has been done.

The Evans function for (7.34) measures the degree of linear dependency between two $2m-$dimensional subspaces of $\mathbb{C}^{4m}$; namely, kerΓ and $\mathcal{E}_-(x',\widehat{\beta},\rho)$. Let $e_1,\dots,e_{2m}$ be the standard basis of $\mathbb{C}^{2m}$.

Definition 7.5. Define the Evans function for (7.34) as the $4m\times 4m$ determinant $\mathbb{D}(x',\widehat{\beta},\rho)=$

$$\det\begin{pmatrix} e_1 & \cdots & e_{2m} & \mathbb{U}^R_{1+} & \cdots & \mathbb{U}^R_{m+} & 0 & \cdots & 0\\ e_{1-} & \cdots & e_{2m-} & 0 & \cdots & 0 & \mathbb{U}^L_{1-} & \cdots & \mathbb{U}^L_{m-}\end{pmatrix}|_{z=0}. \tag{7.35}$$

Remark 7.8. 1. We note that the last $2m$ columns of the above matrix form a basis for $\mathcal{E}_-(x',\widehat{\beta},\rho)$.
2. The x' dependence of $\mathcal{E}_-(x',\widehat{\beta},\rho)$ and $\mathbb{D}(x',\widehat{\beta},\rho)$ enters only through $p^0(x')$.

Recalling the normalization (7.32) we set

$$\mathcal{E}_{-,\phi}(x',\widehat{\beta},\rho)=\text{span}\begin{pmatrix}\mathbb{U}^R_{1+}\\ \mathbb{U}^L_{m-}\end{pmatrix}|_{(x',0,\widehat{\beta},\rho)}. \tag{7.36}$$

For $\kappa>0$ fixed denote by $\mathcal{E}^c_{-,\phi,\kappa}(x',\widehat{\beta},\rho)$ any complementary subspace in $\mathcal{E}_-(x',\widehat{\beta},\rho)$ varying continuously with $(x',\widehat{\beta},\rho)$ such that

$$\mathcal{E}_-(x',\widehat{\beta},\rho)=\mathcal{E}_{-,\phi}(x',\widehat{\beta},\rho)\oplus\mathcal{E}^c_{-,\phi,\kappa}(x',\widehat{\beta},\rho) \tag{7.37}$$

with uniformly bounded projections for $0\leq\rho\leq\kappa$.

The following key Proposition is essentially Proposition 4.1 of [GMWZ1] adapted to curved shocks. The small differences reflect our slightly different reduction of the original problem to a first order system.

Proposition 7.2. *1. Let $\mathcal{D}(x',\widehat{\beta},\rho)$ be the Evans function defined in Lemma 7.1. Then*

$$\mathcal{D}(x',\widehat{\beta},\rho)=\mathbb{D}(x',\widehat{\beta},\rho). \tag{7.38}$$

2. Under the assumptions of section 2 we have
(a) For any choice of $0<\delta<R$ there is a constant $C_{\delta,R}$ such that when $\delta\leq\rho\leq R$,

$$|\Gamma u|\geq C_{\delta,R}|u| \text{ for } u\in\mathcal{E}_-(x',\widehat{\beta},\rho) \text{ for all } (x',\widehat{\beta})\in\mathbb{R}^N\times S^N_+. \tag{7.39}$$

(b) There exist positive constants C_1, C_2, δ such that

$$C_1\rho|u|\leq|\Gamma u|\leq C_2\rho|u| \text{ for } u\in\mathcal{E}_{-,\phi}(x',\widehat{\beta},\rho) \tag{7.40}$$

for $0\leq\rho\leq\delta$ and all $(x',\widehat{\beta})\in\mathbb{R}^N\times S^N_+$.
(c) For $\mathcal{E}^c_{-,\phi,\kappa}(x',\widehat{\beta},\rho)$ as in (7.37) there exists $C>0$ such that

$$|\Gamma u|\geq C|u| \text{ for } u\in\mathcal{E}^c_{-,\phi,\kappa}(x',\widehat{\beta},\rho) \tag{7.41}$$

for $0\leq\rho\leq\kappa$ and all $(x',\widehat{\beta})\in\mathbb{R}^N\times S^N_+$.
(d)There exists $C>0$ such that

$$|\Gamma u|\geq C\rho|u| \text{ for } u\in\mathcal{E}_-(x',\widehat{\beta},\rho) \tag{7.42}$$

for $0\leq\rho\leq\kappa$ and all $(x',\widehat{\beta})\in\mathbb{R}^N\times S^N_+$.

Proof. The proof is given in [GMWZ1], Proposition 7.1. We simply note that the compactness arguments there still work here because of the way we chose the extension of $(U^0_\pm(x'), d\psi_0(x'))$ to all $x' \in \mathbb{R}^N$ in section 2. Also, the degeneracy of the Lopatinski condition expressed by (7.40) reflects the degeneracy of the Evans function for the shock problem at zero frequency.

□

7.7. **Conjugation to remove z dependence.** In this section we construct an [MZ] conjugator to remove the (x', z) dependence from the coefficients of $\mathcal{G}$. The main new point here is that since we can't tolerate $O(|U|_{L^2})$ errors at this stage, we must use the semiclassical calculus to construct a second term $\mathcal{W}_1$ in the conjugator which permits us to attain $O(\epsilon|U|_{L^2})$ errors.

The first step is a construction at the symbol level. In this section $\mathcal{G}(z)$ (resp., $\mathcal{G}_\infty$) denotes the function $\mathcal{G}(\mathcal{V}_\epsilon(x', z), p, \beta)$ (resp., $\mathcal{G}_\infty(p, \beta)$) defined in (7.19).

Lemma 7.2. *Let $\underline{p}$ be as in (7.14) and set $\underline{\beta} = 0$. For δ as in (7.8) let $\mathcal{F}(x', z, \beta) \in C^\infty(\mathbb{R}^N_{x'} \times [0, \infty) \times \mathbb{R}^N \times \overline{\mathbb{R}}_+)$ satisfy*

$$|\mathcal{F}(x', z, \beta)| \le Ce^{-\delta z} \tag{7.43}$$

for C independent of (x', z, β). There is a neighborhood ω of $(\underline{p}, \underline{\beta})$ and matrices $\mathcal{W}^\epsilon_0$, $\mathcal{W}^\epsilon_1$ defined and C^∞ on $\mathbb{R}^N_{x'} \times [0, \infty) \times \omega$ such that (dropping epsilons for now)

1) $\mathcal{W}_0^{-1}$ is uniformly bounded and there is a $\theta > 0$ such that

$$\begin{aligned} &|\mathcal{W}_0(x', z, p, \beta) - I| \le Ce^{-\theta z}\\ &|\mathcal{W}_1(x', z, p, \beta)| \le Ce^{-\theta z}. \end{aligned} \tag{7.44}$$

2) $\mathcal{W}_0$, $\mathcal{W}_1$ satisfy

$$\begin{aligned} &(a)\ \partial_z \mathcal{W}_0 = \mathcal{G}(z)\mathcal{W}_0(z) - \mathcal{W}_0(z)\mathcal{G}_\infty\\ &(b)\ \partial_z \mathcal{W}_1 = \mathcal{G}(z)\mathcal{W}_1(z) - \mathcal{W}_1(z)\mathcal{G}_\infty + \mathcal{F}(x', z, \beta). \end{aligned} \tag{7.45}$$

Proof. The right side of (7.45)(a) can be written

$$\mathcal{L}\mathcal{W}_0 + \Delta\mathcal{G}\mathcal{W}_0, \tag{7.46}$$

where $\mathcal{L}$ is the constant coefficient operator ad $\mathcal{G}_\infty = [\mathcal{G}_\infty, \cdot]$ and $\Delta\mathcal{G}$ is left multiplication by $\mathcal{G} - \mathcal{G}_\infty = O(e^{-\delta z})$.

The eigenvalues of $\mathcal{L}$ are differences of eigenvalues of $\mathcal{G}_\infty(p, \beta)$. Suppose we can choose $\kappa \in (0, \delta)$ such that $\mathcal{L}$ has no eigenvalues on $\Re\mu = -\kappa$ for $(p, \beta) \in \overline{\omega}$. Let $\Pi_+(p, \beta)$ (resp., $\Pi_-(p, \beta)$) be the spectral projector on the sum of the generalized eigenspaces of $\mathcal{L}$ associated with eigenvalues in $\Re\mu > -\kappa$ (resp., $\Re\mu < -\kappa$). Then the "Gap Lemma" estimates of [GZ, Z] show that $\mathcal{W}_0$, $\mathcal{W}_1$ satisfying (7.44) with $\theta < \kappa$ and depending smoothly on parameters can be obtained as solutions of

$$\begin{aligned} &\mathcal{W}_0(z) =\\ &I + \int_0^z e^{(z-s)\mathcal{L}}\Pi_-(s)\Delta\mathcal{G}(s)\mathcal{W}_0(s)ds - \int_z^\infty e^{(z-s)\mathcal{L}}\Pi_+(s)\Delta\mathcal{G}(s)\mathcal{W}_0(s)ds\\ &\mathcal{W}_1(z) = \int_0^z e^{(z-s)\mathcal{L}}\Pi_-(s)(\Delta\mathcal{G}(s)\mathcal{W}_1(s) + \mathcal{F}(s))ds\\ &\qquad\qquad - \int_z^\infty e^{(z-s)\mathcal{L}}\Pi_+(s)(\Delta\mathcal{G}(s)\mathcal{W}_1(s) + \mathcal{F}(s))ds. \end{aligned} \tag{7.47}$$

There is no problem choosing κ as above satisfying the separation condition at the basepoint $(\underline{p}, \underline{\beta})$. The same choice works for $(p, \beta) \in \overline{\omega}$ provided ω is small enough. (In Remark 7.9 we describe a better way of choosing ω).

The uniform boundedness of $\mathcal{W}_0^{-1}$ follows by the argument in [MZ], Lemma 2.6.

□

Remark 7.9 (**Choice of ω and ω_1**). In proving the degenerate L^2 estimate we will see that spatial cutoffs $\phi(x)$ lead to unacceptable $O(|U|_{L^2})$ errors, while frequency cutoffs $\chi(\beta)$ localizing near $\beta = 0$ are allowed. Since we can't localize in x we have to restrict how much the original shock $(U^0_\pm(x), d\psi^0(x'))$ can deviate from $(U^0_\pm(0), d\psi^0(0))$ in order to insure $(p^\epsilon(x), \beta) \in \overline{\omega}$ for all $x \in \overline{\mathbb{R}}^{N+1}_+$. To see how much deviation from flatness can be allowed, set $\epsilon = 0$ in $\mathcal{G}_\infty(p^\epsilon, \epsilon\zeta)$ to obtain a matrix $\mathcal{K}(p^0_{1,+}(x), p^0_{1,-}(x), p^0_2(x'))$ with blocks

$$\begin{pmatrix} 0 & I \\ 0 & C(x')^{-1}\left(\pm\mathcal{A}_N(U^0_\pm(x), d\psi^0(x'))\right) \end{pmatrix}. \tag{7.48}$$

We first choose a connected, relatively compact neighborhood

$$\omega_1 \ni (U^0_+(0), U^0_-(0), d\psi^0(0)) \tag{7.49}$$

as large as possible so that for all $(p_{1,+}, p_{1,-}, p_2) \in \overline{\omega}_1$, differences of eigenvalues of the frozen matrix $\mathcal{K}(p_{1,+}, p_{1,-}, p_2)$ avoid the line $\Re\mu = -\kappa$ for some $\kappa \in (0, \delta)$. Of course, we should take

$$\omega_1 \subset \mathcal{O}_+ \times \mathcal{O}_- \times \mathbb{R}^N.$$

In addition, we need to choose ω_1 so that

$$A_{\infty,\pm}(p_{1,+}, p_{1,-}, p_2, 0, 0) \text{ have uniformly bounded inverses on } \overline{\omega}_1. \tag{7.50}$$

ω_1 specifies how much $(U^0_\pm(x), d\psi^0(x'))$ can deviate from $(U^0_\pm(0), d\psi^0(0))$.

Observe that the larger δ is, that is, the faster the leading profiles $V^0_\pm(x', z)$ decay exponentially to zero, the more the original shock may deviate from flatness. In addition, the more slowly varying the matrix $\mathcal{K}(p_{1,+}, p_{1,-}, p_2)$ is, the larger the neighborhood ω_1 can be chosen.

Now having fixed ω_1, if we choose any small enough neighborhood ω_2 (resp., ω_3) of $p_3 = 0 \in \mathbb{R}^{2m(N-1)+1}$ (resp., of $\beta = 0 \in \mathbb{R}^N \times \overline{\mathbb{R}}_+$), the choice

$$\omega = \omega_1 \times \omega_2 \times \omega_3 \tag{7.51}$$

works in Lemma 7.2.

The above discussion is what motivates the choice of Assumption (H6).

7.8. Hyperbolic and elliptic blocks.

The following lemma separates out the eigenspaces corresponding to small and large eigenvalues of $\mathcal{G}_\infty$.

Let $\underline{p}$ be as in (7.2) and $\omega = \omega_1 \times \omega_2 \times \omega_3$ as in (7.51). Recall that p_3 is the placeholder for $p^\epsilon(x)$ defined in (7.12).

Lemma 7.3. *Shrinking ω_2 and ω_3 if necessary, we can construct a C^∞ invertible matrix $T(p, \beta)$ defined on ω such that $T^{-1}\mathcal{G}_\infty(p, \beta)T$ has the block diagonal form*

$$T^{-1}\mathcal{G}_\infty T = \begin{pmatrix} H_R & 0 & 0 & 0 \\ 0 & P_R & 0 & 0 \\ 0 & 0 & H_L & 0 \\ 0 & 0 & 0 & P_L \end{pmatrix} \equiv \mathcal{G}_{1,\infty}, \tag{7.52}$$

where, with R, L corresponding to $+, -$ respectively

$$\begin{aligned} H_{R,L}(p, \beta) &= -A^{-1}_{\infty,\pm} M_{\infty,\pm} + (O(\beta) + O(p_3))^2 \\ P_{R,L}(p, \beta) &= A_{\infty,\pm} + O(\beta) + O(p_3). \end{aligned} \tag{7.53}$$

T has the form

$$T(p,\beta) = \begin{pmatrix} I & A_{\infty,+}^{-1} & 0 & 0 \\ -A_{\infty,+}^{-1}M_{\infty,+} + \tau_1 & I + \tau_2 & 0 & 0 \\ 0 & 0 & I & A_{\infty,-}^{-1} \\ 0 & 0 & -A_{\infty,-}^{-1}M_{\infty,-} + \tau_3 & I + \tau_4 \end{pmatrix}, \tag{7.54}$$

where

$$\tau_i = \tau_i(p,\beta) = O(\beta) + O(p_3) \tag{7.55}$$

and $O(\beta)$ (resp., $O(p_3)$) represents a smooth function of (p,β) of the form $\beta \cdot f(p,\beta)$ (resp., $p_3 \cdot f(p_3)$).

The eigenvalues of $P_R(p,\beta)$ and P_L satisfy $|\Re\mu| > C > 0$ for $|\beta| + |p_3|$ small.

Proof. The proof is a simple computation. Look for T of the given form and use (7.50) to solve for $\tau_1, \dots, \tau_4$.

The eigenvalues of $P_{R,L}$ have the stated property since the eigenvalues of $A_{\infty,\pm}$ are nonvanishing. □

7.9. **Block structure.**

Notation 7.5. Given a function $f(p) = f(p_{1,+}, p_{1,-}, p_2, p_3)$, set $p' = (p_{1,+}, p_{1,-}, p_2) \in \mathbb{R}^{2m} \times \mathbb{R}^N$ and with slight abuse write

$$f(p') \equiv f(p', 0). \tag{7.56}$$

Observe that we can rewrite $H_{R,L}(p,\beta)$ in (7.53) as

$$H_{R,L}(p,\beta) = H_{R,L}(p',\beta) + O(p_3) + O(p_3)O(\beta) + O(p_3)^2. \tag{7.57}$$

Conjugation by a constant coefficient matrix T_1 (with only zeros and ones) changes $\mathcal{G}_{1,\infty}$ in (7.52) to $T_1^{-1}\mathcal{G}_{1,\infty}T_1 =$

$$\mathcal{G}_{2,\infty}(p,\beta) = \begin{pmatrix} H_R & 0 & 0 & 0 \\ 0 & H_L & 0 & 0 \\ 0 & 0 & P_R & 0 \\ 0 & 0 & 0 & P_L \end{pmatrix} \tag{7.58}$$

In the next Proposition we use the polar coordinate notation introduced in Notation 7.2.

Proposition 7.3 (Block structure). *Let $\underline{p'} \in \overline{\omega}_1$. For all $\underline{\widehat{\beta}}$ with $\underline{\widehat{\gamma'}} \geq 0$ there is a neighborhood $\widehat{\omega}$ of $(\underline{p'}, \underline{\widehat{\beta}}, 0)$ in $(\mathbb{R}^{2m} \times \mathbb{R}^N) \times S_+^N \times \overline{\mathbb{R}}_+$ and there are C^∞ matrices $T_2(p', \widehat{\beta}, \rho)$ on $\widehat{\omega}$ such that $T_2^{-1}\mathcal{G}_{2,\infty}T_2$ has the following block diagonal structure*

$$T_2^{-1}\mathcal{G}_{2,\infty}T_2 = \begin{bmatrix} H_B(p',\widehat{\beta},\rho) & 0 & 0 \\ 0 & P_+(p',\beta) & 0 \\ 0 & 0 & P_-(p',\beta) \end{bmatrix} \equiv \mathcal{G}_{B,\infty}. \tag{7.59}$$

Here the eigenvalues of P_+ (resp. P_-) belong to a compact set in $\Re\mu > 0$ (resp. $\Re\mu < 0$) and in addition

$$\Re P_+ = \frac{1}{2}(P_+ + P_+^*) \geq cI \text{ and } -\Re P_- \geq cI \text{ on } \widehat{\omega} \tag{7.60}$$

for some $c > 0$.

We have $H_B(p',\widehat{\beta},\rho)=\rho\widehat{H}_B(p',\widehat{\beta},\rho)$ *with*

$$\widehat{H}_B(p',\widehat{\beta},\rho)=\begin{bmatrix}Q_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & Q_p\end{bmatrix}(p',\widehat{\beta},\rho). \tag{7.61}$$

The blocks Q_k *are* $\nu_k\times\nu_k$ *matrices which satisfy one of the following conditions:*

i) $\Re Q_k$ *is positive definite.*

ii) $\Re Q_k$ *is negative definite.*

iii) $\nu_k=1$, $\Re Q_k=0$ *when* $\widehat{\gamma}'=\rho=0$, *and* $\partial_{\widehat{\gamma}'}(\Re Q_k)\partial_\rho(\Re Q_k)>0$.

iv) $\nu_k>1$, Q_k *has purely imaginary coefficients when* $\widehat{\gamma}'=\rho=0$, *there is* $\mu_k\in\mathbb{R}$ *such that*

$$Q_k(\underline{p'},\underline{\beta},0)=i\begin{bmatrix}\mu_k & 1 & 0 & \\ 0 & \mu_k & \ddots & 0\\ & \ddots & \ddots & 1\\ & & \cdots & \mu_k\end{bmatrix}, \tag{7.62}$$

and the lower left corner a *of* Q_k *satisfies* $\partial_{\widehat{\gamma}'}(\Re a)\partial_\rho(\Re a)>0$.

Moreover, the matrix T_2 *can be taken of the form*

$$T_2(p',\widehat{\beta},\rho)=\begin{pmatrix}T_H(p',\widehat{\beta},\rho) & 0\\ 0 & T_P(p',\beta)\end{pmatrix}, \tag{7.63}$$

for C^∞ *functions* T_H *and* T_P. *In fact, a single smooth matrix* $T_P(p',\beta)$ *defined for* $|\beta|$ *small and* p' *in a neighborhood of* $\overline{\omega}_1$ *can be chosen to conjugate the* (P_R,P_L) *block of* $\mathcal{G}_{2,\infty}$ *to the* (P_+,P_-) *block of* $\mathcal{G}_{B,\infty}$.

7.10. **Transport Proposition 7.2.** We need to transport the information in Proposition 7.2 about the problem

$$\begin{aligned}&U_z-\mathcal{G}_0(x',z,\beta)U=0\\ &\Gamma U=0 \text{ on } z=0\end{aligned} \tag{7.64}$$

to an appropriately conjugated boundary problem.

Notation 7.6. Corresponding to the p' notation introduced above, set $p'(x)=(p^0_{1,+}(x),p^0_{1,-}(x),p^0_2(x))=(U^0_+(x),U^0_-(x),d\psi_0(x'))$. Note $p^0(x)=(p'(x),0)$ (the last entry is p_3), so sometimes we'll write (abusively) $p^0(x)=p'(x)$.

