Large amplitude high frequency waves for quasilinear hyperbolic systems

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Contents

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1 Introduction

This paper is concerned with the existence and stability of *multi-dimensional large amplitude high frequency waves* associated to a linearly degenerate field. They are families $\{u^\varepsilon; \varepsilon \in]0,1]\}$ of solutions of a hyperbolic system of conservation laws on a fixed domain independent of ε, such that

$$(1.1) \qquad u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \mathbf{U}^\varepsilon\big(\, t, x, \vec{\boldsymbol{\varphi}}(t,x)/\varepsilon\,\big)\,, \qquad \partial_\theta U^\varepsilon(t,x,\theta) = O(1)\,.$$

These $O(1)$ rapid variations are anomalous oscillations in the general context of nonlinear geometric optics, where the standard regime concerns $O(\varepsilon)$ oscillations:

$$(1.2) \qquad u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} u_0(t,x) + \varepsilon \mathbf{U}_1^\varepsilon\big(\, t, x, \vec{\boldsymbol{\varphi}}(t,x)/\varepsilon\,\big)\,.$$

However, when the oscillations are associated to linearly degenerate modes, the equations for $\mathbf{U}_1$ are linear, suggesting that, in this case, oscillations of larger amplitude can be considered.

A strong motivation for studying waves (1.1) is the existence of *simple waves* associated to linearly degenerate modes (see [20]). They are solutions of the form

$$(1.3) \qquad \mathbf{V}\big(h(\mathbf{k}\cdot x - \omega\, t)\big),$$

with $\mathbf{V} \in \mathcal{C}^1(I;\mathbb{R}^N)$ and $(\omega, \mathbf{k}) \in \mathbb{R}^{1+d}$ suitably chosen, and h is an arbitrary function in $\mathcal{C}^1(\mathbb{R};I)$. Fix any $h \in \mathcal{C}^1(\mathbb{R};I)$. The functions

$$(1.4) \qquad u^\varepsilon(t,x) = \mathbf{U}\big(\varphi(t,x)/\varepsilon\big), \qquad \mathbf{U} = \mathbf{V}\circ h, \qquad \varphi(t,x) = \mathbf{k}\cdot x - \omega\, t$$

are exact solutions of the equations, of the form (1.1).

In one space dimension, under assumptions which are satisfied by many physical examples, there are quite complete results at least for solutions which are local in time. The first informations were obtained by W. E [7] for the Euler system of gaz dynamics in Lagrangian coordinates, and extended by A. Heibig [13] to the case of systems admitting a *good symmetrizer*. More recently, these results have been generalized by A. Corli, O. Guès [6] and A. Museux [22] up to the setting of *stratified weak solutions*, which contains for example the case of solutions (1.3) when h is only L^∞, still assuming the existence of a good symmetrizer. Concerning global weak solutions let us quote Peng's results [23] for the Euler system of gaz dynamics.

In several space dimensions, the situation is much more delicate. A first step in the analysis is to determine a set of sufficient conditions leading to formal WKB solutions

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \sum_{j=0}^{\infty} \varepsilon^j \ \mathbf{U}_j\big(t,x,\vec{\boldsymbol{\varphi}}(t,x)/\varepsilon\big), \qquad \partial_\theta U_0(t,x,\theta) \not\equiv 0\,. \tag{1.5}$$

A second step is to determine a set of sufficient conditions, which are in general strictly stronger, insuring the stability of these solutions (see [16] for such an approach in the semilinear case). None of these steps is easy.

There are strong obstructions to the construction of WKB solutions. For instance, D. Serre has shown in [26] that, for the isentropic gaz dynamics, an expansion like (1.5) leads to modulation equations for the $\mathbf{U}_j$, that are *ill posed* with respect to the initial value problem ; more precisely the linearized equations deduced from the modulation equations are not hyperbolic.

Moreover, strong instabilities can be present. For example, in the case of compressible or incompressible isentropic gaz dynamics, the explicit solutions (1.4) are strongly unstable, because of Rayleigh instabilities, as shown in the works of M. Artola, A. Majda [2], S. Friedlander, W. Strauss, M. Vishik [8] and E. Grenier [9]. These results indicate that in space dimension $d > 1$, the existence of a good symmetrizer adapted to a linear degenerate eigenvalue, is in general *not* sufficient to guarantee the stability of large amplitude high frequency waves.

The recent paper [5] gives a better understanding of the problem. First, it contains a discussion on the magnitude of oscillations, between (1.1) and (1.2), that can be expected. Assuming the existence of a good symmetrizer corresponding to some linear degenerate eigenvalue, we proved in [5] that there always exist formal WKB solutions

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} u_0(t,x) \ + \ \sum_{j=1}^{\infty} \varepsilon^{j/2} \ \mathbf{V}_j\big(\,t,x,\varphi(t,x)/\varepsilon\,\big) \tag{1.6}$$

where u_0 is any given smooth local solution of the quasilinear system. Here, the oscillations are of amplitude $O(\sqrt{\varepsilon})$. The resulting equations for the profiles $\mathbf{V}_j$ are *well posed*. Moreover, the equation for the profile V_1 has *non linear* features, which means that the expansion (1.6) is a *relevant regime* for the chosen context. More striking is the instability result obtained in [5]: in general, the approximate solutions obtained by stopping the expansion (1.6) at an arbitrarily order k, are strongly unstable. In fact the linearized evolution may produce *exponential amplifications* of small disturbances of the data (see [16] for a similar situation in the semilinear case). This confirms the instability of large amplitude oscillating waves (1.1), since the regime (1.1) is more singular than (1.6) which is already unstable.

However, we also proved in [5] that in some very particular cases, the *strong oscillations* (1.6) are linearly and nonlinearly stable. For instance, this is true if the linearly degenerate eigenvalue is stationary on the state u_0 (see [5]). But this condition is very restrictive and never satisfied for Euler equations. We also give in [5] less restrictive conditions that insure the *weak linear stability* of waves (1.6). This means that the amplification's rate of the solution is polynomial in t/ε instead of being exponential. In the case of the Euler system of entropic gaz dynamics, these conditions mean exactly that the oscillations are *polarized on the entropy.* This result indicates that the polarization of the oscillations is a strong factor in the stability analysis.

In this paper we push further this idea of looking at waves which have a particular polarization. In situations which extend the case of entropy waves, we prove the existence and the non linear stability *large amplitude waves* (1.1).

• **Large entropy waves for Euler equations.** In the case of the entropic Euler equations, we prove the existence and the stability of non trivial solutions $u^\varepsilon = (\mathbf{v}^\varepsilon, \mathbf{p}^\varepsilon, \mathbf{s}^\varepsilon)$ (velocity, pressure, entropy) of the form

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$$
\begin{aligned}
\mathbf{v}^\varepsilon(t,x) &= \mathbf{v}_0(t,x) + \varepsilon\ \mathbf{V}(t,x,\varphi(t,x)/\varepsilon) + O(\varepsilon^2) \\
\mathbf{p}^\varepsilon(t,x) &= \mathbf{p}_0 + \varepsilon\ \mathbf{P}(t,x) + O(\varepsilon^2) \\
\mathbf{s}^\varepsilon(t,x) &= \mathbf{S}\big(t,x,\varphi(t,x)/\varepsilon\big) + O(\varepsilon)
\end{aligned}
\tag{1.7}
$$

where $\mathbf{v}_0$ satisfies the overdetermined system

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$$
\partial_t \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla_x)\mathbf{v}_0 = 0, \qquad \mathrm{div}_x\, \mathbf{v}_0 = 0\,. \tag{1.8}
$$

Here $\mathbf{p}_0$ is a constant and the phase φ is a smooth real valued function satisfying the eiconal equation $\partial_t\varphi + (\mathbf{v}_0 \cdot \nabla_x)\varphi = 0$. An example with the Euler equations is detailed in the **subsection 3.4** while the **subsection 2.3** is devoted to the system (1.8). For solutions (1.7), the main oscillations are of order $O(1)$ and polarized on the entropy.

• **General systems.** It is interesting to understand what are the structure conditions on a general system that allow the construction of solutions similar to (1.7). We consider a $N \times N$ symmetrizable hyperbolic system of conservation laws in space dimension $d \geq 1$

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$$
\partial_t f_0(u) + \sum_{j=1}^{d} \partial_j f_j(u) \;=\; 0\,. \tag{1.9}
$$

The flux functions $f_j(u)$ are defined in a neighborhood $\mathcal{O}$ of $0 \in \mathbb{R}^N$. We assume that $\det f_0'(u) \neq 0$ for all $u \in \mathcal{O}$. We note

$$A_j(u) := f_0'(u)^{-1}\, f_j'(u), \qquad A(u,\xi) := \sum_{j=1}^{d} \xi_j\, A_j(u)\,.$$

Let $\lambda(u,\xi)$ be a given eigenvalue of the matrix $A(u,\xi)$. We introduce

$$\mathbb{F}(u,\xi) := \ker\, \big(A(u,\xi) - \lambda(u,\xi)\,\mathrm{Id}\big) \,\subset\, \mathbb{R}^N\,.$$

We suppose that $\lambda(u,\xi)$ is *linearly degenerate* with constant multiplicity

F00003000 (1.10) $$r \cdot \nabla_u \lambda(u,\xi) = 0, \qquad \forall\, r \in \mathbb{F}(u,\xi), \qquad \forall\, (u,\xi) \in \mathcal{O} \times (\mathbb{R}^d \setminus \{0\})\,.$$

F00000000 (1.11) $$\exists\, \tilde N > 0\,; \qquad \dim\, \mathbb{F}(u,\xi) = \tilde N, \qquad \forall\, (u,\xi) \in \mathcal{O} \times (\mathbb{R}^d \setminus \{0\})\,.$$

We consider the vector space

$$\mathbb{F}(u) := \,\cap_{\xi \neq 0}\, \mathbb{F}(u,\xi) \,\subset\, \mathbb{R}^N\,.$$

For Euler equation, this space is exactly the polarization space of the entropy. Our main assumptions are first that $\mathbb{F}(u)$ is non trivial has constant dimension

F (1.12) $$\exists\, N' > 0\,; \qquad \dim\, \mathbb{F}(u) = N', \qquad \forall\, u \in \mathcal{O}$$

and second that the system (1.9) admits a *good symmetrizer* with respect to the field $u \mapsto \mathbb{F}(u)$. This last requirement has an intrinsic meaning that we briefly describe. It means that there exists a smooth symmetric positive definite matrix $S(u)$ such that the matrix

$$\mathbf{L}(u,\xi) := S(u)\,\big(A(u,\xi) - \lambda(u,\xi)\,\mathrm{Id}\big)$$

is symmetric for all $(u,\xi) \in \mathcal{O} \times \mathbb{R}^d$ (i.e. S is a symmetrizer) and that, viewing the symmetric matrix $\mathbf{L}$ as a two times covariant tensor, for all smooth vector field $\mathcal{V}$ on $\mathcal{O}$ satisfying $\mathcal{V}(u) \in \mathbb{F}(u)$ for all $u \in \mathcal{O}$, the Lie derivative of $\mathbf{L}$ along $\mathcal{V}$ is 0 (see also [5]). All these assumptions are introduced in the **section 2**, where we show that the system can be put in a canonical form similar to that of the Euler equations, by using suitable non linear change of dependent variables.

• **Oscillations with several phases.** We will also consider the case of oscillating waves with several phases

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \mathbf{U}^\varepsilon\big(t,x,\vec{\boldsymbol{\varphi}}(t,x)/\varepsilon\big)\,, \qquad \vec{\boldsymbol{\varphi}} = (\varphi_1,\cdots,\varphi_\ell)\,. \tag{1.13}$$

The framework is the one introduced by Joly, Métivier and Rauch [16]-[17] in the study of weakly non linear geometrical optics. In particular we make *coherence* assumptions on the phases φ_j, together with *small divisor assumptions* which are used to get high order WKB approximate solutions. New difficulties appear in this context, especially concerning the justification of the asymptotic expansion (1.13). This is the subject of the **section 3**.

• **ε-stratified and ε-conormal waves.** We will pay a special attention to the case of single phase high frequency waves ($\vec{\boldsymbol{\varphi}} = \varphi$). In order to allow more general fluctuations, we consider the larger class of ε-stratified waves. Roughly speaking, it means that $u^\varepsilon(t,x)$ satisfies on an open set Ω of $\mathbb{R}^{1+d}$, a condition like

$$(\varepsilon\,\partial)^\alpha\; \mathcal{T}_1\cdots\mathcal{T}_k\,u^\varepsilon \in L^2(\Omega), \qquad \forall\,\alpha\in\mathbb{N}^{1+d}\,, \qquad \forall\,k\in\mathbb{N} \tag{1.14}$$

for any vector fields $\mathcal{T}_j$ on Ω with $\mathcal{C}^\infty_b(\Omega)$ coefficients[1], which are tangent to the foliation $\{\varphi = cte\}$. In other words, we impose $\mathcal{T}_1\,\varphi = 0,\cdots,\mathcal{T}_k\,\varphi = 0$. Waves like (1.5) with $\vec{\boldsymbol{\varphi}} = \varphi$ provide a natural example of such ε-stratified waves. The ε-stratified waves were introduced in [11] in the context of weakly non linear geometric optics. They are inspired from the classical stratified waves introduced by J. Rauch and M. Reed in [24] and Métivier in [21] for the study of singular solutions to non linear hyperbolic systems.

In the same spirit, we also treat the case of ε-conormal waves which correspond to the case where the vector fields in (1.14) are required to be tangent to only *one hypersurface*, say $\Sigma = \{\varphi = 0\}$. It means that : $(\mathcal{T}_1\,\varphi)_{|\Sigma} = 0,\cdots,(\mathcal{T}_k\,\varphi)_{|\Sigma} = 0$. Hence, it is a special case of ε-stratified wave but where u^ε may vary rapidly in a region which is closed to Σ, like an *inner layer*. For example it may converge to a discontinuous fonction as ε goes to zero. A function like $\chi(t,x)\arctan\big(\varphi(t,x)/\varepsilon\big)$ with $\chi\in\mathcal{C}^\infty_0(\mathbb{R}^{1+d})$ is an example of ε-conormal wave converging to a discontinuous function.

Since we study the Cauchy problem for such ε-stratified or ε-conormal waves, we are lead to the question of the compatibility conditions required on the initial data. We show that there actually exist compatible initial data (see theorem 4.4). All this matter is treated in the **section 4**.

[1] By $\mathcal{C}^\infty_b(\Omega)$ we mean that the functions are in $\mathcal{C}^\infty(\Omega)$ and are bounded with bounded derivatives at any order.

2 Position of the problem

position du probleme

2.1 Structure assumptions

Structure assumptions

Let us consider the $N \times N$ symmetrizable hyperbolic system of conservation laws (1.9) in space dimension $d \geq 1$. The framework is the one described in (1.10)-(1.11)-(1.12). It implies other properties.

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Lemma 2.1. *Under the condition (1.12), the function $\lambda(u,\xi)$ is linear with respect to ξ. Moreover the field $u \mapsto \mathbb{F}(u)$ is locally integrable.*

Proof. We select $r(u) \neq 0$ belonging to $\mathbb{F}(u)$. Differentiating in ξ_j the relation $A(u,\xi)\, r(u) = \lambda(u,\xi)\, r(u)$, yields

$$A_j(u)\, r(u) = \partial_{\xi_j} \lambda(u,\xi)\, r(u)\,.$$

The left hand side is independent of ξ. Thus $\partial_{\xi_j} \lambda(u,\xi)$ does not depend on ξ which proves the linearity. Furthermore, by [3], for all $\xi \neq 0$ the field $\mathbb{F}(\cdot,\xi) : u \mapsto \mathbb{F}(u,\xi)$ is locally integrable. Since this property is preserved by intersection, the result follows for $u \mapsto \mathbb{F}(u)$. □

Example 2.1. *Let us consider the Euler system of entropic gaz dynamics, in space dimension $d = 2$*

euler(v,rho,s)

$$\left\{ \begin{aligned} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla_x)\mathbf{v} + \boldsymbol{\rho}^{-1}\,\nabla_x \mathbf{p} = 0 \\ \partial_t \boldsymbol{\rho} + (\mathbf{v}\cdot\nabla_x)\boldsymbol{\rho} + \boldsymbol{\rho}\ \mathrm{div}_x\, \mathbf{v} = 0 \\ \partial_t \mathbf{s} + (\mathbf{v}\cdot\nabla_x)\mathbf{s} = 0 \end{aligned} \right. \tag{2.1}$$

with $\mathbf{p} = P(\boldsymbol{\rho},\mathbf{s})$. For the unknown $u = (\mathbf{v},\boldsymbol{\rho},\mathbf{s}) \in \mathbb{R}^4$ we have

$$A(u,\xi) = \begin{bmatrix} \mathbf{v}\cdot\xi & 0 & a\,\xi_1 & b\,\xi_1 \\ 0 & \mathbf{v}\cdot\xi & a\,\xi_2 & b\,\xi_2 \\ \boldsymbol{\rho}\,\xi_1 & \boldsymbol{\rho}\,\xi_2 & \mathbf{v}\cdot\xi & 0 \\ 0 & 0 & 0 & \mathbf{v}\cdot\xi \end{bmatrix} \tag{2.2}$$

where $a := \boldsymbol{\rho}^{-1}\, P'_{\boldsymbol{\rho}}(\boldsymbol{\rho},\mathbf{s})$ and $b := \boldsymbol{\rho}^{-1}\, P'_{\mathbf{s}}(\boldsymbol{\rho},\mathbf{s})$. It is assumed that $a(\boldsymbol{\rho},\mathbf{s}) > 0$ for all $\boldsymbol{\rho}$ and all $\mathbf{s}$. The linear degenerate eigenvalue is $\lambda(u,\xi) = \mathbf{v}\cdot\xi$ and the corresponding eigenspace is the plane of $\mathbb{R}^4$ defined by

$$\left\{\, (v',\rho',s') \in \mathbb{R}^4\,;\ a\,\rho' + b\,s' = 0,\quad \xi_1\, v'_1 + \xi_2\, v'_2 = 0 \,\right\}.$$

We find that $\mathbb{F}(u)$ is the line of $\mathbb{R}^4$ defined by

$$\left\{\, (0,\rho',s') \in \mathbb{R}^4\,;\ a\,\rho' + b\,s' = 0 \,\right\}. \tag{2.3}$$

The same calculation with the 3-D equations gives again $N' = 1$. •

We denote by $(e_1, \cdots, e_N)$ the canonical basis of $\mathbb{R}^N$. Let $N'' := N - N'$. Since $\mathbb{F}$ is locally integrable, there exists a smooth diffeomorphism $\chi \in \mathcal{C}^\infty(\tilde{\mathcal{O}}; \mathcal{O})$ between two open sets $\tilde{\mathcal{O}}$ and $\mathcal{O}$ of $\mathbb{R}^N$ both containing 0, with $\chi(0) = 0$, and such that the change of coordinates maps the vector fields $e_{N"+1}, \cdots, e_N$ onto a basis of $\mathbb{F}$. In other words, the N' vectors

$$\frac{\partial \chi}{\partial \tilde{u}_{N"+1}}\,, \quad \cdots\,, \quad \frac{\partial \chi}{\partial \tilde{u}_N}$$

form a basis of the linear space $\mathbb{F}(\chi)$. The conditions to impose on the new variable $\tilde{u} = \chi^{-1}(u)$ are

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$$\partial_t \tilde{f}_0(\tilde{u}) + \sum_{j=1}^{d} \partial_j \tilde{f}_j(\tilde{u}) = 0\,, \qquad \tilde{f}_j = f_j \circ \chi\,. \tag{2.4}$$

For $\mathcal{C}^1$ solutions, this system is equivalent to the quasilinear system

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$$\partial_t \tilde{u} + \sum_{j=1}^{d} \tilde{A}_j(\tilde{u})\; \partial_j \tilde{u} = 0\,, \qquad \tilde{A}_j := D\chi^{-1}\, A_j(\chi)\, D\chi\,. \tag{2.5}$$

We introduce the decomposition

$$\tilde{u} = (v, w)\,, \qquad v := (\tilde{u}_1, \cdots, \tilde{u}_{N"})\,, \qquad w := (\tilde{u}_{N"+1}, \cdots, \tilde{u}_N)\,.$$

The fact that $\lambda(u, \xi)$ is linearly degenerate implies that the new eigenvalue $\tilde{\lambda}(\tilde{u}, \xi) := \lambda\big(\chi(\tilde{u}), \xi\big)$ of the matrix

$$\tilde{\mathcal{A}}(\tilde{u}, \xi) := \sum_{j=1}^{d} \xi_j\, \tilde{A}_j(\tilde{u})$$

does not depend on w. Since we already know that $\tilde{\lambda}$ is linear with respect to ξ, it remains

$$\tilde{\lambda}(\tilde{u}, \xi) \;=\; \mu(v) \cdot \xi\,, \qquad \forall\, (u, \xi) \in \mathbb{R}^N \times \mathbb{R}^d\,. \tag{2.6}$$

In all the sequel we will note $\mathbf{X}_v$ the corresponding characteristic field

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$$\mathbf{X}_v \;:=\; \partial_t \,+\, \mu(v) \cdot \nabla_x\,. \tag{2.7}$$

Furthermore, the linear space

$$\tilde{\mathbb{F}}(\tilde{u}) := \cap_{\xi \neq 0}\, \mathbb{F}(\tilde{u}, \xi)\,, \qquad \mathbb{F}(\tilde{u}, \xi) := \ker\, \big(\tilde{A}(\tilde{u}, \xi) - \mu(v) \cdot \xi \times \mathrm{Id}\big)$$

becomes the *constant* linear subspace of $\mathbb{R}^N$ with equation $\{v = 0\}$.

Now on, we drop the "~". For example we call again u the unknown $\tilde{u}$. The system (2.5) can be put in the following form

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$$\mathbf{X}_v \, u + \mathcal{M}(u, \partial_x) \, u = 0 \tag{2.8}$$

where $\mathcal{M}(u, \partial_x)$ is the $N \times N$ first order linear operator

$$\mathcal{M}(u, \partial_x) = \mathcal{M}_1(u) \, \partial_1 + \cdots + \mathcal{M}_d(u) \, \partial_d \, .$$

By construction, the matrix $\mathcal{M}(u, \xi)$ satisfies

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$$\{v = 0\} \subset \ker \mathcal{M}(u, \xi) \, , \qquad \forall \, (u, \xi) \in \mathcal{O} \times \mathbb{R}^d \, . \tag{2.9}$$

The system (1.9) being symmetrizable, the same is true for the system (2.8). Hence we can find a symmetric positive definite matrix $S(u)$ with $\mathcal{C}^\infty$ coefficients such that

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$$S(u) \, \mathcal{M}(u, \xi) \;\text{ is symmetric for all }\; (u, \xi) \in \mathcal{O} \times \mathbb{R}^d \, . \tag{2.10}$$

To summarize, we consider a system of the form (2.8) satifying (2.9) and we make the following hypothesis.

hyp 1.1
Assumption 2.2. *There exists a good symmetrizer (2.10) such that the coefficients of the skew symmetric differential operator $S(u) \, \mathcal{M}(u, \partial_x)$ are independent on w. We will note in the sequel $\mathbf{L}(v, \partial_x) := S(u) \, \mathcal{M}(u, \partial_x)$.*

The system (2.8) is equivalent to

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$$S(u) \, \mathbf{X}_v \, u \, + \, \mathbf{L}(v, \partial_x) u \; = \; 0 \tag{2.11}$$

and it follows from the symmetry of $\mathbf{L}$ and from the property (2.9) that $\mathbf{L}$ has the following form

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$$\mathbf{L}(v, \xi) \;=\; \begin{bmatrix} \mathbf{L}^\flat(v, \xi) & 0 \\ 0 & 0 \end{bmatrix} \tag{2.12}$$

where the block $\mathbf{L}^\flat(v, \xi)$ is symmetric, of size $N'' \times N''$. For all $(v, \xi) \in \mathbb{R}^{N"} \times (\mathbb{R}^d \setminus \{0\})$, we will note $\mathbf{P}(v, \xi)$ the matrix of the orthogonal projector of $\mathbb{R}^N$ onto $\ker \mathbf{L}(v, \xi)$ written in the canonical basis of $\mathbb{R}^N$, and we will note $\mathbf{P}^\flat(v, \xi)$ the matrix of the orthogonal projector of $\mathbb{R}^{N"}$ onto $\ker \mathbf{L}^\flat(v, \xi)$ written in the canonical basis of $\mathbb{R}^{N"}$. These two operators are linked by

$$\mathbf{P}(v, \xi) \;=\; \begin{bmatrix} \mathbf{P}^\flat(v, \xi) & 0 \\ 0 & \mathrm{Id} \end{bmatrix} . \tag{2.13}$$

The assumption (1.10) implies that $\mathbf{L}(v,\xi)$ has a constant rank when (u,ξ) varies in $\mathcal{O}\times(\mathbb{R}^d\setminus\{0\})$ so that its range and its kernel depend smoothly on (u,ξ). We will make a repeated use of this property.

Lemma 2.3. *The mapping $(u,\xi)\mapsto \mathbf{P}(v,\xi)$ is a $\mathcal{C}^\infty$ function on the open set $\mathcal{O}\times(\mathbb{R}^d\setminus\{0\})$.*

Example 2.2. The entropic Euler equations. *We consider Euler equations of gaz dynamics as it is written in (2.1). A suitable change of dependent coordinates χ consists in choosing the unkown $u=(\mathbf{v},\mathbf{p},\mathbf{s})$, which means to express $\boldsymbol{\rho}$ in terms of $(\mathbf{p},\mathbf{s})$ by a relation of the form $\boldsymbol{\rho}=\rho(\mathbf{p},\mathbf{s})$. In that case, and after being symmetrized, the system writes*

euler(v,p,s)*

$$\left\{\begin{aligned} \rho\,\big(\partial_t\mathbf{v}+(\mathbf{v}\cdot\nabla_x)\mathbf{v}\big)\;+\;\nabla_x\mathbf{p}=0\\ \alpha\,\big(\partial_t\mathbf{p}+(\mathbf{v}\cdot\nabla_x)\mathbf{p}\big)\;+\;\mathrm{div}_x\,\mathbf{v}=0\\ \partial_t\mathbf{s}+(\mathbf{v}\cdot\nabla_x)\mathbf{s}=0\end{aligned}\right. \tag{2.14}$$

with $\alpha(\mathbf{p},\mathbf{s})=\rho'_{\mathbf{p}}(\mathbf{p},\mathbf{s})/\rho(\mathbf{p},\mathbf{s})>0$. We still have $\lambda(u,\xi)=\mathbf{v}\cdot\xi$ but now $\mathbb{F}(u)$ is the constant linear subspace of $\mathbb{R}^4$

$$\mathbb{F}=\big\{\,(0,0,0,s')\,;\;s'\in\mathbb{R}\,\big\}\,.$$

In this example, the variables v and w are given by $v=(\mathbf{v},\mathbf{p})$, $w=\mathbf{s}$, and

$$\mathbf{X}_v\;\equiv\;\partial_t+\mathbf{v}_1\,\partial_1+\mathbf{v}_2\,\partial_2\;\equiv\;\partial_t+\mathbf{v}\cdot\nabla_x\,.$$

The system (2.14) is actually of the form (2.11) with

$$\mathbf{S}(u)=\begin{bmatrix}\rho&0&0&0\\0&\rho&0&0\\0&0&\alpha&0\\0&0&0&1\end{bmatrix},\qquad \mathbf{L}(u,\xi)=\begin{bmatrix}0&0&\xi_1&0\\0&0&\xi_2&0\\\xi_1&\xi_2&0&0\\0&0&0&0\end{bmatrix}.$$

Observe that $\mathbf{S}$ is a good symmetrizer since the matrix $\mathbf{L}(u,\xi)=\mathbf{L}(v,\xi)$ does not depend on the entropy $w=\mathbf{s}$. This analysis extends to any dimension.

2.2 Setting of the problem and motivations

There exists very particular solutions of (2.11) with large amplitude fluctuations. These are solutions $u_0=(v_0,w_0)$ of the overdetermined system

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$$\mathbf{X}_{v_0}\,u_0=0\,,\qquad \mathbf{L}(v_0,\partial_x)u_0=0\,. \tag{2.15}$$

In view of the form of the matrix $\mathbf{L}$, this is equivalent to say that

1.9 (2.16) $$\partial_t v_0 + \mu(v_0)\cdot\nabla_x v_0 \;=\; 0\,, \qquad \mathbf{L}^\flat(v_0,\partial_x)v_0 = 0$$

and that

1.10 (2.17) $$\partial_t w_0 + \mu(v_0)\cdot\nabla_x w_0 = 0\,.$$

The condition (2.15) splits into the two parts (2.16) and (2.17). On the one hand, a non linear overdetermined system on v_0. On the other hand a linear transport equation on w_0, with coefficients depending on v_0. The system (2.16) being overdetermined, it is ill posed for the initial value problem. It admits however solutions like for example the constant solutions or some simple waves (see the following remark). In the special case of the Euler equations, the system (2.16) will be studied with more details in subsection 2.3.

remarqueondessimples **Remark 2.4.** *The dimension of the linear subspace* $\ker \mathbf{L}^\flat(v,\xi)$ *is independent on* (v,ξ). *When it is not* 0, *the system* (2.16) *admits non constant simple wave solutions. In that case, let us fix* $\xi^0\in\mathbb{R}^d\setminus\{0\}$ *such that* $\mu(0)\cdot\xi^0=0$, *and consider* $v\mapsto r(v)$ *a* $\mathcal{C}^\infty$ *vector field on a neighborhood of* 0 *in* $\mathbb{R}^{N"}$ *such that* $r(v)\in\ker\mathbf{L}^\flat(v,\xi^0)\setminus\{0\}$. *Let* γ *be an integral curve of* r. *It is a local smooth solution in a neighborhood* J *of* $0\in\mathbb{R}$ *of*

courbeintegraleder (2.18) $$\frac{d}{ds}\,\gamma(s) = r\big(\gamma(s)\big)\,, \qquad \gamma(0)=0\,, \qquad s\in J\,.$$

Hence, the function $v_0(t,x) := \gamma(\xi^0\cdot x)$ *is a local solution (which is not constant) on a neighborhood of* $0\in\mathbb{R}^{1+d}$ *of the system* (2.16). *Indeed, the fact that* $\mathbf{L}^\flat(v_0,\partial_x)v_0=0$ *follows directly from* (2.18). *For the other relation, the fact that the eigenvalue* $\mu(v)\cdot\xi^0$ *is linearly degenerate implies that*

$$\frac{d}{ds}\left\{\mu\big(\gamma(s)\big)\cdot\xi^0\right\} = \nabla_v(\mu\cdot\xi^0)\big(\gamma(s)\big)\cdot r\big(\gamma(s)\big) = 0\,.$$

It follows that

$$\mu\big(\gamma(s)\big)\cdot\xi^0 = \mu(0)\cdot\xi^0 = 0\,, \qquad \forall\, s\in J\,.$$

It implies that $\mathbf{X}_{v_0}\,v_0=0$ *and shows that* v_0 *is a solution of* (2.16).

We fix a v_0 satisfying (2.16). One can choose

1.12 (2.19) $$w_0(t,x) = \mathbf{w}\big(t,x,\varphi(t,x)/\varepsilon\big)$$

where

$$\partial_t \varphi + \mu(v_0)\cdot \nabla_x \varphi = 0\,, \qquad \varphi \in C^1([0,T]\times \mathbb{R}^d;\mathbb{R})\,.$$
$$\partial_t \mathbf{w} + \mu(v_0)\cdot \nabla_x \mathbf{w} = 0\,, \qquad \mathbf{w} \in C^1([0,T]\times \mathbb{R}^d \times \mathbb{T};\mathbb{R}^{N'})\,.$$

It gives an example of **large amplitude oscillating solution**, with just one phase. One can also consider examples with several phases like

1.15 (2.20) $$w_0^\varepsilon(t,x) = \mathbf{w}\big(t,x,\varphi_1(t,x)/\varepsilon,\cdots,\varphi_\ell(t,x)/\varepsilon\big)\,,$$

with again

$$\partial_t \varphi_j + \mu(v_0)\cdot \nabla_x \varphi_j = 0\,, \qquad \forall\, j \in \{1,\cdots,\ell\}\,.$$
$$\partial_t \mathbf{w} + \mu(v_0)\cdot \nabla_x \mathbf{w} = 0\,, \qquad \mathbf{w} \in C^1([0,T]\times \mathbb{R}^d \times \mathbb{T}^\ell;\mathbb{R}^{N'})\,.$$

It is an example of a several phase oscillating solution generalizing (2.19). Let us insist on the fact that all the phases ϕ_j are eiconal *for the same field*.

One can also vary the nature of the profiles, and consider *jump profiles* instead of periodic profiles. For example, one can choose a function $\mathbf{w}$ having limits in $+\infty$ and in $-\infty$

1.14 (2.21) $$w_0^\varepsilon(t,x) = \mathbf{w}\big(t,x,\varphi(t,x)/\varepsilon\big), \qquad \lim_{z\to\pm\infty} \mathbf{w}(t,x,z) \;=\; \mathbf{w}^\pm(t,x)$$

where we impose

$$\partial_t \mathbf{w} + \mu(v_0)\cdot \nabla_x \mathbf{w} = 0\,, \qquad \mathbf{w} \in C^1([0,T]\times \mathbb{R}^d \times \mathbb{R};\mathbb{R}^{N'})\,.$$
$$\partial_t \mathbf{w}^\pm + \mu(v_0)\cdot \nabla_x \mathbf{w}^\pm = 0\,, \qquad \mathbf{w} \in C^1([0,T]\times \mathbb{R}^d;\mathbb{R}^{N'})\,.$$

Suppose that $\Omega_+ := \{\varphi > 0\}$ and $\Omega_- := \{\varphi < 0\}$ are two connected open subsets of Ω separated by the smooth (and connected) hypersurface $\{\varphi = 0\}$. Denote by $u^\pm$ the function in $L^2_{loc}(\Omega)$ whose restriction to $\Omega_\pm$ is $(v_0,\mathbf{w}^\pm)$. When the limits $\mathbf{w}^+$ and $\mathbf{w}^-$ are different, $u^\pm$ has a discontinuity along the hypersurface $\{\varphi = 0\}$. Observe now that the function $u_0^\varepsilon = \big(v_0(t,x),\mathbf{w}(t,x,\varphi/\varepsilon)\big)$ is an exact $\mathcal{C}^\infty$ solution of the system (2.8), which converges to $u^\pm$ in $L^2_{loc}(\Omega)$ as ε goes to 0. It follows that $u^\pm$ is solution in the sense of distributions of the system of conservation laws (2.4), discontinuous across the characteristic hypersurface $\{\varphi = 0\}$. It is a *contact discontinuity.*

In the sense of the space $L^2_{loc}(\Omega)$, the solution u_0^ε is a small perturbation of this contact discontinuity. It is important to note that the contact discontinuities obtained in this way are preserved by the change of dependent variables χ introduced after the lemma 2.1. Indeed $\chi(u^\pm)$ is still a contact discontinuity solution of (1.9) in the sense of distributions. Actually, for all $\varepsilon \neq 0$ the smooth function $\chi(u_0^\varepsilon)$ is an *exact solution* of (1.9), and one can pass to the limit as before since $\chi(u_0^\varepsilon)$ converges to $\chi(u^\pm)$ in $L^2_{loc}(\Omega)$.

One important question we discuss in this paper is the *stability* of these various solutions (2.19), (2.20) and (2.21). As a matter of fact, one of our goals is to construct non trivial solutions of (2.8) which are perturbations of such kind of particular solutions.

Notations. We fix once for all $T_0 > 0$ and we note $\Omega :=]-T_0, T_0[\times\mathbb{R}^d$. For every $T > -T_0$, we will note

$$\Omega_T \,:=]-T_0, T[\times\mathbb{R}^d$$

and for all $T > 0$ we will note

$$\omega_T \,:=]0, T[\times\mathbb{R}^d\,.$$

Let $v_0 \in H^\infty(\Omega;\mathbb{R}^{N"})$ be a given function satistying the system (2.16) on a neighborhood $\Omega^\flat$ of $0 \in \mathbb{R}^{1+d}$. To fix our mind, we will assume that

$$\Omega^\flat \;=\; \big\{\, (t,x) \in \Omega\,;\; |t| + |x| < \mathbf{r}\,\big\}\,, \qquad \mathbf{r} > 0 \tag{2.22}$$

where $|\cdot|$ is the Euclidian norm in $\mathbb{R}^d$. The symbol H^∞ is for the usual Sobolev space of order ∞.

The results contained in this paper provide essentially with local informations. One more reason for this is the overdetermined system (2.16) which has in general no global solution in the whole domain Ω (excepted the constants). However, in order to simplify the exposition and to avoid the introduction of local domains of determination of the data, we prefer to give results which are global in space. If necessary, the local versions of the theorems can be easily deduced using the local uniqueness and finite speed of propagation. To sum up, *we are given for all the sequel a global solution* $u_0 := (v_0, w_0) \in H^\infty(\Omega;\mathbb{R}^N)$ *of (2.8), such that* v_0 *is subjected to (2.16) in* $\Omega^\flat$*, and such that* $w_0 \equiv 0$ *in* $\Omega^\flat$. Such a framework can be obtained by a usual procedure[2].

Let a and b be real numbers such that $a \leq b$. We introduce the space

$$\begin{aligned} \mathbf{W}^m(a,b) \,:= \big\{\, & u \in \mathcal{C}\big([a,b]; H^m(\mathbb{R}^d)\big)\,; \\ & \partial_t^j u \in \mathcal{C}\big([a,b]; H^{m-j}(\mathbb{R}^d)\big), \quad \forall\, j \in \{1,\ldots,m\}\,\big\}\,. \end{aligned} \tag{2.23}$$

[2]Let v_0 be a smooth solution of (2.16) in a neighborhood of 0 in $\mathbb{R}^{1+d}$. We can extend $v_0(0,\cdot)$ into $\tilde v_0(0,\cdot) \in H^\infty(\mathbb{R}^d;\mathbb{R}^{N"})$. Since (2.8) is symmetric hyperbolic, we can solve the Cauchy problem corresponding to (2.8) associated with the initial data $(\tilde v_0(0,\cdot), 0)$. The wished conditions are then fulfilled by picking T_0 and $\mathbf{r}$ small enough.

For all fixed $\varepsilon > 0$, the classical theory of muldidimensional quasilinear hyperbolic systems applies. Let us recall that for every function $u^0 \in H^m(\mathbb{R}^d)$ with $m > d/2 + 1$, there is $T > 0$ such that the equation (1.9) has a unique solution $u \in \mathbf{W}^m(0, T)$ satisfying the initial condition $u(0, \cdot) = u^0$.

2.3 The overdetermined system for Euler

This subsection is devoted to a more precise analysis of the overdetermined system (2.16), in the case of the Euler equations of gaz dynamics. We note $\mathbf{v}$ the velocity, $\mathbf{p}$ the pressure and $\mathbf{s}$ the entropy. The system (2.16) writes

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla_x)\, \mathbf{v} = 0\,, \qquad \mathrm{div}_x\, \mathbf{v} = 0\,, \qquad \nabla_x \mathbf{p} = 0$$

which means that the pressure $\mathbf{p}$ is a constant say $\underline{\mathbf{p}}$, and that $\mathbf{v}$ is a solution of the system

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla_x) \mathbf{v} = 0\,, \qquad \mathrm{div}_x\, \mathbf{v} = 0\,, \qquad \mathbf{v}(0, x) = h(x)\,. \tag{2.24}$$

Suppose that $\mathbf{v}$ is a $\mathcal{C}^1$ solution of (2.24) in a neighborhood of the origin of $\mathbb{R}^d$. Hence, in a neighborhood of the origin, $\mathbf{v}$ is constant along the integral curves of the field $(1, \mathbf{v})$. This implies in turn that this vector field is constant along this curves which hence are straight lines, and the classical relation follows

$$\mathbf{v}\big(t, x + t\, h(x)\big) = h(x) \tag{2.25}$$

which holds in a neighborhood of 0. Conversely, if $h \in \mathcal{C}^1(\mathbb{R}^d)$ is given, this relation defines $\mathbf{v} \in \mathcal{C}^1(\mathcal{O})$ in an implicit way on a neighborhood $\mathcal{O}$ of 0 sufficiently small so that $\chi(t, x) := \big(t, x + t\, h(x)\big)$ is a $\mathcal{C}^1$-diffeomorphism in a neighborhood of 0 onto $\mathcal{O}$.

We want now to investigate which condition(s) on the data h will imply that the local solution $\mathbf{v}$ $\big($defined by (2.25)$\big)$ satisfies also the divergence free condition of the system (2.24).

Let us note $D_x \mathbf{v}$ the $d \times d$ Jacobian matrix of $\mathbf{v}(t, \cdot)$ and h' that of h. The formula (2.25) leads to

$$(D_x \mathbf{v})(t, y) = h'(x) \left(\mathrm{Id} + t\, h'(x)\right)^{-1}, \qquad (t, y) = \chi(t, x)\,. \tag{2.26}$$

Taking the trace of each side we obtain

$$\mathrm{div}_x\, \mathbf{v}(t, y) \;=\; \mathrm{Tr}\left(h'(x) \left(\mathrm{Id} + t\, h'(x)\right)^{-1} \right). \tag{2.27}$$

This trace can be evaluated with the following lemma.

Lemma 2.5. *Let A be a $d \times d$ matrix with complex entries. The following formula holds*

$$\operatorname{Tr}\left(A\,(\mathrm{Id} + tA)^{-1} \right) \;=\; \mathcal{Q}'_A(t)/\mathcal{Q}_A(t)$$

with $\mathcal{Q}_A(t) := \det(\mathrm{Id} + tA)$. Moreover, the polynomial $\mathcal{Q}_A$ is constant if and only if A is a nilpotent matrix, and in that case $\mathcal{Q}_A \equiv 1$.

Proof. Let us note $\lambda_1, \cdots, \lambda_d$ the eigenvalues of A, repeated according to their multiplicity. There exists an invertible matrix P such that $A = P^{-1}\,T\,P$ where T is a triangular matrix with diagonal $(\lambda_1, \cdots, \lambda_d)$. Hence we have

traexpl (2.28)
$$\operatorname{Tr}\left(A\,(\mathrm{Id} + tA)^{-1}\right) = \operatorname{Tr}\left(T\,(\mathrm{Id} + t\,T)^{-1}\right). \tag{2.28}$$

detexpl (2.29)
$$\mathcal{Q}_A(t) = \det(\mathrm{Id} + t\,T) = \prod_{j=1}^{d} (1 + t\,\lambda_j)\,. \tag{2.29}$$

It follows that

$$\operatorname{Tr}\left(A\,(\mathrm{Id} + tA)^{-1}\right) \;=\; \sum_{j=1}^{d} \frac{\lambda_j}{1 + t\,\lambda_j} \;=\; \mathcal{Q}'_A(t)/\mathcal{Q}_A(t)\,.$$

Observe that $\mathcal{Q}_A(t) = t^d\,\mathcal{P}_A(-1/t)$ where $\mathcal{P}_A(\tau)$ is the characteristic polynomial of A, that is $\mathcal{P}_A(\tau) = \det(A - \tau I)$. Hence $\mathcal{Q}_A$ is a constant if and only if $\mathcal{P}_A(\tau) \equiv (-\tau)^d$ which means that A is nilpotent. The lemma is proved.

By the way, let us point out that expanding each side of the equality (2.29) leads to $\mathcal{Q}_A(t) = \sum_{j=1}^{d} c_j(A)\,t^j$ where the coefficients $c_j(A)$ are polynomial functions of the entries of A. The $c_j(A)$ can be formulated as $c_j(A) = \boldsymbol{\sigma}_j(\lambda_1, \cdots, \lambda_d)$ where $\boldsymbol{\sigma}_j(\cdot)$ is the elementary symmetric polynomial of degree j of d variables

$$c_1 = \sum_{i=1}^{d} \lambda_i\,, \quad c_2 = \sum_{i<j} \lambda_i\lambda_j\,, \quad c_3 = \sum_{i<j<k} \lambda_i\lambda_j\lambda_k\,, \quad \cdots, \quad c_d = \lambda_1 \cdots \lambda_d\,.$$

Hence the condition $\mathcal{Q} \equiv cte$ is equivalent to the relations

itiondenullitedessigma (2.30)
$$c_j(A) = 0\,, \qquad \forall\, j \in \{1, \cdots, d\}\,. \tag{2.30}$$

As a matter of fact, the condition for $j = 1$ means $\operatorname{Tr} A = 0$ and that for $j = d$ means $\det\, A = 0$. □

It follows from this lemma and from the formula (2.27) that $\mathrm{div}_x\,\mathbf{v}(t,x)$ is 0 in a neighborhood of 0 in $\mathbb{R}^{1+d}$ if and only if the polynomial $\mathcal{Q}'_{h'(x)}$ is 0 for all x in a neighborhood of 0, i.e. if and only if the matrix $h'(x)$ is nilpotent on a neighborhood of 0 in $\mathbb{R}^d$. This shows that the condition $D_x\mathbf{v}$ *is nilpotent* is propagated by the $\mathcal{C}^1$ solutions of the multidimensional Burgers equation. In other words, it is satisfied around 0 in $\mathbb{R}^{1+d}$ if and only if it is satisfied at $t=0$ in a neighborhood of the origin of $\mathbb{R}^d$. To sum up, we have proved the following result.

theoremenilpotence **Theorem 2.6.** *Let $h \in \mathcal{C}^1(\mathbb{R}^d;\mathbb{R}^d)$ and let $\mathbf{v}$ be a local $\mathcal{C}^1$ solution on a neighborhood of 0 of the Cauchy problem*

pbdecauchypourburgers
$$\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla_x)\mathbf{v} = 0\,, \qquad \mathbf{v}_{|t=0} = h\,. \tag{2.31}$$

The following properties are equivalent

(1) $\mathrm{div}_x\mathbf{v} = 0$ *in a neighborhood of* 0 *in* $\mathbb{R}^{1+d}$,

(2) $D_x\mathbf{v}$ *is nilpotent in a neighborhood of* 0 *in* $\mathbb{R}^{1+d}$,

(3) $h'(x)$ *is nilpotent in a neighborhood of* 0 *in* $\mathbb{R}^d$.

When $d=2$, the condition (2.30) writes merely

trdet
$$\mathrm{div}_x\, h = 0\,, \qquad \det\, h'(x) = 0\,. \tag{2.32}$$

A generic situation where $\det\, h'(x) \equiv 0$ (with a non constant h) is when h takes its values in a (strict) submanifold of $\mathbb{R}^2$ (i.e. on a curve). One can construct such h in the following way. Let F and G be two functions in $\mathcal{C}^1(\mathbb{R};\mathbb{R})$ and let a be a local solution of the scalar conservation law

$$\partial_1 F(a) + \partial_2 G(a) = 0\,.$$

We take $h(x) := \big(F\circ a(x)\big), G\circ a(x)\big)$ which satisfies actually the two relations required in (2.32). The solution of the corresponding Cauchy problem will hence satisfy the divergence free condition.

When $d \geq 3$, the conditions (2.30) are more complicated to deal with. Nevertheless, the previous construction is still valid and gives again initial data h with nilpotent h'.

Corollary 2.7. *For all $H \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^d)$, if $a(x)$ is a $\mathcal{C}^k$ local solution around* 0 *of the scalar conservation law* $\mathrm{div}_x\,(H\circ a) = 0$, *the function $h := H\circ a$ satisfies* (2.30), *and the local solution of the corresponding Cauchy problem* (2.31) *satisfies* $\mathrm{div}_x\,\mathbf{v} = 0$.

Proof. By Theorem 2.6, it is sufficient to check that the differential of h is nilpotent. Since $h = H \circ a$, for all x in a small neighborhood of 0, the matrix $h'(x)$ has rank 1. There is at most one non zero eigenvalue of $h'(x)$. Since the trace of $h'(x)$ is also 0, all the eigenvalues of $h'(x)$ must be zero. □

When $d = 3$, there is another generic situation. The condition $\det h' = 0$ is also satisfied when h takes its values in a submanifold Σ of dimension 2. For example, assuming that Σ is locally given by the equation $w = f(u,v)$, one looks for $h = \big(u, v, f(u,v)\big)$ where $u(x,y,z)$ and $v(x,y,z)$ are $\mathcal{C}^\infty$ functions of (x,y,z) with values in $\mathbb{R}$. In that case, the condition h' *nilpotent* is equivalent to the following non linear system of two equations with two unknowns

kovalevsky1

$$\partial_x u + \partial_y v + \partial_z f(u,v) \;=\; 0\,, \tag{2.33}$$

kovalevsky2

$$\det \begin{bmatrix} p & \partial_x u & \partial_x v \\ q & \partial_y u & \partial_y v \\ -1 & \partial_z u & \partial_z v \end{bmatrix} = \; 0\,, \tag{2.34}$$

where $p(u,v) := \partial_u f(u,v)$ and $q(u,v) := \partial_v f(u,v)$. By Cauchy-Kovalevsky theorem, there are local real analytic solutions of the system (2.33)-(2.34). More precisely, let $a(y,z)$ and $b(y,z)$ be two analytic functions from a neighborhood of 0 in $\mathbb{R}^2$ with values in $\mathbb{R}$. Suppose that

$$q\big(a(0), b(0)\big)\,\partial_z a(0) + \partial_y a(0) \neq 0\,.$$

Then, the initial surface $\{x = 0\}$ is non characteristic for the system (2.33)-(2.34) with initial data $(u,v)_{|x=0} = (a,b)$. Therefore, there exists a real analytic solution (u,v) on a neighborhood of 0 in $\mathbb{R}^3$.

We end this section with another result involving the polynomial $\mathcal{Q}_{h'(x)}$. It concerns the life span of the classical solutions of the equation (2.31). Let us introduce

$$\begin{aligned} &B(0,M] \;:=\; \big\{\, x \in \mathbb{R}^d\,;\; |x| \le M \,\big\}\,, \\ &\mathcal{C}^k_b(\mathbb{R}^d) := \big\{\, h \in \mathcal{C}^k(\mathbb{R}^d;\mathbb{R}^d)\,;\; \textstyle\sum_{j=0}^k \|h^{(k)}\|_{L^\infty(\mathbb{R}^d)} < \infty \,\big\}\,, \qquad k \in \mathbb{N}\,. \end{aligned}$$

life span

Theorem 2.8. *Let $h \in \mathcal{C}^0_b(\mathbb{R}^d) \cap \mathcal{C}^1(\mathbb{R}^d)$ and $T > 0$. The following properties are equivalent*

(1) the Cauchy problem

etalors

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla_x)\mathbf{v} = 0\,, \qquad \mathbf{v}_{|t=0} = h \tag{2.35}$$

has a solution $\mathbf{v}(t,x)$ *defined on* $[0,T]\times\mathbb{R}^d$ *and this solution* $\mathbf{v}(t,x)$ *belongs to the space* $\mathcal{C}^1\big([0,T]\times\mathbb{R}^d;\mathbb{R}^d\big)$.