Set

$$\mathcal{G}_{HP}(p'(x),\beta)=\begin{pmatrix}H & 0\\ 0 & P\end{pmatrix} \tag{7.65}$$

where

$$H=\begin{pmatrix}H_R & 0\\ 0 & H_L\end{pmatrix},\ P=\begin{pmatrix}P_+ & 0\\ 0 & P_-\end{pmatrix}. \tag{7.66}$$

In previous subsections we have defined symbols $\mathcal{W}^\epsilon_0(x',z,p^\epsilon(x),\beta)$ (7.44), $T(p^\epsilon(x),\beta)$ (7.54), T_1 (7.58), and $T_P(p'(x),\beta)$ (7.63). Set

$$\mathcal{W}_0(x',z,p'(x),\beta)=\mathcal{W}^0_0(x',z,p^0(x),\beta)$$

and define

$$
\begin{aligned}
&T_3(p'(x),\beta) = \begin{pmatrix} I & 0 \\ 0 & T_P \end{pmatrix},\\
&\mathcal{T}(x,z,\beta) = \mathcal{W}_0(x',z,p'(x),\beta)T(p'(x),\beta)T_1T_3(p'(x),\beta),\\
&\mathcal{T}^0(x',z,\beta) = \mathcal{W}_0(x',z,p'(x'),\beta)T(p'(x'),\beta)T_1T_3(p'(x'),\beta)\\
&\Gamma_1(x',\beta) = \Gamma\mathcal{T}^0(x',0,\beta),
\end{aligned}
\tag{7.67}
$$

and observe that $\mathcal{T}^0$ conjugates the problem (7.64) to

$$
\begin{aligned}
&U_z - \mathcal{G}_{HP}(p'(x'),\beta)U = 0\\
&\Gamma_1(x',\beta)U = 0 \text{ on } z = 0.
\end{aligned}
\tag{7.68}
$$

Corresponding to the blocks in (7.68) there is the obvious decomposition of $\mathbb{C}^{4m}$;

$$
\begin{aligned}
&\mathbb{C}^{4m} = E_H \oplus E_{P_+} \oplus E_{P_-}\\
&U = U_H + U_{P_+} + U_{P_-},
\end{aligned}
\tag{7.69}
$$

where the three spaces on the right have dimensions $2m$, $m-1$, $m+1$, respectively, and, if $U = (u_H, u_{P_+}, u_{P_-})$, we have $U_H = (u_H, 0, 0)$, etc..

For $\rho > 0$, $\widehat{\gamma}' > 0$ define $\mathbb{E}_-(x',\widehat{\beta},\rho)$ to be the space of boundary values at $z = 0$ of decaying solutions of

$$
U_z - \mathcal{G}_{HP}(p'(x'),\beta)U = 0 \text{ on } z \geq 0.
\tag{7.70}
$$

These spaces vary smoothly in $\{\rho > 0, \widehat{\gamma}' > 0\}$ and extend continuously to $\{\rho \geq 0, \widehat{\gamma}' \geq 0\}$. In fact, it is not hard to check that

$$
\mathcal{E}_-(x',\widehat{\beta},\rho) = \mathcal{T}^0(x',0,\beta)\mathbb{E}_-(x',\widehat{\beta},\rho),
\tag{7.71}
$$

where $\mathcal{E}_-$ is the corresponding space for (7.64) defined earlier. We have

$$
\mathbb{E}_-(x',\widehat{\beta},\rho) = E_{H_-}(x',\widehat{\beta},\rho) \oplus E_{P_-},
\tag{7.72}
$$

where $E_{H_-}(x',\widehat{\beta},\rho) = \mathbb{E}_-(x',\widehat{\beta},\rho) \cap E_H$.

Next define the subspace $E_{P_{1,-}}(x',\beta)$ of E_{P_-} by

$$
\mathcal{E}_{-,\phi}(x',\beta) = \mathcal{T}^0(x',0,\beta)E_{P_{1,-}}(x',\beta),
\tag{7.73}
$$

where we have used the regularity property of fast decaying modes explained in Remark 7.5 to rewrite $\mathcal{E}_{-,\phi}(x',\widehat{\beta},\rho) = \mathcal{E}_{-,\phi}(x',\beta)$.

For $\kappa > 0$ fixed choose a smoothly varying subspace $E_{P_{2,-,\kappa}}(x',\beta)$ orthogonal to $E_{P_{1,-}}(x',\beta)$ such that

$$
\begin{aligned}
&E_{P_-} = E_{P_{1,-}}(x',\beta) \oplus E_{P_{2,-,\kappa}}(x',\beta)\\
&U_{P_-} = U_{P_{1,-}}(x',\beta) + U_{P_{2,-,\kappa}}(x',\beta).
\end{aligned}
\tag{7.74}
$$

Then

$$
\mathcal{E}^c_{-,\phi,\kappa}(x',\widehat{\beta},\rho) \equiv \mathcal{T}^0(x',0,\beta)(E_{H_-}(x',\widehat{\beta},\rho) \oplus E_{P_{2,-,\kappa}}(x',\beta))
\tag{7.75}
$$

is a choice of complementary space that works in Proposition 7.2.

Having defined $E_{H_-}(x',\widehat{\beta},\rho)$ we take $E_{H_+}(x',\widehat{\beta},\rho)$ to be any continuously varying subspace of E_H such that

$$
E_H = E_{H_+}(x',\widehat{\beta},\rho) \oplus E_{H_-}(x',\widehat{\beta},\rho)
\tag{7.76}
$$

with uniformly bounded projections. (In [GMWZ1], (3.25) a particular choice of E_{H_+}, denoted there by $E_{H_{+,c}}$, is made, but here any choice as above will do.)

This gives a more refined decomposition,

$$\begin{aligned}&\mathbb{C}^{4m} = \\ &E_{H_+}(x',\widehat{\beta},\rho)\oplus E_{H_-}(x',\widehat{\beta},\rho)\oplus E_{P_+}\oplus E_{P_{1,-}}(x',\beta)\oplus E_{P_{2,-,\kappa}}(x',\beta), \\ &U = U_{H_+}+U_{H_-}+U_{P_+}+U_{P_{1,-}}+U_{P_{2,-,\kappa}}.\end{aligned} \tag{7.77}$$

The next Corollary is then an immediate consequence of Proposition 7.2.

Corollary 7.1. *There exist positive constants $C_1,\dots,C_4$ and κ such that for $0\le\rho\le\kappa$, all $(x',\widehat{\beta})$, and $U\in\mathbb{C}^{4m}$*

$$\begin{aligned}&(a)\ C_1\rho|U_{P_{1,-}}|\le|\Gamma_1(x',\widehat{\beta},\rho)U_{P_{1,-}}|\le C_2\rho|U_{P_{1,-}}| \\ &(b)\ |\Gamma_1(x',\widehat{\beta},\rho)(U_{H_-}+U_{P_{2,-,\kappa}})|\ge C_3(|U_{H_-}|+|U_{P_{2,-,\kappa}}|) \\ &(c)\ |\Gamma_1(x',\widehat{\beta},\rho)U_-|\ge C_4\rho|U_-|.\end{aligned} \tag{7.78}$$

Again, in making these statements for all $x'\in\mathbb{R}^N$, we are using the compactness properties of our choice of extensions in section 2.

Part (a) of the Corollary shows that Γ_1 fails to satisfy the uniform Lopatinski condition at $\rho=0$.

Definition 7.6. A boundary operator $\Gamma_a(x',\widehat{\beta},\rho)$ depending continuously on $(x',\widehat{\beta},\rho)$ satisfies the *uniform Lopatinski condition* at $(\underline{x'},\underline{\widehat{\beta}},\underline{\rho})$ if there exists a $C>0$ such that

$$|\Gamma_a(x',\widehat{\beta},\rho)u|\ge C|u| \tag{7.79}$$

for $u\in\mathbb{E}_-(x',\widehat{\beta},\rho)$ uniformly near $(\underline{x'},\underline{\widehat{\beta}},\underline{\rho})$.

The following simple consequence of Corollary 7.1 gives a more precise version of (7.1)(c) and is essential for the construction of degenerate symmetrizers. It is proved in [GMWZ1], Lemma 4.1.

Lemma 7.4. *There exists a constant $\delta>0$ such that for ρ sufficiently small, all $(x',\widehat{\beta})$ and all $U\in\mathbb{C}^{4m}$ we have*

$$|\Gamma_1(x',\widehat{\beta},\rho)U_{-,c}|\ge\delta(|U_{H_{-,c}}|+\rho|U_{P_-}|). \tag{7.80}$$

7.11. **Standard and degenerate symmetrizers.** In this section p' denotes the frozen variable corresponding to $p'(x)$.

Observe that $H_{R,L}(p',\widehat{\beta},\rho)$ can be written $H_{R,L}=\rho\widehat{H}_{R,L}$ for $\widehat{H}_{R,L}(p',\widehat{\beta},\rho)$ smooth. Set

$$\begin{aligned}&H(p',\widehat{\beta},\rho)=\begin{pmatrix}H_R & 0\\ 0 & H_L\end{pmatrix}=\rho\begin{pmatrix}\widehat{H}_R & 0\\ 0 & \widehat{H}_L\end{pmatrix}, \\ &P(p',\beta)=\begin{pmatrix}P_+ & 0\\ 0 & P_-\end{pmatrix}.\end{aligned} \tag{7.81}$$

Proposition 7.4. *1. Let $\underline{p}'\in\overline{\omega}_1$. There is a C^∞ matrix $S_H(p',\widehat{\beta},\rho)$ on a neighborhood ω^* of $\{\underline{p}'\}\times S^N_+\times\{0\}$ such that $S_H=S_H^*$ and*

$$\Re(S_H\widehat{H})(p',\widehat{\beta},\rho)=\sum(V_l)^*K_lV_l \tag{7.82}$$

where $\{V_l\}$ is a finite collection of C^∞, invertible, $2m \times 2m$ matrices having the following block structure

$$K_l = \begin{bmatrix} B_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_p \end{bmatrix} (p', \widehat{\beta}, \rho). \tag{7.83}$$

with either $B_j = B_j^*$ positive definite, or $B_j = \widehat{\gamma}' B_{j,0} + \rho B_{j,1}$ with $B_{j,0}$ and $B_{j,1}$ positive definite. The number of blocks p can vary with l. Moreover, $\sum V_l^* V_l$ is positive definite.

2. ***Standard symmetrizer.*** *Let $\{(x, \widehat{\beta}, \rho) : (p'(x), \widehat{\beta}, \rho) \in \omega^*\}$ be denoted (abusively) by $p'^{-1}(\omega^*)$ and set $p_0'^{-1}(\omega^*) = \{(x', \widehat{\beta}, \rho) : (p'(x'), \widehat{\beta}, \rho) \in \omega^*\}$*

Given a boundary condition Γ_a that satisfies

$$|\Gamma_a(x', \widehat{\beta}, \rho)u| \geq C|u| \text{ for } u \in \mathbb{E}_-(x', \widehat{\beta}, \rho) \text{ on } p_0'^{-1}(\omega^*) \tag{7.84}$$

choose

$$S_P(p', \beta) = \begin{pmatrix} CI & 0 \\ 0 & -I \end{pmatrix}, \tag{7.85}$$

where the blocks correspond to those in the matrix P. Then for C large enough we have

$$\Re(S_P P)(p', \beta) \geq I \text{ for } p' \in \overline{\omega}_1,\ |\beta| \text{ small}. \tag{7.86}$$

Set

$$S(x, \widehat{\beta}, \rho) = \begin{pmatrix} S_H(p'(x), \widehat{\beta}, \rho) & 0 \\ 0 & S_P \end{pmatrix} \tag{7.87}$$

on $p'^{-1}(\omega *)$.

Then S_H as above can be chosen so that in addition we have for C large enough and $U \in \mathbb{C}^{4m}$,

$$((S + \Gamma_a^* \Gamma_a)(x', \widehat{\beta}, \rho)U, U) \geq c|U|^2 \text{ on } p_0'^{-1}(\omega^*) \tag{7.88}$$

for some $c > 0$.

3. ***Degenerate symmetrizer***

Consider Γ_1 which is degenerate in the way specified in Corollary 7.1. Now choose

$$S_P = \begin{pmatrix} CI & 0 \\ 0 & -\rho^2 I \end{pmatrix}. \tag{7.89}$$

Then for C large enough we have for $u = (u_{P_+}, u_{P_-}) \in \mathbb{C}^{2m}$

$$(\Re(S_P P)(p', \beta)u, u) \geq C|u_{P_+}|^2 + \rho^2 |u_{P_-}|^2 \tag{7.90}$$

p' in a neighborhood of $\overline{\omega}_1$ and $|\beta|$ small.

Define $S(x', \widehat{\beta}, \rho)$ as above with the new S_P. Then S_H as above can be chosen so that in addition we have for C large enough and $U \in \mathbb{C}^{4m}$

$$\begin{aligned} &(a)((S + \Gamma_1^* \Gamma_1)(x', \widehat{\beta}, \rho)U, U) \geq c_1(|U_H|^2 + |U_{P_+}|^2 + |U_{P_{2,-,\kappa}}|^2) + c_2 \rho^2 |U_{P_{1-}}|^2, \\ &(b) c_1 \rho^2 |U|^2 \leq ((S + \Gamma_1^* \Gamma_1)(x', \widehat{\beta}, \rho)U, U) \leq c_2 \rho^2 |U|^2 \text{ for } U \in E_{P_{1,-}} \end{aligned} \tag{7.91}$$

on $p_0'^{-1}(\omega^*)$ for some positive c_1, c_2.

Proof. In the case where the uniform Lopatinski condition holds the construction of S follows by the same argument as in [MZ] (see also the section on standard symmetrizers in [GMWZ1]).

In the degenerate case the construction of [GMWZ1], Proposition 7.4, gives S satisfying (7.91), except for the upper bound in part (b), which follows from the definition of S_P and Corollary 7.1(a).

We note also that the construction of S_H in [GMWZ1] is close to the original method of [K] and involves localization on S^N_+ and use of the refined block structure of Proposition 7.3. The boundary estimate in (7.91) (a) follow by incorporating the information in Corollary 7.1 into the Kreiss argument.

□

8. Semiclassical and mixed calculi

8.1. **Semiclassical calculus.** Our proof of the L^2 estimate in the small frequency region requires the use of classical, semiclassical, and mixed-type pseudodifferential operators. We begin with a simple calculus of semiclassical operators with finite regularity in x'.

We have had to do a fair amount of bookkeeping to state precisely how much regularity in x' is needed to make the calculi work. The motivation is to be able to apply the calculus to problems in which the inviscid shock is not piecewise C^∞ but just piecewise C^M for M big enough. Attention to regularity hypotheses also helps clarify what determines the size of the constants appearing in our estimates. In this paper our linearized problem has C^∞ coefficients since the inviscid shock is assumed piecewise C^∞ (for convenience), so the regularity hypotheses needed for applying the calculi are always satisfied.

The reader may wonder why we don't paralinearize as in [MZ], thereby eliminating much of the bookkeeping and allowing us to assume much less regularity in x'. The reason is that this introduces $O(|U|_{L^2})$ errors at a stage when they are too big to be absorbed by the left side of our degenerate symmetrizer L^2 estimate (9.1).

Notation 8.1. 1. Let $\zeta' = (\zeta_0, \zeta'') \in \mathbb{R}^N$ denote variables dual to the tangential variables $x' = (x_0, x'')$, and set $\zeta = (\zeta', \gamma)$, where we always take $\gamma \geq 1$. Set $\langle\zeta\rangle = \sqrt{|\zeta|^2} = \sqrt{|\zeta', \gamma|^2}$ and, with slight abuse, $\langle\zeta'\rangle = \sqrt{|\zeta', 1|^2}$.

2. For $\epsilon > 0$ let $\beta \in \mathbb{R}^N \times \overline{\mathbb{R}}_+$ (resp. $\beta' \in \mathbb{R}^N$) denote a placeholder for $\epsilon\zeta$ (resp. $\epsilon\zeta'$).

3. We will ignore powers of 2π in all formulas involving pseudodifferential operators and Fourier transforms. We write $\hat{p}(\xi', \zeta)$ for the partial Fourier transform of $p(x', \zeta)$ with respect to x'.

4. On $H^s(\mathbb{R}^N)$ define the norms $|u|_{s,\gamma} = |\langle\zeta\rangle \hat{u}|_{L^2}$.

5. When we write

$$T_{\epsilon,\gamma} : \mathcal{X} \to \mathcal{Y}$$

for a family of linear operators mapping one function space into another, we mean that the operator norm is uniformly bounded with respect to ϵ, γ for $0 < \epsilon < 1$ and $\gamma \geq 1$. For a particular $s \in \mathbb{R}$ we say $T_{\epsilon,\gamma}$ is *of order k on H^s* if

$$T_{\epsilon,\gamma} : H^s(\mathbb{R}^N) \to H^{s-k}(\mathbb{R}^N). \tag{8.1}$$

When $T_{\epsilon,\gamma}$ is of order k on H^k, we shall simply say it is of order k. When the domain and target spaces of T are clear from the context, we'll write simply $|T|$ for the operator norm.

6. Constants C that appear in the proofs are always uniform with respect to $\gamma \geq 1$, $0 < \epsilon \leq 1$, but may change from line to line (even term to term).

7. We'll sometimes denote spaces like $C^M(\mathbb{R}^N_{x'}, C^\infty(\mathbb{R}^N \times \overline{\mathbb{R}}_+))$ by $C^M(x', C^\infty(\beta))$ when the domains of the variables involved are clear.

Remark 8.1. Our pseudodifferential operators are defined by symbols with finite regularity in x'. Such an operator is generally of order k on H^s only for s in a proper subinterval of $\mathbb{R}$. See Proposition 8.9.

The semiclassical operators are built from "symbols" in the set

$$\mathcal{S}_M = \{p(x',\beta) \in C^M(\mathbb{R}^N_{x'}, C^\infty(\mathbb{R}^N \times \overline{\mathbb{R}}_+)) : \\ p \text{ is independent of } x' \text{ for } |x'| \text{ large and } \sup_{|\mu|\le M} |\partial^\mu_{x'}\partial^\nu_{\beta'} p(x',\beta)| \le C_\nu\}. \tag{8.2}$$

Let

$$\mathcal{S}_\infty = \cap_M \mathcal{S}_M. \tag{8.3}$$

Define symbol norms

$$|p|_{M,K} = \sup_{|\mu|\le M} \sup_{|\nu|\le K} \sup_{(x',\beta)} |\partial^\mu_{x'}\partial^\nu_{\beta'} p(x',\beta)|. \tag{8.4}$$

Remark 8.2. Often we will use symbols in

$$\{p(x',\beta) \in C^M(x', C^\infty_0(\beta)) : \\ p \text{ is independent of } x' \text{ for } |x'| \text{ large}\} \subset S_M. \tag{8.5}$$

(The subscript 0 indicates compact support in β.)

To each $p(x',\beta) \in \mathcal{S}_M$ we associate the operator defined by

$$p(x',\epsilon D)u = \int e^{ix'\zeta'} p(x',\epsilon\zeta)\hat{u}(\zeta')d\zeta'. \tag{8.6}$$

The following propositions are proved in Appendix A.

Proposition 8.1. *If $p \in \mathcal{S}_M$ and $M \ge N+1$ then*

$$p(x',\epsilon D) : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N).$$

Definition 8.1. A family of linear operators $r_{\epsilon,\gamma}$ is said to be *of order* ϵ^k if $r_{\epsilon,\gamma} = \epsilon^k \mathcal{R}_{\epsilon,\gamma}$ where

$$\mathcal{R}_{\epsilon,\gamma} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N).$$

Proposition 8.2 (Products)**.** *Suppose $p \in \mathcal{S}_{M_1}$ and $q \in \mathcal{S}_{M_2}$, where $M_1 \ge N+1$ and $M_2 \ge M_1 + (N+1) + k + 1$ for some $k \ge 1$. Set*

$$t(x',\beta) = \sum_{|\alpha|\le k-1} \frac{1}{\alpha!}\epsilon^{|\alpha|}\partial^\alpha_{\beta'} p(x',\beta) D^\alpha_{x'} q(x',\beta). \tag{8.7}$$

Then $t(x',\beta) \in \mathcal{S}_{M_1}$ and

$$A \equiv p(x',\epsilon D)q(x',\epsilon D) = t(x',\epsilon D) + r_{\epsilon,\gamma}, \tag{8.8}$$

where $r_{\epsilon,\gamma}$ is of order ϵ^k. Precisely, $r_{\epsilon,\gamma} = \epsilon^k T$, where

$$|T| \le C|p|_{N+1,k}|\partial_{x'} q|_{M_2-1,0}.$$

Proposition 8.3 (Adjoints)**.** *Suppose $p \in \mathcal{S}_M$, where $M \ge (N+1) + k + 1$, for some $k \ge 1$. Set*

$$t(x',\beta) = \sum_{|\alpha|\le k-1} \frac{1}{\alpha!}\epsilon^{|\alpha|}\partial^\alpha_{\beta'} D^\alpha_{x'} p^*(x',\beta). \tag{8.9}$$

Then $t \in \mathcal{S}_{M-k+1}$ and

$$p(x',\epsilon D)^* = t(x',\epsilon D) + r_{\epsilon,\gamma},$$

where $r_{\epsilon,\gamma}$ is of order ϵ^k. We have $r_{\epsilon,\gamma} = \epsilon^k T$, where

$$|T| \le C|\partial_{x'} p|_{M-1,k}.$$

8.2. **Mixed calculus.** We'll sometimes need to compose classical with semiclassical pseudodifferential operators. For $m \in \mathbb{R}$ define the classical ($\mathcal{C}$) and mixed ($\mathcal{M}$) symbol classes

$$\begin{aligned}\mathcal{C}^m_M = \{&p(x',\zeta) \in C^M(\mathbb{R}^N_{x'}, C^\infty(\mathbb{R}^N \times \{\gamma \geq 1\}) : p \text{ is independent of } x' \\ &\text{for } |x'| \text{ large and } \sup_{|\mu|\leq M} |\partial^\mu_{x'}\partial^\nu_{\zeta'} p(x',\zeta)| \leq C_\nu \langle\zeta\rangle^{m-|\nu|}\},\end{aligned} \tag{8.10}$$

$$\begin{aligned}\mathcal{M}^m_M = \{&a(x',\beta,\zeta) \in C^M(x', C^\infty(\beta,\zeta)) : a \text{ is independent of } x' \\ &\text{for } |x'| \text{ large and } \sup_{|\mu|\leq M} |\partial^\mu_{x'}\partial^\nu_{\beta'}\partial^\tau_{\zeta'} a(x',\beta,\zeta)| \leq C_{\nu,\tau} \langle\zeta\rangle^{m-|\tau|}\},\end{aligned} \tag{8.11}$$

Set

$$\mathcal{C}^m_\infty = \cap_M \mathcal{C}^m_M, \qquad \mathcal{M}^m_\infty = \cap_M \mathcal{M}^m_M.$$

Define associated symbol norms

$$\begin{aligned}|p|_{M,K} &= \sup_{|\mu|\leq M}\sup_{|\nu|\leq K}\sup_{(x',\zeta)} |\partial^\mu_{x'}\partial^\nu_{\zeta'} p(x',\zeta)|\langle\zeta\rangle^{|\nu|-m} \\ |a|_{M,K_1,K_2} &= \sup_{|\mu|\leq M}\sup_{|\nu|\leq K_1}\sup_{|\tau|\leq K_2}\sup_{(x',\beta,\zeta)} |\partial^\mu_{x'}\partial^\nu_{\beta'}\partial^\tau_{\zeta'} a(x',\beta,\zeta)|\langle\zeta\rangle^{|\tau|-m}.\end{aligned} \tag{8.12}$$

Remark 8.3. 1. Clearly, $\mathcal{S}_M \subset \mathcal{M}^0_M$ and $\mathcal{C}^m_M \subset \mathcal{M}^m_M$.
2. If $p(x,\beta) \in \mathcal{S}_M$ and $q(x',\zeta) \in \mathcal{C}^m_M$, then $p(x',\beta)q(x',\zeta) \in \mathcal{M}^m_M$.

To an element $a(x',\beta,\zeta) \in \mathcal{M}^m_M$ we associate the mixed operator

$$a(x',\epsilon D, D)u = \int e^{ix'\zeta'} a(x',\epsilon\zeta,\zeta)\hat{u}(\zeta')d\zeta'. \tag{8.13}$$

In the proof of the Garding inequality (Proposition 8.10), we'll need to compose and take adjoints of mixed type operators. The next few propositions give a mixed calculus that extends the semiclassical calculus.

Proposition 8.4. *If $a \in \mathcal{M}^m_M$ and $M \geq N+1$ then*

$$a(x',\epsilon D, D) : H^m(\mathbb{R}^N) \to L^2(\mathbb{R}^N).$$

Definition 8.2. 1. Let $\langle D\rangle^m$ be the operator defined by the Fourier multiplier $\langle\zeta\rangle^m$.

2. We say that an operator $r_{\epsilon,\gamma}$ is *of order* $\epsilon^k\langle D\rangle^m$ if it has the form $\epsilon^k T_{\epsilon,\gamma}$ for some operator $T_{\epsilon,\gamma}$ of order m. In that case we write $r_{\epsilon,\gamma} = O(\epsilon^k\langle D\rangle^m)$ and define the *mixed degree* of $r_{\epsilon,\gamma}$ to be $m-k$.

3. For $\gamma \geq 1$ define $\langle D,\gamma\rangle^m_{max} = \begin{cases}\langle D\rangle^m, & m \geq 0 \\ \gamma^m, & m \leq 0\end{cases}$.

Proposition 8.5 (Mixed products)**.** *Suppose $a \in \mathcal{M}^{m_1}_{M_1}$ and $b \in \mathcal{M}^{m_2}_{M_2}$, where $M_1 \geq N+1$ and $M_2 \geq M_1 + (N+1) + m_1 + k + 1$ for some $k \geq 1$. Then*

$$A \equiv a(x',\epsilon D, D)b(x',\epsilon D, D) = t(x',\epsilon D, D) + r_{\epsilon,\gamma}, \tag{8.14}$$

where for appropriate constants $C_{\alpha,\mu,\nu}$, we have $t(x',\beta,\zeta) =$

$$\sum_{|\alpha|\leq k-1}\frac{1}{\alpha!}\left(\sum_{\mu+\nu=\alpha} C_{\alpha,\mu,\nu}(\epsilon\partial_{\beta'})^\mu\partial^\nu_{\zeta'} a(x',\beta,\zeta)\right) D^\alpha_{x'} b(x',\beta,\zeta) \in \mathcal{M}^{m_1+m_2}_{M_1} \tag{8.15}$$

and

$$r_{\epsilon,\gamma} = \sum_{0\le l\le k} r_{l,\epsilon,\gamma} \text{ with } r_{l,\epsilon,\gamma} = O(\epsilon^l \langle D\rangle^{m_2}\langle D,\gamma\rangle_{max}^{m_1-k+l}). \tag{8.16}$$

We have $|\epsilon^{-l} r_{l,\epsilon,\gamma}| \le C|a|_{N+1,k,k}|\partial_{x'}b|_{M_2-1,0,0}$.

Note that when $k \le m_1$ each of the terms $r_{l,\epsilon,\gamma}$ has mixed degree m_1+m_2-k.

Remark 8.4. 1. Suppose $a \in \mathcal{M}^{m_1}_{M_1}$ for $M_1 \ge N+1$ and $b \in \mathcal{M}^{m_2}_{M_2}$ where M_2 satisfies only $M_2 \ge 2(N+1)+m_1+k+1$ for some $k\ge 1$. Since $\mathcal{M}^{m_1}_{M_1} \subset \mathcal{M}^{m_1}_{N+1}$, we can still apply Proposition 8.5 obtaining $t \in \mathcal{M}^{m_1+m_2}_{N+1}$.

2. When a is independent of β in Proposition 8.5, $r_{l,\epsilon,\gamma}=0$ for $l \ne 0$.

Proposition 8.6 (Mixed adjoints)**.** *Suppose* $a \in \mathcal{M}^m_M$ *where* $M \ge (N+1)+m+k+1$ *for some* $k \ge 1$. *Then*

$$a(x',\epsilon D, D)^* = t(x',\epsilon D, D) + r_{\epsilon,\gamma},$$

where for appropriate constants $C_{\alpha,\mu,\nu}$, *we have* $t(x',\beta,\zeta) =$

$$\sum_{|\alpha|\le k-1} \frac{1}{\alpha!}\left(\sum_{\mu+\nu=\alpha} C_{\alpha,\mu,\nu}(\epsilon\partial_{\beta'})^\mu \partial^\nu_{\zeta'} D^\alpha_{x'} a^*(x',\beta,\zeta)\right) \in \mathcal{M}^m_{M-k+1} \tag{8.17}$$

and

$$r_{\epsilon,\gamma} = \sum_{l\le k} r_{l,\epsilon,\gamma} \text{ with } r_{l,\epsilon,\gamma} = O(\epsilon^l\langle D,\gamma\rangle^{m-k+l}_{max}). \tag{8.18}$$

We have $|\epsilon^{-l} r_{l,\epsilon,\gamma}| \le C|\partial_{x'}a|_{M-1,k,k}$.

8.3. **Classical calculus.** The results on mixed products and adjoints contain results for classical products and adjoints as special cases, but we need to supplement those results with the following.

Proposition 8.7 (Classical products)**.** *Suppose*

$$p(x',\zeta) \in \mathcal{C}^{m_1}_{M_1} \text{ and } q(x',\zeta) \in \mathcal{C}^{m_2}_{M_2},$$

where $M_1 \ge N+1$ *and* $M_2 \ge 2(N+1)+|m_1|+3$. *Set* $t(x',\zeta) = p(x',\zeta)q(x',\zeta)$. *Then* $t \in \mathcal{C}^{m_1+m_2}_{M_1}$ *and*

$$p(x',D)q(x',D) = t(x',D) + r, \tag{8.19}$$

where r *is of order* m_1+m_2-1. *We have*

$$|r| \le C|p|_{N+1,1}|\partial_{x'}q|_{M_2-1,0}.$$

Proposition 8.8 (Classical adjoints)**.** *Suppose*

$$p(x',\zeta) \in \mathcal{C}^m_M,$$

where $M \ge (N+1)+|m|+3$. *Set* $t(x',\zeta) = p^*(x',\zeta)$. *Then*

$$p(x',D)^* = t(x',D) + r_{\epsilon,\gamma}, \tag{8.20}$$

where r *is of order* $m-1$ *and* $|r| \le C|\partial_{x'}p|_{M-1,1}$.

8.4. **Remainder terms.** Sometimes, for example in the proof of the Garding inequality in the next subsection, we need to know the mapping properties on H^s of the remainder terms that arise in the mixed calculus. The proofs show that these operators, while not actually given by symbols in $\mathcal{M}$, are still given by superpositions of Fourier multipliers converging in an appropriate sense. The next Proposition makes more precise the mapping properties of such superpositions.

Proposition 8.9. *Fix $s \in \mathbb{R}$ and suppose $a(x', \beta, \zeta) \in C^M(x', C^\infty(\beta, \zeta))$ has compact support in x' and satisfies*

$$|\partial_{x'}^\alpha a(x', \beta, \zeta)| \leq C\langle\zeta\rangle^k$$

for $|\alpha| \leq M$. Suppose $M \geq N + 1 + [|s - k|]$, where $[x]$ denotes the least integer $\geq x$. Then the operator

$$A(x', \epsilon D, D) \equiv \int e^{ix'\xi'} \hat{a}(\xi', \epsilon D, D) d\xi' \tag{8.21}$$

is of order k on H^s.

8.5. **Garding inequality.**

Notation 8.2. 1. (u, v) denotes the L^2 pairing, which can be extended as the duality pairing on $H^s \times H^{-s}$.

2. For a matrix a (symbol or operator) set $\Re a = \frac{a+a^*}{2}$.

The following Garding inequality will be used in the proof of the L^2 estimate to obtain bounds from below both in the interior and on the boundary.

Proposition 8.10 (Garding inequality)**.** *Consider $n \times n$ matrix symbols $a \in \mathcal{M}^m_{M_1}$, $w \in \mathcal{M}^0_{M_2}$, where $M_1 \geq 2(N + 1) + \max(\frac{m}{2}, m) + 2 + [|\frac{m}{2}|]$ and $M_2 \geq 2(N + 1) + m + 2 + [|\frac{m}{2}|]$. Suppose there is a scalar symbol $\chi \in \mathcal{M}^0_{M_1}$ and $c > 0$ such that $\chi^2 w = w$ and*

$$\Re a(x', \beta, \zeta) \geq c\langle\zeta\rangle^m \text{ on supp } \chi. \tag{8.22}$$

Let $A = a(x', \epsilon D, D)$ and $W = w(x', \epsilon D, D)$. Then there exists $C > 0$ such that for all $u \in H^{\frac{m}{2}}$

$$\frac{c}{2}|Wu|^2_{\frac{m}{2},\gamma} \leq \Re(AWu, Wu) + C(|u|^2_{\frac{m}{2}-1,\gamma} + \epsilon^2|u|^2_{\frac{m}{2},\gamma}). \tag{8.23}$$

C depends on symbol norms of a, w, and χ.

Part 4. **Stability estimates**

9. L^2 ESTIMATE

9.1. **The main estimates.**

Notation 9.1. 1. For $u(x) \in L^2(\overline{\mathbb{R}}_+, H^s(\mathbb{R}^N_{x'}))$ and $\zeta = (\zeta', \gamma) = (\ \zeta_0, \zeta'', \gamma)$, set

$$|u|_{s,\gamma} = |\langle\zeta\rangle^s \hat{u}(\zeta', x_N)|_{L^2(\zeta', x_N)}.$$

2. For $u(x') \in H^s(\mathbb{R}^N)$ set $\langle u\rangle_{s,\gamma} = |\langle\zeta\rangle^s \hat{u}|_{L^2(\zeta')}$.
3. When $s = 0$ we write $|u|_{0,\gamma} = |u|_0$, $\langle v\rangle_{0,\gamma} = \langle v\rangle_0$.
4. Let $\Lambda(\epsilon\zeta) = (1 + (\epsilon\gamma)^2 + (\epsilon\zeta_0)^2 + |\epsilon\zeta''|^4)^{\frac{1}{4}}$. For $u(x)$, $v(x')$ set

$$|u|_\Lambda = |\Lambda(\epsilon\zeta)\hat{u}(\zeta', x_N)|_{L^2(\zeta', x_N)},\ \langle v\rangle_\Lambda = |\Lambda(\epsilon\zeta)\hat{v}(\zeta')|_{L^2(\zeta')},$$

and similarly define $|u|_\phi$, $\langle v\rangle_\phi$ for other weights ϕ.
5. For $u(x)$ set $\langle u(0)\rangle_\phi = \langle u(x', 0)\rangle_\phi$.

Our goal is to prove the following (degenerate) L^2 estimate for solutions to the doubled boundary problem (7.6)

$$\begin{aligned}&\partial_N U - \frac{1}{\epsilon}\mathcal{G}U = F\\ &\Gamma U = 0 \text{ on } x_N = 0\\ &U = 0 \text{ in } x_0 < 0:\end{aligned} \tag{9.1}$$

Theorem 9.1 (Main L^2 estimate)**.** *Under the assumptions of section 2, there exist positive constants C, ϵ_0, γ_0 such that for all $\gamma > \gamma_0$, $0 < \epsilon < \epsilon_0$ with $\epsilon\gamma \leq 1$, solutions to* (9.1) *satisfy*

$$\sqrt{\epsilon}|U|_0 + \epsilon\langle U(0)\rangle_0 \leq C|F|_0. \tag{9.2}$$

The preceding estimate is a composite of three more precise estimates corresponding to the three natural frequency regimes in the problem, the regimes in which $\epsilon\zeta$ is of small, medium, or large size.

Recall $\beta = (\beta', \gamma') \in \mathbb{R}^N \times \mathbb{R}_+$ is a placeholder for $\epsilon\zeta$. We shall localize with respect to the size of β using cutoff functions $\chi_j(\beta) \in \mathcal{S}_\infty$ (8.2), $j = S, M, L$, such that

$$\chi_S(\beta) + \chi_M(\beta) + \chi_L(\beta) = 1, \tag{9.3}$$

where for some constants R_1 (sufficiently small), R_2 (sufficiently large)

$$\begin{aligned}&\text{supp } \chi_S \subset \{0 \leq |\beta| \leq R_1\}\\ &\text{supp } \chi_M \subset \{\frac{3}{4}R_1 \leq |\beta| \leq R_2\}\\ &\text{supp } \chi_L \subset \{\frac{3}{4}R_2 \leq |\beta|\}.\end{aligned} \tag{9.4}$$

Notation 9.2. 1. Our calculi involve semiclassical ($a(x', \beta)$), mixed ($b(x', \beta, \zeta)$), and classical ($c(x', \zeta)$) symbols, sometimes depending on parameters like x_N, ϵ, or $z = \frac{x_N}{\epsilon}$. When the nature of the symbol is clear from the context, we'll often write simply aU, bU, cU in place of $a(x', \epsilon D)U$, $b(x', \epsilon D, D)U$, $c(x', D)U$, respectively. When composing operators, we need to distinguish, for example, $b_1(x', \epsilon D, D)b_2(x', \epsilon D, D)$ from $(b_1b_2)(x', \epsilon D, D)$. To avoid ambiguity we add then the subscript D and write these two operators as $b_{1,D}b_{2,D}$, $(b_1b_2)_D$ respectively.

2. We will occasionally use the symbol χ_M to denote a cutoff distinct from the one in (9.4), but also supported in a bounded region strictly away from the origin. Similar statements apply as well to χ_S, χ_L.

3. The symbol r_0 will always denote a symbol or operator of order zero. It may change from line to line or even from term to term, entry to entry, etc..

4. Write the solution to (9.1) as $U = (U_+, U_-) = (u_+, v_+, u_-, v_-)$. Define

$$U_\Lambda = (\Lambda u_+, v_+, \Lambda u_-, v_-), \tag{9.5}$$

where $\Lambda(\epsilon D)$ is the multiplier associated to the symbol defined in Notation 9.1.

5. Let $\Pi_1(x', \hat{\beta}, \rho)$ and $\Pi_2(x', \hat{\beta}, \rho)$ be the projections defined in (9.68) satisfying $\Pi_1 + \Pi_2 = I$.

Proposition 9.1. *Using the notation just introduced, we have the following estimates for solutions to* (9.1)*. Let R_1, R_2 be as in* (9.4)*. For R_1 sufficiently small and R_2 sufficiently large, there exist* *constants C, γ_1, ϵ_1 such that for all $\gamma > \gamma_1$, $0 < \epsilon < \epsilon_1$ with $\epsilon\gamma \le 1$*

$$\begin{aligned}
&(a)\ \sqrt{\epsilon}|\chi_S U|_{1,\gamma} + \epsilon\langle \Pi_{1,D}\chi_S U(0)\rangle_{1,\gamma} + \langle \Pi_{2,D}\chi_S U(0)\rangle_0 \le \\
&\qquad C\left(|F|_0 + |(\partial_{\beta'}\chi_S)U|_0 + \epsilon|U|_0 + \epsilon\langle U(0)\rangle_0\right) \\
&(b)\ |\chi_M U|_0 + \sqrt{\epsilon}\langle \chi_M U(0)\rangle_0 \le \\
&\qquad C\left(\epsilon|F|_0 + \epsilon|U|_0 + \epsilon\langle U(0)\rangle_0\right) \\
&(c)\ |\chi_L U_\Lambda|_\Lambda + \sqrt{\epsilon}\langle \chi_L U_\Lambda(0)\rangle_{\sqrt{\Lambda}} \le \\
&\qquad C\left(\epsilon|F|_0 + \epsilon|U_\Lambda|_0 + \epsilon\langle U_\Lambda(0)\rangle_0\right).
\end{aligned} \tag{9.6}$$

Assuming Proposition 9.1 for the moment we prove Theorem 9.1.

Proof of Theorem 9.1. Let A denote the sum of the three left hand sides in (9.6). We have

$$\sqrt{\epsilon}|U|_0 + \epsilon\langle U(0)\rangle_0 \le A \le C|F|_0 + B, \tag{9.7}$$

where B represents the sum of the terms on the three right sides not involving F. Of these terms only 3 are not obviously dominated by other terms on the right, namely

$$|(\partial_{\beta'}\chi_S)U|_0,\ \epsilon|U_\Lambda|_0,\ \epsilon\langle U_\Lambda(0)\rangle_0. \tag{9.8}$$

We need to check that each of these 3 terms can be absorbed by A, at least for ϵ small enough and γ big enough. The second and third terms are easily handled by noting that in the small and midfrequency regions, the symbol $\Lambda \sim 1$. Finally, on supp $\partial_{\beta'}\chi_S$ we have $|\beta| \ge C > 0$, so to absorb this term we can first estimate it using (9.6)(b). ∎

The use of pseudodifferential calculi in the following arguments gives rise to many error terms that we need to absorb in the estimates. Having stated Proposition 9.1, we can now make the following definition:

Definition 9.1. We will call an error term *acceptable* if: (a) it can be dominated by a finite sum in which each term has the same form as one of those appearing on the right sides in (9.6), or

(b) it can be absorbed by the sum of the left sides of (9.6) for small enough ϵ and large enough γ satisfying $\epsilon\gamma \le 1$.