(2) For all $M\in\mathbb{R}^+$, *the following minoration holds*

min (2.36) $$\inf\,\{\,|\mathcal{Q}_{h'(x)}(t)|\,;\;(t,x)\in[0,T]\times B(0,M)\,\}\;>\;0\,.$$

Theorem 2.8 has the following consequence. If in addition, $h\in\mathcal{C}^0_b(\mathbb{R}^d)\cap\mathcal{C}^1(\mathbb{R}^d)$ satisfies

(2.37) $$h'(x)\ \text{ is nilpotent for all } x \text{ in } \mathbb{R}^d,$$

then the solution $\mathbf{v}$ is *global* in time. Indeed, in that case $\mathcal{Q}_{h'(x)}\equiv 1$ and the condition (2.36) is verified for all T.

The proof below shows that Theorem 2.8 has an analogue when h is defined only locally in space, replacing $[0,T]\times\mathbb{R}^d$ by an appropriate domain of determination.

Proof. Assume first that *(1)* is satisfied. Let us consider the system of ordinary differential equations

$$\frac{d}{dt}\,\chi(t,x)=\mathbf{v}\big(t,\chi(t,x)\big),\qquad \chi(0,x)=x\,.$$

The solution is defined (and $\mathcal{C}^1$) on $[0,T]$. For all $t\in[0,T]$, the application $\chi(t,\cdot)$ is a $\mathcal{C}^1$ diffeomorphism of $\mathbb{R}^d$. For all $M\in\mathbb{R}^+$, we have

$$\inf\,\big\{\,|\det D_x\chi(t,x)|\,;\;(t,x)\in[0,T]\times B[0,M]\,\big\}\;>\;0\,.$$

Since by construction $\chi(t,x)=x+t\,h(x)$, we find

$$\det\,D_x\chi(t,x)=\det\big(\mathrm{Id}+t\,h'(x)\big)=\mathcal{Q}_{h'(x)}(t)$$

and the condition (2.36) follows.

Conversely assume that *(2)* holds. Fix any $M\in\mathbb{R}^+$. Let T^* be the supremum of the $\bar T$ such that (2.35) has a solution on the domain

$$\mathcal{D}(\bar T,M)\;:=\;\big\{\,(t,x)\,;\;|x|+t\;\|\,h\,\|_{L^\infty(\mathbb{R}^d)}\le M\,,\;t\in[0,\bar T]\,\big\}\,.$$

For $t\in[0,T^*[$ and $y=\chi(t,x)$ the formula (2.26) can be written

diffcal (2.38) $$D_x\mathbf{v}(t,y)=\mathcal{Q}_{h'(x)}(t)^{-1}\;h'(x)\;\mathrm{co}\big(\mathrm{Id}+t\,h'(x)\big)$$

where $\mathrm{co}(M)$ is the co-matrix of M. Then (2.36) and (2.38) imply

majochi (2.39) $$\sup\,\big\{\,|D_x\mathbf{v}(t,x)|\,;\;(t,x)\in\mathcal{D}(\bar T,M)\,\big\}\;<\;\infty\,.$$

This estimate contradicts the definition of T^*, because it allows to extend the solution $\mathbf{v}$ beyond T^* (see [20]). Therefore $T^*=T$. Since M is arbitrary, this implies *(1)*. □

2.4 Reduction of the system

We consider the equation (2.8). By using the property (2.9), we get

$$\mathcal{M}(u,\partial_x) = \begin{bmatrix} \mathcal{M}_1(u,\partial_x) & 0 \\ \mathcal{M}_2(u,\partial_x) & 0 \end{bmatrix} . \tag{2.40}$$

Let $S(u)$ be a symmetrizer for the system (2.8). We have

$$S(u) = {}^tS(u) = \begin{bmatrix} E(u) & {}^tF(u) \\ F(u) & G(u) \end{bmatrix} \gg 0 . \tag{2.41}$$

The matrices $E(u)$ and $G(u)$ are symmetric positive definite, thus invertible. Moreover, by (2.10),

$$F\,\mathcal{M}_1 + G\,\mathcal{M}_2 \;=\; 0\,, \qquad E\,\mathcal{M}_1 + {}^tF\,\mathcal{M}_2 \;\text{ is skew symmetric.} \tag{2.42}$$

Thus

$$\mathcal{M}_2(u,\partial_x) \;=\; -\,C(u)\,\mathcal{M}_1(u,\partial_x)\,, \qquad C := G^{-1}\,F\,. \tag{2.43}$$

By construction, the operator $\mathbf{L}^\flat(v,\partial_x)$ involved in (2.11)-(2.12) is

$$\mathbf{L}^\flat(v,\partial_x) \;=\; (E\,\mathcal{M}_1 + {}^tF\,\mathcal{M}_2)(v,\partial_x)\,.$$

Therefore, (2.43) implies:

$$\mathbf{L}^\flat(v,\partial_x) \;=\; \Sigma(u)\;\mathcal{M}_1(u,\partial_x) \quad \text{with} \quad \Sigma := (E - {}^tF\,G^{-1}\,F) = {}^t\Sigma \gg 0\,.$$

In this paper, we are interested in solutions u^ε which can be put in the form $u^\varepsilon = (v_0 + \varepsilon\,V^\varepsilon, W^\varepsilon)$ where

$$\text{the supports of } V^\varepsilon \text{ and } W^\varepsilon \text{ are contained in } \;\Omega^\flat\,. \tag{2.44}$$

Since v_0 is fixed, the true unknown is the couple $(V^\varepsilon, W^\varepsilon)$. It turns out that the system (2.11) is equivalent to

$$\left\{ \begin{aligned} & E\;X_{v_0+\varepsilon V}\,V + \mathbf{L}^\flat(v_0+\varepsilon V,\partial_x)\,V + \varepsilon^{-1}\;{}^tF\;X_{v_0+\varepsilon V}\,W \\ & \qquad\qquad = \,-\,\varepsilon^{-1}\,\big\{E\;X_{v_0+\varepsilon V}\,v_0 + \mathbf{L}^\flat(v_0+\varepsilon V,\partial_x)\,v_0\big\}\,, \\ & \\ & \varepsilon\;C\;X_{v_0+\varepsilon V}\,V + X_{v_0+\varepsilon V}\,W \;=\; -\,G^{-1}\,F\,X_{v_0+\varepsilon V}\,v_0\,. \end{aligned} \right. \tag{2.45}$$

Since v_0 satisfies (2.16), we have

$$\begin{aligned} & X_{v_0+\varepsilon V}\,v_0 \;=\; \varepsilon\;\Big(\int_0^1 (V\cdot\nabla_v)\mu(v_0+\varepsilon\,s\,V)\;ds\Big)\cdot\nabla_x v_0\,. \\ & \mathbf{L}^\flat(v_0+\varepsilon V,\partial_x)\,v_0 \;=\; \varepsilon\;\int_0^1 (V\cdot\nabla_v)\mathbf{L}^\flat(v_0+\varepsilon\,s\,V,\partial_x)\,v_0\;ds\,. \end{aligned} \tag{2.46}$$

The second equation in (2.45) yields

$$
\begin{aligned}
\varepsilon^{-1}\ {}^tF\, X_{v_0+\varepsilon V} W \ = \ & -{}^tF\, G^{-1}\, F\ X_{v_0+\varepsilon V} V \\
& -{}^tF\, G^{-1}\, F\ \Big(\int_0^1 (V\cdot\nabla_v)\mu(v_0+\varepsilon\, s\, V)\ ds\Big)\cdot\nabla_x v_0\,.
\end{aligned}
$$

We can use this last identity in order to interpret the first equation in (2.45). Then we can put (2.45) in a symmetric form to get

5.1

$$
\begin{aligned}
\mathbf{S}^\varepsilon(v_0+\varepsilon V, W)\, \mathbf{X}_{v_0+\varepsilon V}\, U \ &+ \mathbf{L}(v_0+\varepsilon V,\partial_x)\, U \\
&+ \mathbf{K}^\varepsilon(v_0,\partial v_0, U)\, U = 0\,.
\end{aligned}
\tag{2.47}
$$

Here, the matrix K^ε is a $\mathcal{C}^\infty$ function of its arguments, including ε. The operator $\mathbf{L}(v,\partial_x)$ is as in (2.11). The matrix $S^\varepsilon(u)$ is given by

$$
\mathbf{S}^\varepsilon(u)\ :=\ \begin{bmatrix} \Sigma(u)+\varepsilon^2\ {}^tC\, C & \varepsilon\ {}^tC \\ \varepsilon\ C & \mathrm{Id} \end{bmatrix}. \tag{2.48}
$$

Observe that, for ε small enough, the matrix $\mathbf{S}^\varepsilon(u)$ is still symmetric positive definite. In all the sequel we will note $\mathcal{H}^\varepsilon(t,x,U,\partial)$ with $U=(V,W)$ the linear first order symmetric operator

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$$
\begin{aligned}
\mathcal{H}^\varepsilon(t,x,U,\partial)\ :=\ & \mathbf{S}^\varepsilon(v_0+\varepsilon V, W)\, \mathbf{X}_{v_0+\varepsilon V} \\
& + \mathbf{L}(v_0+\varepsilon V,\partial_x)\ +\ \mathbf{K}^\varepsilon(v_0,\partial v_0, U)\,.
\end{aligned}
\tag{2.49}
$$

3 Oscillating solutions and the WKB expansions

and the WKB expansions

The goal of this section is to construct solutions $u^\varepsilon(t,x)$ of (2.8) admitting an asymptotic expansion of the form

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$$
u^\varepsilon(t,x)\ \underset{\varepsilon\to 0}{\sim}\ \sum_{n\geq 0} \varepsilon^n\ \mathbf{U}_n\big(t,x,\varphi_1(t,x)/\varepsilon,\cdots,\varphi_\ell(t,x)/\varepsilon\big) \tag{3.1}
$$

where the profiles $\mathbf{U}_n(t,x,\theta_1,\cdots,\theta_\ell)$ are smooth functions which are $(2\pi\mathbb{Z})^\ell$-periodic with respect to the *fast* variable $\theta=(\theta_1,\cdots,\theta_\ell)$

$$
\mathbf{U}_n(t,x,\theta)\in H^\infty(\Omega\times\mathbb{T}^\ell;\mathbb{R}^N)\,,\qquad \mathbb{T}:=\mathbb{R}/2\pi\mathbb{Z},\qquad \forall\, n\geq 0\,.
$$

The *phases* $\varphi_1(t,x),\cdots,\varphi_\ell(t,x)$ are real valued functions in $\mathcal{C}^\infty_b(\Omega;\mathbb{R})$. They are all solutions of the same eiconal equation

$$
\mathbf{X}_{v_0}\,\varphi_k\equiv 0\,,\qquad \forall\,(t,x)\in\Omega\,,\qquad \forall\, k\in\{1,\cdots,\ell\}\,.
$$

Introduce the notations $\vec{\boldsymbol{\varphi}}:=(\varphi_1,\cdots,\varphi_\ell)$; $\langle\alpha\cdot\alpha'\rangle$ denotes the Euclidian scalar product in $\mathbb{R}^\ell$; accordingly, $\alpha\cdot\partial_x\vec{\boldsymbol{\varphi}}=\nabla_x\langle\alpha\cdot\vec{\boldsymbol{\varphi}}\rangle$ where the term on the left is the usual product of the line matrix α and the Jacobian matrix of the mapping $\vec{\boldsymbol{\varphi}}(t,\cdot)$.

We call $\mathbf{\Phi}$ the $\mathbb{R}$-linear subspace of $\mathcal{C}^\infty(\Omega,\mathbb{R})$ generated by $\{\varphi_1,\cdots,\varphi_\ell\}$. It follows from the assumptions that for all $\psi\in\mathbf{\Phi}$, we have $\mathbf{X}_{v_0}\psi\equiv 0$. We add conditions which are usual in the context of multiphase geometrical optics (see [15], [16] and [17]).

strong coherence **Assumption 3.1. (strong coherence)** *We have* $1\notin\mathbf{\Phi}$. *Moreover, for all* $\psi\in\mathbf{\Phi}$, $\partial_x\psi(t,x)$ *nowhere vanishes in* $\overline{\Omega}$ *or is identically* 0 *in* Ω.

The first assumption is satisfied in most applications. If Φ contains nontrivial constants, then extra factors $e^{ic/\varepsilon}$, with c constant, have to be added in the expansions below. Here, we avoid this unessential technicality. On the contrary, the second part of the assumption is essential to the construction of WKB solutions. When there is only one phase φ, it means that $\partial_x\varphi$ never vanishes on *Omega*. In general, Φ is a finite dimensional subspace of $C_b^\infty(\Omega)$, of dimension $\ell'\leq\ell$. Taking a basis $\{\psi_1,\ldots,\psi_{\ell'}\}$, the second condition means that the differential $\partial_x\psi_1,\ldots,\partial_x\psi_{\ell'}$ are linearly independent in $\mathbb{R}^d$ at every point of Ω. In particular, $\ell'\leq d$.

It was shown in [16], [17], [18] that a small divisor condition is necessary for the contruction of arbitrary order asymptotic WKB solutions. Therefore we include :

petits diviseurs **Assumption 3.2. (small divisors)** *There are two constants* $c>0$ *and* $\rho\geq -1$ *such that for all* $\alpha\in\mathbb{Z}^\ell\setminus\{0\}$, *there holds for all* $(t,x)\in\Omega$*:*

sd (3.2)
$$|\,\alpha\cdot\partial_x\vec{\boldsymbol{\varphi}}(t,x)\,|\;\geq\;c\,/\,|\alpha|^\rho\,.$$

This assumption involves only the phases $\alpha\cdot\partial_x\vec{\boldsymbol{\varphi}}$ with $\alpha\in\mathbb{Z}^\ell$. There are two parts in this assumption: first, for all $\alpha\neq 0$, $\alpha\cdot\partial_x\vec{\boldsymbol{\varphi}}(t,x)$ never vanishes on Ω, which by Assumption 3.1 means that $\alpha\cdot\vec{\boldsymbol{\varphi}}$ is not a constant and thus not zero; this implies that the φ_j are linearly independent over $\mathbb{Q}$. Second, taking a basis $\{\psi_1,\ldots,\psi_{\ell'}\}$ of Φ and writing the φ_j in this basis, that is, with obvious notations, $\varphi_j=k_j\cdot\vec{\psi}$, (3.2) is an arithmetic condition on the $k_j\in\mathbb{R}^{\ell'}$.

Example 3.1. *When* $\ell=1$, *there is only one phase* φ. *The strong coherence and small divisor assumptions reduce to the constraint*

$$\inf_{(t,x)\in\Omega}\;|\nabla_x\varphi(t,x)|\;>\;0\,.$$

Example 3.2. *Suppose that $\vec{\varphi}$ satisfies*

$$\alpha \cdot \partial_x \vec{\varphi}(t,x) \neq 0\,, \qquad \forall\, \alpha \in \mathbb{R}^\ell \setminus \{0\}\,, \qquad \forall\,(t,x) \in \overline{\Omega}\,. \tag{3.3}$$

Then the strong coherence Assumption 3.1 is fullfiled. Moreover, by homogeneity, the small divisor Assumption 3.2 holds with $\rho = -1$.

Example 3.3. *Suppose that v_0 is constant and consider linear phases*

$$\varphi_j(t,x) = a_j\, t + k_j \cdot x \qquad \forall\, j \in \{1, \cdots, \ell\}\,.$$

The Assumption 3.1 is satisfied, since for all $\psi \in \Phi$, $\partial_x \psi$ is a constant. The condition (3.3) are satisfied if and only if the vectors $k_1, \cdots, k_\ell$ are linearly independent in $\mathbb{R}^d$.

The small divisors condition 3.2 means that the vectors $k_j, \cdots, k_\ell$ in $\mathbb{R}^d$ are linearly independent over $\mathbb{Q}$, and satisfy an arithmetic condition. It is generically satisfied when the k_j are independent over $\mathbb{Q}$ (see e.g. [16]).

In the expansion (3.1) the profiles $\mathbf{U}_n$ can be decomposed into $(\mathbf{V}_n, \mathbf{W}_n)$ with $\mathbf{V}_n(t,x,\theta) \in \mathbb{R}^{N''}$ and $\mathbf{W}_n(t,x,\theta) \in \mathbb{R}^{N'}$. We assume that the first profile satisfies the relation $\mathbf{V}_0 = v_0$ where v_0 satisfies (2.16). In particular, $\mathbf{V}_0$ does not depend on θ.

It is also interesting to consider oscillatory source terms. Hence we consider the following system.

$$S(u^\varepsilon)\, \mathbf{X}_{v^\varepsilon}\, u^\varepsilon + \mathbf{L}(v^\varepsilon, \partial_x) u^\varepsilon \;=\; \begin{bmatrix} \varepsilon \tilde{f}^\varepsilon \\ \varepsilon \tilde{g}^\varepsilon \end{bmatrix} \tag{3.4}$$

with $\tilde{f}^\varepsilon(t,x) = \tilde{\mathbf{f}}^\varepsilon(t,x,\vec{\varphi}/\varepsilon)$ and $\tilde{g}^\varepsilon(t,x) = \tilde{\mathbf{g}}^\varepsilon(t,x,\vec{\varphi}/\varepsilon)$. Here the profiles $\tilde{\mathbf{f}}^\varepsilon$ and $\tilde{\mathbf{g}}^\varepsilon$ are $\mathcal{C}^\infty$ functions of the parameter $\varepsilon \in]0,1]$ with values respectively in the spaces $H^\infty(\Omega \times \mathbb{T}^\ell; \mathbb{R}^{N''})$ and $H^\infty(\Omega \times \mathbb{T}^\ell; \mathbb{R}^{N'})$.

3.1 Formal solutions

The first interesting result is the existence of formal or WKB solutions of the system (3.4) of the form (3.1). Let us first explain what is meant by *formal solutions*. Plugging the expansion (3.1) into the system (3.4), using by Taylor expansions and ordering the terms in powers of ε, we obtain a formal expansion in power series of ε:

$$\sum_{j=-1}^{\infty} \varepsilon^j\ \mathcal{F}_j(t,x,\vec{\varphi}/\varepsilon)$$

with profiles $\mathcal{F}_j$ in $H^\infty(\Omega \times \mathbb{T}^\ell; \mathbb{R}^N)$. We say that (3.1) is a formal solution when all the resulting $\mathcal{F}_j$ are indentically zero.

Introduce first some notations. Every function $u(t,x,\theta)$ in the space $H^\infty(\Omega_T \times \mathbb{T}^\ell; \mathbb{R}^N)$ has a Fourier expansion

$$u(t,x,\theta) = \sum_{\alpha \in \mathbb{Z}^\ell} \widehat{u}_\alpha(t,x)\; e^{i\langle \alpha \cdot \theta \rangle} \tag{3.5}$$

where $\widehat{u}_\alpha \in H^\infty(\Omega_T; \mathbb{R}^N)$ for all $\alpha \in \mathbb{Z}^\ell$ and

$$\sum |\alpha|^p \; \|\widehat{u}_\alpha\|_{H^q(\Omega_T)} \,<\, \infty\,, \qquad \forall\, p > 0\,, \qquad \forall\, q > 0\,. \tag{3.6}$$

Conversely, the property (3.6) and the formula (3.5) characterize the elements of $H^\infty(\Omega_T \times \mathbb{T}^\ell; \mathbb{R}^N)$. We remark that $\widehat{u}_0$ is the averaged value (in θ) of u, that is

$$\widehat{u}_0(t,x) = \frac{1}{(2\pi)^\ell} \int_0^{2\pi} \cdots \int_0^{2\pi} u(t,x,\theta)\; d\theta_1 \cdots d\theta_\ell\,.$$

Recall that $\mathbf{P}^\flat(v,\xi)$ is the matrix for the orthogonal projector of $\mathbb{R}^{N''}$ onto $\ker \mathbf{L}^\flat(v,\xi)$, in the canonical basis of $\mathbb{R}^{N''}$. By For $(t,x) \in \Omega$ and $\alpha \in \mathbb{R}^\ell$, we will note

$$\Pi^\flat_\alpha(t,x) := \mathbf{P}^\flat\big(v_0(t,x), \alpha \cdot \partial_x \vec{\boldsymbol{\varphi}}(t,x)\big)\,.$$

Following Joly, Métivier and Rauch [16], we introduce the operator $\mathcal{E}(t,x,\partial_\theta)$ defined by the following formula applied to $\mathbf{V} \in H^\infty(\Omega_T \times \mathbb{T}^\ell; \mathbb{R}^{N"})$

$$\mathcal{E}(t,x,\partial_\theta)\, \mathbf{V}(t,x,\theta) := \widehat{\mathbf{V}}_0 + \sum_{\alpha \in \mathbb{Z}^\ell \setminus \{0\}} \Pi^\flat_\alpha\, \widehat{\mathbf{V}}_\alpha(t,x)\; e^{i\langle \alpha \cdot \theta \rangle}\,. \tag{3.7}$$

Proposition 3.3. *$\mathcal{E}(t,x,\partial_\theta)$ is a linear continuous operator from $H^\infty(\Omega_T \times \mathbb{T}^\ell; \mathbb{R}^{N''})$ into itself.*

It is a consequence of the more general Proposition 3.6 below.

The following theorem states that the system (3.4) has formal solutions of the form (3.1), and that one can *prescribe arbitrary initial values to $\mathbf{W}_n$ (for $n \geq 0$) and to $\mathcal{E}\mathbf{V}_n$ (for $n \geq 1$).*

Theorem 3.4. *Let $\{\boldsymbol{a}_k(x,\theta)\}_{k \in \mathbb{N}_*}$ in $H^\infty(\mathbb{R}^d \times \mathbb{T}^\ell; \mathbb{R}^{N''})^{\mathbb{N}_*}$ and $\{\mathbf{b}_\ell(x,\theta)\}_{l \in \mathbb{N}}$ in $H^\infty(\mathbb{R}^d \times \mathbb{T}^\ell; \mathbb{R}^{N'})^{\mathbb{N}}$ be two sequences of profiles, such that $\mathcal{E}(0,x,\partial_\theta)\boldsymbol{a}_k = \boldsymbol{a}_k$ for all k. There exist $T > 0$ and a sequence of profiles $\{\mathbf{U}_n\}_{n \in \mathbb{N}}$ in*

$H^\infty(\Omega_T \times \mathbb{T}^\ell; \mathbb{R}^N)^{\mathbb{N}}$ *with* $\mathbf{U}_n = (\mathbf{V}_n, \mathbf{W}_n)$, *satisfying* $\mathbf{V}_0 = v_0$ *together with the initial conditions*

$$\big(\,\mathcal{E}(t,x,\partial_\theta)\mathbf{V}_k\,\big)_{|t=0} = \boldsymbol{a}_k\,, \qquad \mathbf{W}_{\ell|t=0} = \mathbf{b}_\ell, \tag{3.8}$$

and such

$$\sum_{n\geq 0} \varepsilon^n\ \mathbf{U}_n\big(t,x,\varphi(t,x)/\varepsilon\big)$$

is a formal solution of (3.4) *on* Ω_T. *Moreover, the profile* $\mathbf{V}_1$ *satisfies the polarization condition*

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$$\mathcal{E}(t,x,\partial_\theta)\mathbf{V}_1 = \mathbf{V}_1\,, \qquad \forall\,(t,x)\in\Omega_T\times\mathbb{T}\,. \tag{3.9}$$

We refer to the section 3.4 for an example related to the Euler equations of gaz dynamics.

3.2 Proof of the theorem 3.4

The material used in the analysis of the profile equations is closed to that of the paper [16]. However, since the hypothesis are not exactly the same, we give a self contained demonstration of the technical lemmas.