9.2. **Proof of Proposition 9.1.** This proof will occupy the rest of section 9. We note first that the estimates on $\chi_M U$ and $\chi_L U$ in (9.6) are taken from [MZ]. Indeed, estimates (9.6)(b),(c) are essentially estimates (4.37),(4.28), respectively, in [MZ]. We say "essentially" because, although [MZ] considers Dirichlet boundary conditions, the same argument in the medium and large frequency regions yields estimates for any boundary condition satisfying the uniform Lopatinski condition in those regions. In particular, note the extra gain in regularity in the high frequency region. This plays an important role in the final nonlinear stability argument.

We refer the reader to [GMWZ1], section 3, for more detail on how the standard symmetrizer argument works in those regions for such boundary conditions.

Thus, we shall focus henceforth on the estimate for $\chi_S U$ in (9.6)(a).

1. Localize to small frequency region.

In (9.1) $\mathcal{G} = \mathcal{G}(\mathcal{V}(x', \frac{x_N}{\epsilon}), p^\epsilon(x), \epsilon D)$. $\chi_S U$ satisfies

$$\begin{aligned}
&\chi_S U_{x_N} - \frac{1}{\epsilon}\mathcal{G}\chi_S U = \chi_S F + \frac{1}{\epsilon}[\chi_S, \mathcal{G}] \\
&\Gamma\chi_S U = 0 \text{ on } x_N = 0.
\end{aligned} \tag{9.9}$$

There is a high frequency contribution to the commutator because of the x' dependence of $\mathcal{G}$, and to get a good estimate for this we use the semiclassical calculus. Although $\mathcal{G}$ does not belong to

$\mathcal{S}_M$, it is simply a differential operator and we can use and directly estimate the expression for remainders in (12.15). Since

$$\chi_{S,D}\mathcal{G}_D = (\chi_S\mathcal{G})_D + \frac{\epsilon}{i}(\partial_{\beta'}\chi_S\partial_{x'}\mathcal{G})_D + \epsilon^2 R_D, \tag{9.10}$$

where R_D is of order zero, we have

$$\frac{1}{\epsilon}[\chi_{S,D}, \mathcal{G}_D] = \frac{1}{i}(\partial_{\beta'}\chi_S\partial_{x'}\mathcal{G})_D U + \epsilon R_D U. \tag{9.11}$$

Thus $U_S = \chi_S U$ satisfies

$$\begin{aligned} \partial_{x_N} U_S - \frac{1}{\epsilon}\mathcal{G}U_S &= F' \\ \Gamma U_S = 0 \text{ on } x_N &= 0, \end{aligned} \tag{9.12}$$

where

$$|F'|_0 \leq C|F|_0 + |(\partial_{\beta'}\chi_S)U|_0 + \epsilon|U|_0. \tag{9.13}$$

2. Extend and invert conjugator. It will be convenient first to extend the symbols $\mathcal{W}_j(x', z, p, \beta)$, $j = 0, 1$ defined in Lemma 7.2 smoothly to all $\beta \in \mathbb{R}^N \times \overline{\mathbb{R}}_+$, so that the estimates in (7.44) continue to hold. However, the ODEs (7.45) hold only on the original domain. For now we suppose that $\mathcal{W}_1$ satisfies (7.45)(b) for some $\mathcal{F}$ to be chosen.

The symbols $\mathcal{W}_j$ belong to $\mathcal{S}_\infty$, and we'll use the semiclassical calculus to construct approximate right and left inverses, $W_{-1,R,D}$ and $W_{-1,L,D}$ for the conjugator

$$\mathcal{W}_D = \mathcal{W}_0(x', \frac{x_N}{\epsilon}, p^\epsilon(x), \epsilon D) + \epsilon\mathcal{W}_1(x', \frac{x_N}{\epsilon}, p^\epsilon(x), \epsilon D). \tag{9.14}$$

These operators will satisfy

$$\begin{aligned} \mathcal{W}_D W_{-1,R,D} &= I + \epsilon^2 r_0 \\ W_{-1,L,D}\mathcal{W}_D &= I + \epsilon^2 r_0 \end{aligned} \tag{9.15}$$

where $r_0 = r_0(x', \frac{x_N}{\epsilon}, p^\epsilon(x), \epsilon D)$ has order zero.

Construct $W_{-1,R}(x', \frac{x_N}{\epsilon}, p^\epsilon(x), \beta)$ of the form

$$W_{-1,R} = w_0 + \epsilon w_1, \tag{9.16}$$

and use the calculus to find that w_0, w_1 should satisfy

$$\begin{aligned} w_0 &= \mathcal{W}_0^{-1} \\ w_1 &= \mathcal{W}_0^{-1}(\mathcal{W}_1 w_0 + \frac{1}{i}\partial_{\beta'}\mathcal{W}_0\partial_{x'}w_0). \end{aligned} \tag{9.17}$$

The construction of $W_{-1,L,D}$ is similar.

3. Second order conjugation to $\mathcal{G}_\infty$. Choose smooth cutoffs $\chi_1(\beta)$, $\chi_2(\beta)$, $\chi_S(\beta)$ compactly supported in ω_3 (recall (7.51)) and such that

$$\chi_1\chi_2 = \chi_1, \quad \chi_S\chi_1 = \chi_S. \tag{9.18}$$

Later, we'll use the same notation to denote a possibly different set of three cutoffs with the properties (9.18).

Note that (9.12) still holds with $\mathcal{G}$ replaced by $\mathcal{G}_D\chi_{2,D}$ and henceforth set

$$\mathcal{G}_D\chi_{2,D} = G_D, \quad \mathcal{G}_{\infty,D}\chi_{2,D} = G_{\infty,D}. \tag{9.19}$$

$\mathcal{G}$, for example, does not belong to $\mathcal{S}_\infty$, but G does.

Define $V = W_{-1,R,D}U_S$ and use (9.15) to see

$$\mathcal{W}_D V = U_S + \epsilon^2 r_0 U_S. \tag{9.20}$$

For later reference observe also that the equation and the calculus imply for each $L > 0$

$$(9.21)\quad \begin{aligned} &(a)\ \epsilon^2\partial_{x_N}U_S = \epsilon G_D U_S + \epsilon^2 F' = \epsilon r_0 U_S + \epsilon^2 F' \\ &(b)\ (1-\chi_{1,D})V = (1-\chi_{1,D})W_{-1,R,D}\ \chi_{S,D}U = \epsilon^L r_{L,0}U_S, \\ &(c)\ \partial_N U_S = \frac{1}{\epsilon}G_D\mathcal{W}_D V + F' + \epsilon r_0 U \end{aligned}$$

for some operator $r_{L,0}$ of order zero.

Computing $\partial_N(\mathcal{W}_D V)$ in two ways we have

$$(9.22)\quad \begin{aligned} &\partial_N(\mathcal{W}_D V) = \\ &(a)\ \frac{1}{\epsilon}\partial_z\mathcal{W}_{0,D}V + \partial_z\mathcal{W}_{1,D}V + (\partial_p\mathcal{W}_0\partial_N p^\epsilon)_D V + \epsilon r_0 V + \mathcal{W}_D\partial_N V = \\ &(b)\ \partial_N U_S + \epsilon^2\partial_N(r_0 U_S) = \frac{1}{\epsilon}G_D\mathcal{W}_D V + F' + \epsilon r_0 U. \end{aligned}$$

Using Lemma 7.2 and (9.21)(b), we have

$$(9.23)\quad \begin{aligned} \partial_z\mathcal{W}_{0,D}V &= (G\mathcal{W}_0 - \mathcal{W}_0 G_\infty)_D\ \chi_{1,D}V + \epsilon^L r_{0,L}U \\ \partial_z\mathcal{W}_{1,D}V &= (G\mathcal{W}_1 - \mathcal{W}_1 G_\infty + \mathcal{F})_D\ \chi_{1,D}V + \epsilon^L r_{0,L}U. \end{aligned}$$

Using the calculus and (9.21)(b) we find

$$(9.24)\quad \begin{aligned} G_D\mathcal{W}_D V &= (G\mathcal{W}_0 + \frac{\epsilon}{i}\partial_{\beta'}G\partial_{x'}\mathcal{W}_0 + \epsilon G\mathcal{W}_1)_D\ \chi_{1,D}V + \epsilon^2 r_0 U \\ \mathcal{W}_D G_{\infty,D}\ (\chi_{1,D}V) &= (\mathcal{W}_0 G_\infty + \frac{\epsilon}{i}\partial_{\beta'}\mathcal{W}_0\partial_{x'}G_\infty + \epsilon\mathcal{W}_1 G_\infty)_D\ \chi_{1,D}V + \epsilon^2 r_0 U. \end{aligned}$$

Finally, substitute (9.23) and (9.24) into (9.22) and observe that provided we choose

$$(9.25)\quad \mathcal{F} = \frac{1}{i}(\partial_{\beta'}G\partial_{x'}\mathcal{W}_0 - \partial_{\beta'}\mathcal{W}_0\partial_{x'}G_\infty) - \partial_p\mathcal{W}_0\partial_N p^\epsilon$$

in Lemma 7.2, then $\chi_{1,D}V$ satisfies

$$(9.26)\quad \mathcal{W}_D\partial_N(\chi_{1,D}V) - \frac{1}{\epsilon}\mathcal{W}_D G_{\infty,D}\ (\chi_{1,D}V) = F' + \epsilon r_0 U.$$

Apply $W_{-1,L,D}$ to get

$$(9.27)\quad \partial_N(\chi_{1,D}V) - \frac{1}{\epsilon}G_{\infty,D}(\chi_{1,D})V = W_{-1,L,D}F' + \epsilon r_0 U.$$

Since

$$(9.28)\quad \begin{aligned} 0 = \Gamma U_S = \Gamma\mathcal{W}_D V + \epsilon^2\Gamma r_0 U_S = \\ \Gamma\mathcal{W}_D\chi_{1,D}V + \Gamma\mathcal{W}_D(1-\chi_{1,D})V + \epsilon^2\Gamma r_0 U_S, \end{aligned}$$

we find using (9.21)(b) and (9.20)

$$(9.29)\quad \begin{aligned} &U_S = \mathcal{W}_D\chi_{1,D}V + \epsilon^2 r_0 U_S \text{ and} \\ &\Gamma\mathcal{W}_D(\chi_{1,D}V) = \epsilon^2 r_0 U_S. \end{aligned}$$

In view of (9.27) and (9.29) we have reduced the proof of Proposition 9.1 to showing that the estimate in (9.6)(a) holds with U_S on the left replaced by $U_{1,S}$ satisfying for some new F

$$\begin{aligned}&\partial_N U_{1,S} - \frac{1}{\epsilon} G_{\infty,D} U_{1,S} = F\\&\Gamma \mathcal{W}_D U_{1,S} = \epsilon^2 r_0 U_S \text{ on } x_N = 0\\&U_{1,S} = \chi_{1,D} r_0 U_S.\end{aligned} \tag{9.30}$$

Remark 9.1. In proving the estimate (9.6)(a) for $U_{1,S}$ one should replace $\mathcal{T}(x',0,0,\beta)$ (for $\mathcal{T}$ as in (7.67)) by $\mathcal{W}_0^{-1}(x',0,p'(x',0),\beta)\mathcal{T}(x',0,0,\beta)$ in the definition of Π_1, Π_2 (Definition 9.2). Similar remarks apply to the reductions that follow.

4. Conjugation to G^ϵ_{HP}. Here we'll reduce to a problem where $G_{\infty,D}$ in (9.30) is replaced by an operator $G^\epsilon_{HP,D}$ close to $\mathcal{G}_{HP,D}$ (recall (7.65). Until we achieve that reduction (9.46), the only way we can deal with $O(|U|_0)$ errors is to "incorporate them back into the system". The remaining acceptable errors are simply incorporated into the new forcing term F.

Notation 9.3. Denote by $O(\epsilon D)$ a semiclassical operator with symbol $s(x,\beta)$ such that $s = \beta\cdot f(x,\beta)$ for some smooth f. $O(\epsilon)$ denotes an operator with symbol $s = \epsilon f(x,\beta) \in \mathcal{S}_\infty$. In a similar way define $O(\epsilon^2)$, $O((\epsilon D)^2)$, etc.. When speaking of symbols instead of operators we'll use, as before, the notation $O(\epsilon\zeta)$, $O(\epsilon)$, etc.. In ambiguous cases like $O(\epsilon)$, the intent (symbol or operator) should be clear from the context.

Remark 9.2. Terms like $|O(\epsilon D)\chi_{2,D} U_{1,S}|_0$ are acceptable errors in the sense of Definition 9.1.

The main step is to conjugate to $\mathcal{G}_{1,\infty,D}$ with acceptable errors, since, for example, the conjugation by T_1 from $\mathcal{G}_{1,\infty,D}$ to $\mathcal{G}_{2,\infty,D}$ is exact.

Let $T_+(p,\beta)$ denote the upper left block of the matrix $T(p,\beta)$ defined in (7.54) and set

$$\begin{aligned}&G^+_\infty(p^\epsilon(x),\beta) = \begin{pmatrix} 0 & I \\ M_{\infty,+} & A_{\infty,+}\end{pmatrix}\chi_2,\\&G^+_{1,\infty} = \begin{pmatrix} H_R & 0 \\ 0 & P_R\end{pmatrix}\chi_2.\end{aligned} \tag{9.31}$$

As we did earlier with $\mathcal{W}_j$, extend the semiclassical symbol $T_+(p^\epsilon(x),\beta)$ smoothly to all $\beta \in \mathbb{R}^N \times \overline{\mathbb{R}}_+$ so that the extension has a uniformly bounded inverse, and use the calculus to construct right and left inverses satisfying

$$\begin{aligned}&T_{+,D} T_{-1,r,D} = I + \epsilon^2 r_0\\&T_{-1,l,D} T_{+,D} = I + \epsilon^2 r_0.\end{aligned} \tag{9.32}$$

The symbol $T_{-1,l}$ has the form

$$T_{-1,l}(p^\epsilon(x),\beta) = T_+^{-1} + \epsilon r_0, \tag{9.33}$$

and so does $T_{-1,r}$.

Write $U_{1,S} = (U_{1,S,+}, U_{1,S,-})$, set $U_s = U_{1,S,+}$, and define $V = T_{-1,r,D} U_s$. We have with $F = (F_+, F_-)$

$$\begin{aligned}&(a)\ T_{+,D} V = U_s + \epsilon^2 r_0 U_s\\&(b)\ (\partial_N T_{+,D})V + T_{+,D}\partial_N V = \partial_N U_s + O(\epsilon)(r_0 U_s + \epsilon F_+) =\\&\frac{1}{\epsilon} G^+_{\infty,D} T_{+,D} V + F_+ + O(\epsilon)(r_0 U_s + \epsilon F_+).\end{aligned} \tag{9.34}$$

We have the following symbol equalities

$$
\begin{aligned}
&(a)\ T_+ = \begin{pmatrix} I & A^{-1}_{\infty,+} \\ 0 & I \end{pmatrix} + O(\epsilon\zeta) + O(\epsilon) \\
&(b)\ T_{-1,l} = \begin{pmatrix} I & -A^{-1}_{\infty,+} \\ 0 & I \end{pmatrix} + O(\epsilon\zeta) + O(\epsilon) \\
&(c)\ T_{-1,l}\partial_N T_+ = \begin{pmatrix} 0 & r_0 \\ 0 & 0 \end{pmatrix} + O(\epsilon\zeta) + O(\epsilon) \\
&(d)\ G^+_\infty T_+ = \begin{pmatrix} 0 & I \\ 0 & A_{\infty,+} \end{pmatrix} \chi_2(\epsilon\zeta) + O(\epsilon\zeta) + O(\epsilon) \\
&(e)\ \partial_{\zeta'} T_{-1,l} = \epsilon O(\epsilon\zeta) + O(\epsilon) \\
&(f)\ \frac{1}{\epsilon}(\partial_{\zeta'} T_{-1,l})\partial_{x'}(G^+_\infty T_+) = \begin{pmatrix} 0 & r_0 \\ 0 & r_0 \end{pmatrix} \chi_2(\epsilon\zeta) + O(\epsilon\zeta) + O(\epsilon) \\
&(g)\ \frac{1}{\epsilon} T_{-1,l} G^+_\infty T_+ = \frac{1}{\epsilon} G^+_{1,\infty} + \begin{pmatrix} 0 & r_0 \\ 0 & r_0 \end{pmatrix} \chi_2(\epsilon\zeta) + O(\epsilon\zeta) + O(\epsilon).
\end{aligned} \tag{9.35}
$$

For (9.35)(g) we used (9.33), (7.52), and (9.35)(d).

Applying the operator $T_{-1,l,D}$ to (9.34)(b) and using the semiclassical calculus, we obtain in view of the symbol equalities (9.35):

$$
\begin{aligned}
\partial_N V = \frac{1}{\epsilon} \begin{pmatrix} H_{R,D} & \epsilon r_0 \\ 0 & P_{R,D} + \epsilon r_0 \end{pmatrix} \chi_{2,D} V + r_0 F_+ + \\
O(\epsilon) U_s + O(\epsilon D) V + O(\epsilon) V.
\end{aligned} \tag{9.36}
$$

Observe that terms on the right in (9.35)(c),(f), and (g) all make contributions to the r_0 entries of the first matrix on the right in (9.36).

Using the analogue of (9.21)(b) we obtain

$$
\partial_N(\chi_{1,D} V) = \frac{1}{\epsilon} \begin{pmatrix} H_{R,D} & \epsilon r_0 \\ 0 & P_{R,D} + \epsilon r_0 \end{pmatrix} (\chi_{1,D} V) + r_0 F_+ + E_{acc}, \tag{9.37}
$$

where E_{acc} is an acceptable error in the sense of Definition 9.1.

After performing the same sort of manipulations on the lower right block of $G_{\infty,D}$ we reduce to proving the desired estimate for

$$
\begin{aligned}
&\partial_N U_{2,S} - \frac{1}{\epsilon} \begin{pmatrix} H_{R,D} & \epsilon r_0 & 0 & 0 \\ 0 & P_{R,D} + \epsilon r_0 & 0 & 0 \\ 0 & 0 & H_{L,D} & \epsilon r_0 \\ 0 & 0 & 0 & P_{L,D} + \epsilon r_0 \end{pmatrix} U_{2,S} + F \\
&\Gamma \mathcal{W}_D T_D U_{2,S} = \epsilon^2 r_0 U_S \text{ on } x_N = 0 \\
&U_{2,S} = \chi_{1,D} r_0 U_S,
\end{aligned} \tag{9.38}
$$

in place of the problem (9.30).

Define

$$
G^+_{1,\infty,\epsilon}(p^\epsilon(x), \beta) = \begin{pmatrix} H_R & \epsilon r_0 \\ 0 & P_R + \epsilon r_0 \end{pmatrix} \chi_2. \tag{9.39}
$$

A direct computation using the invertibility of P_R shows that for $\beta \in \omega_3$ one can choose a matrix symbol $T_{a,+}$ of the form

$$T_{a,+}(p^\epsilon(x), \beta) = \begin{pmatrix} I & \epsilon r_0 \\ 0 & I \end{pmatrix} \tag{9.40}$$

such that

$$T_{a,+}^{-1} G_{1,\infty,\epsilon}^+ T_{a,+} = \begin{pmatrix} H_R & 0 \\ 0 & P_R + \epsilon r_0 \end{pmatrix}. \tag{9.41}$$

As before we extend and invert $T_{a,+,D}$. The operator $T_{-1,a,D}$ associated to the symbol

$$T_{-1,a} = \begin{pmatrix} I & -\epsilon r_0 \\ 0 & I \end{pmatrix} \tag{9.42}$$

is easily seen to be a right and left inverse satisfying the analogue of (9.32).

Write $U_{2,S} = (U_{2,S,+}, U_{2,S,-})$, set $U_s = U_{2,S,+}$, and define $V = T_{-1,a,D} U_s$. Now repeat the preceding argument line for line, but note, for example, that now instead of (9.35)(c),(e),(f) we have, respectively,

$$\begin{aligned} &T_{-1,a} \partial_N T_{a,+} = O(\epsilon) \\ &\partial_{\zeta'} T_{-1,a} = \begin{pmatrix} 0 & \epsilon^2 r_0 \\ 0 & 0 \end{pmatrix} \\ &\frac{1}{\epsilon} (\partial_{\zeta'} T_{-1,a}) \partial_{x'} (G_{1,\infty,\epsilon}^+ T_{a,+}) = O(\epsilon). \end{aligned} \tag{9.43}$$

Similarly choose $T_{a,-}$ corresponding to the lower right block of the matrix in (9.38), define T_a to be the matrix with blocks $(T_{a,+}, T_{a,-})$, recall T_1 from (7.58), and use the calculus as before to reduce to proving the desired estimate for

$$\begin{aligned} &\partial_N U_{3,S} - \frac{1}{\epsilon} G_{2,\infty,D}^\epsilon U_{3,S} + F \\ &\Gamma \mathcal{W}_D T_D T_{a,D} T_1 U_{3,S} = \epsilon^2 r_0 U_S \text{ on } x_N = 0 \\ &U_{3,S} = \chi_{1,D} r_0 U_S \end{aligned} \tag{9.44}$$

where

$$G_{2,\infty}^\epsilon(p^\epsilon(x), \beta) = \begin{pmatrix} H_R & & & \\ & H_L & & \\ & & P_R + \epsilon r_0 & \\ & & & P_L + \epsilon r_0 \end{pmatrix} \chi_2(\beta). \tag{9.45}$$

Finally, a similar but easier version of the above arguments allows us to use $T_{3,D}$ for $T_3(p'(x), \beta)$ as in (7.67) to reduce to the problem

$$\begin{aligned} &\partial_N U_{4,S} - \frac{1}{\epsilon} G_{HP,D}^\epsilon U_{4,S} + F \\ &\Gamma \mathcal{W}_D T_D T_{a,D} T_1 T_{3,D} U_{4,S} = \epsilon^2 r_0 U_S \text{ on } x_N = 0 \\ &U_{4,S} = \chi_{1,D} r_0 U_S \end{aligned} \tag{9.46}$$

where (recall (7.65))

$$G_{HP}^\epsilon(p^\epsilon(x), \beta) = \left(\mathcal{G}_{HP}(p'(x), \beta) + \begin{pmatrix} \epsilon r_0 & 0 \\ 0 & \epsilon r_0 \end{pmatrix} \right) \chi_2(\beta). \tag{9.47}$$

Here we have used the fact that $p^\epsilon(x) = (p'(x) + \epsilon r_0, \epsilon r_0)$.

5. Quantize the degenerate symmetrizer.

In this paragraph we'll use notation introduced in Notation 7.6 and Proposition 7.4, as well as the $O(\epsilon^l\langle D\rangle^k)$ notation introduced in the section on the mixed calculus. Recall $\zeta=(\zeta',\gamma)$ where $\gamma\geq 1$ and that β is a placeholder for $\epsilon\zeta$.

In Proposition 7.4 we constructed the symbol of a degenerate symmetizer

$$S(x,\widehat{\beta},\rho)=\begin{pmatrix}S_H(p'(x),\widehat{\beta},\rho) & 0\\ 0 & S_P(\beta)\end{pmatrix} \tag{9.48}$$

on $p^{'-1}(\omega^*)$. Let $\kappa_2(x)$ and $\chi_3(\beta)$ be smooth cutoffs such that

$$\text{supp }\kappa_2(x)\chi_3(\beta)\subset p^{'-1}(\omega^*). \tag{9.49}$$

Set $\mathcal{A}^\epsilon_1=\{(x,\zeta):(x,\epsilon\zeta)\in p^{'-1}(\omega^*)\}$ and define symbols supported in $\mathcal{A}^\epsilon_1$:

$$\begin{aligned}&s^\epsilon_h(x,\zeta)=S_H(p'(x),\frac{\zeta}{|\zeta|},\epsilon|\zeta|)\kappa_2(x)\chi_3(\epsilon\zeta)\\ &s^\epsilon_p(x,\zeta)=S_P(\epsilon\zeta)\kappa_2(x)\chi_3(\epsilon\zeta)=\begin{pmatrix}s^\epsilon_{p+} & 0\\ 0 & s^\epsilon_{p-}\end{pmatrix}, \text{ where}\\ &s^\epsilon_{p+}(x,\zeta)=CI\kappa_2(x)\chi_3(\epsilon\zeta),\ s^\epsilon_{p-}(x,\zeta)=-\epsilon^2(|\zeta|^2\kappa_2(x)\chi_3(\epsilon\zeta)I).\end{aligned} \tag{9.50}$$

For $H(p'(x),\beta)$, $P(p'(x),\beta)$ as in (7.65) define

$$\begin{aligned}&\mathcal{H}^\epsilon(x,\zeta)=H(p'(x),\epsilon\zeta)\chi_2(\epsilon\zeta)\kappa_1(x)\\ &\mathcal{P}^\epsilon(x,\zeta)=P(p'(x),\epsilon\zeta)\chi_2(\epsilon\zeta)\kappa_1(x)=\begin{pmatrix}\mathcal{P}^\epsilon_+ & 0\\ 0 & \mathcal{P}^\epsilon_-\end{pmatrix},\end{aligned} \tag{9.51}$$

where the cutoffs satisfy $\kappa_1\kappa_2=\kappa_1$, $\chi_3\chi_2=\chi_2$.

Remark 9.3. 1. Observe that since $\chi_3(\beta)$ has compact support, $s^\epsilon_h(x,\zeta)$ represents a bounded family in the classical symbol class $\mathcal{C}^0_\infty$. On the other hand the operators $s^\epsilon_{p,D}$, $\mathcal{H}^\epsilon_D$, and $\mathcal{P}^\epsilon_D$ can be viewed as semiclassical, mixed, or classical operators corresponding to symbols in $\mathcal{S}_\infty$, $\mathcal{M}^0_\infty$, or $\mathcal{C}^0_\infty$, respectively.

2. As indicated in (9.50) we can also view $s^\epsilon_{p-}(x,\zeta)$ as the product of $-\epsilon^2$ with the mixed symbol of order two $|\zeta|^2\kappa_2(x)\chi_3(\epsilon\zeta)I\in\mathcal{M}^2_\infty$.

Let

$$\mathcal{A}^\epsilon_0=\{(x,\zeta):\kappa_1(x)\chi_2(\epsilon\zeta)=1\}\subset\mathcal{A}^\epsilon_1. \tag{9.52}$$

Lemma 9.1 (Interior properties). *The symbols just defined satisfy the following properties on $\mathcal{A}^\epsilon_0$ for $\epsilon\in(0,1]$:*

1. s^ϵ_h is self-adjoint and

$$\Re(\frac{1}{\epsilon}s^\epsilon_h\mathcal{H}^\epsilon)=\sum(v^\epsilon_l)^*k^\epsilon_lv^\epsilon_l, \tag{9.53}$$

where

(a)the $v^\epsilon_l(x,\zeta)$ are bounded families (parametrized by ϵ) in $\mathcal{C}^0_\infty$ such that the finite sum $\sum(v^\epsilon_l)^*v^\epsilon_l>C>0$;

(b)the k^ϵ_l are bounded families in $\mathcal{C}^1_\infty$ having the block structure

$$k^\epsilon_l(x,\zeta)=\begin{pmatrix}b^\epsilon_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & b^\epsilon_p\end{pmatrix}. \tag{9.54}$$

The number of blocks p can vary with l. There is $c > 0$ such that either

$$b_j^\epsilon \geq c|\zeta| \tag{9.55}$$

or $b_j^\epsilon = \gamma b_{j,0}^\epsilon + \epsilon b_{j,2}^\epsilon$, where $b_{j,k}^\epsilon$ are bounded families in $\mathcal{C}_\infty^k$ satisfying

$$b_{j,0}^\epsilon \geq c \text{ and } b_{j,2}^\epsilon \geq c|\zeta|^2. \tag{9.56}$$

2. s_p^ϵ is self-adjoint and

$$\begin{aligned} &\Re(\frac{1}{\epsilon} s_{p+}^\epsilon \mathcal{P}_+^\epsilon) \geq \frac{C}{\epsilon} \\ &\Re(\frac{1}{\epsilon} s_{p-}^\epsilon \mathcal{P}_-^\epsilon) = \epsilon b_-^\epsilon(x, \zeta, \epsilon\zeta), \end{aligned} \tag{9.57}$$

where $b_-^\epsilon(x, \zeta, \beta)$ is a bounded family of mixed symbols in $\mathcal{M}_\infty^2$ such that

$$b_-^\epsilon \geq c|\zeta|^2 \tag{9.58}$$

for some $c > 0$.

Proof. The proof is an immediate consequence of Proposition 7.4. □

9.3. **Block diagonalization of $S + \Gamma_1^*\Gamma_1$.** In order to describe the properties of the symmetrizer on the boundary we first rewrite the decomposition of $\mathbb{C}^{4m}$ in (7.77):

$$C^{4m} = E_{P_{1,-}}(x', \beta) \oplus E_{P_{1,-}}^c(x', \beta), \tag{9.59}$$

where $E_{P_{1,-}}^c(x', \beta)$ is the $4m - 1$ dimensional space

$$E_{P_{1,-}}^c(x', \beta) = E_H \oplus E_{P_+} \oplus E_{P_{2,-,\kappa}}(x', \beta). \tag{9.60}$$

Choose a smooth basis vector $f_{4m}(x', \beta)$ for $E_{P_{1,-}}(x', \beta)$ of unit length. Note that f_{4m} will be some scalar multiple of the image under $(\mathcal{T}^0)^{-1}(x', 0, \beta)$ (7.67) of the "doubled profile" (recall (7.32))

$$\begin{pmatrix} \mathbb{U}_{1+}^R \\ \mathbb{U}_{m-}^L \end{pmatrix} |_{(x',0,\widehat{\beta},\rho)}.$$

Next choose an orthonormal basis

$$\{f_1(x', \beta), \ldots, f_{4m-1}(x', \beta)\} \text{ for } E_{P_{1,-}}^c(x', \beta)$$

so that

$$\{f_1(x', \beta), \ldots, f_{4m}(x', \beta)\} \tag{9.61}$$

is a smoothly varying orthonormal basis for $\mathbb{C}^{4m}$. The f_j are initially defined for all x' and $|\beta|$ small, but can be smoothly extended to all β as an orthonormal basis.

Let M denote the positive (for $\rho > 0$) self-adjoint matrix $S + \Gamma_1^*\Gamma_1$. Since f_{4m} is generally not an eigenvector of M, we need to modify the above basis. The estimates (7.91) together with a Rayleigh-Ritz argument show that M has a simple small eigenvalue $\lambda_s(x', \hat{\beta}, \rho) \sim \rho^2$ isolated from the rest, and hence varying smoothly with $(x', \hat{\beta}, \rho)$. We write

$$\lambda_s(x', \hat{\beta}, \rho) = \rho^2 \mu(x', \hat{\beta}, \rho). \tag{9.62}$$

The estimate (7.91) implies that the smoothly varying eigenspace $H_1(x', \hat{\beta}, \rho)$ corresponding to λ_s is spanned by a vector of size ~ 1 of the form

$$e_{4m}(x', \hat{\beta}, \rho) = f_{4m}(x', \beta) + \rho g_{4m}(x', \hat{\beta}, \rho) \tag{9.63}$$

for some smooth $g_{4m} \in f_{4m}^{\perp}$ of size $O(1)$. Thus, $H_2(x', \hat{\beta}, \rho) \equiv e_{4m}^{\perp}$ is an invariant subspace for M spanned by vectors of the form

$$e_j(x', \hat{\beta}, \rho) = f_j(x', \beta) + \rho g_j(x', \hat{\beta}, \rho), \text{ for } j = 1, \ldots, 4m-1. \tag{9.64}$$

The smooth dependence of the f_j on β (unlike g_j) plays an important role in the following arguments. Normalize and relabel so that the e_j give an orthonormal basis for $\mathbb{C}^{4m}$.

Notation 9.4. Thinking of e_j as a column vector and e_j^* as its row vector adjoint, we'll write $e_j e_j^*$ for the $4m \times 4m$ matrix which orthogonally projects $x \in \mathbb{C}^{4m}$ onto e_j. Thus,

$$e_j e_j^* x = (x, e_j) e_j, \tag{9.65}$$

where (,) is the complex inner product on $\mathbb{C}^{4m}$.

Definition 9.2. Define $\pi_1(x', \hat{\beta}, \rho)$ and $\pi_2(x', \hat{\beta}, \rho)$ to be the smoothly varying orthogonal projections of $\mathbb{C}^{4m}$ onto the first and second components, respectively, of

$$C^{4m} = H_1(x', \hat{\beta}, \rho) \oplus H_2(x', \hat{\beta}, \rho). \tag{9.66}$$

In terms of the basis just chosen we have

$$\begin{aligned} \pi_1(x', \hat{\beta}, \rho) &= e_{4m} e_{4m}^* \\ \pi_2(x', \hat{\beta}, \rho) &= \sum_{j=1}^{4m-1} e_j e_j^*. \end{aligned} \tag{9.67}$$

In the statement of Proposition 9.1 Π_1, Π_2 are defined as follows:

$$\Pi_j(x', \hat{\beta}, \rho) = \mathcal{T}(x', 0, 0, \beta) \pi_j(x', \hat{\beta}, \rho) \mathcal{T}^{-1}(x', 0, 0, \beta) \tag{9.68}$$

for $\mathcal{T}$ as in (7.67).

Lemma 9.2 (Boundary properties)**.**
Set $s^\epsilon(x, \zeta) = \begin{pmatrix} s_h^\epsilon & 0 \\ 0 & s_p^\epsilon \end{pmatrix}$. *Let* $T_4(x', \hat{\zeta}, \epsilon|\zeta|)$ *be the matrix whose columns are* $(e_1, \ldots, e_{4m})$, *define* $V \in \mathbb{C}^{4m}$ *by* $U = T_4 V$, *and set* $V' = (v_1, \ldots, v_{4m-1})$ *and*

$$\mathcal{A}_{0,0}^\epsilon = \{(x', \zeta) : (x', 0, \zeta) \in \mathcal{A}_0^\epsilon\}. \tag{9.69}$$

Then on $\mathcal{A}_{0,0}^\epsilon$ *we have*

$$\begin{aligned} &(a)((s^\epsilon + \Gamma_1^* \Gamma_1) \pi_2 U, \pi_2 U) = (B_2^\epsilon(x', \zeta) V', V') \\ &\qquad \text{where } B_2^\epsilon \in \mathcal{C}_\infty^0 \text{ is } (4m-1) \times (4m-1) \text{ and } B_2^\epsilon \geq cI. \\ &(b)((s^\epsilon + \Gamma_1^* \Gamma_1) \pi_1 U, \pi_1 U) = (B_1^\epsilon(x', \zeta) v_{4m}, v_{4m}) \\ &\qquad \text{where } B_1^\epsilon = \epsilon^2 b_1^\epsilon(x', \epsilon\zeta, \zeta) \text{ is } 1 \times 1 \text{ and } b_1^\epsilon \in \mathcal{M}_\infty^2 \text{ satisfies } b_1^\epsilon \geq c\langle\zeta\rangle^2. \end{aligned} \tag{9.70}$$

Proof. The equality in (a) is proved by a simple computation using the explicit formulas for T_4 and π_2. B_2 is positive since H_2 is a strictly positive invariant subspace for M.

(b) is proved by the same kind of computation as (a). The form of B_1^ϵ follows from (9.62). Use Remark 9.3 to see that B_1^ϵ, B_2^ϵ are in the stated symbol classes. ☐

Our symmetrizer is the operator

$$s_D^\epsilon = \begin{pmatrix} s_{h,D}^\epsilon & 0 \\ 0 & s_{p,D}^\epsilon \end{pmatrix}. \tag{9.71}$$

6. Reduce to $(G_{\mathcal{H}^\epsilon \mathcal{P}^\epsilon}, \Gamma_1)$**.**

Choose cutoffs $\kappa(x)$ and $\chi_1(\beta)$ such that $\kappa\kappa_1 = \kappa$, $\chi_1\chi_2 = \chi_1$, and set $U_{5,S} = \kappa(x)U_{4,S}$. Having reduced to the block form (9.46), we are now in a position to absorb $O(|U|_0)$ errors (see (9.76)).

Observe that the boundary condition in (9.46) satisfies

$$\Gamma \mathcal{W}_D T_D T_{a,D} T_1 T_{3,D} = \Gamma_{1,D} + \epsilon r_0 \tag{9.72}$$

for $\Gamma_1(x', \beta)$ as in (7.67).

Also, we see by using the semiclassical calculus that commuting the cutoff $\kappa(x)$ through (9.46) produces an $O(1)$ error in the interior and an $O(\epsilon)$ error on the boundary. Thus, using a partition of unity in x we reduce to

$$\begin{aligned} &\partial_N U_{5,S} - \frac{1}{\epsilon} G_{\mathcal{H}^\epsilon \mathcal{P}^\epsilon, D} U_{5,S} = F_0 \\ &\Gamma_{1,D} U_{5,S} = \epsilon r_0 U_S \text{ on } x_N = 0 \\ &U_{5,S} = \kappa(x) U_{4,S} = \kappa(x) \chi_{1,D} r_0 U_S, \end{aligned} \tag{9.73}$$

where

$$\begin{aligned} &G_{\mathcal{H}^\epsilon \mathcal{P}^\epsilon} = \begin{pmatrix} \mathcal{H}^\epsilon(x,\zeta) & 0 \\ 0 & \mathcal{P}^\epsilon(x,\zeta) \end{pmatrix} \\ &F_0 = F + \begin{pmatrix} r_0 & 0 \\ 0 & r_0 \end{pmatrix} U_{4,S}. \end{aligned} \tag{9.74}$$

Remark 9.4. Since U_5 is cutoff by $\chi_{1,D}$, we may replace Γ_1 by $\Gamma_1\chi_0$ for some cutoff such that $\chi_0\chi_1 = \chi_0$ without affecting (9.73). From now on we assume this has been done.

7. L^2 estimate.

Notation 9.5. 1. Let (f, g) denote the inner product of $L^2(x)$ and $\langle f, g\rangle$ that of $L^2(x')$.
2. Set $U_{5,S} = (u, v) = (u, v_{p+}, v_{p-})$.
3. Set $F_0 = (f_h, f_p) = (f_h, f_{p+}, f_{p-})$.
4. Let $\phi = \sqrt{\gamma} + \sqrt{\epsilon}|\zeta|$ and set $|u|_\phi = |\phi u|_0$.

Begin from the identities

$$\begin{aligned} &(a)\ \langle s^\epsilon_{h,D} u, u\rangle + \Re \frac{1}{\epsilon}(s^\epsilon_{h,D} \mathcal{H}^\epsilon_D u, u) = -((\partial_N s^\epsilon_{h,D})u, u) - 2\Re(f_h, s^\epsilon_{h,D} u) \\ &(b)\ \langle s^\epsilon_{p,D} v, v\rangle + \Re \frac{1}{\epsilon}(s^\epsilon_{p,D} \mathcal{P}^\epsilon_D v, v) = -((\partial_N s^\epsilon_{p,D})v, v) - 2\Re(f_p, s^\epsilon_{p,D} v). \end{aligned} \tag{9.75}$$

The right sides of (9.75)(a),(b) are dominated, respectively, by

$$\begin{aligned} &C(|u|_0^2 + |f_h|_0^2) \\ &C(|v_{p+}|_0^2 + |f_{p+}|_0^2 + \delta\epsilon |v_{p-}|^2_{1,\gamma} + C_\delta \epsilon |f_{p-}|_0^2). \end{aligned} \tag{9.76}$$

Here we've used $|\epsilon\zeta| \le C$ and the special form of s^ϵ_{p-}.

Remark 9.5. Recall that $r_0 U_{4,S}$ is one contribution (of size $O(|U|_{L^2})$ to f_{p-}. Fortunately, the ϵ on the f_{p-} term in the previous estimate allows us to absorb that contribution. In this sense $O(|U|_{L^2})$ errors can be tolerated once HP block structure has been achieved.

A similar remark applies to f_h, which has an $O(|U|_{L^2})$ contribution that can be absorbed using the estimate (9.77) below.

The argument of [MZ] Proposition 4.8 shows

$$\Re \frac{1}{\epsilon}(s^\epsilon_{h,D} \mathcal{H}^\epsilon_D u, u) \ge c_1 |u|^2_\phi - c_2 |u|_0^2. \tag{9.77}$$

This uses Lemma 9.1(1) and the Garding inequality for classical operators.