We start with formulating the problem in terms of the new unknown $U^\varepsilon = (V^\varepsilon, W^\varepsilon)$ such that $u^\varepsilon = (v_0 + \varepsilon V^\varepsilon, W^\varepsilon)$. Using the notations in (2.49), the new system reads

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$$\mathcal{H}^\varepsilon(t,x,U^\varepsilon,\partial)\,U^\varepsilon\;=\;h^\varepsilon\,, \qquad h^\varepsilon\;:=\;\begin{bmatrix} f^\varepsilon \\ g^\varepsilon \end{bmatrix} = \sum_{n\geq 0}\varepsilon^n\;h_n\,. \tag{3.10}$$

We are looking for formal solutions of (3.10) of the form

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$$U^\varepsilon(t,x)\;=\;\sum_{n\geq 0}\varepsilon^n\;\mathcal{U}_n(t,x,\vec{\boldsymbol{\varphi}}/\varepsilon) \tag{3.11}$$

where the profiles $\mathcal{U}_n(t,x,\theta)$ belong to $H^\infty(\omega_T\times\mathbb{T}^\ell;\mathbb{R}^N)$ for some $T > 0$. Plugging the expansion (3.11) into the system (3.10), and ordering the resulting expansion in powers of ε, one gets formally

$$\mathcal{H}^\varepsilon(t,x,U^\varepsilon,\partial)U^\varepsilon - h^\varepsilon\;=\;\sum_{j\geq -1}\;\varepsilon^j\;\Phi_j(t,x,\vec{\boldsymbol{\varphi}}/\varepsilon)\,.$$

We want to solve the cascade of equations $\{\Phi_j \equiv 0\}_{j\geq -1}$. By a first order Taylor expansion in ε, the hyperbolic operator $\mathcal{H}^\varepsilon(t,x,U,\partial_x)$ reads

$$\mathcal{H}^\varepsilon(t,x,U,\partial_{t,x})\;\equiv\;\mathcal{H}^0(t,x,U,\partial_{t,x}) + \sum_{j=1}^{d}\mathbf{B}^\varepsilon_j(v_0,U)\,\varepsilon\,\partial_j\;+\;\varepsilon\,\mathbf{M}^\varepsilon(v_0,\partial v_0,U)\,.$$

Here $\mathcal{H}^0(t,x,U,\partial_{t,x})$ means $\mathcal{H}^\varepsilon(t,x,U,\partial_{t,x})$ with $\varepsilon = 0$, that is

$$\mathcal{H}^0(t,x,U,\partial_{t,x}) = \mathbf{S}^0(v_0,W)\,\mathbf{X}_{v_0} + \mathbf{L}(v_0,\partial_x) + \mathbf{K}^0(v_0,\partial v_0,v_0,U)\,.$$

Moreover $\mathbf{B}_j^\varepsilon(v,U)$ and $\mathbf{M}^\varepsilon(v,v',U)$ are $N \times N$ matrices depending in a $\mathcal{C}^\infty$ way on their arguments ε, v, v', U (up to $\varepsilon = 0$), the matrices $\mathbf{B}_j^\varepsilon$ being symmetric. To write the profile equations, we need the following notations

$$\begin{aligned}
\mathcal{B}(t,x,U,\partial_\theta) &:= \sum_{j=1}^{d}\sum_{k=1}^{\ell} \partial_j\varphi_k\ \mathbf{B}_j^0\big(v_0(t,x),U\big)\ \partial_{\theta_k}\\
\mathcal{L}(t,x,\partial_\theta) &:= \sum_{k=1}^{\ell} \mathbf{L}\big(v_0(t,x),\partial_x\varphi_k(t,x)\big)\ \partial_{\theta_k}\,,\\
\mathbf{L}(v_0,\partial_x) &:= \sum_{j=1}^{d} L_j(t,x)\ \partial_{x_j}\,,\\
\mathbb{H}(t,x,U,\partial_{t,x,\theta}) &:= \mathcal{H}^0(t,x,U,\partial_{t,x}) + \mathcal{B}(t,x,U,\partial_\theta)\,.
\end{aligned}$$

With these notations, there holds:

$$\Phi_{-1}(t,x,\theta) \equiv \mathcal{L}(t,x,\partial_\theta)\mathcal{U}_0, \tag{3.12}$$

$$\Phi_0(t,x,\theta) \equiv \mathbb{H}(t,x,\mathcal{U}_0,\partial_{t,x,\theta})\mathcal{U}_0 + \mathcal{L}(t,x,\partial_\theta)\mathcal{U}_1 - h_0, \tag{3.13}$$

and for $j \geq 1$,

$$\Phi_j(t,x,\theta) \equiv \mathbb{H}_0(t,x,\partial_{t,x,\theta})\mathcal{U}_j + \mathcal{L}(t,x,\partial_\theta)\mathcal{U}_{j+1} - q_j(t,x) \tag{3.14}$$

where $\mathbb{H}_0(t,x,\partial_{t,x,\theta})$ means the linearized operator of $\mathbb{H}(t,x,U,\partial_{t,x,\theta})$ with respect to U on the state $\mathcal{U}_0$, and q_j is a term depending only on the right hand side h^ε and on the profiles $\mathcal{U}_k$ with $k \leq j-1$. More precisely

$$q_j = Q_j(v_0,\partial v_0,\mathcal{U}_k,\partial\mathcal{U}_k\,;\,k \leq j-1) + \frac{1}{j!}\,\Big(\frac{\partial^j h^\varepsilon}{\partial\varepsilon^j}\Big)_{|\varepsilon=0}\,. \tag{3.15}$$

The averaging operator. For all $(v,\xi) \in \mathbb{R}^{N''} \times \mathbb{R}^d$, recall that $\mathbf{P}(v,\xi)$ is the matrix of the orthogonal projector of $\mathbb{R}^N$ onto ker $\mathbf{L}(v,\xi)$ written in the canonical basis of $\mathbb{R}^N$. For $(t,x) \in \Omega$ and $\alpha \in \mathbb{R}^\ell$, we note

$$\Pi_\alpha(t,x) := \mathbf{P}\big(v_0(t,x),\alpha\cdot\partial_x\vec{\varphi}(t,x)\big)\,.$$

Lemma 3.5. *For all $\alpha \in \mathbb{Z}^\ell$, the entries of the matrix $\Pi_\alpha(\cdot,\cdot)$ belong to the space $\mathcal{C}_b^\infty(\Omega)$. Moreover the following inequality holds*

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$$\| \, \partial_{t,x}^\beta \Pi_\alpha(\cdot,\cdot) \, \|_{L^\infty(\Omega)} \, \leq \, c_\beta \, , \qquad \forall \, \beta \in \mathbb{N}^{1+d} \tag{3.16}$$

where c_β is a constant independent on α.

Proof. Since $\mathbf{P}(\cdot,\cdot)$ is $\mathcal{C}^\infty$ on $\mathcal{O} \times (\mathbb{R}^d \setminus \{0\})$, the Assumption 3.2 implies that $\Pi_\alpha(t,x)$ is $\mathcal{C}^\infty$ in Ω for all α in $\mathbb{Z}^d \setminus \{0\}$. Let us introduce an $\mathbb{R}$-basis $\psi_1, \cdots, \psi_{\ell'}$ of $\mathbf{\Phi}$, and denote $\vec{\boldsymbol{\psi}}$ the function $(\psi_1, \cdots, \psi_{\ell'})$ from Ω to $\mathbb{R}^{\ell'}$, so that

$$|\alpha' \cdot \partial_x \vec{\boldsymbol{\psi}}| \, \geq \, C \, |\alpha'| \, , \qquad \forall \, \alpha' \in \mathbb{R}^{l'} \, .$$

Moreover, we can write

$$(\varphi_1, \cdots, \varphi_\ell) = (\psi_1, \cdots, \psi_{\ell'}) \, R$$

where R is a constant real $(\ell' \times \ell)$-matrix. According to these notations, we have $\alpha \cdot \partial_x \vec{\boldsymbol{\varphi}} = \alpha \, {}^t R \cdot \partial_x \vec{\boldsymbol{\psi}}$. From the coherence Assumption 3.1, we deduce

$$\alpha' \cdot \partial_x \vec{\boldsymbol{\psi}}(t,x) \neq 0 \, , \qquad \forall \, (t,x) \in \Omega \, , \qquad \forall \, \alpha' \in \mathbb{R}^{\ell'} \setminus \{0\} \, .$$

The function $\mathbf{P}$ is homogeneous of degree zero with respect to ξ (as well as Π_α with respect to α). Thus

$$\Pi_\alpha(t,x) = \mathbf{P}\Big(v_0(t,x), \frac{\alpha' \cdot \partial_x \vec{\boldsymbol{\psi}}(t,x)}{|\alpha' \cdot \partial_x \vec{\boldsymbol{\psi}}(t,x)|} \Big) = \Pi_{\frac{\alpha}{|\alpha|}}(t,x)$$

with $\alpha' = \alpha \, {}^t R$. Therefore the image of $\Omega \times (\mathbb{R}^{\ell'} \setminus \{0\})$ by the mapping

$$(t,x,\alpha') \mapsto \Big(v_0(t,x), \frac{\alpha' \cdot \partial_x \vec{\boldsymbol{\psi}}(t,x)}{|\alpha' \cdot \partial_x \vec{\boldsymbol{\psi}}(t,x)|} \Big)$$

is contained in a *compact* set of $\mathbb{R}^N \times \mathbf{S}^{\ell'}$ ($\mathbf{S}^{\ell'}$ means the unit sphere of $\mathbb{R}^{\ell'}$). It yields the inequality (3.16) for $\beta = 0$. For $\beta \neq 0$, we compute with the chain rule the quantity $\partial_{t,x}^\beta \Pi_\alpha(t,x)$. By using again arguments of homogeneity, we can then obtain (3.16). □

For all $\alpha \in \mathbb{R}^\ell$, it holds

$$\mathcal{L}(t,x,\partial_\theta) \; e^{i\langle \alpha \cdot \theta \rangle} = i \, \mathbf{L}(v_0, \alpha \cdot \partial_x \vec{\boldsymbol{\varphi}}) \; e^{i\langle \alpha \cdot \theta \rangle}$$

and it follows that for all $u \in H^\infty(\omega_T \times \mathbb{T}^\ell; \mathbb{R}^N)$

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$$\mathcal{L}(t,x,\partial_\theta)u = \sum_{\alpha \in \mathbb{Z}^\ell \setminus \{0\}} i\, \mathbf{L}(v_0, \alpha \cdot \partial_x \vec{\boldsymbol{\varphi}})\, \widehat{u}_\alpha(t,x)\; e^{i\langle \alpha \cdot \theta \rangle} \tag{3.17}$$

where the serie is summable in the sense of (3.6). One can deduce from the relation (3.17) that $\mathcal{L}(t,x,\partial_\theta)u = 0$ if and only if all the Fourier coefficients $\mathbf{L}(v_0, \alpha \cdot \partial_x \vec{\boldsymbol{\varphi}})\widehat{u}_\alpha(t,x)$ vanish. In other words

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$$\big(\mathrm{Id} - \Pi_\alpha(t,x)\big)\widehat{u}_\alpha(t,x) = 0, \qquad \forall\, \alpha \in \mathbb{Z}^\ell \setminus \{0\}, \qquad \forall\, (t,x) \in \Omega_T\,. \tag{3.18}$$

Let us introduce the mean operator

$$\mathbb{E}(t,x,\partial_\theta)u(x,\theta) := \widehat{u}_0 + \sum_{\alpha \in \mathbb{Z}^\ell \setminus \{0\}} \Pi_\alpha(t,x)\, \widehat{u}_\alpha(t,x)\; e^{i\langle \alpha \cdot \theta \rangle}\,. \tag{3.19}$$

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Proposition 3.6. *$\mathbb{E}(t,x,\partial_\theta)$ is a linear continuous operator from $H^\infty(\omega_T \times \mathbb{T}^\ell)$ into itself. It is the projector on the kernel of $\mathcal{L}(t,x,\partial_\theta)$ parallel to the range of $\mathcal{L}(t,x,\partial_\theta)$. Moreover $\mathbb{E}(t,x,\partial_\theta)$ extends as a linear continuous operator on $L^2(\omega_T \times \mathbb{T}^\ell)$, which is an orthogonal projector on $L^2(\omega_T \times \mathbb{T}^\ell)$. Extended in this way, $\mathbb{E}$ maps continuously $H^s(\omega_T \times \mathbb{T}^\ell)$ into itself for any real $s \geq 0$.*

Proof. The fact that $\Pi_\alpha(t,x)$ depends on (t,x) in a $\mathcal{C}_b^\infty$ way together with the uniform inequalities (3.16) implies that for any integer m and for all $u \in H^\infty(\omega_T \times \mathbb{T}^\ell)$

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$$\|\Pi_\alpha \widehat{u}_\alpha\|_{H^m(\omega_T)} \leq c_m\, \|\widehat{u}_\alpha\|_{H^m(\omega_T)}\,, \qquad \forall\, \alpha \in \mathbb{Z}^\ell \tag{3.20}$$

where c_m is a constant depending only on m (and not on α). This proves the continuity of $\mathbb{E}(t,x,\partial_\theta)$ on $H^\infty(\omega_T \times \mathbb{T}^\ell)$. By density of $H^\infty(\omega_T \times \mathbb{T}^\ell)$ in $H^m(\omega_T \times \mathbb{T}^\ell)$ for any $m \in \mathbb{N}$, the inequality (3.20) shows that $\mathbb{E}(t,x,\partial_\theta)$ extends as a linear continuous operator on $H^m(\omega_T \times \mathbb{T}^\ell)$ for any $m \in \mathbb{N}$. The fact that $\Pi_\alpha(t,x) \circ \Pi_\alpha(t,x) = \Pi_\alpha(t,x)$ implies that $\mathbb{E}(t,x,\partial_\theta) \circ \mathbb{E}(t,x,\partial_\theta) = \mathbb{E}(t,x,\partial_\theta)$. Therefore $\mathbb{E}(t,x,\partial_\theta)$ is a projector on $H^\infty(\omega_T \times \mathbb{T}^\ell)$. We also deduce from the relation (3.18) that for all u in $H^\infty(\omega_T \times \mathbb{T}^\ell)$

$$\mathcal{L}(t,x,\partial_\theta)\, u = 0 \;\text{ if and only if }\; \mathbb{E}(t,x,\partial_\theta)\, u = u\,. \tag{3.21}$$

When $m = 0$ in (3.20), one can take $c_0 = 1$ and the fact that the Π_α are orthogonal projectors shows that $\mathbb{E}(t,x,\partial_\theta)$ is symmetric for the inner product of $L^2(\omega_T \times \mathbb{T}^\ell)$. Again, the density of $H^\infty(\omega_T \times \mathbb{T}^\ell)$ in $L^2(\omega_T \times \mathbb{T}^\ell)$, shows that $\mathbb{E}$ extends as *an orthogonal projector* on $L^2(\omega_T \times \mathbb{T}^\ell)$.

It remains to prove that the range of $\mathrm{Id} - \mathbb{E}$ is exactly the range of $\mathcal{L}$. Let $u \in H^\infty(\omega_T \times \mathbb{T}^\ell)$ such that $\mathbb{E}\, u = 0$. We want to show that there exists $v \in H^\infty(\omega_T \times \mathbb{T}^\ell)$ such that $\mathcal{L}\, v = u$. Passing to Fourier coefficients and noting $\Pi_\alpha^\perp := \mathrm{Id} - \Pi_\alpha$, this is equivalent to

$$\mathbf{L}(v_0, \partial_x \langle \alpha \cdot \vec{\boldsymbol{\varphi}} \rangle)\, \widehat{v}_\alpha = \widehat{u}_\alpha$$

which is again equivalent, since $\Pi_\alpha^\perp\, \widehat{u}_\alpha = \widehat{u}_\alpha$, to solve

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$$\big(\Pi_\alpha^\perp\; \mathbf{L}(v_0, \partial_x \langle \alpha \cdot \vec{\boldsymbol{\varphi}} \rangle)\, \Pi_\alpha^\perp + \Pi_\alpha\big)\; \Pi_\alpha^\perp\, \widehat{v}_\alpha \;=\; \widehat{u}_\alpha\,. \tag{3.22}$$

For all $(v, \xi) \in \mathcal{O} \times (\mathbb{R}^d \setminus \{0\})$ the determinant of the matrix

$$\mathbf{P}^\perp(v,\xi)\, \mathbf{L}(v,\xi)\, \mathbf{P}^\perp(v,\xi) + \mathbf{P}(v,\xi)$$

where $\mathbf{P}^\perp := \mathrm{Id} - \mathbf{P}$, calculated in a basis of the form (basis of $\mathrm{Im}\, \mathbf{P}^\perp$, basis of $\ker\, \mathbf{P}^\perp$) is

$$\det \begin{bmatrix} M(v,\xi) & 0 \\ 0 & \mathrm{Id} \end{bmatrix} = \det M(v,\xi)$$

where $M(v,\xi)$ is an *invertible* matrix of size $r = \mathrm{rank}\, \mathbf{L}$ with coefficients in $\mathcal{C}^\infty\big(\mathbb{R}^{N"} \times (\mathbb{R}^d \setminus \{0\})\big)$ homogeneous of degree 1 in ξ. Hence this determinant is subjected to

$$|\det\, M\big(v_0(t,x), \xi\big)| \;\geq\; c\; |\xi|^r$$

where c is a constant independent on ξ.

For all $\alpha \in \mathbb{Z}^d \setminus \{0\}$, we deduce from the small divisor hypothesis 3.2 and from the definition of $\Pi_\alpha(t,x)$, that the determinant $d_\alpha(t,x)$ of the matrix

matriceelliptique
$$\Pi_\alpha^\perp\, \mathbf{L}(v_0, \partial_x \langle \alpha \cdot \vec{\boldsymbol{\varphi}} \rangle)\, \Pi_\alpha^\perp + \Pi_\alpha \tag{3.23}$$

is in $\mathcal{C}_b^\infty(\Omega)$ and do satisfy

inorationdudeterminant
$$|\, d_\alpha(t,x)\, | \;\geq\; c_0 \,/\, |\alpha|^{\rho\, r} \;>\; 0 \tag{3.24}$$

for all $(t,x) \in \Omega$, the constant c_0 in the inequality (3.24) being independent on $\alpha \in \mathbb{Z}^d \setminus \{0\}$. Hence, for all $\alpha \in \mathbb{Z}^d \setminus \{0\}$, there is a matrix $R_\alpha(t,x)$, whith coefficients in $\mathcal{C}_b^\infty(\Omega)$, such that

$$\widehat{v}_\alpha \;=\; R_\alpha(t,x)\, \widehat{u}_\alpha \tag{3.25}$$

is the unique solution of the equation (3.22). The relations (3.23) and (3.24) imply that for all $\beta \in \mathbb{N}^{1+d}$, the matrix R_α satisfy the estimates

$$\|\, \partial_{t,x}^\beta R_\alpha \,\|_{L^\infty(\Omega)} \;\leq\; c_\beta\; (1 + |\alpha|)^{m(\beta)}\,, \qquad \forall\, \alpha \in \mathbb{Z}^d$$

where the constant c_β does not depend on α. Using (3.6), we see that

$$v(t,x,\theta) := \sum_{\alpha \in \mathbb{Z}^d \setminus \{0\}} R_\alpha \, \widehat{u}_\alpha(t,x) \; e^{i\langle \alpha \cdot \theta \rangle}$$

belongs to $H^\infty(\Omega \times \mathbb{T}^\ell)$. It is the unique solution of the problem

$$\mathcal{L}(t,x,\partial_\theta) v = u, \qquad \mathbb{E}(t,x,\partial_\theta) v = 0\,.$$

The proposition is proved. □

inversionelliptique **Corollary 3.7.** *For every $f \in H^\infty(\omega_T \times \mathbb{T}^\ell)$ such that $\mathbb{E}\, f = 0$, there is a unique $\mathcal{U} \in H^\infty(\omega_T \times \mathbb{T}^\ell)$ such that*

$$\mathcal{L}(t,x,\partial_\theta)\, \mathcal{U} = f\,, \qquad \mathbb{E}(t,x,\partial_\theta)\, \mathcal{U} = 0\,.$$

Denoting by $\mathcal{U} := \mathcal{Q}(t,x,\theta)\, f$ the solution, this defines a continuous operator $\mathcal{Q}$ from $\ker \mathbb{E} \equiv \operatorname{Im} \mathcal{L}$ *into itself, for the topology of H^∞.*

The next theorem states that there exist sequences of profiles $\mathcal{U}_n$ satisfying all the equations $\Phi_j \equiv 0$, together with arbitrary given initial values for $(\mathbb{E}\mathcal{U}_n)_{|t=0}$. The theorem 3.4 is then a direct consequence of this result, since the profiles $\mathbf{V}_{n+1}$, $\mathbf{W}_n$ $(n \geq 0)$ and $\mathcal{U}_n$ are related by $(\mathbf{V}_{n+1}, \mathbf{W}_n) = \mathcal{U}_n$.

solutionsformellespourU **Theorem 3.8.** *Let $\{a_n(x,\theta)\}_{n\in\mathbb{N}}$ be a sequence in $H^\infty(\mathbb{R}^d \times \mathbb{T}^\ell; \mathbb{R}^N)^{\mathbb{N}}$ of profiles satisfying $\mathbb{E}(0,x,\partial_\theta) a_n = a_n$. There exist $T > 0$ and a unique sequence of profiles $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ in $H^\infty(\omega_T \times \mathbb{T}^\ell; \mathbb{R}^N)^{\mathbb{N}}$ satisfying the initial conditions*

$$(\mathbb{E}\mathcal{U}_n)_{|t=0} \;=\; a_n\,, \qquad \forall\, n \in \mathbb{N} \tag{3.26}$$

and such that $\Phi_j \equiv 0$ on $\omega_T \times \mathbb{T}^\ell$ for all $j \geq -1$.

For the proof we show that the infinite sequence of systems

$$(\mathrm{Id} - \mathbb{E})\, \Phi_{n-1} = 0\,, \quad \mathbb{E}\, \Phi_n = 0\,, \quad (\mathbb{E}\mathcal{U}_n)_{|t=0} = a_n\,, \quad n \in \mathbb{N} \tag{3.27}$$

can actually be solved by induction.

We start with $n = 0$. We get the first profile $\mathcal{U}_0$ by the following result which also determines the time $T > 0$ of the theorem 3.4.

tion du premier profil **Theorem 3.9.** *Let $h \in H^\infty(\omega \times \mathbb{T}^\ell; \mathbb{R}^N)$ and $a \in H^\infty(\mathbb{R}^d \times \mathbb{T}^\ell; \mathbb{R}^N)$. There exist $T > 0$ and a unique $\mathcal{U}_0 \in H^\infty(\omega_T \times \mathbb{T}^\ell; \mathbb{R}^N)$ satisfying on $\omega_T \times \mathbb{T}^\ell$*

tion du premier profil

$$\left\{ \begin{array}{l} (\mathrm{Id} - \mathbb{E})\,\mathcal{U}_0 \;=\; 0 \\ \mathbb{E}\,\mathbb{H}(t, x, \mathcal{U}_0, \partial_{t,x,\theta})\,\mathcal{U}_0 \;=\; \mathbb{E}\,h \end{array} \right. \tag{3.28}$$

with the initial condition

$$\mathcal{U}_0(0, \cdot) = \mathbb{E}(0, x, \partial_\theta)\, a_0 \,.$$

The proof of this theorem relies on classical arguments in non linear geometrical optics (see for example [19], [16] or [17]). In fact, for all U in $H^\infty(\omega_T \times \mathbb{T}^\ell; \mathbb{R}^N)$, *the linear operator* $\mathbb{E}\,\mathbb{H}(t, x, U, \partial_{t,x,\theta})$ coincides on $\ker(\mathrm{Id} - \mathbb{E}) = \mathrm{Im}\,\mathbb{E}$ with the operator $\mathbb{E}\,\mathbb{H}(t, x, U, \partial_{t,x,\theta})\,\mathbb{E}$, and acts like a (non local) *symmetric* hyperbolic operator. More precisely, one can solve uniquely the *linear* Cauchy problem with initial data in $\mathrm{Im}\,\mathbb{E}$, together with usual energy estimates, according to the following proposition.

prop6.3 **Proposition 3.10.** *(see [16], [17] and [19]) Fix any $\underline{U} \in H^\infty(\omega_{T_0} \times \mathbb{T}^\ell)$ with $\|\underline{U}\|_{W^m(0,T)} \le R$. Let $u_0 \in H^\infty(\mathbb{R}^d \times \mathbb{T}^\ell)$ such that $\mathbb{E}(0, x, \partial_\theta) u_0 = u_0$. Let $h \in H^\infty(\omega_{T_0} \times \mathbb{T}^\ell)$ satisfying $\mathbb{E}(t, x, \partial_\theta) h = h$. Then, there exists a unique $\mathcal{U} \in H^\infty(\omega_{T_0} \times \mathbb{T})$ such that $(\mathrm{Id} - \mathbb{E})\,\mathcal{U} = 0$ and*

$$\mathbb{E}\,\mathbb{H}(t, x, \underline{U}, \partial_{t,x,\theta})\,\mathbb{E}\,\mathcal{U} = h, \qquad \mathcal{U}_{|t=0} = u_0 \,.$$

Furthermore, for all $m \ge m_0$ where m_0 is big enough, we have

$$\|\mathcal{U}\|_{W^m(0,T)} \;\le\; c_m(R)\ \big(T\ \|h\|_{W^m(0,T)} \;+\; \|u_0\|_{H^m(\mathbb{R}^d)}\,\big) \tag{3.29}$$

where $c_m(\cdot)$ is an increasing function on $[0, +\infty[$ and

$$\|V\|_{W^m(0,T)} \;:=\; \sup_{0 \le t \le T}\ \sup_{j \le m}\ \|\partial_t^j V(t, \cdot)\|_{H^m(\mathbb{R}^d \times \mathbb{T}^\ell)} \,.$$

The non linear problem can then be solved classically by a simple *Picard* iterative scheme, the convergence following from the estimations of the proposition 3.10. The other profile equations are linear (steps $n \ge 1$), and can be solved by induction using the proposition 3.10 to determine $\mathbb{E}\,\mathcal{U}_n$, and the elliptic inversion of corollary 3.7 to get $(I - \mathbb{E})\,\mathcal{U}_n$.

3.3 Exact oscillating solutions

In this section we are interested in the existence of *exact* oscillating solutions, asymptotic to the formal solutions constructed in the previous section. We assume that

$$\sum_{n\geq 0} \varepsilon^n \, \mathbf{U}_n(t,x,\vec{\boldsymbol{\varphi}}/\varepsilon)$$

is a formal solution on $\omega_T =]0,T[\times\mathbb{R}^d$ given byTheorem 3.4, with $\mathbf{U}_n = (\mathbf{V}_n, \mathbf{W}_n) \in H^\infty(\omega_T \times \mathbb{T}^\ell)$ and $\mathbf{V}_0 = v_0$. We obtain approximate solutions

$$u^\varepsilon_{app} = (v^\varepsilon_{app}, w^\varepsilon_{app}) = (v^0 + \varepsilon V^\varepsilon_{app}, W^\varepsilon_{app})$$

of the system (3.4), choosing

$$\begin{aligned} V^\varepsilon_{app}(t,x) &= \textstyle\sum_{n=1}^M \varepsilon^{n-1} \, \mathbf{V}_n\big(t,x,\vec{\boldsymbol{\varphi}}(t,x)/\varepsilon\big) \, . \\ W^\varepsilon_{app}(t,x) &= \textstyle\sum_{n=0}^M \varepsilon^{n} \, \mathbf{W}_n\big(t,x,\vec{\boldsymbol{\varphi}}(t,x)/\varepsilon\big) \, . \end{aligned} \tag{3.30}$$

They satisfy

$$S(u^\varepsilon_{app})\, \mathbf{X}_{v^\varepsilon_{app}} u^\varepsilon_{app} + \mathbf{L}(v^\varepsilon_{app}, \partial_x) u^\varepsilon_{app} \; - \; \begin{bmatrix} \varepsilon f^\varepsilon \\ g^\varepsilon \end{bmatrix} \; = \; \varepsilon^M \begin{bmatrix} \varepsilon R^\varepsilon_I \\ R^\varepsilon_{II} \end{bmatrix} \tag{3.31}$$

with $R^\varepsilon(t,x) := {}^t(R^\varepsilon_I, R^\varepsilon_{II}) = \mathbf{R}^\varepsilon(t,x,\vec{\boldsymbol{\varphi}}/\varepsilon)$ and profiles $\mathbf{R}^\varepsilon(t,x,\theta)$ bounded in $H^\infty(\omega_T \times \mathbb{T}^\ell)$.

Let us now consider the Cauchy problem for (3.4) with the initial data

$$u^\varepsilon_{|t=0} \; = \; u^\varepsilon_{app|t=0} \, . \tag{3.32}$$

Theorem 3.11. *Define*

$$M_0 \; := \; \min \, \big\{ \, d+2 \, ; \, (d+\ell+2)/2 \, \big\} \, .$$

There exists $\varepsilon_0 > 0$ *small enough such that, if* $M \in \mathbb{N}$ *with* $M > M_0$, *and if* $0 < \varepsilon < \varepsilon_0$, *the Cauchy problem* (3.4) − (3.32) *has a local solution* $u^\varepsilon = (v_0 + \varepsilon V^\varepsilon, W^\varepsilon) \in H^\infty(\omega_T)$. *Moreover, for all* $s > 0$, *the components* V^ε *and* W^ε *satisfy*

$$\| V^\varepsilon - V^\varepsilon_{app} \|_{H^s(\omega_T)} = O(\varepsilon^{M-s}) \, , \qquad \| W^\varepsilon - W^\varepsilon_{app} \|_{H^s(\omega_T)} = O(\varepsilon^{M-s}) \, .$$

When $M_0 = d+2$, the proof is based on estimates in the domain ω_T with suitable weighted norms involving the $\mathbf{X}^k_{v_0} \, (\varepsilon \, \partial_x)^\alpha$ derivatives. The demonstration is in the spirit of the approximation theorem given in [12]. When $M_0 = (d+\ell+2)/2$ the proof relies on a *singular system* approach with Sobolev estimates on the enlarged domain $\omega_T \times \mathbb{T}^\ell$. It is in the spirit of [4] and [17]. It relies on the special structure of the approximate solution and especially on the coherence assumption.

Theorem 3.11 concerns the *Cauchy problem* with oscillatory data. Compatibility conditions on the initial data are necessary to kill the oscillations on the other modes. These compatibility conditions are hidden in the choice of the Cauchy data (3.32). The larger is M, the more compatible are the data and the higher we can take Sobolev index s. When dealing with *continuation* results from the past to the future, one can work under the weaker assumption $M > d/2 + 1$. This is the aim of the next theorem. First introduce the following conditions imposed on the exact solutions $u^\varepsilon = (v_0 + \varepsilon V^\varepsilon, W^\varepsilon)$

$$\begin{aligned} \| \mathbf{X}^k_{v_0} \, \partial^\alpha_x \, (V^\varepsilon - V^\varepsilon_{app}, W^\varepsilon - W^\varepsilon_{app}) \|_{L^2(\omega_t)} \;=\; \bigcirc(\varepsilon^{M-|\alpha|}) \,, \\ \text{for all } (k,\alpha) \in \mathbb{N} \times \mathbb{N}^d \text{ such that } k + |\alpha| \leq M \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} \| \mathbf{X}^k_{v_0} \, \partial^\alpha_x \, (V^\varepsilon - V^\varepsilon_{app}, W^\varepsilon - W^\varepsilon_{app}) \|_{L^\infty(\omega_t)} \;=\; \bigcirc(\varepsilon^{1-|\alpha|}) \,, \\ \text{for all } (k,\alpha) \in \mathbb{N} \times \mathbb{N}^d \text{ such that } k + |\alpha| \leq 1 \,. \end{aligned} \tag{3.34}$$

Theorem 3.12. *Assume $M \in \mathbb{N}$ and $M > d/2 + 1$. Let τ be such that $0 < \tau < T$. Suppose that for all $\varepsilon \in]0,1]$, $u^\varepsilon \in H^\infty(\omega_\tau)$ is an exact solution of (3.4) on ω_τ, of the form $u^\varepsilon = (v_0 + \varepsilon V^\varepsilon, W^\varepsilon)$ where $(V^\varepsilon, W^\varepsilon)$ satisfies (3.33) and (3.34) with $t = \tau$. Then, there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$ the solution u^ε extends as a solution $\tilde u^\varepsilon \in H^\infty(\omega_T)$ of (3.4) on ω_T. Moreover, $\tilde u^\varepsilon = (v_0 + \varepsilon \widetilde{V}^\varepsilon, \widetilde{W}^\varepsilon)$ where $(\widetilde{V}^\varepsilon, \widetilde{W}^\varepsilon)$ satisfies the estimations (3.33) and (3.34) with $t = T$.*

3.4 The example of Euler equations

Consider the entropic Euler equations, for simplicity of notations, in space dimension two. We use the variables $(\mathbf{v}, \mathbf{p}, \mathbf{s})$ as in (2.14). To fix the ideas, we take $v_0 = 0$ and $p_0 = \underline{\mathbf{p}}_0$. Then $\mathbf{X}_{v_0} \equiv \partial_t$ and and we choose a single phase function $\varphi(t,x) \equiv x_1$, which is linear. Then, we have

$$\mathbf{L}\big(v_0, (\xi_1, 0)\big) = \begin{bmatrix} 0 & 0 & \xi_1 & 0 \\ 0 & 0 & 0 & 0 \\ \xi_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \qquad \mathbf{P}\big(v_0, (\xi_1, 0)\big) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ,$$

and

$$\mathcal{L}(t, x, \partial_\theta) \;=\; \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \partial_\theta \,.$$

Moreover, the averaging operator $\mathbb{E}(t,x,\partial_\theta)$ is

$$\mathbb{E}(t,x,\partial_\theta)\begin{bmatrix}\mathbf{V}_1(t,x,\theta)\\ \mathbf{V}_2(t,x,\theta)\\ \mathbf{P}(t,x,\theta)\\ \mathbf{S}(t,x,\theta)\end{bmatrix} = \begin{bmatrix}\langle\mathbf{V}_1\rangle(t,x)\\ \mathbf{V}_2(t,x,\theta)\\ \langle\mathbf{P}\rangle(t,x)\\ \mathbf{S}(t,x,\theta)\end{bmatrix}$$

where the notation $\langle u\rangle$ means the average value in θ of the function $u(t,x,\theta)$

$$\langle u\rangle(t,x) := \frac{1}{2\pi}\int_0^{2\pi} u(t,x,\theta)\;\mathrm{d}\theta\,.$$

The general results of the previous section provide non trivial large amplitude oscillating exact solutions of the system (2.14). By the polarization condition (3.9), they satisfy

$$\begin{aligned}\mathbf{v}_1^\varepsilon(t,x) &= \varepsilon\, V_1(t,x) + O(\varepsilon^2),\\ \mathbf{v}_2^\varepsilon(t,x) &= \varepsilon\, V_2(t,x,x_1/\varepsilon) + O(\varepsilon^2),\\ \mathbf{p}^\varepsilon(t,x) &= \mathbf{p}_0 + \varepsilon\, P(t,x) + O(\varepsilon^2),\\ \mathbf{s}^\varepsilon(t,x) &= S(t,x,x_1/\varepsilon) + O(\varepsilon).\end{aligned} \tag{3.35}$$

The profiles $V_1(t,x)$, $V_2(t,x,\theta)$, $P(t,x)$ and $S(t,x,\theta)$ are given by the quasilinear integro-differential hyperbolic system (3.28):

$$\begin{cases} \langle\rho(\mathbf{p}_0,S)\rangle\,\partial_t V_1 + \partial_1 P = 0\,, & V_{1|t=0} = a_1(x)\,,\\ \rho(\mathbf{p}_0,S)\,(\partial_t V_2 + V_1\,\partial_\theta V_2) + \partial_2 P = 0\,, & V_{2|t=0} = a_2(x,\theta)\,,\\ \langle\alpha(\mathbf{p}_0,S)\rangle\,\partial_t P + \partial_1 V_1 + \langle\partial_2 V_2\rangle = 0\,, & P_{|t=0} = a_3(x)\,,\\ \partial_t S + V_1\partial_\theta S = 0\,, & S_{|t=0} = a_4(x,\theta)\,. \end{cases} \tag{3.36}$$

Theorem 3.9 implies that we can solve locally (3.36) for all data a_1 and a_3 in $H^\infty(\mathbb{R}^2)$ and a_2 and a_4 in $H^\infty(\mathbb{R}^2\times\mathbb{T})$.

3.5 Proof of the theorem 3.11 when $M_0 = d+2$.

We prove a slightly more general result, forgetting the origin of the approximate solutions and the assumptions used for their constructions. We assume that $u^\varepsilon_{app} = (v_0 + \varepsilon V^\varepsilon_{app}, W^\varepsilon_{app})$ satisfies the equation (3.31) with an error term $R^\varepsilon = (R_I^\varepsilon, R_I^\varepsilon I)$ and that $(V_{app}^\varepsilon, W_{app}^\varepsilon, R^\varepsilon)$ satisfy the estimates : for all $k \in \mathbb{N}$ and for all $\beta \in \mathbb{N}^d$, there is $c_{k,\beta}$ such that forall $\varepsilon \in]0,1]$

le de la sol approchee (3.37) $$\|\mathbf{X}_{v_0}^k (\varepsilon\,\partial_x)^\beta (V_{app}^\varepsilon, W_{app}^\varepsilon, R^\varepsilon)\|_{L^2(\omega_T)\cap L^\infty(\omega_T)} \le c_{k,\beta}\,.$$

The space $L^2(\omega_T) \cap L^\infty(\omega_T)$ is equipped with the norm $\|\cdot\|_{L^2} + \|\cdot\|_{L^\infty}$. Since $\mathbf{X}_{v_0}\vec{\boldsymbol{\varphi}} = 0$, these estimate are satisfied by any family $\mathbf{U}^\varepsilon(t,x,\vec{\boldsymbol{\varphi}}/\varepsilon)$, if $\mathbf{U}^\varepsilon$ is bounded in $H^\infty(\Omega \times \mathbb{T}^\ell)$. In particular, the approximate solution defined by (3.30) satisfy (3.37). Thus Theorem 3.11 follows from the next result.

solutionsexactes **Theorem 3.13.** *Let $M > d+2$. Assume that for all $\varepsilon \in]0,1]$, $u_{app}^\varepsilon := (v_0 + \varepsilon V_{app}^\varepsilon, W_{app}^\varepsilon)$ and $R^\varepsilon = (R_I^\varepsilon, R_I^\varepsilon I)$ satisfy (3.31), and the estimate (3.37). Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$, the Cauchy problem (3.4) − (3.32) has a unique solution $u^\varepsilon \in H^\infty(\Omega_T)$ which, for all $s > 0$, satisfies $u^\varepsilon = (v_0 + \varepsilon V^\varepsilon, W^\varepsilon)$ and*

$$\|V^\varepsilon - V_{app}^\varepsilon\|_{H^s(\omega_T)} = O(\varepsilon^{M-s})\,, \qquad \|W^\varepsilon - W_{app}^\varepsilon\|_{H^s(\omega_T)} = O(\varepsilon^{M-s})\,.$$

First, we reformulate the problem in terms of the unknown $U^\varepsilon = (V^\varepsilon, W^\varepsilon)$. With notations as in (2.47) (2.49), the equation for U^ε reads

systrd (3.38) $$\mathcal{H}^\varepsilon(t,x,U^\varepsilon,\partial)U^\varepsilon = h^\varepsilon\,.$$

where $h^\varepsilon = {}^t(f^\varepsilon, g^\varepsilon)$. Define $U_a^\varepsilon = (V_{app}^\varepsilon, W_{app}^\varepsilon)$. It is an approximate solution of the problem in the sense that

mozarella e prosciuto (3.39) $$\mathcal{H}^\varepsilon(t,x,U_a^\varepsilon,\partial)U_a^\varepsilon = h^\varepsilon + \varepsilon^M R^\varepsilon\,.$$

We look for a solution U^ε of the form $U^\varepsilon = U_a^\varepsilon + \varepsilon\,\mathbf{u}_{err}^\varepsilon$. The equation for $\mathbf{u}_{err}^\varepsilon$ reads

systemepourl'erreur (3.40) $$\begin{cases} \mathcal{H}^\varepsilon(t,x,U_a^\varepsilon + \varepsilon\,\mathbf{u}_{err}^\varepsilon,\partial)\mathbf{u}_{err}^\varepsilon + J^\varepsilon(\boldsymbol{a}^\varepsilon, \mathbf{u}_{err}^\varepsilon)\mathbf{u}_{err}^\varepsilon = \varepsilon^{M-1} R^\varepsilon \\ \mathbf{u}_{err|t=0}^\varepsilon = 0 \end{cases}$$

where J is some $N \times N$ matrix with $\mathcal{C}^\infty$ entries (up to $\varepsilon = 0$), and

(3.41) $$\boldsymbol{a}^\varepsilon := \big(t,x,v_0,\partial_x v_0, U_a^\varepsilon, \mathbf{X}_{v_0} U_a^\varepsilon, \varepsilon\,\partial_x U_a^\varepsilon\big)\,.$$

Using the notations

$$\widetilde{\mathcal{H}^\varepsilon}(a^\varepsilon, \mathbf{u}^\varepsilon, \partial) := \mathcal{H}^\varepsilon(t,x,U_a^\varepsilon + \varepsilon\,\mathbf{u}^\varepsilon,\partial) + J^\varepsilon(\boldsymbol{a}^\varepsilon, \mathbf{u}^\varepsilon)$$

the proof of Theorem 3.13 is based on priori estimates for the linear problem

systemeerreurlineaire (3.42) $$\begin{cases} \widetilde{\mathcal{H}^\varepsilon}(a^\varepsilon, \underline{\mathbf{u}}^\varepsilon, \partial)\mathbf{u}^\varepsilon = \varepsilon^{M-1} R^\varepsilon\,, \\ \mathbf{u}_{|t=0}^\varepsilon = 0\,. \end{cases}$$

3.5.1 Weighted norms and anisotropic regularity

anisotropic regularity

Consider the ε dependent vector fields on Ω

$$\mathfrak{X}_{0,\varepsilon} := \mathbf{X}_{v_0}\,, \quad \mathfrak{X}_{1,\varepsilon} := \varepsilon\,\partial_1\,, \quad \cdots \quad , \quad \mathfrak{X}_{d,\varepsilon} := \varepsilon\,\partial_d\,.$$

For all multi-index $\alpha \in \mathbb{N}^{1+d}$, $\mathfrak{X}_\varepsilon^\alpha := \mathfrak{X}_{0,\varepsilon}^{\alpha_0} \cdots \mathfrak{X}_{d,\varepsilon}^{\alpha_d}$. Note that the commutator of two such fields is a linear combination with $\mathcal{C}_b^\infty(\Omega)$ coefficients of the $\mathfrak{X}_{j,\varepsilon}$. This property is due to the fact that the coefficient of ∂_t in $\mathbf{X}_{v_0}$ is constant. There holds:

$$[\mathfrak{X}_{i,\varepsilon};\mathfrak{X}_{j,\varepsilon}] \;=\; \mathfrak{X}_{i,\varepsilon}\circ\mathfrak{X}_{j,\varepsilon} - \mathfrak{X}_{j,\varepsilon}\circ\mathfrak{X}_{i,\varepsilon} \;=\; \sum_{0\le k\le d} a_k\,\mathfrak{X}_{k,\varepsilon}\,, \qquad a_k \in \mathcal{C}_b^\infty(\Omega)\,.$$

For $\lambda > 0$ we define the weighted norms $\|u\|_{0,\lambda} := \|e^{-\lambda t}\,u\|_{L^2(\omega_T)}$ and for $\varepsilon \in\,]0,1]$ and $m \in \mathbb{N}$

$$\|u\|_{m,\lambda,\varepsilon} := \sum_{|\alpha|\le m} \lambda^{m-|\alpha|}\;\|\mathfrak{X}_\varepsilon^\alpha\,u\|_{0,\lambda}$$

and

$$|u|_{*,\varepsilon} := \sum_{|\alpha|\le 1} \|\mathfrak{X}_\varepsilon^\alpha\,u\|_{L^\infty(\omega_T)}\,. \tag{3.43}$$

To estimate the traces on $t = 0$ we also use the following norms

$$|v|_{m,\lambda,\varepsilon} := \sum_{|\alpha|\le m} \lambda^{m-|\alpha|}\;\|(\varepsilon\,\partial_x)^\alpha\,v\|_{L^2(\mathbb{R}^d)}\,.$$

We use the following Gagliardo-Nirenberg estimates.

Lemma 3.14. *Let $m \in \mathbb{N}_*$. There is $c_m > 0$ such that for all $u \in L^\infty(\Omega_T) \cap H^m(\omega_T)$, for all $\varepsilon \in\,]0,1]$ and for all $\alpha \in \mathbb{N}^{1+d}$ such that $|\alpha| \le m$*

$$\|e^{-\lambda t}\,\mathfrak{X}_\varepsilon^\alpha\,u\|_{L^{2m/|\alpha|}(\omega_T)} \;\le\; c_m\;\|u\|_{L^\infty(\Omega_T)}^{1-|\alpha|/m}\;\|u\|_{m,\lambda,\varepsilon}^{|\alpha|/m}\,. \tag{3.44}$$

Proof. It is a special case of inequality (Ap-II-3) given in [10], p. 643. □

This implies the following Moser's type inequality.

MoserI **Lemma 3.15.** *Let m an integer. There is $c_m > 0$ such that for all $\varepsilon \in\,]0,1]$ and for all functions $a_1, \cdots, a_p$ in $H^m(\omega_T) \cap L^\infty(\omega_T)$*

$$\lambda^{m-k}\;\|\mathfrak{X}_\varepsilon^\alpha\,a\|_{0,\lambda,\varepsilon} \;\le\; c_m\;\sum_k\;\Big(\prod_{j\ne k}\|a_j\|_{L^\infty(\Omega_T)}\Big)\;\|a_k\|_{m,\lambda,\varepsilon} \tag{3.45}$$

where $\alpha = (\alpha_1, \cdots, \alpha_p) \in \mathbb{N}^p$ and

$$\mathfrak{X}_\varepsilon^\alpha\,a \;:=\; \mathfrak{X}_\varepsilon^{\alpha_1}a_1 \times \cdots \times \mathfrak{X}_\varepsilon^{\alpha_p}a_p\,, \qquad |\alpha_1| + \cdots + |\alpha_p| \le k \le m\,.$$

lemme Sobolev I **Lemma 3.16.** *For all $m > d/2+1$ there is $c_{sob} > 0$ such that for all $\varepsilon \in]0,1]$ and for all $u \in H^m(\omega_T)$*

plongement de Sobolev
$$|u|_{*,\varepsilon} \leq c_{sob}\, \varepsilon^{-d/2}\, e^{\lambda T}\, \|u\|_{m,\lambda,\varepsilon}\,, \qquad \forall\, \lambda \geq 1\,. \tag{3.46}$$

Proof. It is a consequence of the usual Sobolev embedding applied to the function $x \mapsto u(t, \varepsilon\, x)$. □

3.5.2 Traces estimates on the exact solution

For all fixed $\varepsilon \in]0,1]$, there is $T^\varepsilon > 0$ such that to the Cauchy problem (3.38) has a solution $U^\varepsilon \in H^\infty(\omega_{T^\varepsilon})$. In particular, the terms $(\mathbf{X}_{v_0}^k\, U^\varepsilon)_{|t=0}$ are well defined in $H^\infty(\mathbb{R}^d)$. Our main objective is to show that $T^\varepsilon = T$. We start the analysis by looking at the traces of U^ε at $t = 0$.

ilfallaitleno **Lemma 3.17.** *For all $m \in \mathbb{N}$, there is an increasing function $p_m : \mathbb{R}^+ \longmapsto \mathbb{R}^+_*$ such that for all $\varepsilon \in]0,1]$, all k and m' such that $k + m' \leq m$ and for all $\lambda \in [1, +\infty[$, the following estimate holds:*

estimtraces
$$\big|\big(\mathbf{X}_{v_0}^k\, (U^\varepsilon - U_a^\varepsilon)\big)_{|t=0}\big|_{m',\lambda,\varepsilon} \leq p_m(\lambda)\; \varepsilon^{M-k+1}. \tag{3.47}$$

Proof. By construction, we have $(U^\varepsilon - U_a^\varepsilon)_{|t=0} \equiv 0$ thus (3.47) is trivial for $m = 0$. We prove (3.47) for $k = 1$. The equation (3.38) on U^ε can be formulated as

$$\mathbf{X}_{v_0} U^\varepsilon \;=\; \mathcal{G}\big(\varepsilon, t, x, (\partial_x^\alpha U^\varepsilon)_{|\alpha|\leq 1}\big) \;+\; \mathbf{T}(\varepsilon, t, x, U^\varepsilon) h^\varepsilon$$

where $\mathcal{G}$ is a smooth function of its arguments. It implies that

$$\big(\mathbf{X}_{v_0} U^\varepsilon\big)_{|t=0} = \mathcal{G}\big(\varepsilon, 0, x, (\partial_x^\alpha U_a^\varepsilon)_{|\alpha|\leq 1}\big)_{|t=0} + \mathbf{T}(\varepsilon, 0, x, U_a^\varepsilon)(h^\varepsilon)_{|t=0}$$

Since U_a^ε satisfies the equation (3.39) which only differs by the additional source term $\varepsilon^M R^\varepsilon$, we have

$$\big(\mathbf{X}_{v_0}\,(U^\varepsilon - U_a^\varepsilon)\big)_{|t=0} \;=\; \varepsilon^M\; \mathbf{T}(\varepsilon, 0, x, U_a^\varepsilon)(R^\varepsilon)_{|t=0}\,.$$

Estimating the L^2-norm of the right hand side, as well as the L^2-norm of its $\varepsilon\,\partial_x-$derivative, implies (3.47) when $k = 1$.

The other cases $k > 1$ are proved by induction. applying $\mathbf{X}_{v_0}^{k-1}$ to the equations above, causing a loss of ε^{-k+1} in the right hand side. □

3.5.3 A priori estimates for the linear problem

In this paragraph we prove the following result.

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Proposition 3.18. *For all $m \in \mathbb{N}$, there is $\lambda_m \geq 1$ and there is a positive function $C_m : \mathbb{R}^+ \longmapsto \mathbb{R}^+_*$ such that the following holds. For all $\varepsilon \in]0,1]$, for all $\lambda \geq \lambda_m$, and for all functions $\mathbf{u}$ and $\underline{\mathbf{u}}$ which belong to $H^m(\omega_T) \cap Lip(\omega_T)$ and satisfy the system (3.42), one has*

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(3.48)

$$\|\mathbf{u}\|_{m,\lambda,\varepsilon} \leq \frac{C_m(|\underline{\mathbf{u}}|_{*,\varepsilon})}{\lambda^{1/2}} \Big(\|\mathbf{u}\|_{m,\lambda,\varepsilon} + \varepsilon^{-d/2}\, e^{\lambda T}\, \|\underline{\mathbf{u}}\|_{m,\lambda,\varepsilon}\, \|\mathbf{u}\|_{m,\lambda,\varepsilon}$$

$$+\, \varepsilon^{M-1}\, \|R^\varepsilon\|_{m,\lambda,\varepsilon} + \textstyle\sum_{k+m' \leq m} \lambda^{m-k-m'} |\, (\mathbf{X}^k_{v_0}\mathbf{u})_{|t=0}|_{m',\lambda,\varepsilon} \Big)\,.$$

Proof. The proof is in several steps.