The operator $\frac{1}{\epsilon^2}s^\epsilon_{p-,D}$ is of order two in the mixed calculus. Using that calculus gives

$$\frac{1}{\epsilon^2}s^\epsilon_{p-,D}\mathcal{P}^\epsilon_{-,D} = (\frac{1}{\epsilon^2}s^\epsilon_{p-}\mathcal{P}^\epsilon_{-})_D + O(\langle D\rangle) + O(\epsilon\langle D\rangle^2). \tag{9.78}$$

Thus, using (9.57), (9.58) and applying the Garding inequality in the mixed calculus we obtain,

$$\begin{aligned} c\epsilon|v_{p-}|^2_{1,\gamma} \leq \Re\frac{1}{\epsilon}(s^\epsilon_{p-,D}\mathcal{P}^\epsilon_{-,D}v_{p-}, v_{p-}) + \\ C(\epsilon|U_S|^2_0 + \epsilon^3|U_S|^2_{1,\gamma}) + C\epsilon|v_{p-}|^2_{\frac{1}{2},\gamma}. \end{aligned} \tag{9.79}$$

Here the U_S terms on the right are errors from Garding's inequality, while the v_{p-} terms correspond to the errors in (9.78). (We've used $|\epsilon\zeta| \leq C$ on supp $\chi_1(\epsilon\zeta)$).

Similarly, but more easily we obtain

$$c\frac{|v_{p+}|^2_0}{\epsilon} \leq \Re\frac{1}{\epsilon}(s^\epsilon_{p+,D}\mathcal{P}^\epsilon_{+,D}v_{p+}, v_{p+}) + C(|v_{p+}|^2_0 + \epsilon|U_S|^2_0). \tag{9.80}$$

Boundary terms This is the most delicate part, since we'll have only weak trace control on $\pi_{1,D}U_{5,S}$.

Notation 9.6. 1. Set $U_{5,S} = U_5$.

2. Let $\mathcal{F}(x', \hat{\zeta}, \epsilon|\zeta|)$ denote an element of $\mathcal{C}^0_\infty$ which may change from line to line.

3. Using (2) and (9.64) we may write for $j = 1, 2$

$$\pi_j(x', \hat{\zeta}, |\epsilon\zeta|) = \pi_{j,s}(x', \epsilon\zeta) + \epsilon|\zeta|\mathcal{F}, \tag{9.81}$$

where $\pi_{1,s} = f_{4m}f^*_{4m}$, for example. Recall

$$f_{4m} = (0_h, 0_{p+}, f_{p-}). \tag{9.82}$$

4. Set $M^\flat(x, \epsilon\zeta) = (s^{\epsilon,\flat} + \Gamma_1^*\Gamma_1)(x', \epsilon\zeta)$, where

$$s^{\epsilon,\flat} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & s^\epsilon_{p-}(x', \epsilon\zeta) \end{pmatrix}. \tag{9.83}$$

Recall M itself is not smooth in β.

5. When $a(x', \hat{\zeta}, \epsilon|\zeta|)$ defines a bounded family in $\mathcal{C}^0_\infty$ for ϵ small, we'll write just $a(x', \hat{\zeta}, \epsilon|\zeta|) \in \mathcal{C}^0_\infty$.

The next Lemma is used repeatedly below.

Lemma 9.3. *For $i = 1, 2$ let $A_i = a_i(x', \hat{\zeta}, \epsilon|\zeta|) \in \mathcal{C}^0_\infty$, $B = \epsilon|\zeta|b(x', \hat{\zeta}, \epsilon|\zeta|) \in \mathcal{C}^0_\infty$, and $C = c(x', \epsilon\zeta) \in \mathcal{S}_\infty$. Then*

$$\begin{aligned} &(a)\ A_D B_D = (AB)_D + \epsilon r_0, \ \ B_D A_D = (BA)_D + \epsilon r_0 \\ &(b)\ C_D A_D = (CA)_D + \epsilon r_0 \\ &(c)\ A_{1,D}A_{2,D} = (A_1A_2)_D + O(\langle D\rangle^{-1}) + \epsilon r_0 \\ &(d)\ (C_D)^* = (C^*)_D + \epsilon r_0; \ \ (B_D)^* = (B^*)_D + \epsilon r_0 \end{aligned} \tag{9.84}$$

Proof. The lemma follows immediately from the product and adjoint theorems in the semiclassical, mixed, and classical calculi. ∎

Setting $\mathcal{M}_D = s^\epsilon_D + (\Gamma_{1,D})^*\Gamma_{1,D}$ and $U_5 = \pi_{1,D}U_5 + \pi_{2,D}U_5$ we'll estimate the four terms

$$\begin{aligned}&(a)\langle \mathcal{M}_D\pi_{1,D}U_5, \pi_{1,D}U_5\rangle,\\ &(b)\langle \mathcal{M}_D\pi_{1,D}U_5, \pi_{2,D}U_5\rangle,\\ &(c)\langle \mathcal{M}_D\pi_{2,D}U_5, \pi_{1,D}U_5\rangle,\\ &(d)\langle \mathcal{M}_D\pi_{2,D}U_5, \pi_{2,D}U_5\rangle.\end{aligned} \tag{9.85}$$

The left side of our L^2 estimate for U_5 will contain the boundary terms

$$\epsilon^2\langle \pi_{1,D}U_5\rangle^2_{1,\gamma} + C\langle \pi_{2,D}U_5\rangle^2_0. \tag{9.86}$$

These terms, along with the others on the left in Proposition 9.1, determine which errors arising from the calculus can be absorbed. The following errors, for example, are acceptable:

$$\begin{aligned}&(a)\langle \epsilon\pi_{2,D}U_5, U_5\rangle \le \delta\langle \pi_{2,D}U_5\rangle^2_0 + C_\delta\epsilon^2\langle \pi_{1,D}U_5\rangle^2_0,\\ &(b)\epsilon^2\langle U_S\rangle^2_0 + \langle \epsilon O(\epsilon D)U_S, U_S\rangle \le C_\delta\epsilon^2\langle U_S\rangle^2_0 + \delta\epsilon^2\langle U_S\rangle^2_{1,\gamma}.\end{aligned} \tag{9.87}$$

Observe that we cannot tolerate $\epsilon\langle U_5\rangle^2_0$ or $\langle\langle D\rangle^{-1}U_5\rangle^2_0$ errors.

Notation 9.7. We'll denote the errors on the left in (9.87)(a),(b) by Err_1, Err_2, respectively.

Remark 9.6. The replacement of $\mathcal{M}_D$ by M_D in (9.85)(b),(c),(d) results by Lemma 9.3 only in an error of type Err_1. This replacement has to be done differently for (9.85)(a) (Lemma 9.4).

Mixed boundary terms.

Proposition 9.2.
$(\mathcal{M}_D\pi_{1,D}U_5, \pi_{2,D}U_5) + (\mathcal{M}_D\pi_{2,D}U_5, \pi_{1,D}U_5) = Err_1 + Err_2.$

Proof. In view of Remark 9.6 we may replace $\mathcal{M}$ by M. Lemma 9.3 gives

$$\begin{aligned}&(\mathcal{M}_D\pi_{2,D}U_5, \pi_{1,D}U_5) = (\pi_{1,D}\mathcal{M}_D\pi_{2,D}U_5, U_5) + Err_1\\ &\langle(\pi_1 M)_D\pi_{2,D}U_5, U_5\rangle + Err_1 =\\ &\langle(\lambda_s\pi_1)_D\pi_{2,D}U_5, U_5\rangle + Err_1 =\\ &\langle(\lambda_s\pi_1\pi_2)_D U_5, U_5\rangle + Err_1 + Err_2,\end{aligned} \tag{9.88}$$

where we've used the form (9.62) of λ_s in the last equality. Since $\pi_1\pi_2 = 0$ this gives the result for this term. The other term is handled the same way.

□

Positive boundary terms. Consider first the more difficult term $\langle \mathcal{M}_D\pi_{1,D}U_5, \pi_{1,D}U_5\rangle$.

Lemma 9.4. $\langle \mathcal{M}_D\pi_{1,D}U_5, \pi_{1,D}U_5\rangle = \langle(\pi_1 M\pi_1)_D U_5, U_5\rangle + Err_1 + Err_2.$

Proof. **1.** Using Lemma 9.3 again, we obtain

$$\begin{aligned}&\langle(\Gamma_{1,D})^*\Gamma_{1,D}\pi_{1,D}U, \pi_{1,D}U\rangle = \langle \Gamma_{1,D}\pi_{1,D}U, \Gamma_{1,D}\pi_{1,D}U\rangle =\\ &\langle(\Gamma_1\pi_1)_D U, (\Gamma_1\pi_1)_D U\rangle + Err_1 + Err_2 =\\ &\langle((\Gamma_1\pi_1)_D)^*(\Gamma_1\pi_1)_D U, U\rangle + Err_1 + Err_2 =\\ &\langle(\pi_1\Gamma_1^*\Gamma_1\pi_1)_D U, U\rangle + Err_1 + Err_2.\end{aligned} \tag{9.89}$$

Here for the second, third, and fourth equalities we have used

$$\Gamma_1\pi_1 = \rho\mathcal{F} \;\; (\mathcal{F} \text{ as in Notation 9.6}). \tag{9.90}$$

2. Next consider $\langle s^\epsilon_D\pi_{1,D}U,\pi_{1,D}U\rangle$. Writing π_1 as in (9.81), we have four terms to examine. We begin with

$$\begin{aligned}&\langle s^\epsilon_D\pi_{1,s,D}U,\pi_{1,s,D}U\rangle=\langle s^{\epsilon,\flat}_D\pi_{1,s,D}U,\pi_{1,s,D}U\rangle=\\&\langle(s^{\epsilon,\flat}\pi_{1,s})_DU,\pi_{1,s,D}U\rangle+Err_2=\langle\pi_{1,s,D}(s^{\epsilon,\flat}\pi_{1,s})_DU,U\rangle+Err_2=\\&\langle(\pi_{1,s}s^{\epsilon,\flat}\pi_{1,s})_DU,U\rangle+Err_2=\langle(\pi_{1,s}s^\epsilon\pi_{1,s})_DU,U\rangle+Err_2.\end{aligned}\tag{9.91}$$

In the first equality we've used (9.83) and

$$\pi_{1,s}=\begin{pmatrix}0&&\\&0&\\&&f_{p-}f^*_{p-}\end{pmatrix},\tag{9.92}$$

while the second, third, and fourth equalities follow from Lemma 9.3 and the fact that $s^{\epsilon,\flat}=O(\rho^2)$ (9.50).

3. Next consider

$$\begin{aligned}&\langle s^\epsilon_D(\rho\mathcal{F})_DU,\pi_{1,s,D}U\rangle=\langle\pi_{1,s,D}s^\epsilon_D(\rho\mathcal{F})_DU,U\rangle+Err_2=\\&\langle(\pi_{1,s}s^{\epsilon,\flat})_D(\rho\mathcal{F})_DU,U\rangle+Err_2=\langle(\pi_{1,s}s^{\epsilon,\flat}\rho\mathcal{F})_DU,U\rangle+Err_2=\\&(\pi_{1,s}s^\epsilon\rho\mathcal{F})_DU,U\rangle+Err_2\end{aligned}\tag{9.93}$$

Here again we've used Lemma 9.3 and the special form of $\pi_{1,s}$ and $s^{\epsilon,\flat}$.

4. The remaining two terms are handled quite similarly. Adding up we obtain the result. ☐

Proposition 9.3. *There exist positive constants c, C such that*

$$c\epsilon^2\langle\pi_{1,D}U_5\rangle^2_{1,\gamma}\le\langle\mathcal{M}_D\pi_{1,D}U_5,\pi_{1,D}U_5\rangle+Err_1+Err_2.\tag{9.94}$$

Proof. **1.** Let $T_4=T$ (from Lemma 9.2) and define $V=(V',V_{4m})=(T^*)_DU$. As we did with π_j, we can write

$$T=T_s+\rho\mathcal{F}\tag{9.95}$$

where $T_s(x',\beta)$ is smooth in β. By Lemma 9.3 we have

$$\begin{aligned}&(a)(T^*)_DT_D=I+\epsilon r_0,\ T_D(T^*)_D=I+\epsilon r_0\\&(b)T_DV=U_5+\epsilon r_0U_5\\&(c)\pi_{1,D}U_5=T_D\begin{pmatrix}0\\V_{4m}\end{pmatrix}+\epsilon r_0U_5\\&(d)\pi_{2,D}U_5=T_D\begin{pmatrix}V'\\0\end{pmatrix}+\epsilon r_0U_5.\end{aligned}\tag{9.96}$$

2. In view of Lemma 9.4 we just need to consider

$$\begin{aligned}&\langle(\pi_1M\pi_1)_DU_5,U_5\rangle=\langle(\lambda_s\pi_1)_DT_DV,T_DV\rangle+Err_2=\\&\langle(T^*\lambda_s\pi_1T)_DV,V\rangle+Err_2.\end{aligned}\tag{9.97}$$

We've used Lemma 9.3, (9.95), and the special form of λ_s (9.62).

Now $T^*\lambda_s\pi_1T$ is a $4m\times4m$ matrix with all entries zero except for the $(4m,4m)$ entry, which equals $\lambda_s=\epsilon^2b^\epsilon_1$ for b^ϵ_1 as in (9.70)(b). Applying the Garding inequality for the mixed calculus and using (9.96) gives the result. ☐

Proposition 9.4. *There exist positive constants c, C such that for γ large and ϵ small*

$$c\langle\pi_{2,D}U_5\rangle^2_{1,\gamma}\le\langle\mathcal{M}_D\pi_{2,D}U_5,\pi_{2,D}U_5\rangle+Err_2.\tag{9.98}$$

Proof.

$$\begin{aligned}(9.99)\quad &\langle M_D\pi_{2,D}U_5,\pi_{2,D}U_5\rangle = \left\langle M_D(T_D\begin{pmatrix}V'\\0\end{pmatrix}+O(\epsilon)U_5), T_D\begin{pmatrix}V'\\0\end{pmatrix}+O(\epsilon)U_5\right\rangle = \\ &\langle B^\epsilon_D V',V'\rangle + \langle O(\langle D\rangle^{-1}+\epsilon)V',V'\rangle + \\ &\langle O(1)V',O(\epsilon)U_5\rangle + \langle O(\epsilon)U_5,O(1)V'\rangle + \langle O(\epsilon)U_5,O(\epsilon)U_5\rangle.\end{aligned}$$

The classical Garding inequality gives in view of the positivity of B^ϵ

$$\begin{aligned}(9.100)\quad c\langle V'\rangle_0^2 \le \langle M_D\pi_{2,D}U_5,\pi_{2,D}U_5\rangle + \frac{C}{\gamma}\langle V'\rangle_0^2 + C\epsilon\langle V'\rangle_0^2 \\ +\epsilon^2 C_\delta\langle U_5\rangle_0^2 + \delta\langle V'\rangle_0^2 + C\epsilon^2\langle U_5\rangle_0^2.\end{aligned}$$

Using (9.96) the result follows easily. □

Adding up the estimates and absorbing terms by taking ϵ small and γ large, we obtain the degenerate L^2 estimate for the problem (9.73) satisfied by $U_{5,S}=(u,v_{p+},v_{p-})$:

$$\begin{aligned}(9.101)\quad |u|^2_\phi + \frac{|v_{p+}|_0^2}{\epsilon} + \epsilon|v_{p-}|^2_{1,\gamma} + \epsilon^2\langle\pi_{1,D}U_5\rangle^2_{1,\gamma} + \langle\pi_{2,D}U_5\rangle_0^2 \le \\ C(|F|^2 + \epsilon|U_S|_0^2 + \epsilon^3|U_S|^2_{1,\gamma}) + Err_1 + Err_2\end{aligned}$$

The errors on the right are acceptable in the sense of Definition 9.1, so this concludes the proof of Proposition 9.1 and the degenerate L^2 estimate of Theorem 9.1.

10. Higher derivative estimates

In this section we'll use the notation for norms introduced in section 9. We use ∂ to denote some tangential derivative, one of $\partial_0,\dots,\partial_{N-1}$. Sometimes ∂U will denote the tangential gradient of U, instead of just a single partial derivative of U.

Notation 10.1. 1. For $k=1,2,\dots$ let $U^{*,k}=((\frac{\gamma}{\epsilon^2})^kU,(\frac{\gamma}{\epsilon^2})^{k-1}\partial U,\dots,\partial^kU)$. Here ∂^jU represents all possible tangential derivatives of U order j.

2. Define $U^{*,k}_\Lambda$ simply by replacing U by U_Λ (9.5) in the definition of $U^{*,k}$.

Proposition 10.1. *Under the assumptions of section 2, there exist positive constants C, ϵ_0, γ_0 such that for all $\gamma>\gamma_0$, $0<\epsilon<\epsilon_0$ with $\epsilon\gamma\le 1$, solutions to* (9.1) *satisfy*

$$(10.1)\quad |U^{*,k}|_0 + \sqrt{\epsilon}\langle U^{*,k}\rangle_0 \le \frac{C}{\sqrt{\epsilon}}|F^{*,k}|_0.$$

This follows immediately from the following more precise estimates by an argument parallel to the proof of Theorem 9.1.

Proposition 10.2. *Using the notation just introduced, we have the following estimates for solutions to* (9.1)*. Let R_1, R_2 be as in* (9.4)*. For R_1 sufficiently small and R_2 sufficiently large, there* *exist constants* C, γ_1, ϵ_1 *such that for all* $\gamma > \gamma_1$, $0 < \epsilon < \epsilon_1$ *with* $\epsilon\gamma \leq 1$

$$
\begin{aligned}
&(a)\ |\chi_S U^{*,k}|_{1,\gamma} + \sqrt{\epsilon}\langle \chi_S U^{*,k}\rangle_{1,\gamma} \leq \\
&\qquad \frac{C}{\sqrt{\epsilon}}\left(|F^{*,k}|_0 + |(\partial_{\beta'}\chi_S)U^{*,k}|_0 + \epsilon|U^{*,k}|_0 + \epsilon\langle U^{*,k}\rangle_0\right)\\
&(b)\ |\chi_M U^{*,k}|_0 + \sqrt{\epsilon}\langle \chi_M U^{*,k}\rangle_0 \leq \\
&\qquad C\left(\epsilon|F^{*,k}|_0 + \epsilon|U^{*,k}|_0 + \epsilon\langle U^{*,k}\rangle_0\right)\\
&(c)\ |\chi_L U^{*,k}_\Lambda|_\Lambda + \sqrt{\epsilon}\langle \chi_L U^{*,k}_\Lambda\rangle_{\sqrt{\Lambda}} \leq \\
&\qquad C\left(\epsilon|F^{*,k}|_0 + \epsilon|U^{*,k}_\Lambda|_0 + \epsilon\langle U^{*,k}_\Lambda\rangle_0\right).
\end{aligned}
\tag{10.2}
$$

Proof. The estimates in (b) and (c) follow directly from the higher derivative estimates of [MZ] in the medium and large freqency regions. These are estimates with γ weights for the linearized problem, so one can simply apply them to the problems satisfied by $\frac{U}{(\epsilon^2)^j}$ for various j.

As usual, therefore, we focus on the small frequency region. If we simply differentiate the equation and throw commutators on the right as forcing, those new forcing terms are too large to absorb in a straightforward way. To get around this problem we reprove L^2 estimates for an appropriate enlarged system.