• **step 1: the L^2 estimate.** Expanding by the Taylor formula the coefficients of the operator $\widetilde{\mathcal{H}^\varepsilon}(a^\varepsilon, \underline{\mathbf{u}}^\varepsilon, \partial)$, we get

$$\widetilde{\mathcal{H}^\varepsilon}(a^\varepsilon, \underline{\mathbf{u}}^\varepsilon, \partial) \equiv \sum_{0 \leq j \leq d} \mathbf{S}^\varepsilon_j(\boldsymbol{a}^\varepsilon, \underline{\mathbf{u}}^\varepsilon)\, \mathfrak{X}_{j,\varepsilon} + \mathbf{L}(v_0, \partial_x) + \tilde{\mathbf{J}}^\varepsilon(\boldsymbol{a}^\varepsilon, \underline{\mathbf{u}}^\varepsilon) \tag{3.49}$$

where the matrices $\mathbf{S}^\varepsilon_j$ and $\tilde{\mathbf{J}}^\varepsilon$ are $\mathcal{C}^\infty$ functions of their arguments ε, $\boldsymbol{a}$ and $\underline{\mathbf{u}}$. Moreover, the $\mathbf{S}^\varepsilon_j$ are symmetric, with $\mathbf{S}^\varepsilon_0$ positive definite. Introduce the new unknown $\tilde{\mathbf{u}} := e^{-\lambda t}\, \mathbf{u}$ which satisfies

equationpourutilde

$$\sum_{0 \leq j \leq d} \mathbf{S}^\varepsilon_j(\boldsymbol{a}^\varepsilon, \underline{\mathbf{u}}^\varepsilon)\, \mathfrak{X}_{j,\varepsilon}\tilde{\mathbf{u}} + \mathbf{L}(v_0, \partial_x)\tilde{\mathbf{u}} + \lambda\, \mathbf{S}^\varepsilon_0(\boldsymbol{a}^\varepsilon, \underline{\mathbf{u}}^\varepsilon)\tilde{\mathbf{u}} + \tilde{\mathbf{J}}^\varepsilon(\boldsymbol{a}^\varepsilon, \underline{\mathbf{u}}^\varepsilon)\tilde{\mathbf{u}} = \varepsilon^{M-1}\, e^{-\lambda t}\, R^\varepsilon\,. \tag{3.50}$$

Forming the product of $\tilde{\mathbf{u}}$ with the equation (3.50), integrating by parts on ω_T and using the symmetry of the matrices $\mathbf{S}_j$ and $\mathbf{L}_j$, we obtain the following inequality, exact for $\lambda \geq \lambda_0$ and λ_0 big enough

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$$\|\mathbf{u}\|_{0,\lambda,\varepsilon} \leq \frac{C_0(|\underline{\mathbf{u}}|_{*,\varepsilon})}{\lambda^{1/2}} \big(\|\mathbf{u}\|_{0,\lambda,\varepsilon} + |\mathbf{u}(0)|_{0,\lambda,\varepsilon} + \varepsilon^{M-1}\, \|R^\varepsilon\|_{0,\lambda,\varepsilon} \big)\,. \tag{3.51}$$

• **step 2: end of the proof when v_0 is constant.** It is interesting to treat the special case v_0 *is constant* because the proof is simpler, involving however some commutator estimates that will be useful in the general situation. So, suppose that v_0 is a constant vector in $\mathbb{R}^N$. We want to estimate the higher derivatives of $\mathbf{u}$, i.e. the $\|\mathfrak{X}_\varepsilon^\alpha\,\mathbf{u}\|_{0,\lambda,\varepsilon}$ for $|\alpha|\le m$. As usual, we compose (3.50) on the left with $\mathfrak{X}_\varepsilon^\alpha$ and we perform energy estimates

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$$\begin{aligned}\lambda^{m-|\alpha|}\ \|\mathfrak{X}_\varepsilon^\alpha\,\mathbf{u}\|_{0,\lambda,\varepsilon}\ \le\ &\frac{C_0(|\underline{\mathbf{u}}|_{*,\varepsilon})}{\lambda^{1/2}}\\ &\times\big(\lambda^{m-|\alpha|}\ \|\,[\,\widetilde{\mathcal{H}}^\varepsilon(\boldsymbol{a}^\varepsilon,\underline{\mathbf{u}},\partial);\mathfrak{X}_\varepsilon^\alpha\,]\,\mathbf{u}\,\|_{0,\lambda,\varepsilon}\\ &\quad+\|\mathbf{u}\|_{m,\lambda,\varepsilon}+\lambda^{m-|\alpha|}\ |\mathfrak{X}_\varepsilon^\alpha\,\mathbf{u}(0)|_{0,\lambda,\varepsilon}+\varepsilon^{M-1}\,\|R^\varepsilon\|_{m,\lambda,\varepsilon}\big)\,.\end{aligned}\tag{3.52}$$

We are lead to estimate the commutator in the right hand side of (3.52).

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Lemma 3.19. *Let $\alpha\in\mathbb{N}^{1+d}$ such that $|\alpha|\le m$. Suppose that v_0 is constant. Then for all $\varepsilon\in]0,1]$, for all $\lambda\in[1,+\infty[$, and for all $\underline{\mathbf{u}}$ and $\mathbf{u}$ in $H^\infty(\omega_T)$, one has*

mmutateur cas constant

$$\begin{aligned}\lambda^{m-|\alpha|}\ \|\,[\,\widetilde{\mathcal{H}}^\varepsilon(\boldsymbol{a}^\varepsilon,\underline{\mathbf{u}},\partial);\,\mathfrak{X}_\varepsilon^\alpha\,]\,\mathbf{u}\,\|_{0,\lambda,\varepsilon}\ \le&\\ c(|\underline{\mathbf{u}}|_{*,\varepsilon})\ &\big(\|\mathbf{u}\|_{m,\lambda,\varepsilon}+|\mathbf{u}|_{*,\varepsilon}\ \|\underline{\mathbf{u}}\|_{m,\lambda,\varepsilon}\big)\,.\end{aligned}\tag{3.53}$$

Proof. Since v_0 is constant

$$[\,\mathbf{L}(v_0,\partial_x);\mathfrak{X}_\varepsilon^\alpha\,]=0\,,\qquad\forall\,\alpha\in\mathbb{N}^{1+d}\,.\tag{3.54}$$

Thus the commutator that we want to estimate writes

$$\sum\,[\,\mathbf{S}_j^\varepsilon(\boldsymbol{a}^\varepsilon,\underline{\mathbf{u}})\,\mathfrak{X}_{j,\varepsilon};\mathfrak{X}_\varepsilon^\alpha\,]\,+\,\lambda\,[\,\mathbf{S}_0^\varepsilon(\boldsymbol{a}^\varepsilon,\underline{\mathbf{u}});\mathfrak{X}_\varepsilon^\alpha\,]\,+\,[\,\tilde{\mathbf{J}}^\varepsilon(\boldsymbol{a}^\varepsilon,\underline{\mathbf{u}});\mathfrak{X}_\varepsilon^\alpha\,]\,.$$

With this simplification, the estimate (3.53) is a classical estimate for commutator and follows from Moser's estimates of Lemma 3.15. ☐

Using estimate (3.53) in (3.52) and Lemma 3.16 to control $|\mathbf{u}|_{*,\varepsilon}$, we get the inequality (3.48) and the proposition is proved in this special case.

• **step 3: reduction to the case where the field $\mathfrak{X}_{0,\varepsilon}$ is constant.** We perform a change of variables which reduces $\mathbf{X}_{v_0}$ to ∂_t. Since $\mathbf{X}_{v_0}v_0=0$, v_0 is constant along the the integral curves of the field and the integral curve $s\mapsto\big(s,\gamma(s;t,x)\big)$ issued from (t,x) at time $s=t$ is

$$\gamma(s;t,x)\ =\ \big(\,t,x+(s-t)\,\mu(v_0(t,x))\,\big).$$

We perform the change of variables $(t,x)\mapsto\Phi(t,x):=\big(\,t,\gamma(0;t,x)\,\big)$ with

$$\Phi(t,x)=\big(\,t,x-t\,\mu(v_0(t,x))\,\big)\,,\qquad\Phi^{-1}(t,x)=\big(\,t,x+t\,\mu(v_0(0,x))\,\big)\,.$$

This is a $\mathcal{C}^\infty$ diffeomorphism from Ω and $D\Phi$ and $D\Phi^{-1}$ belong to $\mathcal{C}_b^\infty(\Omega)$.

In the new coordinates

$$(t', x') := \Phi(t,x) = \big(t, \Phi_1(t,x), \cdots, \Phi_d(t,x)\big)$$

the functions $f(t,x)$ become $f'(t',x')$ with f' is defined by $f(t,x) = f'\big(\Phi(t,x)\big)$. The system (3.42) is transformed to

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$$\mathbf{S}_0'(\boldsymbol{a}', \underline{\mathbf{u}}')\, \partial_{t'} \mathbf{u}' + \sum_{j=1}^{d} \mathbf{S}_j'(\boldsymbol{a}', \underline{\mathbf{u}}')\, \varepsilon\, \partial_{x_j'} \mathbf{u}' + \mathfrak{L}(t', x', \partial_{x'}) \mathbf{u}' + \mathbf{J}'(\boldsymbol{a}', \underline{\mathbf{u}}') \mathbf{u}' = \varepsilon^{M-1} (R^\varepsilon)' \tag{3.55}$$

where

$$\mathfrak{L}(t', x', \partial_{x'}) = \sum_{j=1}^{d} \mathfrak{L}_j\ \partial_{x_j'}\,, \qquad \mathfrak{L}_j := \sum_{k=1}^{d} L_k'\ (\partial_{x_k} \Phi_j) \circ \Phi^{-1}\,. \tag{3.56}$$

$\mathfrak{L}(t', x', \partial_{x'})$ is the differential operator with symbol

$$\mathfrak{L}(t', x', i\,\xi) = i\ \mathbf{L}\big(\, v_0'\,,\ (\xi \cdot \partial_x \Phi) \circ \Phi^{-1} \,\big)\,. \tag{3.57}$$

The new system (3.55) obtained in the new coordinates (t', x') has the same structure as (3.42), with $\mathfrak{L}(t', x', \partial_{x'})$ in place of $\mathbf{L}(v_0, \partial_x)$ and $\partial_{t'}$ in place of $\mathfrak{X}_{0,\varepsilon}$. It is equivalent to prove the estimates for $\mathbf{u}$ or for $\mathbf{u}'$. Thus, dropping the $'$, it is sufficient to prove a priori estimates for the solutions of

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$$\sum_{0 \le j \le d} \mathbf{S}_j(\boldsymbol{a}, \underline{\mathbf{u}})\, \mathfrak{X}_{j,\varepsilon}\, \mathbf{u} + \mathfrak{L}(t, x, \partial_x) \mathbf{u} + \mathbf{J}(\boldsymbol{a}, \underline{\mathbf{u}}) \mathbf{u} = \varepsilon^{M-1} R^\varepsilon \tag{3.58}$$

when $\mathfrak{X}_{0,\varepsilon} \equiv \partial_t$.

• **step 4: reduction to the case where $\mathfrak{L}(t,x,\partial_x)$ has constant coefficients.** Since Φ is a diffeomorphism

$$\xi \cdot \partial_x \Phi^{-1}(t,x) \neq 0\,, \qquad \forall\, (t,x,\xi) \in \Omega \times (\mathbb{R}^d \setminus \{0\})\,. \tag{3.59}$$

It follows that the matrix $\mathfrak{L}(t,x,\xi)$ has a *constant rank* on $\Omega \times (\mathbb{R}^d \setminus \{0\})$. Hence there is a $N \times N$ matrix $\boldsymbol{\Psi}(t,x,\xi)$ with real entries in $\mathcal{C}_b^\infty(\Omega \times \mathbb{R}^d)$, which is invertible for all $(t,x,\xi) \in \Omega \times \mathbb{R}^d$, which is homogenous of degree 0 in ξ for $|\xi|$ large enough (say for $|\xi| > r \gg 0$), and which is such that for all $(t,x,\xi) \in \omega_T \times \mathbb{R}^d$

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$${}^t\boldsymbol{\Psi}(t,x,\xi)\, \mathfrak{L}(t,x,\xi)\, \boldsymbol{\Psi}(t,x,\xi) = \mathfrak{p}(\xi)\ \underline{\mathbf{L}} + \rho(t,x,\xi) \tag{3.60}$$

where $\underline{\mathbf{L}}$ is a constant $N \times N$ real symmetric matrix, the function $\mathfrak{p}(\cdot)$ belongs to $\mathcal{C}_b^\infty(\mathbb{R}^d;\mathbb{R})$ and is homogenous of degree 1 in ξ for $|\xi| > r$, and $\rho(\cdot)$ is a $N \times N$ symmetric matrix whose coefficients are in $\mathcal{C}_b^\infty(\omega_T \times \mathbb{R}^d;\mathbb{R})$ and are supported in $\{|\xi| \leq r'\}$ for some $r' > 0$.

The functions $\Psi(t,x,\xi)$ and $\mathfrak{p}(\xi)$ are symbols of pseudo-differential operators on $\mathbb{R}^d$, depending in a $\mathcal{C}_b^\infty$ way on the parameter $t \in]-T_0, T_0[$. The symbol $\Psi(t,x,\xi)$ is of order 0, $\mathfrak{p}(\xi)$ of order 1 and $\rho(t,x,\xi)$ is a symbol of order $-\infty$. For the corresponding operators, the relation (3.60) implies that

$$\boldsymbol{\Psi}(t,x,D_x)^*\, \mathfrak{L}(t,x,D_x)\, \boldsymbol{\Psi}(t,x,D_x) = \underline{\mathbf{L}}\, \mathfrak{p}(D_x) \,+\, \mathfrak{r}(t,x,D_x) \tag{3.61}$$

where we use the notation $D_x \equiv -i\partial_x$, $\boldsymbol{\Psi}(t,x,D_x)^*$ denotes the adjoint of the operateur op$(\boldsymbol{\Psi})$ and $\mathfrak{r}$ is a pseudo-differential operator of order 0, depending smoothly on the parameter t). Moreover, $\boldsymbol{\Psi}(t,x,D_x)^*$ is also a pseudo-differential operator of order 0; we denote by $\boldsymbol{\Psi}'(t,x,\xi)$ its symbol, so that $\boldsymbol{\Psi}(t,x,D_x)^* = \boldsymbol{\Psi}'(t,x,\partial_x)$. Let us emphasize the fact that, by construction, there exists a constant $c > 0$ such that

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$$0 \,<\, c \,<\, |\det \boldsymbol{\Psi}(t,x,\xi)\,| \,<\, c^{-1}\,, \qquad \forall\,(t,x,\xi) \in \Omega \times \mathbb{R}^d. \tag{3.62}$$

Denote by $\mathfrak{H}^\varepsilon(\boldsymbol{a}^\varepsilon, \underline{\mathbf{u}}, \mathfrak{X})$ the operator defined by

$$\mathfrak{H}^\varepsilon(\boldsymbol{a}^\varepsilon, \underline{\mathbf{u}}, \mathfrak{X}) \equiv \sum_{0 \leq j \leq d} \mathbf{S}_j(\boldsymbol{a}^\varepsilon, \underline{\mathbf{u}})\, \mathfrak{X}_{j,\varepsilon} + \mathbf{J}(\boldsymbol{a}^\varepsilon, \underline{\mathbf{u}})\,. \tag{3.63}$$

We still note $\mathbf{u}$ the unknown of the system (3.58) obtained at the step 3. The idea is to introduce a new function $\mathfrak{u} \in H^\infty(\Omega)$ which is defined by $\mathbf{u} = \boldsymbol{\Psi}(t,x,D_x)\mathfrak{u}$ and which satisfy the system

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$$\begin{aligned} \mathrm{op}(\boldsymbol{\Psi}')\, \mathfrak{H}^\varepsilon(\boldsymbol{a}^\varepsilon, \underline{\mathbf{u}}, \mathfrak{X})\, \mathrm{op}(\boldsymbol{\Psi})\mathfrak{u} \,+ i\, \underline{\mathbf{L}}\, \mathrm{op}(\mathfrak{p})\mathfrak{u} \,&= \\ -\, \mathrm{op}(\mathfrak{r})\mathfrak{u} \,&+\, \varepsilon^{M-1}\, \mathrm{op}(\boldsymbol{\Psi}')R^\varepsilon \end{aligned} \tag{3.64}$$

and the condition $\mathfrak{u}_{|t=0} = 0$.

lemme choix de Psi

Lemma 3.20. *For all $m \in \mathbb{N}$, one can choose the function $\boldsymbol{\Psi}(t,x,\xi)$ such that $\boldsymbol{\Psi}(t,x,\partial_x)$ is an isomorphism from $H^m(\omega_T)$ to $H^m(\omega_T)$.*

Proof. Let $\boldsymbol{\Psi}(t,x,\xi)$ satisfy (3.60) and (3.62). In general, the corresponding operator $\boldsymbol{\Psi}(t,x,\partial_x)$ is not invertible in $L^2(\omega_T)$ (or in $H^m(\omega_T)$ for a given m). But, with $\boldsymbol{\Psi}_\delta(t,x,\xi) := \boldsymbol{\Psi}(t,x,\delta\,\xi)$, (3.62) implies that $\boldsymbol{\Psi}_\delta(t,x\partial_x)$ is an isomorphism of $H^m(\omega_T)$ for $\delta > 0$ small enough, (see for example [1], exercise 5.14, p.75). Since $\mathcal{L}$ is homogenous of degree 1 in ξ, the following relation holds

$$
{}^t\mathbf{\Psi}_\delta(t,x,\xi)\,\mathfrak{L}(t,x,\xi)\,\mathbf{\Psi}_\delta(t,x,\xi) \;=\; \delta^{-1}\,\mathfrak{p}(\delta\,\xi)\,\underline{\mathbf{L}} + \delta^{-1}\,\rho(t,x,\delta\,\xi) \tag{3.65}
$$

and this shows that the function $\mathbf{\Psi}_\delta$ solves the question. □

From now on, we fix a function $\mathbf{\Psi}$ such that the conclusion of Lemma 3.20 holds.

Let $\mathfrak{v} \in H^\infty(\Omega_T)$ such that $\mathfrak{v}_{|t=0} = 0$ and

$$
\mathfrak{f} := \operatorname{op}(\mathbf{\Psi})^*\,\mathfrak{H}^\varepsilon(\boldsymbol{a}^\varepsilon, \underline{\mathbf{u}}, \mathfrak{X})\,\operatorname{op}(\mathbf{\Psi})\mathfrak{v} \;+\; i\,\underline{\mathbf{L}}\operatorname{op}(\mathfrak{p})\mathfrak{v}\,.
$$

If we note $\mathbf{v} = \operatorname{op}(\mathbf{\Psi})\mathfrak{v}$, the symbolic calculus shows that

$$
\mathfrak{v} = \operatorname{op}(\mathbf{\Psi}^{-1})\mathbf{v} + \operatorname{op}_{-1}\mathbf{v} \tag{3.66}
$$

where op_{-1} is a pseudo-differential operator of degree -1. Applying the energy estimate of the step 1 to $\mathbf{v} = \operatorname{op}(\mathbf{\Psi})\mathfrak{v}$ and using the relation (3.66), we deduce that $\mathfrak{v}$ satisfies

$$
\|\mathfrak{v}\|_{0,\lambda,\varepsilon} \;\le\; \frac{C_0(|\underline{\mathbf{u}}|_{*,\varepsilon})}{\lambda^{1/2}}\,\big(\,\|\mathfrak{v}\|_{0,\lambda,\varepsilon} + |\mathfrak{v}(0)|_{0,\lambda,\varepsilon} + \|\mathfrak{f}\|_{0,\lambda,\varepsilon}\,\big) \tag{3.67}
$$

for all $\lambda \ge \lambda_0$, λ_0 being fixed large enough, and for some positive increasing function $C_0(\cdot)$.

In order to estimate the derivatives of $\mathfrak{v}$ we apply the energy estimate (3.67) to $\mathfrak{X}_\varepsilon^\alpha\,\mathfrak{v}$. Since by construction

$$
[\,\operatorname{op}(\mathfrak{p}); \mathfrak{X}_\varepsilon^\alpha\,] \;=\; 0\,, \tag{3.68}
$$

we get

$$
\begin{aligned}
\lambda^{m-|\alpha|}\;\|\mathfrak{X}_\varepsilon^\alpha\mathfrak{v}\|_{0,\lambda,\varepsilon} \;\le\; & C_0(|\underline{\mathbf{u}}|_{*,\varepsilon})\;\lambda^{-1/2} \\
& \times\,\big(\,\|\mathfrak{v}\|_{m,\lambda\varepsilon} + \lambda^{m-|\alpha|}\;|\mathfrak{X}_\varepsilon^\alpha\mathfrak{v}(0)|_{0,\lambda,\varepsilon} + \|\mathfrak{f}\|_{m,\lambda,\varepsilon} \\
& +\;\lambda^{m-|\alpha|}\;\|\;[\,\operatorname{op}(\mathbf{\Psi}')\,\mathfrak{H}^\varepsilon\operatorname{op}(\mathbf{\Psi})\,;\,\mathfrak{X}_\varepsilon^\alpha\,]\mathfrak{v}\;\|_{0,\lambda,\varepsilon}\,\big)\,.
\end{aligned} \tag{3.69}
$$

Lemma 3.21. *Let $\alpha \in \mathbb{N}^{1+d}$ such that $|\alpha| \le m$. Then*

$$
\begin{aligned}
&\|\;[\,\operatorname{op}(\mathbf{\Psi}')\,\mathfrak{H}^\varepsilon\operatorname{op}(\mathbf{\Psi})\,;\,\mathfrak{X}_\varepsilon^\alpha\,]\;\mathfrak{v}\;\|_{0,\lambda,\varepsilon} \;\le \\
&\qquad C(|\underline{\mathbf{u}}|_{*,\varepsilon})\;\big(\,\|\mathfrak{v}\|_{m,\lambda,\varepsilon} + |\operatorname{op}(\mathbf{\Psi})\mathfrak{v}|_{*,\varepsilon}\,\|\underline{\mathbf{u}}\|_{m,\lambda,\varepsilon}\,\big)\,.
\end{aligned} \tag{3.70}
$$

Proof. The commutator writes as the sum

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$$\operatorname{op}(\mathbf{\Psi}')\,[\mathfrak{H}^\varepsilon;\mathfrak{X}_\varepsilon^\alpha]\,\operatorname{op}(\mathbf{\Psi})\mathfrak{v} + \operatorname{op}(\mathbf{\Psi}')\,\mathfrak{H}^\varepsilon\,[\operatorname{op}(\mathbf{\Psi});\mathfrak{X}_\varepsilon^\alpha]\mathfrak{v} + [\operatorname{op}(\mathbf{\Psi}');\mathfrak{X}_\varepsilon^\alpha]\,\mathfrak{H}^\varepsilon\,\operatorname{op}(\mathbf{\Psi})\mathfrak{v}\,. \tag{3.71}$$

1) The first term satisfies

$$\|\operatorname{op}(\mathbf{\Psi}')\,[\mathfrak{H}^\varepsilon;\mathfrak{X}_\varepsilon^\alpha]\,\operatorname{op}(\mathbf{\Psi})\mathfrak{v}\|_{o,\lambda,\varepsilon} \le c\,\|[\mathfrak{H}^\varepsilon;\mathfrak{X}_\varepsilon^\alpha]\,\operatorname{op}(\mathbf{\Psi})\mathfrak{v}\|_{0,\lambda,\varepsilon}.$$

The term on the right can be treated as was the commutator $[\widetilde{\mathcal{H}}^\varepsilon;\mathfrak{X}_\varepsilon^\alpha]\mathbf{u}$ in step 2. This implies that it is controlled by the right hand side of (3.70).

2) The second term satisfies

$$\begin{aligned}\|\operatorname{op}(\mathbf{\Psi}')\,\mathfrak{H}^\varepsilon\,[\operatorname{op}(\mathbf{\Psi});\mathfrak{X}_\varepsilon^\alpha]\mathfrak{v}\|_{0,\lambda,\varepsilon} &\le c\,\|\mathfrak{H}^\varepsilon\,[\operatorname{op}(\mathbf{\Psi});\mathfrak{X}_\varepsilon^\alpha]\mathfrak{v}\|_{0,\lambda,\varepsilon}\\ &\le C(|\underline{\mathbf{u}}^\varepsilon|_{*,\varepsilon})\sum_{0\le j\le d}\|\mathfrak{X}_{j,\varepsilon}\,[\operatorname{op}(\mathbf{\Psi});\mathfrak{X}_\varepsilon^\alpha]\mathfrak{v}\|_{0,\lambda,\varepsilon}.\end{aligned} \tag{3.72}$$

Using the notation $\mathfrak{X}_\varepsilon^\alpha = \partial_t^{\alpha_0}(\varepsilon\partial_x)^\beta$, the commutator $[\operatorname{op}(\mathbf{\Psi});\mathfrak{X}_\varepsilon^\alpha]$ writes as a sum

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$$\sum_{0\le j\le\alpha_0}\varepsilon^{|\beta|}\,\mathbf{\Psi}_j(t,x,D_x)\,\partial_t^j\mathfrak{v} \tag{3.73}$$

where the $\mathbf{\Psi}_j(t,x,D_x)$ are pseudo-differential operators on $\mathbb{R}^d$, depending as before on the parameter t, of order p_j such that $p_j \le |\beta|-1$. It follows that for $\Xi^\varepsilon = h^\varepsilon$ or $\Xi^\varepsilon = \mathfrak{X}_{j,\varepsilon}$

$$\lambda^{m-|\alpha|}\,\|\operatorname{op}(\mathbf{\Psi}')\,\Xi^\varepsilon\,[\operatorname{op}(\mathbf{\Psi});\mathfrak{X}_\varepsilon^\alpha]\mathfrak{v}\|_{0,\lambda,\varepsilon} \le C(|\underline{\mathbf{u}}^\varepsilon|_{L^\infty(\omega_T)})\,\|\mathfrak{v}\|_{m,\lambda,\varepsilon}\,. \tag{3.74}$$

This quantity is obviously dominated by the right hand side of (3.70).

3) The third term is similar to the second. The commutator $[\operatorname{op}(\mathbf{\Psi}'),\mathfrak{X}_\varepsilon^\alpha]$ is a sum (3.73) and it is sufficient to estimate terms of the form

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$$\lambda^{m-|\alpha|}\,\|\varepsilon^{|\beta|}\,\mathbf{\Psi}_j(t,x;D_x)\,\partial_t^j\mathfrak{H}^\varepsilon\,\operatorname{op}(\mathbf{\Psi})\,\mathfrak{v}\|_{0,\lambda,\varepsilon}\,. \tag{3.75}$$

Each of these terms (3.75) is controlled by

$$\|\,\mathfrak{H}^\varepsilon\,\operatorname{op}(\mathbf{\Psi})\mathfrak{v}\,\|_{|\alpha|-1,\lambda,\varepsilon} \le \|\,\mathfrak{H}^\varepsilon\,\operatorname{op}(\mathbf{\Psi})\mathfrak{v}\,\|_{m-1,\lambda,\varepsilon}$$

which can be estimated using again Gagliardo-Nirenberg estimates as in step 2. The lemma is proved. □

Using lemma 3.16, we replace the term $|\operatorname{op}(\mathbf{\Psi})\mathfrak{v}|_{*,\varepsilon}$ in the right hand side of (3.70) by

$$c\,\varepsilon^{-d/2}\,e^{\lambda T}\,\|\operatorname{op}(\mathbf{\Psi})\mathfrak{v}\|_{m,\lambda,\varepsilon} \le c'\,\varepsilon^{-d/2}\,e^{\lambda T}\,\|\mathfrak{v}\|_{m,\lambda,\varepsilon}.$$

Pluging the result in the right hand side of (3.69), we obtain the claimed estimation. The proposition 3.18 is proved. □

3.5.4 End of the proof of theorem 3.13

the proof of theorem I

Let $\underline{\mathbf{u}}^\varepsilon \in H^\infty(\omega_T)$ such that

$$\exists\, \tau \in]0,T]\,; \qquad \underline{\mathbf{u}}^\varepsilon(t,x) = \mathbf{u}^\varepsilon_{err}(t,x)\,, \qquad \forall\,(t,x) \in]0,\tau] \times \mathbb{R}^d\,.$$

We consider the solution $\mathbf{u}^\varepsilon \in H^\infty(\omega_T)$ of the linear system (3.42). By uniqueness, we are sure that

$$\mathbf{u}^\varepsilon(t,x) = \mathbf{u}^\varepsilon_{err}(t,x) = \varepsilon^{-1}\,(U^\varepsilon - U^\varepsilon_a)(t,x)\,, \qquad \forall\,(t,x) \in]0,\tau] \times \mathbb{R}^d\,.$$

By applying lemma 3.17, we get

$$\sum_{k+m' \leq m} \lambda^{m-k-m'}\; |(\mathbf{X}^k_{v_0} \mathbf{u}^\varepsilon)_{|t=0}|_{m',\lambda,\varepsilon} \;\leq\; p_m(\lambda)\; \varepsilon^{M-m}\,.$$

By assumption $M > d+2$. Therefore, we can find $m \in \mathbb{N}$ such that

$$d/2 + 1 \;<\; m \;<\; M - d/2\,.$$

Let $\delta > 0$ be a given arbitrary positive real number, and fix $\underline{\lambda} \geq 1$ such that $\underline{\lambda}^{-1/2}\, C_m(\delta) < 1/2$. We define the application

$$\sigma \;:\; \lambda \longmapsto \sigma(\lambda) \;:=\; 4\; C_m(\delta)\; \lambda^{-1/2}\; \big(\|R_\varepsilon\|_{m,\lambda,\varepsilon} + p_m(\lambda)\big) + 1\,.$$

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Lemma 3.22. *There is $\varepsilon_0 > 0$ such that if $\underline{\mathbf{u}}^\varepsilon$ satisfies*

$$|\underline{\mathbf{u}}^\varepsilon|_{*,\varepsilon} \;\leq\; \delta\,, \qquad \|\underline{\mathbf{u}}^\varepsilon\|_{m,\underline{\lambda},\varepsilon} \;\leq\; \sigma(\underline{\lambda})\; \varepsilon^{M-m}\,, \qquad \forall\, \varepsilon \in]0,\varepsilon_0]\,, \tag{3.76}$$

then $\mathbf{u}^\varepsilon$ satisfies the same estimates

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$$|\mathbf{u}^\varepsilon|_{*,\varepsilon} \;\leq\; \delta\,, \qquad \|\mathbf{u}^\varepsilon\|_{m,\underline{\lambda},\varepsilon} \;\leq\; \sigma(\underline{\lambda})\; \varepsilon^{M-m}\,, \qquad \forall\, \varepsilon \in]0,\varepsilon_0]\,. \tag{3.77}$$

Proof. It follows from Proposition 3.18, absorbing in the left hand side the term $\underline{\lambda}^{-1/2}\, C_m(\delta)\, \|\mathbf{u}^\varepsilon\|_{m,\underline{\lambda},\varepsilon}$. This implies

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$$\begin{aligned} \|\mathbf{u}^\varepsilon\|_{m,\underline{\lambda},\varepsilon} \leq 2\; C_m(\delta)\; \underline{\lambda}^{-1/2}\; \big(&\sigma(\underline{\lambda})\, \varepsilon^{M-m-d/2}\, e^{\underline{\lambda} T}\, \|\mathbf{u}^\varepsilon\|_{m,\underline{\lambda},\varepsilon} \\ &+ \varepsilon^{M-1}\, \|R^\varepsilon\|_{m,\underline{\lambda},\varepsilon} + p_m(\underline{\lambda})\, \varepsilon^{M-m}\big)\,. \end{aligned} \tag{3.78}$$

Since $M - m - d/2 > 0$, we can find $\varepsilon_0 > 0$ small enough such that

$$2\, C_m(\delta)\, \underline{\lambda}^{-1/2}\, \sigma(\underline{\lambda})\, e^{\underline{\lambda} T}\, \varepsilon_0^{M-m-d/2} \;<\; 1/2\,.$$

We absorb again in the left hand side the term $\|\mathbf{u}^\varepsilon\|_{m,\underline{\lambda},\varepsilon}$ which yields

$$\|\mathbf{u}^\varepsilon\|_{m,\underline{\lambda},\varepsilon} \;\leq\; 4\, C_m(\delta)\, \underline{\lambda}^{-1/2}\, \big(\varepsilon^{M-1}\, \|R^\varepsilon\|_{m,\underline{\lambda},\varepsilon} + p_m(\lambda)\, \varepsilon^{M-m}\big) \;\leq\; \sigma(\underline{\lambda})\, \varepsilon^{M-m}\,.$$

It gives the second control in (3.77). Then, decreasing ε_0 if necessary, we find

$$c_{sob}\, e^{\underline{\lambda} T}\, \varepsilon^{M-m-d/2}\, \sigma(\underline{\lambda}) \,\leq\, \delta\,, \qquad \forall\, \varepsilon \in]0, \varepsilon_0]\,.$$

Lemma 3.16 implies

$$|\mathbf{u}|_{*,\varepsilon} \,\leq\, c_{sob}\, e^{\underline{\lambda} T}\, \varepsilon^{M-m-d/2}\, \sigma(\underline{\lambda}) \,\leq\, \delta\,.$$

This finishes the proof of Lemma 3.22. □

Theorem 3.13 is now a classical consequence of Lemma 3.22. The exact solution $\mathbf{u}$ is obtained on $[0, T_\varepsilon]$ by a simple Picard iteration scheme, taking $\mathbf{u}^n$ as $\underline{\mathbf{u}}$ and $\mathbf{u}^{n+1}$ as $\mathbf{u}$ and starting the induction with $\mathbf{u}^0 \equiv 0$. Since $\mathbf{u}^0$ obviously satisfies the estimates of Lemma 3.22, all the $\mathbf{u}^n$ satisfy the same estimates, implying that the limit $\mathbf{u}$ exists on the whole interval $[0, T]$ and also satisfies the estimates.

3.5.5 Proof of the theorem 3.12

It is similar to the proof of Theorem 3.13, thus we only give the key ingredients and point out where the new condition on M is used. Let $m \in \mathbb{N}$ such that $d/2 + 1 < m \leq M$. Without loss of generality, we can assume that the approximate solution is defined on the whole domain $\Omega =]-T_0, T_0[\times\mathbb{R}^d$, and that the exact solution is known in the past $\Omega \cap \{t < 0\}$. Ω_T denotes the strip $]-T_0, T[\times\mathbb{R}^d$ for all $T > 0$.

As in the proof of Theorem 3.13, we look for $U^\varepsilon = U^\varepsilon_{app} + \varepsilon\, \mathbf{u}^\varepsilon_{err}$, where $\mathbf{u}^\varepsilon_{err}$ is now given in the past Ω_0 instead of being given at $\{t = 0\}$. We are lead to find a priori estimates for the problem (3.42) where the condition $\mathbf{u}^\varepsilon_{|t=0} \equiv 0$ is now replaced by $\mathbf{u}^\varepsilon_{|t<0} = \varepsilon^{M-1}\, r^\varepsilon$ with $r^\varepsilon = \varepsilon^{-1}\, (U^\varepsilon - U^\varepsilon_a)_{|\Omega_0}$ given and satsifying

$$\sup_{0<\varepsilon\leq 1} \sum_{|\alpha|\leq m} \|\mathfrak{X}^\alpha_\varepsilon r^\varepsilon\|_{L^2(\Omega_0)} \,+\, \sup_{0<\varepsilon\leq 1} \sum_{|\beta|\leq 1} \|\mathfrak{X}^\beta_\varepsilon r^\varepsilon\|_{L^\infty(\Omega_0)} \,<\, \infty\,.$$

We replace the norms $\|.\|_{m,\lambda,\varepsilon}$ defined in section 3.5.1 by the following ones.

$$|||v|||_{m,\lambda,\varepsilon} \,:=\, \sum_{|\alpha|\leq m} \lambda^{m-|\alpha|}\, \|e^{-\lambda t}\, \mathfrak{X}^\alpha_\varepsilon v\|_{L^2(\Omega)}\,,$$

$$\|v\|_{*,\varepsilon} \,:=\, \sum_{|\alpha|\leq 1} \lambda^{m-|\alpha|}\, \|e^{-\lambda t}\, \mathfrak{X}^\alpha_\varepsilon v\|_{L^\infty(\Omega)}\,,$$

$$\mathbf{r}_\varepsilon(\lambda) \,:=\, \sum_{|\alpha|\leq m} \lambda^{m-|\alpha|}\, \|e^{-\lambda t}\, \mathfrak{X}^\alpha_\varepsilon r^\varepsilon\|_{L^2(\Omega_0)}\,.$$

The substitute for (3.48) is :

$$
(3.79)\qquad |||\mathbf{u}|||_{m,\lambda,\varepsilon} \;\le\; \lambda_m\, \varepsilon^{M-1}\, \mathbf{r}_\varepsilon(\lambda) \;+\; \frac{C_m(\|\underline{\mathbf{u}}\|_{*,\varepsilon})}{\lambda^{1/2}} \Big(|||\mathbf{u}|||_{m,\lambda,\varepsilon} + \varepsilon^{-d/2}\, e^{\lambda T}\, |||\underline{\mathbf{u}}|||_{m,\lambda,\varepsilon}\, |||\mathbf{u}|||_{m,\lambda,\varepsilon} + \varepsilon^{M-1}\, |||R^\varepsilon|||_{m,\lambda,\varepsilon} \Big) .
$$

for all $\lambda \ge \lambda_m$ with some λ_m large enough. The proof is a consequence of the estimates (3.48). Indeed, suppose first that $r^\varepsilon \equiv 0$. Then the estimate (3.79) follows from (3.48) since all the terms $(\mathbf{X}_{v_0}^k \mathbf{u})_{|t=0}$ vanish. In the general case we use a cut-off function $\chi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(t) = 0$ if $t < -2\,T_0/3$ and $= 1$ if $t > -T_0/3$. We write $\mathbf{u}^\varepsilon = \chi(t)\,\mathbf{u}^\varepsilon + (1-\chi(t))\,\mathbf{u}^\varepsilon$, and we can apply the previous case to $\chi(t)\,\mathbf{u}^\varepsilon$ since it vanishes in the past $t < -2\,T_0/3$, and this gives the estimate, up to a translation in the t coordinates.

The difference between (3.79) and (3.48) is that the terms involving the traces $(\mathbf{X}_{v_0}^k \mathbf{u})_{|t=0}$ have been replaced by $\varepsilon^{M-1}\, \mathbf{r}_\varepsilon(\lambda) = O(\varepsilon^{M-1})$. This is a gain since, for a fixed λ, the terms involving the traces are $O(\varepsilon^{M-m})$. Now, we can prove by induction a lemma similar to Lemma 3.22, where the conditions

$$
(3.80)\qquad \|\underline{\mathbf{u}}^\varepsilon\|_{*,\varepsilon} \le \delta\,, \qquad |||\underline{\mathbf{u}}^\varepsilon|||_{m,\underline{\lambda},\varepsilon} \le \sigma(\underline{\lambda})\,\varepsilon^{M-1}\,, \qquad \forall\,\varepsilon \in]0, \varepsilon_0]
$$

imply that $\mathbf{u}$ satisfies the same estimate. Indeed, to prove this, we use the a priori estimate (3.79) which implies the following substitute for (3.78)

$$
(3.81)\qquad |||\mathbf{u}^\varepsilon|||_{m,\underline{\lambda},\varepsilon} \le 2\; C_m(\delta)\; \underline{\lambda}^{-1/2}\; \Big(\sigma(\underline{\lambda})\,\varepsilon^{M-1-d/2}\, e^{\underline{\lambda} T}\, |||\mathbf{u}^\varepsilon|||_{m,\underline{\lambda},\varepsilon} + \varepsilon^{M-1}\, \|R^\varepsilon\|_{m,\underline{\lambda},\varepsilon} \Big) + c_\lambda\, \varepsilon^{M-1}\,.
$$

Taking $\varepsilon > 0$ small enough, we can now absorb in left hand side the term $|||\mathbf{u}^\varepsilon|||_{m,\underline{\lambda},\varepsilon}$ and obtain the expected bound for $|||\mathbf{u}^\varepsilon|||_{m,\lambda,\varepsilon}$. The control of $\|\mathbf{u}^\varepsilon\|_{*,\varepsilon}$ follows then from Lemma 3.16. Theorem 3.12 follows, along the lines developed at the end of the section 3.5.4.

3.6 Proof of theorem 3.11 whith $M_0 = (d + \ell + 2)/2$.

We set the problem in the general context of singular equations as treated by G. Browning and H.O. Kreiss [4] or by J.-L. Joly, G. Métiver and J. Rauch [17]. We prove first the existence of solutions and justify next their asymptotic expansions.

We consider again the problem for the unknown $U^\varepsilon = (V^\varepsilon, W^\varepsilon)$ defined by $u^\varepsilon = (v_0 + \varepsilon V^\varepsilon, W^\varepsilon)$. The equations are

$$
(3.82)\qquad \mathcal{H}^\varepsilon(U^\varepsilon, \partial) U^\varepsilon = h^\varepsilon
$$

with the initial condition

$$U^\varepsilon(0,x) = \mathbf{U}_0^\varepsilon\big(x, \vec{\boldsymbol{\varphi}}^0(x)/\varepsilon\big) \tag{3.83}$$

where $\vec{\boldsymbol{\varphi}}^0(x) := \vec{\boldsymbol{\varphi}}(0,x)$ and

$$\mathbf{U}_0^\varepsilon = \sum_{0\le n\le M} \varepsilon^n \, (\mathbf{V}_{n+1}, \mathbf{W}_n)_{|t=0}\,. \tag{3.84}$$

We look for a solution $U^\varepsilon(t,x)$ as $\mathbf{U}^\varepsilon\big(t,x,\vec{\boldsymbol{\varphi}}(t,x)/\varepsilon\big)$. A sufficient condition for U^ε to be a solution of (3.82) is that $\mathbf{U}^\varepsilon(t,x,\theta)$ be a solution of a Cauchy problem that can be written in the condensated form

7.3

$$\mathbf{H}^\varepsilon(\mathbf{U}^\varepsilon, \partial')\mathbf{U}^\varepsilon \, + \, \frac{1}{\varepsilon}\,\mathcal{L}(t,x,\partial_\theta)\mathbf{U}^\varepsilon \, = \, \mathbf{h}^\varepsilon, \tag{3.85}$$

with

7.4

$$\mathbf{U}^\varepsilon_{|t=0} = \mathbf{U}_0^\varepsilon, \tag{3.86}$$

oublicau

$$\mathbf{h}^\varepsilon := \begin{bmatrix} \mathbf{f}^\varepsilon \\ \mathbf{g}^\varepsilon \end{bmatrix}. \tag{3.87}$$

Here, $\mathbf{H}^\varepsilon(\mathbf{U}, \partial')$ denotes a first order symmetric hyperbolic operator of the form

$$\mathbf{H}^\varepsilon(\mathbf{U}, \partial') \, \equiv \, \sum_{j=0}^{d'} \mathcal{A}_j^\varepsilon\big(v_0, \mathbf{U}\big)\,\partial_j \, + \, \mathcal{C}^\varepsilon(v_0, \mathbf{U})\,.$$

In this formula, ∂_j with $j > d$ denotes the derivative with respect to the variables θ_k:

$$\partial_{d+k} = D_{\theta_k}\,, \qquad 1 \le k \le \ell\,.$$

The matrices $\mathcal{A}_j^\varepsilon(v,U)$ and $\mathcal{C}^\varepsilon(v,U)$ are $N \times N$ and are $\mathcal{C}^\infty$ functions of ε, v, U up to $\varepsilon = 0$. The matrices $\mathcal{A}_j^\varepsilon(v,U)$ are symmetric with $\mathcal{A}_0^\varepsilon(v,U)$ positive definite. Introduce

$$\mathbf{U}^\varepsilon_{app} \, := \, \sum_{j=0}^{M} \varepsilon^j \, \left(\mathbf{V}_j, \mathbf{W}_j\right)$$

which is bounded in $H^\infty(\omega_T \times \mathbb{T}^\ell)$ and satisfies

7.3app

$$\mathbf{H}^\varepsilon(\mathbf{U}^\varepsilon_{app}, \partial')\mathbf{U}^\varepsilon_{app} \, + \, \frac{1}{\varepsilon}\,\mathcal{L}(t,x,\partial_\theta)\mathbf{U}^\varepsilon_{app} = \mathbf{h}^\varepsilon + \varepsilon^M \mathbf{R}^\varepsilon\,. \tag{3.88}$$

thm323 **Theorem 3.23.** *Let $m \in \mathbb{N}$ with $(d+\ell)/2 + 1 < m \leq M$. There is $T > 0$ such that for all $\varepsilon \in]0,1]$ the solution $\mathbf{U}^\varepsilon$ of the Cauchy problem (3.85) – (3.87) exists on $\omega_T \times \mathbb{T}^\ell$ in $\mathcal{C}\big([0,T]; H^m(\mathbb{R}^d \times \mathbb{T}^\ell)\big)$. Moreover, for all $j \in \{0, \cdots, m\}$, we have*

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$$\sup_{t \in [0,T]} \|\partial_t^j (\mathbf{U}^\varepsilon - \mathbf{U}^\varepsilon_{app})(t)\|_{H^{m-j}(\mathbb{R}^d \times \mathbb{T}^\ell)} = \bigcirc(\varepsilon^{M-j}) . \tag{3.89}$$

Proof. Consider the *linear* singular system

7.5
$$\mathbf{H}^\varepsilon(\underline{\mathbf{U}}^\varepsilon, \partial')\mathbf{U} + \frac{1}{\varepsilon} \mathcal{L}(t,x,\partial_\theta)\mathbf{U} = \mathbf{h} , \qquad \mathbf{U}^\varepsilon_{|t=0} = \mathbf{U}^\varepsilon_0 . \tag{3.90}$$

We look for a priori estimates in the space

$$W^m(T) := \bigcap_{0 \leq j \leq m} C^j\big([0,T], H^{m-j}(\mathbb{R}^d \times \mathbb{T}^\ell)\big)$$

endowed with its natural norm $\| \cdot \|_{W^m(T)}$. Fix m with $m > (d+\ell)/2 + 1$. Suppose that $\underline{\mathbf{U}}^\varepsilon \in W^m(T)$ and we choose a constant $R > \| \underline{\mathbf{U}}^\varepsilon \|_{W^m(T)}$. We fix a bounded neighborood $\mathcal{K}$ of 0 in $\mathbb{R}^N$ such that $\underline{\mathbf{U}}^\varepsilon$ and $\mathbf{U}^\varepsilon_0$ take their values in $\mathcal{K}$. We also assume that $\underline{\mathbf{U}}^\varepsilon$ satisfies

traceiterecompatible
$$\underline{\mathbf{U}}^\varepsilon_{|t=0} = \mathbf{U}^\varepsilon_0 . \tag{3.91}$$

1) L^2 **estimate.** By symmetry and integration by parts, using the fact that the coefficients of $\mathcal{L}(t,x,\partial_\theta)$ do not depend on θ, we get

$$\|\mathbf{U}^\varepsilon\|_{C([0,T];L^2)} \leq C(R)\; T\; \|\mathbf{h}\|_{C([0,T];L^2)} + c_0\; \|\mathbf{U}^\varepsilon_0\|_{L^2} \tag{3.92}$$

where $C(R)$ is a function of R and c_0 is a constant independent on R.

2) **Estimates of the derivatives.** Here the analysis relies strongly on the small divisor Assumption 3.2 and on the relation (3.91). Since $L(v,\xi)$ has a constant rank for $\xi \neq 0$, there exists a $N \times N$ invertible matrix $\mathbf{\Gamma}(v,\xi)$, defined for all $(v,\xi) \in \mathbb{R}^{N''} \times \mathbb{R}^d$, homogeneous of degree 0 in ξ and $\mathcal{C}^\infty$ on $\mathbb{R}^{N''} \times (\mathbb{R}^d \setminus \{0\})$, such that $\mathbf{\Gamma}(v,0) = \mathrm{Id}_{N \times N}$ and

$${}^t\mathbf{\Gamma}(v,\xi)\, \mathbf{L}(v,\xi)\, \mathbf{\Gamma}(v,\xi) = |\xi|\, \underline{\mathbf{L}} , \qquad \forall\, (v,\xi) \in \mathbb{R}^{N''} \times \mathbb{R}^d .$$

Here $\underline{\mathbf{L}}$ is a constant symmetric matrix. For $(t,x) \in \Omega$ and $\alpha \in \mathbb{Z}^\ell$, let

$$\boldsymbol{\gamma}(t,x,\alpha) := \mathbf{\Gamma}\big(v_0(t,x), \alpha \cdot \partial_x \vec{\boldsymbol{\varphi}}(t,x)\big) .$$

We denote by $\boldsymbol{\gamma}(t,x,\partial_\theta)$ the corresponding Fourier multiplier. It is defined on $L^2(\Omega\times\mathbb{T}^\ell;\mathbb{R}^N)$ by

operateurgamma

$$
\boldsymbol{\gamma}(t,x,\partial_\theta)\,U(t,x,\theta) \;:=\; \widehat{U}_0(t,x) + i\sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} \boldsymbol{\gamma}(t,x,\alpha)\,\widehat{U}_\alpha(t,x)\; e^{i\langle\alpha\cdot\theta\rangle}\,. \tag{3.93}
$$

The properties of $\boldsymbol{\gamma}(t,x,\partial_\theta)$ are summarized in the next proposition. Note that the small divisors assumption 3.2 (petits diviseurs) implies that $\boldsymbol{\gamma}(t,x,\alpha)$ is invertible for all $(t,x)\in\Omega$ and for all $\alpha\in\mathbb{Z}^\ell$.

Proposition 3.24. *The operator $\boldsymbol{\gamma}(t,x,\partial_\theta)$ is an isomorphism from $H^s(\Omega\times\mathbb{T}^\ell;\mathbb{R}^N)$ onto itself for all $s\geq 0$. Its inverse is the Fourier multiplier $\boldsymbol{\gamma}^{-1}(t,x,\partial_\theta)$ defined by*

$$
\boldsymbol{\gamma}^{-1}(t,x,\partial_\theta)\,U(t,x,\theta) \;:=\; \widehat{U}_0(t,x) + i\sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} \boldsymbol{\gamma}(t,x,\alpha)^{-1}\,\widehat{U}_\alpha(t,x)\; e^{i\langle\alpha\cdot\theta\rangle}\,.
$$

Moreover, the operators $(\partial^\beta_{t,x}\boldsymbol{\gamma})(t,x,\partial_\theta)$ and $(\partial^\beta_{t,x}\boldsymbol{\gamma}^{-1})(t,x,\partial_\theta)$ are continuous from $H^s(\Omega\times\mathbb{T}^\ell;\mathbb{R}^N)$ into itself, for all $\beta\in\mathbb{N}^{1+d}$ and $s\geq 0$.