1. Enlarging the system. We begin with a solution U of the doubled boundary problem

$$
\begin{aligned}
&\partial_N U - \frac{1}{\epsilon}\mathcal{G}U = F\\
&\Gamma U = 0 \text{ on } x_N = 0\\
&U = 0 \text{ in } x_0 < 0:
\end{aligned}
\tag{10.3}
$$

Let ∂ denote one of $\partial_0, \ldots, \partial_{N-1}$. Observe that $(\frac{\gamma}{\epsilon^2}U, \partial U)$ satisfies the enlarged system

$$
\begin{aligned}
&\partial_N\begin{pmatrix}\frac{\gamma}{\epsilon^2}U\\ \partial U\end{pmatrix} - \frac{1}{\epsilon}\begin{pmatrix}\mathcal{G} & 0\\ 0 & \mathcal{G}\end{pmatrix}\begin{pmatrix}\frac{\gamma}{\epsilon^2}U\\ \partial U\end{pmatrix} = \begin{pmatrix}\frac{\gamma}{\epsilon^2}F\\ \partial F\end{pmatrix} + \begin{pmatrix}0\\ \frac{\epsilon}{\gamma}[\partial, \mathcal{G}]\left(\frac{\gamma}{\epsilon^2}U\right)\end{pmatrix},\\
&\begin{pmatrix}\Gamma & 0\\ 0 & \Gamma\end{pmatrix}\begin{pmatrix}\frac{\gamma}{\epsilon^2}U\\ \partial U\end{pmatrix} = 0 \text{ on } x_N = 0,\\
&\begin{pmatrix}\frac{\gamma}{\epsilon^2}U\\ \partial U\end{pmatrix} = 0 \text{ in } x_0 < 0.
\end{aligned}
\tag{10.4}
$$

2. Localize to small frequency region. Let $\chi_S(\epsilon\zeta)$ be a small frequency cutoff as before. Commuting $\chi_{S,D}$ through (10.4) we obtain (writing χ_S for $\chi_{S,D}$)

$$
\begin{aligned}
&\partial_N(\chi_S U^{*,1}) - \frac{1}{\epsilon}\begin{pmatrix}\mathcal{G} & 0\\ 0 & \mathcal{G}\end{pmatrix}(\chi_S U^{*,1}) = \\
&\qquad \chi_S F^{*,1} + \chi_S\begin{pmatrix}0\\ \frac{\epsilon}{\gamma}[\partial, \mathcal{G}](\frac{\gamma}{\epsilon^2}U)\end{pmatrix} + \frac{1}{\epsilon}\left[\chi_S, \begin{pmatrix}\mathcal{G} & 0\\ 0 & \mathcal{G}\end{pmatrix}\right]U^{*,1} = F',
\end{aligned}
\tag{10.5}
$$

where

$$
|F'|_0 \leq C(|F^{*,1}|_0 + |(\partial_{\beta'}\chi_S)U^{*,1}|_0 + \epsilon|U^{*,1}|_0).
\tag{10.6}
$$

The second commutator was computed like the corresponding term in the previous section (9.11).

The boundary condition is

$$
\begin{pmatrix}\Gamma & 0\\ 0 & \Gamma\end{pmatrix}\chi_S U^{*,1} = 0.
\tag{10.7}
$$

The problem (10.5),(10.7) can be treated just like (9.12). We may now repeat the (entire) argument of the previous section to obtain the desired estimate of $U^{*,1}$. Iteration completes the proof.

$\square$

Remark 10.1. If $U^{*,1}$ had been defined instead as $\begin{pmatrix} \frac{\gamma}{\epsilon}U \\ \partial U \end{pmatrix}$, the first commutator in (10.5) would have produced an unacceptable $O(|U^{*,1}|_0)$ error.

11. Nonlinear stability

Notation 11.1. 1. Suppose u, v, w are vectors in $\mathbb{R}^{m_1}$, $\mathbb{R}^{m_2}$, $\mathbb{R}^{m_3}$, respectively. Denote by $uf(v)w$ a function with values in $\mathbb{R}^m$ whose components are finite sums of terms of the form

$$u_j f_k(v) w_l$$

where f_k is a C^∞ function of v and u_j, w_l represent components of u and w.

2. Recall $|u|_{k,\gamma} = |\langle\zeta\rangle^k \hat{u}(\zeta, x_N)|_0$. For $k \in \mathbb{N}$ we have the equivalence of norms

$$|u|_{k,\gamma} \sim \sum_{|\alpha|\le k} \gamma^{k-|\alpha|} |\partial^\alpha u|_0. \tag{11.1}$$

3. Set $|u|_* = |u|_{L^\infty}$.

4. Define

$$\|u\|_{k,\gamma} = |u|_{k,\gamma} + |\epsilon \partial u|_{k,\gamma}. \tag{11.2}$$

5. Let M and $L < M$ be the positive integers appearing in the nonlinear error equation (6.14). They can be taken arbitrarily large as long as the approximate solution $(\tilde{u}, d\Psi)$ is constructed with sufficiently many terms. Let $b_\pm$ and $F_\pm$ be the functions appearing in (6.14). They are bounded in $H^M(\overline{\mathbb{R}}^{N+1}_+)$ uniformly with respect to ϵ. With $\nabla = (\partial_1, \dots, \partial_N)$ and $z = \frac{x_N}{\epsilon}$, set $b = (b_+, b_-)$ and

$$\begin{aligned} B &= (b, \nabla b) \\ \tilde{\mathcal{U}} &= (\tilde{u}, \partial_z \tilde{u}, \partial \tilde{u}, d\Psi). \end{aligned} \tag{11.3}$$

6. $\phi(\gamma)$ always denotes an increasing function of γ. It may change from term to term.

7. Set $\partial'' = (\partial_1, \dots, \partial_{N-1})$.

Let us first rewrite the error equation in the doubled form corresponding to the linear problem (10.3). As before let

$$U_\pm = e^{-\gamma x_0}(w_\pm, \epsilon \partial_N w_\pm) \text{ and } U = (U_+, U_-).$$

Let $\kappa(x_0)$ be a smooth cutoff which is identically one on $[0, T_0]$. After computing out the divergence term in (6.14) and inserting the subscript n to denote the nth iterate, we obtain an error equation of the form

$$\begin{aligned} &\partial_N U_{n+1} - \frac{1}{\epsilon}\mathcal{G} U_{n+1} = e^{-\gamma x_0}\kappa(x_0)\mathcal{F}_\epsilon(U_n, \partial'' U_n), \\ &\Gamma U_{n+1} = 0 \text{ on } x_N = 0, \\ &U_{n+1} = 0 \text{ in } x_0 < 0, \end{aligned} \tag{11.4}$$

where in the notation just described,

$$\begin{aligned}\mathcal{F}_n \equiv \mathcal{F}_\epsilon(U_n,\partial''U_n) &= \epsilon^{L-3}(\epsilon U_n)f(\tilde{\mathcal{U}},\epsilon^M B,\epsilon^L U_n,\epsilon,e^{\gamma x_0})(\epsilon U_n)\\ &+\epsilon^{L-3}(\epsilon U_n)f(\tilde{\mathcal{U}},\epsilon^M B,\epsilon^L U_n,\epsilon,e^{\gamma x_0})(\epsilon\partial''U_n)\\ &+\epsilon^{M-2}Bf(\tilde{\mathcal{U}},\epsilon^M B,\epsilon^L U_n,\epsilon,e^{\gamma x_0})(\epsilon U_n,\epsilon\partial''U_n)\\ &+\epsilon^{M-L}C(x')^{-1}F\\ &=\mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}.\end{aligned}\tag{11.5}$$

For $\mathbb{F}(U,\partial''U)\equiv e^{-\gamma x_0}\kappa(x_0)\mathcal{F}_\epsilon(U,\partial''U)$ consider the nonlinear error equation

$$\begin{aligned}&\partial_N U-\frac{1}{\epsilon}\mathcal{G}U=\mathbb{F}(U,\partial''U),\\ &\Gamma U=0 \text{ on } x_N=0,\\ &U=0 \text{ in } x_0<0.\end{aligned}\tag{11.6}$$

Theorem 11.1. *Recall N is the number of space dimensions. Suppose the constants k,L,M satisfy*

$$\begin{aligned}&k-3>\frac{N}{2}\\ &M-L-2k-\frac{1}{2}>1\\ &L-3-2k-\frac{1}{2}>1.\end{aligned}\tag{11.7}$$

Then there exist constants ϵ_0, γ_0 such that for all $0<\epsilon\le\epsilon_0$, $\gamma\ge\gamma_0$ satisfying $\epsilon\gamma\le 1$, the error equation (11.6) *has a unique solution U satisfying the estimates*

$$\begin{aligned}&\|U\|_{k,\gamma}\le\epsilon^{M-L-2k-\frac{1}{2}}\phi(\gamma)\\ &|U|_*\le 1\\ &|\partial U|_*\le 1\end{aligned}\tag{11.8}$$

for some $\phi(\gamma)$, an increasing function of γ.

Proof. The first few points are some preliminaries.

1. Sobolev inequalities. For $k-3>\frac{N}{2}$ we have

$$\begin{aligned}&(a)\epsilon|\partial U|_*\le C(\gamma)(\epsilon|U|_{k-2,\gamma}+\epsilon|\partial_N U|_{k-2,\gamma})\\ &(b)\epsilon|U|_*\le C(\gamma)(\epsilon|U|_{k-3,\gamma}+\epsilon|\partial_N U|_{k-3,\gamma}).\end{aligned}\tag{11.9}$$

2. Moser inequalities.

For $k\in\mathbb{N}$ let $\alpha=(\alpha_1,\dots,\alpha_r)$ with $|\alpha|=\alpha_1+\cdots+\alpha_r\le k$, $\alpha_i\in\mathbb{N}$. Suppose $|v_i|_{k,\gamma}+|v_i|_*<\infty$. Then

$$\gamma^{k-|\alpha|}|(\partial^{\alpha_1}v_1)\cdots(\partial^{\alpha_r}v_r)|_0\le C\sum_{i=1}^r|v_i|_{k,\gamma}(\prod_{j\ne i}|v_i|_*)$$

3. Relations between norms. Directly from the definitions we see

$$\begin{aligned}&(a)|U|_{k,\gamma}\le C|U^{*,k}|_0\\ &(b)|U^{*,k}|_0\le\frac{C}{\epsilon^{2k}}|U|_{k,\gamma}.\end{aligned}\tag{11.10}$$

Let $\chi_L(\epsilon\zeta)$ be a high frequency cutoff like the one in (10.2)(c). Observe that

$$\|U\|_{k,\gamma}\sim|U|_{k,\gamma}+|\chi_L(\epsilon\partial U)|_{k,\gamma}.\tag{11.11}$$

4. High frequency estimate. We can absorb the high frequency pieces of $U_\Lambda^{*,k}$ in the two terms on the right in (10.2)(c) to obtain

$$|\chi_L U_\Lambda^{*,k}|_\Lambda \leq C\left(\epsilon|F^{*,k}|_0 + \epsilon|U^{*,k}|_0 + \epsilon\langle U^{*,k}\rangle_0\right). \tag{11.12}$$

Use the main L^2 estimate (10.1) to replace the right side of the above inequality by $C|F^{*,k}|_0$.

When $|\epsilon\zeta|$ is large, we have $\frac{\Lambda^2}{\epsilon} \geq C\langle\zeta\rangle$. Thus, we may conclude

$$|\chi_L(\epsilon\partial U)|_{k,\gamma} \leq C|F^{*,k}|_0. \tag{11.13}$$

5. Induction assumption. Let the first iterate U_1 be 0. Assume there exist $\epsilon_1(\gamma)$, γ_1 such that for $0<\epsilon\leq\epsilon_1$, $\gamma\geq\gamma_1$, and some $\phi(\gamma)$

$$\begin{aligned} &\|U_n\|_{k,\gamma} \leq 2\epsilon^{M-L-2k-\frac{1}{2}}\phi(\gamma)\\ &|U_n|_* \leq 1\\ &|\partial U_n|_* \leq 1 \end{aligned} \tag{11.14}$$

The main step is to show, after decreasing ϵ_1 if necessary, that U_{n+1} satisfies the same estimates.

6. Estimate $\mathbb{F}_n \equiv \mathbb{F}(U_n, \partial'' U_n)$. Set $\mathbb{A} = e^{-\gamma x_0}\kappa(x_0)\mathcal{A}$ and define $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ similarly. Applying the Moser inequalities we have

$$|\mathbb{A}|_{k,\gamma} \leq C(\gamma)\epsilon^{L-2}|U_n|_{k,\gamma}, \tag{11.15}$$

where $C(\gamma)$ depends on L^∞ norms of $\tilde{\mathcal{U}}$, $\epsilon^M B$, and ϵU_n.

Write $\epsilon\partial U_n = (1-\chi_L)(\epsilon\partial U_n) + \chi_L(\epsilon\partial U_n)$, and corresponding to this decomposition set $\mathbb{B} = \mathbb{B}_1 + \mathbb{B}_2$. Since $|\epsilon\zeta| \leq C$ on supp $(1-\chi_L(\epsilon\zeta))$, we have just as above

$$|\mathbb{B}_1|_{k,\gamma} \leq C(\gamma)\epsilon^{L-2}|U_n|_{k,\gamma}. \tag{11.16}$$

For $\mathbb{B}_2$ we have

$$|\mathbb{B}_2|_{k,\gamma} \leq C(\gamma)(\epsilon^{L-2}|U_n|_{k,\gamma} + \epsilon^{L-3}|\chi_L(\epsilon\partial U_n)|_{k,\gamma}). \tag{11.17}$$

Similarly, we have

$$\begin{aligned} &|\mathbb{C}|_{k,\gamma} \leq C(\gamma)(\epsilon^{M-1}|U_n|_{k,\gamma} + \epsilon^{M-2}|\chi_L(\epsilon\partial U_n)|_{k,\gamma}), \text{ and}\\ &|\mathbb{D}|_{k,\gamma} \leq \phi(\gamma)\epsilon^{M-L}. \end{aligned} \tag{11.18}$$

Summing the above estimates we obtain

$$|\mathbb{F}_n|_{k,\gamma} \leq C(\gamma)(\epsilon^{L-2}|U_n|_{k,\gamma} + \epsilon^{L-3}|\chi_L(\epsilon\partial U_n)|_{k,\gamma}) + \epsilon^{M-L}\phi(\gamma). \tag{11.19}$$

7. Estimate $\|U_{n+1}\|_{k,\gamma}$. In view of the main L^2 estimate, (11.10), and (11.19) we have

$$\begin{aligned} |U_{n+1}|_{k,\gamma} &\leq C|U_{n+1}^{*,k}|_0 \leq \frac{C}{\sqrt{\epsilon}}|\mathbb{F}_n^{*,k}|_0 \leq \frac{C}{\epsilon^{2k+\frac{1}{2}}}|\mathbb{F}_n|_{k,\gamma}\\ &\leq C(\gamma)(\epsilon^{L-2-2k-\frac{1}{2}}|U_n|_{k,\gamma} + \epsilon^{L-3-2k-\frac{1}{2}}|\chi_L(\epsilon\partial U_n)|_{k,\gamma}) + \epsilon^{M-L-2k-\frac{1}{2}}\phi(\gamma). \end{aligned} \tag{11.20}$$

From (11.13) and (11.19) we obtain

$$\begin{aligned} |\chi_L(\epsilon\partial U_{n+1})|_{k,\gamma} &\leq C|\mathbb{F}_n^{*,k}|_0 \leq \frac{C}{\epsilon^{2k}}|\mathbb{F}_n|_{k,\gamma}\\ &\leq C(\gamma)(\epsilon^{L-2-2k}|U_n|_{k,\gamma} + \epsilon^{L-3-2k}|\chi_L(\epsilon\partial U_n)|_{k,\gamma}) + \epsilon^{M-L-2k}\phi(\gamma). \end{aligned} \tag{11.21}$$

Adding the previous two estimates we find

$$\|U_{n+1}\|_{k,\gamma} \leq \epsilon^{L-3-2k-\frac{1}{2}}C(\gamma)\|U_n\|_{k,\gamma} + \epsilon^{M-L-2k-\frac{1}{2}}\phi(\gamma). \tag{11.22}$$

Provided $\epsilon_1(\gamma)$ is chosen so that $\epsilon^{L-3-2k-\frac{1}{2}}C(\gamma) \leq \frac{1}{2}$, the induction assumption and (11.22) imply

$$\|U_{n+1}\|_{k,\gamma} \leq 2\epsilon^{M-L-2k-\frac{1}{2}}\phi(\gamma). \tag{11.23}$$

8. L^∞ **estimates.** The equation gives

$$\epsilon|\partial_N U_{n+1}|_{k-2,\gamma} \leq C|U_{n+1}|_{k,\gamma} + \epsilon|\mathbb{F}_n|_{k-2,\gamma}. \tag{11.24}$$

From (11.19) we get

$$|\mathbb{F}_n|_{k,\gamma} \leq \epsilon^{L-3}C(\gamma)\|U_n\|_{k,\gamma} + \epsilon^{M-L}\phi(\gamma). \tag{11.25}$$

Thus,

$$\epsilon|\partial_N U_{n+1}|_{k-2,\gamma} \leq 2\epsilon^{M-L-2k-\frac{1}{2}}\phi(\gamma). \tag{11.26}$$

This together with the inequalities (11.9) and the assumption (11.7) immediately implies that for ϵ_1 small enough

$$\begin{aligned} &\epsilon|U_{n+1}|_* \leq \epsilon \\ &\epsilon|\partial U_{n+1}|_* \leq \epsilon. \end{aligned} \tag{11.27}$$

This completes the inductive step.

9. Contraction Thus, the sequence of iterates satisfies the estimates (11.14). One can now consider the problem satisfied by $U_{n+1} - U_n$ and use estimates like those above (but simpler) to show that for ϵ_1 small enough, the sequence converges to some U in the $\| \|_{0,\gamma}$ norm. A standard argument (involving interpolation and weak convergence) implies that U solves the error equation (11.6) and satisfies the estimates (11.8) in Theorem 11.1.

This completes the proof of Theorem 11.1.

□

12. Long time versus small viscosity

In this brief section we remark on some of the relations between the question of long time stability of planar shocks studied in [GMWZ1] and the small viscosity limit for curved shocks on a finite time interval studied in this paper.

I. Consider the following constant endstates, long-time stability error problem, similar to the one studied in [GMWZ1]:

$$\begin{aligned} &v_t + (f'(U)v)_x + \delta g_x(x,v) = v_{xx} \\ &v(0,x) = v_0(x), \end{aligned} \tag{12.1}$$

Here $v = v(t,x)$, $x \in \mathbb{R}^d$, $U(x_1)$ is a profile, v_{xx} means Laplacian of v, $g(x,v) = O(|v|^2)$, and F_x means div F. δ is a parameter that we are free to make small, but we don't want to confuse it later with viscosity ϵ. We suppose

$$\lim_{x_1 \to \pm\infty} U(x_1) = U_\pm. \tag{12.2}$$

Think of the profile $U(x_1)$ as being initially perturbed to $U(x_1) + \delta v_0(x)$. To show the profile is stable as $t \to \infty$ one wants to show the solution $v(t,x)$ to (12.1) satisfies

$$|v(t,x)|_{L^\infty(x)} \to 0 \text{ as } t \to \infty. \tag{12.3}$$

To do this it suffices to show for any $p > \frac{d}{2}$

$$|v|_{L^2(t,H^{p+1}(x))} + |v_t|_{L^2(t,H^{p-1}(x))} \leq C. \tag{12.4}$$

Observe that (12.4) holds if and only if for some $T_0 > 0$

$$|v|_{L^2([0,\frac{T_0}{\epsilon}],H^{p+1}(x))} + |v_t|_{L^2([0,\frac{T_0}{\epsilon}],H^{p-1}(x))} \leq C \tag{12.5}$$

uniformly with respect to ϵ.

12.1. **Rescale.** Set

$$\epsilon t = \tau, \epsilon x = y, z(\tau, y) = v(\frac{\tau}{\epsilon}, \frac{y}{\epsilon}), \tag{12.6}$$

so the problem (12.1) becomes (after dividing the parabolic eqn by epsilon)

$$\begin{aligned} &z_\tau + (f'(U(\frac{y_1}{\epsilon}))z)_y + \delta(g(\frac{y}{\epsilon}, z))_y = \epsilon z_{yy} \\ &z(0,y) = v_0(\frac{y}{\epsilon}), \end{aligned} \tag{12.7}$$

a small viscosity problem.

Observe that for any $T_0 > 0$

$$|z|_{L^2([0,T_0],y)} = \epsilon^{\frac{d+1}{2}} |v|_{L^2([0,\frac{T_0}{\epsilon}],x)}, \tag{12.8}$$

and $\epsilon\partial_{\tau,y} = \partial_{t,x}$.

Thus, v satisfies (12.5) if and only if z satisfies

$$\sum_{k=0}^{p+1} \epsilon^{k-\frac{d+1}{2}} |\partial_y^k z|_{L^2([0,T_0],y)} + \sum_{k=0}^{p-1} \epsilon^{k+1-\frac{d+1}{2}} |\partial_y^k z_\tau|_{L^2([0,T_0],y)} \leq C \tag{12.9}$$

for some $T_0 > 0$ and some $p > \frac{d}{2}$, with C independent of ϵ.

So if we show (12.9), we have a long time stability result.

Can one prove (12.9) using methods like those in this paper: double the problem, construct an approximate solution, etc.? We think not.

Note first some obvious differences between (12.7) and the error equation (6.4). The initial data of $\tilde{u} + w$ for w as in (6.4) depends on $(x', \frac{x_N}{\epsilon})$ and corresponds to a boundary layer near $x_N = 0$, while in (12.7) the data depends on $\frac{y}{\epsilon}$, where $y = (y_1, \ldots, y_d)$.

In addition the $\frac{1}{\epsilon}$ dependence in the coefficients of (6.4) enters only through $\frac{x_N}{\epsilon}$, while in (12.7) it enters through $\frac{y}{\epsilon}$.

In short the small viscosity problem has a boundary layer, while the long time problem does not.

II. Consider the problem studied in this paper in the case when the original inviscid shock is actually planar, that is, when $(U^0_\pm, d\psi_0)$ is constant. The problem is trivial in this case since the profile is already an exact solution to the parabolic problem (1.2). The sense in which the solution to the parabolic problem converges to the inviscid shock is immediately clear.

III. On the other hand consider the question of long time stability in the case when the original inviscid shock is curved. Clearly, for “most” curved shocks this problem does not even make sense, since curved inviscid shocks generally don't live forever. This suggests the interesting question of how viscous shocks evolve near times when the corresponding inviscid shock becomes unstable, a question we don't address in this paper.

Part 5. Appendix: Proofs for section 8

12.2. Semiclassical calculus.

We use the notation introduced in section ().

Proof of Proposition 8.1. When $p(x',\epsilon\zeta)$ independent of x', $p(x',\epsilon D)$ is just a bounded Fourier multiplier on L^2. Reduce the general case to this case by writing

$$p(x',\epsilon\zeta)=p(0,\epsilon\zeta)+h(x',\epsilon\zeta), \tag{12.10}$$

where $h\in\mathcal{S}_M$ has compact support in x'. We have

$$h(x',\epsilon D)=\int e^{ix'\xi'}\hat{h}(\xi',\epsilon D)d\xi', \tag{12.11}$$

and

$$|\hat{h}(\xi',\epsilon\zeta)|\leq C\langle\xi'\rangle^{-M}. \tag{12.12}$$

Thus, $h(x',\epsilon D)$ is an absolutely convergent superposition of bounded Fourier multipliers on L^2, and so is bounded on L^2.

□

Proof of Proposition 8.2. **1.** The assertion is trivial if $q(x',\beta)$ is independent of x', so since $q(x',\beta)=q(0,\beta)+Q(x',\beta)$ where Q has compact support in x', we can reduce to the case where q has compact support in x'. This uses up one x' derivative of q and accounts for the appearance of $\partial_{x'}q$ in the upper bound for $|T|$. Below we relabel Q as q.

2. Write

$$Au(x)=\int e^{ix'\zeta'}d(x',\epsilon\zeta)\hat{u}(\zeta')d\zeta', \tag{12.13}$$

where

$$d(x',\epsilon\zeta)=\int e^{i(x'-y')\xi'}p(x',\epsilon(\zeta'+\xi'),\epsilon\gamma)q(y',\epsilon\zeta',\epsilon\gamma)dy'd\xi'. \tag{12.14}$$

Expand $p(x',\epsilon(\zeta'+\xi'),\epsilon\gamma)$ about $\epsilon\zeta'$,

$$\sum_{|\alpha|\leq k-1}\frac{1}{\alpha!}\partial^\alpha_{\beta'}p(x',\epsilon\zeta',\epsilon\gamma)\epsilon^{|\alpha|}\xi'^\alpha+$$
$$\epsilon^k\sum_{|\alpha|=k}\int_0^1\frac{k(1-t)^{k-1}}{\alpha!}\partial^\alpha_{\beta'}p(x',\epsilon\zeta'+\epsilon t\xi',\epsilon\gamma)\xi'^\alpha dt,$$

and use basic properties of the Fourier transform to obtain

$$d(x',\epsilon\zeta)=t(x',\epsilon\zeta)+\epsilon^kR(x',\epsilon\zeta)$$

where $R(x',\epsilon\zeta',\epsilon\gamma)=$

$$\sum_{|\alpha|=k}\int\int_0^1 e^{ix'\xi'}\frac{k(1-t)^{k-1}}{\alpha!}\partial^\alpha_{\beta'}p(x',\epsilon\zeta'+\epsilon t\xi',\epsilon\gamma)\xi'^\alpha\hat{q}(\xi',\epsilon\zeta)dtd\xi'. \tag{12.15}$$

3. It remains to show that $R(x',\epsilon D)$ is bounded on L^2. Write

$$\partial^\alpha_{\beta'}p(x',\epsilon\zeta'+\epsilon t\xi',\epsilon\gamma)=\partial^\alpha_{\beta'}p(0,\epsilon\zeta'+\epsilon t\xi',\epsilon\gamma)+h(x',\epsilon\zeta'+\epsilon t\xi',\epsilon\gamma), \tag{12.16}$$

where h is C^{M_1} and has compact support and in x'. Corresponding to (12.16)

$$R(x',\epsilon\zeta)=R_1(x',\epsilon\zeta)+R_2(x',\epsilon\zeta),$$

where

$$|\hat{R}_1(\xi', \epsilon\zeta)| \le C\langle\xi'\rangle^{-(M_2-1-k)}.$$

Thus, $R_1(x', \epsilon D)$ is an absolutely convergent superposition of bounded Fourier multipliers on L^2.

To see that $R_2(x', \epsilon D)$ can also be expressed as such a superposition, it suffices to show

$$|\hat{R}_2(\xi', \epsilon\zeta)| \le C\langle\xi'\rangle^{-(N+1)}. \tag{12.17}$$

Since $M_2 - k - (N+1) \ge N+1$, this follows easily from the estimate (for $|\mu| + |\nu| \le N+1$)

$$|\partial^\mu_{x'} h(x', \epsilon\zeta' + \epsilon t\xi', \epsilon\gamma)\xi'^\nu \xi'^\alpha \hat{q}(\xi', \epsilon\zeta)| \le C\langle\xi'\rangle^{-(M_2-1-k-|\nu|)}. \tag{12.18}$$

□

Proof of Proposition 8.3. We'll consider scalar symbols. The proof for matrix symbols requires no essential changes.

The Proposition is clear when $p(x', \epsilon\zeta)$ is independent of x', so we write $p(x', \beta) = p(0, \beta) + P(x', \beta)$ and reduce as before to the case where p has compact support in x'. This uses up one x' derivative of p. Below we relabel P as p.

On the Fourier side the kernel of $p(x', \epsilon D)$ is $\hat{p}(\zeta' - \xi', \epsilon\xi', \epsilon\gamma)$. That is to say,

$$\widehat{p(x', \epsilon D)}u(\zeta') = \int \hat{p}(\zeta' - \xi', \epsilon\xi', \epsilon\gamma)\hat{u}(\xi')d\xi'. \tag{12.19}$$

Thus, the kernel of its adjoint is $\overline{\hat{p}}(\xi' - \zeta', \epsilon\zeta', \epsilon\gamma) = \hat{\overline{p}}(\zeta' - \xi', \epsilon\zeta', \epsilon\gamma)$. That is,

$$\widehat{p(x', \epsilon D)^*}u(\zeta') = \int \hat{\overline{p}}(\zeta' - \xi', \epsilon\zeta', \epsilon\gamma)\hat{u}(\xi')d\xi'. \tag{12.20}$$

Therefore, the adjoint $p(x', \epsilon D)^*$ is the operator with symbol d defined by $\hat{d}(\zeta' - \xi', \epsilon\xi', \epsilon\gamma) = \hat{\overline{p}}(\zeta' - \xi', \epsilon\zeta', \epsilon\gamma)$, or relabeling,

$$\hat{d}(\xi', \epsilon\zeta', \epsilon\gamma) = \hat{\overline{p}}(\xi', \epsilon(\zeta' + \xi'), \epsilon\gamma). \tag{12.21}$$

Expand $\hat{\overline{p}}(\xi', \epsilon(\zeta' + \xi'), \epsilon\gamma)$ about $\epsilon\zeta'$,

$$\sum_{|\alpha|\le k-1} \frac{1}{\alpha!}\partial^\alpha_{\beta'}\hat{\overline{p}}(\xi', \epsilon\zeta', \epsilon\gamma)\epsilon^{|\alpha|}\xi'^\alpha +$$
$$\epsilon^k \sum_{|\alpha|=k} \int_0^1 \frac{k(1-t)^{k-1}}{\alpha!}\partial^\alpha_{\beta'}\hat{\overline{p}}(\xi', \epsilon\zeta' + \epsilon t\xi', \epsilon\gamma)\xi'^\alpha dt,$$

and use properties of the Fourier transform to obtain

$$d(x', \epsilon\zeta) = t(x', \epsilon\zeta) + \epsilon^k R(x', \epsilon\zeta), \tag{12.22}$$

where $\hat{R}(\xi', \epsilon\zeta', \epsilon\gamma)$ is the factor multiplying ϵ^k in the second term above. We have

$$|\hat{R}(\xi', \epsilon\zeta', \epsilon\gamma)| \le C\langle\xi'\rangle^{-(M-1-k)},$$

so arguing as before, $R(x', \epsilon D)$ is bounded on L^2. □

12.3. **Mixed calculus.**

Proofs of Propositions 8.4, 8.5, and 8.6. **1.** The proofs of these propositions are similar to the proofs of the corresponding semiclassical propositions in the preceding subsection. We indicate here the changes that are needed. The proof of Proposition 8.4 is just like before, except that now one obtains an absolutely convergent superposition of Fourier multipliers of order m.

2. Proposition 8.5. In place of p and q in (12.14) we now have $a(x', \epsilon(\zeta'+\xi'), \epsilon\gamma, \zeta'+\xi', \gamma)$ and $b(y', \epsilon\zeta', \epsilon\gamma, \zeta', \gamma)$. The formula for $t(x', \beta, \zeta)$ arises as before from expanding $a(x', \epsilon(\zeta'+\xi'), \epsilon\gamma, \zeta'+\xi', \gamma)$ about ζ'.

The analogue of $R(x', \epsilon\zeta)$ in (12.15) is, for some constants $K_{\alpha,\mu,\nu}$, $R(x', \epsilon\zeta, \zeta) =$

$$\sum_{|\alpha|=k} \int\int_0^1 e^{ix'\xi'} \frac{k(1-t)^{k-1}}{\alpha!} \cdot \left(\sum_{\mu+\nu=\alpha} K_{\alpha,\mu,\nu} ((\epsilon\partial_{\beta'})^\mu \partial_{\zeta'}^\nu a)(x', \epsilon\zeta' + \epsilon t\xi', \epsilon\gamma, \zeta' + t\xi', \gamma) \right) \xi'^\alpha \hat{b}(\xi', \epsilon\zeta, \zeta) dt d\xi'. \tag{12.23}$$

The replacement for $\epsilon^k R_1$ in the proof of Proposition 8.5 can now be written with obvious notation as

$$R_1 = \sum_{|\alpha|=k} \sum_{\mu+\nu=\alpha} R_{1,\alpha,\mu,\nu}. \tag{12.24}$$

For terms with $|\mu| = l$, $|\nu| = k - l$ we have the estimate

$$\hat{R}_{1,\alpha,\mu,\nu}(\xi', \epsilon\zeta, \zeta) \leq C\epsilon^l \left(\int_0^1 \langle \zeta' + t\xi', \gamma \rangle^{m_1-k+l} dt \right) \langle \xi' \rangle^{-(M_2-1-k)} \langle \zeta \rangle^{m_2}. \tag{12.25}$$

For $|\xi'| \leq \frac{1}{2}|\zeta'|$ (12.25) is $\leq$

$$C\epsilon^l \langle \zeta \rangle^{m_1+m_2-k+l} \langle \xi' \rangle^{-(M_2-1-k)}. \tag{12.26}$$

For $|\xi'| \geq \frac{1}{2}|\zeta'|$, when $m_1 - k + l \geq 0$ (12.25) is $\leq$

$$C\epsilon^l \langle \zeta \rangle^{m_2} \left(\langle \xi' \rangle^{-(M_2-1-m_1-l)} + \gamma^{m_1-k+l} \langle \xi' \rangle^{-(M_2-1-k)} \right), \tag{12.27}$$

and when $m_1 - k + l < 0$ (12.25) is $\leq$

$$C\epsilon^l \langle \xi' \rangle^{-(M_2-1-k)} \langle \zeta \rangle^{m_2} \gamma^{m_1-k+l}. \tag{12.28}$$

Since $l \leq k$ and $M_2 - 1 - m_1 - k \geq N + 1$, the estimates (12.26), (12.27), and (12.28) show that $R_{1,\alpha,\mu,\nu}(x', \epsilon D, D)$ is an absolutely convergent superposition of operators of order $O(\epsilon^l \langle D \rangle^{m_2} \langle D, \gamma \rangle_{max}^{m_1-k+l})$.

Similarly, define

$$R_2 = \sum_{|\alpha|=k} \sum_{\mu+\nu=\alpha} R_{2,\alpha,\mu,\nu}$$

parallel to $\epsilon^k R_2$ in Proposition 8.5, and use

$$M_2 - 1 - m_1 - k - (N+1) \geq N + 1,$$

the argument above, and an estimate like (12.18) to show that for $|\mu| = l$, $|\nu| = k-l$, $R_{2,\alpha,\mu,\nu}(x', \epsilon D, D)$ is also such a superposition.

3. Proposition 8.6. The adjoint $a(x', \epsilon D, D)^*$ is the operator with symbol d defined (recall (12.21)) by

$$\hat{d}(\xi', \epsilon\zeta', \epsilon\gamma, \zeta', \gamma) = \hat{\overline{a}}(\xi', \epsilon(\zeta' + \xi'), \epsilon\gamma, \zeta' + \xi', \gamma). \tag{12.29}$$

The remaining deviations from the proof of Proposition 8.3 parallel the argument just given in the proof of Proposition 8.5.

□

12.4. **Classical calculus.**

Proofs of Propositions 8.7 and 8.8. The proofs are quite similar to the corresponding results in the mixed calculus. Let us begin with the result for adjoints.

In place of $\epsilon^k R$ in (12.22)we have now

$$\hat{R}(\xi', \zeta', \gamma) = \sum_{|\alpha|=1} \int_0^1 \partial^\alpha_{\beta'} \hat{\bar{p}}(\xi', \zeta' + t\xi', \gamma) {\xi'}^\alpha dt, \tag{12.30}$$

where the integrand is $\leq$

$$\langle \zeta' + t\xi', \gamma \rangle^{m-1} \langle \xi' \rangle^{M-2}. \tag{12.31}$$

When $|\xi'| \leq \frac{1}{2}|\zeta'|$, we obtain the desired estimate on $\hat{R}$ as long as $M - 2 \geq N + 1$.

When $|\xi'| \geq \frac{1}{2}|\zeta'|$ and $m - 1 < 0$ (12.31) is $\leq$

$$\gamma^{m-1} \langle \xi' \rangle^{-(M-2)} \leq C \langle \zeta \rangle^{m-1} \langle \xi' \rangle^{-(M-2+m-1)}, \tag{12.32}$$

which works as long as $M - 2 + m - 1 \geq N + 1$.

Finally, when $|\xi'| \geq \frac{1}{2}|\zeta'|$ and $m - 1 \geq 0$ (12.31) is $\leq$

$$(\langle \xi' \rangle^{m-1} + \gamma^{m-1}) \langle \xi' \rangle^{-(M-2)} \leq C \langle \xi' \rangle^{-(M-2-m+1)} \langle \zeta \rangle^{m-1}, \tag{12.33}$$

which works provided $M - m - 1 \geq N + 1$

The proof for products uses the same breakdown into cases, and is simpler than the corresponding mixed calculus result.

To see the reason for the appearance of symbol norms of $\partial_{x'} q$ and $\partial_{x'} p$ in the upper bounds for error terms, recall the beginning of the proof of Proposition 8.2. □

12.5. **Remainder terms.**

Proof of Proposition 8.9. Let $Au = A(x', \epsilon D, D)u$. Then by Plancherel

$$\begin{aligned} |Au|_{s-k,\gamma} &= |\langle D \rangle^{s-k} \left(\int\int e^{ix'(\xi'+\zeta')} \hat{a}(\xi', \epsilon\zeta, \zeta) \hat{u}(\zeta') d\zeta' d\xi' \right) |_{L^2(x')} \\ &\leq \int |\langle \xi' + \zeta', \gamma \rangle^{s-k} \hat{a}(\xi', \epsilon\zeta, \zeta) \hat{u}(\zeta')|_{L^2(\zeta')} d\xi'. \end{aligned} \tag{12.34}$$

By Peetre's inequality

$$\langle \xi' + \zeta', \gamma \rangle^{s-k} \leq C_{s-k} \langle \zeta', \gamma \rangle^{s-k} (1 + |\xi'|)^{|s-k|}, \tag{12.35}$$

so the integrand on the right in (12.34) is $\leq$

$$\left(\sup_{\zeta'} (1 + |\xi'|)^{|s-k|} |\hat{a}(\xi', \epsilon\zeta, \zeta)| \langle \zeta \rangle^{-k} \right) |u|_{s,\gamma}. \tag{12.36}$$

The result now follows from

$$|\hat{a}(\xi', \epsilon\zeta, \zeta)| \leq C \langle \xi' \rangle^{-M} \langle \zeta \rangle^k.$$

□

Remark 12.1. Fix $s \in \mathbb{R}$. In view of Proposition 8.9 if we assume $M \geq N+1+[|s-m|]$ in Proposition 8.4, we obtain an operator of order m on H^s.

Set $[|s-m|]_k = \max([|s-m|],[|s-(m-k)|])$. Similarly, if we assume $M_2 \geq M_1+(N+1)+m_1+k+1+[|s-(m_1+m_2)|]_k$ in Proposition 8.5 (resp., $M \geq (N+1)+m+k+1+[|s-m|]_k$ in Proposition 8.6), the error operators $r_{l,\epsilon,\gamma}$ in (8.16) (resp. (8.18)) have the indicated order on H^s.

12.6. **Garding inequality.**

Proof of Proposition 8.10. We adapt the argument of [MZ], Theorem B.16 to our calculus.

Since $\Re a - \frac{3c}{4}\langle\zeta\rangle^m$ is positive definite on the support of χ, we can define

$$b = b^* = \chi(\Re a - \frac{3c}{4}\langle\zeta\rangle^m)^{\frac{1}{2}} \in \mathcal{M}^{\frac{m}{2}}_{M_1}. \tag{12.37}$$

Thus,

$$\Re a = b^*b + \frac{3c}{4}\langle\zeta\rangle^m + a', \text{ with } a' = (1-\chi^2)(\Re a - \frac{3c}{4}\langle\zeta\rangle^m). \tag{12.38}$$

Set $B = b(x',\epsilon D, D)$. Now $\Re(Av,v) = \frac{1}{2}((A+A^*)v,v)$, so the calculus implies

$$\begin{aligned}&\frac{1}{2}(A+A^*) = \\ &(\Re a)(x',\epsilon D,D) + r_1 = r_1 + (B^*B + r_2) + \frac{3c}{4}\langle D\rangle^m + a'(x',\epsilon D, D).\end{aligned} \tag{12.39}$$

Since $M_1 \geq (N+1)+m+2+[|\frac{m}{2}|]$, Proposition 8.6 with $k=1$ and Remark 12.1 imply that

$$\begin{aligned}&r_1 = r_{1,a} + r_{1,b} \text{ where} \\ &r_{1,a} = O(\langle D\rangle^{m-1}) \text{ on } H^{\frac{m}{2}-1} \text{ and } r_{1,b} = O(\epsilon\langle D\rangle^m) \text{ on } H^{\frac{m}{2}}.\end{aligned} \tag{12.40}$$

Similarly, since $M_1 \geq 2(N+1)+\frac{m}{2}+2+[|\frac{m}{2}|]$, Propositions 8.6, 8.5, and Remark 12.1 imply that r_2 is a sum of two terms with the same mapping properties as in (12.40).

Set $r_3 = r_1 + r_2$. This gives

$$\Re(Av,v) = |Bv|_0^2 + \frac{3c}{4}|v|^2_{\frac{m}{2},\gamma} + \Re(a'(x',\epsilon D,D)v,v) + \Re(r_3v,v). \tag{12.41}$$

where

$$|(r_3v,v)| \leq C_1|v|_{\frac{m}{2}-1,\gamma}|v|_{\frac{m}{2},\gamma} + \epsilon C_2|v|^2_{\frac{m}{2},\gamma}. \tag{12.42}$$

Here we have used the extension of the L^2 pairing to $H^{-\frac{m}{2}} \times H^{\frac{m}{2}}$.

Next apply these estimates to $v = Wu$. Since $a'w = 0$ and $M_2 \geq 2(N+1)+m+2+[|\frac{m}{2}|]$, $a(x',\epsilon D,D)W = r_4$, where r_4 is a sum of two terms as in (12.40). Thus, the a' term in (12.41) is $\leq$

$$C_1|u|_{\frac{m}{2}-1,\gamma}|Wu|_{\frac{m}{2},\gamma} + \epsilon C_2|u|_{\frac{m}{2},\gamma}|Wu|_{\frac{m}{2},\gamma}. \tag{12.43}$$

Finally, use (12.42) and (12.43) and absorb terms in the usual way to obtain the result. □

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