Proof. Since $\boldsymbol{\Gamma}(v,\xi)$ is homogeneous of degree 0 with respect to ξ, there is $c>0$ such that

$$
c < |\det\boldsymbol{\gamma}(t,x,\alpha)| < c^{-1}\,,\qquad \forall\,(t,x,\alpha)\in\Omega\times(\mathbb{Z}^\ell\setminus\{0\})\,.
$$

Hence the entries of the inverse matrix $\boldsymbol{\gamma}^{-1}(t,x,\alpha)$ are bounded on the domain $\Omega\times(\mathbb{Z}^\ell\setminus\{0\})$. This implies that $\boldsymbol{\gamma}(t,x,\partial_\theta)$ is an isomorphism from L^2 to L^2 with inverse $\boldsymbol{\gamma}^{-1}(t,x,\partial_\theta)$. That $\boldsymbol{\gamma}(t,x,\partial_\theta)$ and $\boldsymbol{\gamma}(t,x,\partial_\theta)^{-1}$ map H^s to H^s when $s>0$ is a consequence of the homogeneity of $\boldsymbol{\Gamma}$ and of the coherence assumption, as shown for the the operator $\mathbb{E}$ in Proposition 3.6 (operationdeEE); we do not repeat the details here. □

Introducing the new unknown $\mathbf{V} := \boldsymbol{\gamma}(t,x,\partial_\theta)^{-1}\,\mathbf{U}$ yields for $\mathbf{V}$ the equation

7.8

$$
\gamma^*(t,x,\partial_\theta)\;\mathbf{H}(\underline{\mathbf{U}},\partial')\;\boldsymbol{\gamma}(t,x,\partial_\theta)\,\mathbf{V} + \frac{1}{\varepsilon}\,\underline{\mathbf{L}}\,|\partial_\theta|\,\mathbf{V} = f\,. \tag{3.94}
$$

Applying $\partial_t^k\partial_x^\alpha\partial_\theta^\beta$ to the equation (with $k+|\alpha|+|\beta|\leq m$) and using that $\underline{\mathbf{L}}$ is constant, we prove the following estimate

$$
\|\mathbf{V}\|_{W^m(T)} \;\leq\; C_m(R)\;T\;(\|\mathbf{V}\|_{W^m(T)} + 1) + c_0\sup_{0\leq j\leq m}\|\partial_t^j\boldsymbol{\gamma}(t,x,\partial_\theta)^{-1}\,\mathbf{U}^\varepsilon{}_{|t=0}\|_{H^{m-j}(\mathbb{R}^d\times\mathbb{T}^\ell)}\,. \tag{3.95}
$$

To obtain the desired energy estimate for $\mathbf{V}$ and thus for $\mathbf{U}$ it remains to show that for all $j \in \{0, \cdots, m\}$,

$$\|\partial_t^j \mathbf{U}^\varepsilon\|_{|t=0}\|_{H^{m-j}} \;=\; O(1). \tag{3.96}$$

This is proved by induction on j, as a consequence of the estimates

$$\|\partial_t^j (\mathbf{U}^\varepsilon - \mathbf{U}^\varepsilon_{app})_{|t=0}\|_{H^{m-j}} = O(\varepsilon^{m-j})\,.$$

When $j = 0$ it is obvious. For the higher order derivatives, we use (3.85) to replace the time derivatives by ∂_x and ∂_θ derivatives. There is a loss of at most one power ε at each step, because of the factor $\varepsilon^{-1}\mathcal{L}$ in the equation and the estimate follows.

Going back to the unknown $\mathbf{U}$, this shows that

$$\|\mathbf{U}\|_{W^m(T)} \;\le\; \tilde{C}_m(R)\; T\; (\|\mathbf{U}\|_{W^m(T)} \,+\, 1) \,+\, \tilde{c}_0 \tag{3.97}$$

for some new constants $\tilde{c}_0$ and $\tilde{C}_m(R)$ independent on ε and T.

Applying this inequality with $\underline{\mathbf{U}} \equiv \mathbf{U}$, with $R > 0$ large enough and T small enough, we obtain $\|\mathbf{U}\|_{W^m(T)} \le R$, for all $\varepsilon \in]0, 1]$. This implies the existence of $\mathbf{U}^\varepsilon$ on the time interval $[0, T]$, for all $\varepsilon \in]0, 1]$. Furthermore, if the initial data is in the space $H^\infty(\mathbb{R}^d \times \mathbb{T})$ the solution is also in $H^\infty(\Omega_T \times \mathbb{T})$.

Once the existence of $\mathbf{U}^\varepsilon$ is proved, we can compare $\mathbf{U}^\varepsilon$ and $\mathbf{U}^\varepsilon_{app}$. The difference $\mathbf{V}^\varepsilon := \mathbf{U}^\varepsilon - \mathbf{U}^\varepsilon_{app}$ satisfies

$$\mathbf{H}^\varepsilon(\mathbf{U}^\varepsilon, \partial')\mathbf{V}^\varepsilon \,+\, \frac{1}{\varepsilon}\,\mathcal{L}(t, x, \partial_\theta)\mathbf{V}^\varepsilon \,+\, G\,\mathbf{V}^\varepsilon \,=\, -\,\varepsilon^M\,\mathbf{R}^\varepsilon \tag{3.98}$$

with

$$G \;=\; \sum_{j=0}^{d+\ell} G_j(\mathbf{U}^\varepsilon, \mathbf{U}^\varepsilon_{app})\,\partial_j \mathbf{U}^\varepsilon_{app}$$

and the G_j are smooth matrices. Moreover, $\mathbf{V}^\varepsilon_{|t=0} = 0$. Using the energy estimates already proved for the operator

$$\mathbf{H}^\varepsilon(\mathbf{U}^\varepsilon, \partial') \,+\, \frac{1}{\varepsilon}\,\mathcal{L}(t, x, \partial_\theta)\,.$$

one obtains the estimates (3.89) and Theorem 3.23 is proved. $\square$

4 ε-stratified and ε-conormal waves

solutionsstratifiees

In this section, we consider expansions with a single phase $\varphi \in \mathcal{C}^\infty(\Omega;\mathbb{R})$, which is fixed and satisfies the eiconal equation $\mathbf{X}_{v_0}\varphi = 0$. Moreover, we assume that $d\varphi \in \mathcal{C}_b^\infty(\Omega;\mathbb{R}^{1+d})$ and $\inf_\Omega |\partial_x \varphi| > 0$.

Inspired by the one dimensional analysis of [6] we want to treat more general fluctuations such as almost periodic oscillations or jump profiles (2.21), which enter in the more general context of ε-stratified or ε-conormal waves.

We introduce first several notations. Let $\mathcal{T}_0, \cdots, \mathcal{T}_{d-1}$ denote smooth vector fields on Ω tangent to the foliation $\{\varphi = cte\}$ which means that $\mathcal{T}_j\varphi \equiv 0$ for all j. We assume that the family has a constant rank d. We also assume that the fields $\mathcal{T}_j$ have bounded coefficients, together with all their derivatives, that is $\mathcal{T}_j \in \mathcal{C}_b^\infty(\Omega;\mathbb{R}^d)$.

We denote by $\mathcal{T}_d$ a vector field on Ω, tangent to the hypersurface $\{\varphi = 0\}$, with coefficients in $\mathcal{C}_b^\infty(\Omega;\mathbb{R})$. We assume that the family $\mathcal{T}_0, \cdots, \mathcal{T}_{d-1}, \mathcal{T}_d$ is a generator of vector fields tangent to $\{\varphi = 0\}$, and that this family has rank $d+1$ when $\{\varphi(t,x) \neq 0\}$.

Example 4.1. When the phase $\varphi(t,x)$ is linear or more generally can be reduced to $\varphi = x_d$ after a change of coordinates, we can choose, in the new coordinates,

$$\mathcal{T}_0 = \partial_t, \quad \mathcal{T}_j = \partial_j \ \text{ for } j \in \{1, \cdots, d-1\}, \quad \mathcal{T}_d = h(x_d)\,\partial_d,$$

where $h \in \mathcal{C}_b^\infty(\mathbb{R};\mathbb{R})$, $|h(x_d)| = 1$ if $|x_d| \geq 2$, $h(x_d) = x_d$ if $|x_d| \leq 1$ and $h(x_d) \neq 0$ if $x_d \neq 0$.

Because φ satisfies the eiconal equation, we can choose

eq41

$$\mathcal{T}_0(t,x,\partial_t,\partial_x) = \partial_t + \mu(v_0)\cdot\nabla_x \tag{4.1}$$

and assume that the other fields $\mathcal{T}_1, \cdots, \mathcal{T}_d$ contain only x-derivatives:

tangent to t=0

$$\mathcal{T}_j = \mathcal{T}_j(t,x,\partial_x) = \tau_j(t,x)\cdot\nabla_x\,, \qquad \tau_j \in \mathcal{C}_b^\infty(\Omega;\mathbb{R}^d)\,, \qquad 1 \leq j \leq d\,. \tag{4.2}$$

4.1 ε-stratified and ε-conormal regularity.

We consider two sets of vector fields $\mathbf{Z}_\varepsilon$. When dealing with the ε-stratified case, we take

vees eps tangentielles

$$\mathbf{Z}_\varepsilon^{strat} := \left\{\, \mathcal{T}_0\,, \cdots, \mathcal{T}_{d-1}\,, \varepsilon\,\partial_0\,, \varepsilon\,\partial_1\,, \cdots, \varepsilon\,\partial_d \,\right\}. \tag{4.3}$$

To describe ε-conormal smoothness, we consider

erivees eps conormales

$$\mathbf{Z}_\varepsilon^{con} := \left\{ \mathcal{T}_0 \,, \cdots \,, \mathcal{T}_{d-1} \,, \mathcal{T}_d \,, \varepsilon\, \partial_0 \,, \varepsilon\, \partial_1 \,, \cdots \,, \varepsilon\, \partial_d \right\} . \tag{4.4}$$

Usually, we simply use the notation $\mathbf{Z}_\varepsilon$, the choice being clear from the context.

In both cases, an important property is that the commutator of two elements of $\mathbf{Z}_\varepsilon$ is a linear combination of elements of $\mathbf{Z}_\varepsilon$ with coefficients in $\mathcal{C}_b^\infty(\Omega)$:

$$X \in \mathbf{Z}_\varepsilon \,,\; Y \in \mathbf{Z}_\varepsilon \implies [X;Y] = XY - YX = \sum_{\mathcal{Z} \in \mathbf{Z}_\varepsilon} a_{\mathcal{Z}}\, \mathcal{Z} \,, \qquad a_{\mathcal{Z}} \in \mathcal{C}_b^\infty(\Omega) \,.$$

For all $k \in \mathbb{N}$ the notation $\mathbf{Z}_\varepsilon^k$ will denote the set of all the operators $\mathcal{Z}_1 \circ \cdots \circ \mathcal{Z}_k$ such that $\mathcal{Z}_j \in \mathbf{Z}_\varepsilon$ for $j \in \{1, \cdots, k\}$. We will also note $\mathcal{Z}_\varepsilon^k$ a general element in the set $\mathbf{Z}_\varepsilon^k$.

• **Interior regularity.** For all reals a, b such that $-T_0 < a < b < T_0$, we define the space $\mathbb{A}^m(a,b)$ of families of functions $u^\varepsilon(t,x)$, $0 < \varepsilon \le 1$ such that $\partial_t^j \partial_x^\alpha u^\varepsilon \in \mathcal{C}\big([a,b]; L^2(\mathbb{R}^d)\big)$ for $j + |\alpha| \le m$ and such that

$$\sum_{0 \le k \le m} \; \sum_{\mathcal{Z}_\varepsilon^k \in \mathbf{Z}_\varepsilon^k} \; \sup_{0 < \varepsilon \le 1} \; \| \mathcal{Z}_\varepsilon^k u^\varepsilon \|_{\mathcal{C}([a,b];L^2)} \, < \, \infty \,.$$

In a similar way, replacing L^2 by L^∞ we define the space $\mathbb{B}^m(a,b)$ of (families of) functions u^ε such that $\partial_t^j \partial_x^\alpha u^\varepsilon \in \mathcal{C}\big([a,b]; L^\infty(\mathbb{R}^d)\big)$ for $j + |\alpha| \le m$ and such that

$$\sum_{0 \le k \le m} \; \sum_{\mathcal{Z}_\varepsilon^k \in \mathbf{Z}_\varepsilon^k} \; \sup_{0 < \varepsilon \le 1} \; \| \mathcal{Z}_\varepsilon^k u^\varepsilon \|_{\mathcal{C}([a,b];L^\infty)} \, < \, \infty \,.$$

We also introduce the spaces

$$\Lambda^m(a,b) := \mathbb{A}^m(a,b) \cap \mathbb{B}^1(a,b) \,. \tag{4.5}$$

tangent to t=0

• **Regularity of initial data.** Because of (4.2), every field $\mathcal{Z}^\varepsilon$ in $\mathbf{Z}_\varepsilon$ excepted $\mathcal{T}_0$ and $\varepsilon\, \partial_0$, is tangent to the hypersurfaces $\{t = cte\}$ and in particular to $\{t = 0\}$. Thus if $\mathcal{Z} \in \mathbf{Z}_\varepsilon \setminus \{\mathcal{T}_0, \varepsilon\, \partial_0\}$, the first order operator $\mathcal{Z}_{|t=0}$ is a well defined vector field on $\mathbb{R}^d$. Let us denote by $\mathbf{I}(\mathbf{Z}_\varepsilon)$ the set of such fields

$$\mathbf{I}(\mathbf{Z}_\varepsilon) := \left\{ \mathcal{Z}_{\varepsilon\,|t=0} \,;\; \mathcal{Z}_\varepsilon \in \mathbf{Z}_\varepsilon \setminus \{\mathcal{T}_0, \varepsilon\, \partial_0\} \right\} .$$

We denote by $\mathbb{A}_0^m$ the space of families of functions $u^\varepsilon \in H^m(\mathbb{R}^d)$ such that

initial norms 1

$$\sum_{0 \le k \le m} \; \sum_{\{\mathcal{Z}_1, \cdots, \mathcal{Z}_k\} \in \mathbf{I}(\mathbf{Z}_\varepsilon)^k} \; \sup_{0 < \varepsilon \le 1} \; \| \mathcal{Z}_1 \circ \cdots \circ \mathcal{Z}_k u^\varepsilon \|_{L^2(\mathbb{R}^d)} \, < \, \infty \,. \tag{4.6}$$

According to the choice of the set $\mathbf{Z}_\varepsilon$, the functions belonging to $\mathbb{A}_0^m$ have an ε-stratified (resp. ε-conormal) regularity with respect to the foliation $\{\varphi = cte\} \cap \{t = 0\}$ of $\mathbb{R}^d$ (resp. the hypersurface $\{\varphi = 0\} \cap \{t = 0\}$).

Similarly, we denote by $\mathbb{B}_0^m$ the space of the $u^\varepsilon(x)$ such that

initial norms 2

$$\sum_{0\le k\le m} \sum_{\{\mathcal{Z}_1,\cdots,\mathcal{Z}_k\}\in \mathbf{I}(\mathbf{Z}_\varepsilon)^k} \sup_{0<\varepsilon\le 1} \|\mathcal{Z}_1 \circ \cdots \circ \mathcal{Z}_k u^\varepsilon\|_{L^\infty(\mathbb{R}^d)} < \infty\,. \tag{4.7}$$

We will note $\Lambda_0^m := \mathbb{A}_0^m \cap \mathbb{B}_0^1$.

section3.3

4.2 The Cauchy problem and compatibility conditions.

Our goal is to solve, locally in time, the Cauchy problem for ε-stratified or ε-conormal initial data. We also include source terms and consider the equation

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$$\begin{cases} S(u^\varepsilon)\, \mathbf{X}_{v^\varepsilon} u^\varepsilon + \mathbf{L}(v^\varepsilon, \partial_x) u^\varepsilon \ = \begin{bmatrix} \varepsilon f^\varepsilon \\ g^\varepsilon \end{bmatrix}, \qquad (t,x) \in \omega_T \\ u^\varepsilon_{|t=0} = u_0^\varepsilon, \end{cases} \tag{4.8}$$

where the data $h^\varepsilon := {}^t(f^\varepsilon, g^\varepsilon)$ and u_0^ε belong to $\Lambda^m(0,T_0)$ and Λ_0^m respectively. The goal is to show that, if m is large enough the problem (4.8) has a solution in $\Lambda^m(0,T)$ for some $T > 0$ independent on $\varepsilon > 0$. But, in general, this requires *compatibility conditions*, which ensure that the solutions do not develop singularities associated to other modes of the system and propagating in other directions.

First, we describe these necessary compatibility conditions. Assume that $u^\varepsilon = (v_0 + \varepsilon\, v^\varepsilon, w^\varepsilon) \in \Lambda^m(0,T)$ is a solution of

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$$S(u)\, \mathbf{X}_{v_0+\varepsilon v}\, u \ + \ \mathbf{L}(v, \partial_x) u \ = \begin{bmatrix} \varepsilon\, f^\varepsilon \\ g^\varepsilon \end{bmatrix}. \tag{4.9}$$

Introduce $V^\varepsilon := (v^\varepsilon, w^\varepsilon)$ and $h^\varepsilon := (f^\varepsilon, g^\varepsilon)$. Since $\mathbf{X}_{v_0+\varepsilon v^\varepsilon} - \mathbf{X}_{v_0}$ does not contain time derivatives and because of (2.15, V^ε satisfies an equation

$$\mathbf{X}_{v_0} V^\varepsilon \ = \ \mathcal{F}_1\big(\ \varepsilon, t, x, (\partial_x^\alpha V^\varepsilon)_{|\alpha|\le 1}\,,\ h^\varepsilon\big)\,. \tag{4.10}$$

By induction, for $1 \le k \le m$

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$$\mathbf{X}_{v_0}^k V^\varepsilon \ = \ \mathcal{F}_k\big(\varepsilon, t, x,\ (\partial_x^\alpha V^\varepsilon)_{|\alpha|\le k}\,,\ (\partial_{t,x}^\beta h)_{|\beta|\le k-1}\big) \tag{4.11}$$

where the functions $\mathcal{F}_k\big(\varepsilon, t, x, \, (V_\alpha)_{|\alpha|\le k}\, , \, (H_\beta)_{|\beta|\le k-1}\big)$ are $\mathcal{C}^\infty$ functions of their arguments, up to $\varepsilon = 0$. Using the special structures of the $\mathcal{F}_k$, we see that for all $V^\varepsilon \in \Lambda^m(0,T)$, all real numbers a, b such that $0 \le a < b \le T$ and for all k such that $1 \le k \le m$, there holds

$$\mathcal{F}_k\big(\, \varepsilon, t, x, \, (\partial_x^\alpha V^\varepsilon)_{|\alpha|\le k}\, , \, (\partial_{t,x}^\beta h)_{|\beta|\le k-1}\big) \,\in\, \mathbb{A}^{m-k}(a,b)\,. \tag{4.12}$$

Moreover

$$\mathcal{F}_1\big(\, \varepsilon, t, x, (\partial_x^\alpha V^\varepsilon)_{|\alpha|\le 1}\, , \, h^\varepsilon\big) \,\in\, \mathbb{B}^0(a,b)\,.$$

In particular, taking $a = b = 0$ and denoting by $V_0^\varepsilon = V_{|\{t=0\}}$, we have

3.11
$$\mathcal{F}_1\big(\, \varepsilon, 0, x, \, (\partial_x^\alpha V_0^\varepsilon)_{|\alpha|\le 1}\, , \, h^\varepsilon_{|t=0}\big) \,\in\, \mathbb{A}_0^{m-1} \cap \mathbb{B}_0^0\,, \tag{4.13}$$

and for k such that $2 \le k \le m$

3.12
$$\mathcal{F}_k\big(\, \varepsilon, 0, x, (\partial_x^\alpha V_0^\varepsilon)_{|\alpha|\le k}\, , \, (\partial_{t,x}^\beta h^\varepsilon_{|t=0})_{|\beta|\le k-1}\big) \,\in\, \mathbb{A}_0^{m-k}\,. \tag{4.14}$$

compatible **Definition 4.1.** *Let $h^\varepsilon = (f^\varepsilon, g^\varepsilon)$ be given in $\Lambda^m(0,T_0)$. Consider a family of Cauchy data u_0^ε of the form $u_0^\varepsilon = (v_{0|t=0} + \varepsilon\, a^\varepsilon, b^\varepsilon)$ where $V_0^\varepsilon := (a^\varepsilon, b^\varepsilon)$ is in Λ_0^m. We say that the data u_0^ε and h^ε are* **compatible up to order m** *if the m conditions (4.13) and (4.14) are satisfied.*

If $u^\varepsilon = (v_0 + \varepsilon\, v^\varepsilon, w^\varepsilon)$ is a solution of (4.9) with $(v^\varepsilon, w^\varepsilon) \in \Lambda^m(0,T)$, the trace $u^\varepsilon_{|t=0}$ is *necessarily* compatible up to order m. Conversely, the next theorem asserts that these compatibility conditions are also *sufficient* to solve the Cauchy problem in $\Lambda^m(0,T)$ for some $T > 0$ (independent on ε).

theo 1.2 **Theorem 4.2.** *Let $m \in \mathbb{N}$ such that $m > d/2 + 2$. For all $h^\varepsilon = {}^t(f^\varepsilon, g^\varepsilon)$ in $\Lambda^m(0,T_0)$. and $(a^\varepsilon, b^\varepsilon)$ in Λ_0^m such that the data $u_0^\varepsilon = (v_{0|t=0} + \varepsilon\, a^\varepsilon, b^\varepsilon)$ and h^ε are compatible up to order m, there exists $T > 0$ such that for all $\varepsilon \in]0,1]$ the Cauchy problem (4.8) has a (unique) solution $u^\varepsilon \in \mathcal{C}\big([0,T]; H^m(\mathbb{R}^d)\big)$ in ω_T. Moreover, u^ε has the form $u^\varepsilon = (v_0 + \varepsilon\, v^\varepsilon, w^\varepsilon)$ with $(v^\varepsilon, w^\varepsilon) \in \Lambda^m(0,T)$.*

A consequence of the nessary and sufficient character of the compatibility conditions is the *propagation of the Λ^m regularity* for solutions of the form $u^\varepsilon = (v_0 + \varepsilon\, v^\varepsilon, w^\varepsilon)$. For all $T > T_0$, we denote $\Omega_T :=]-T_0, T[\times\mathbb{R}^d$.

etpourqpas **Corollary 4.3.** $(m > d/2 + 2)$. *Let $h^\varepsilon = (f^\varepsilon, g^\varepsilon)$ be given in $\Lambda^m(-T_0, T_0)$. assume that $u^\varepsilon = (v_0 + \varepsilon\, v^\varepsilon, w^\varepsilon)$ is a family of solutions of (4.9) in Ω_0, such that $(v^\varepsilon, w^\varepsilon) \in \Lambda^m(-T_0, 0)$. Then there exists $T > 0$ such that, for all $\varepsilon \in]0,1]$, u^ε extends (in a unique way) as a solution $\tilde u^\varepsilon \in \mathcal{C}\big([-T_0,T]; H^m(\mathbb{R}^d)\big)$ of system (4.9) on Ω_T. Moreover, $\tilde u^\varepsilon$ has the form $\tilde u^\varepsilon = (v_0 + \varepsilon\, \tilde v^\varepsilon, \tilde w^\varepsilon)$ with $(\tilde v^\varepsilon, \tilde w^\varepsilon) \in \Lambda^m(-T_0, T)$.*

For a given $\tau > 0$, consider $\tilde{u}_0(t,\cdot) := u_0(t-\tau,\cdot)$. We can apply Corollary (4.3) to u_0 as unperturbed state and data

$$\begin{pmatrix} v^\varepsilon \\ w^\varepsilon \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \Lambda^m(-T_0,0)\,, \qquad h^\varepsilon = \begin{pmatrix} f^\varepsilon \\ g^\varepsilon \end{pmatrix} \in \Lambda^m(-T_0,T_0)$$

where h^ε is chosen such that

$$h^\varepsilon(t,\cdot) \equiv 0\,, \quad \forall\, t \in]-T_0,0] \qquad \text{and} \qquad h^\varepsilon(\tau,\cdot) \not\equiv 0\,.$$

The solution $\tilde{u}^\varepsilon(\cdot,\cdot)$ has a non trivial trace $\tilde{u}^\varepsilon(\tau,\cdot)$ at time $t=\tau$, which necessarily satisfies the compatibility condition. This remark shows that Corollary (4.3) can be used to construct the existence of non trivial initial compatible datas. In the framework of oscillations discussed in section 3, BKW formal solutions is another source of compatible data.

In the more general context of stratified or conormal waves, non trivial compatible initial data can be explicitly constructed, as shown in the next result. Let $\mathbf{C}$ be a given compact set in $\mathbb{R}^d$. We denote by $\mathbf{J_C}(\mathbb{R}^d;\mathbb{R}^k)$ the set of functions $u(x,z) \in \mathcal{C}_b^\infty(\mathbb{R}^d_x \times \mathbb{R}_z;\mathbb{R}^k)$ such that $u(x,z)=0$ if $x \notin \mathbf{C}$. Let us recall that the function $\varphi(t,x)$ is scalar. We introduce

$$\Pi_0^\flat(x) \,:=\, \mathbf{P}^\flat\big(v_0(0,x),\partial_x\varphi(0,x)\big)\,.$$

Theorem 4.4. *Let $\{\boldsymbol{a}_j\}_{j\in\mathbb{N}} \in \mathbf{J_C}(\mathbb{R}^d;\mathbb{R}^{N"})^{\mathbb{N}}$ and $\{\mathbf{b}_j\}_{j\in\mathbb{N}} \in \mathbf{J_C}(\mathbb{R}^d;\mathbb{R}^{N'})^{\mathbb{N}}$ be two given sequences of profiles, the first one satisfying the polarization condition $\Pi_0^\flat\,\boldsymbol{a}_j = \boldsymbol{a}_j$ for all $j\in\mathbb{N}$. For any given $m\in\mathbb{N}_*$ there exists a function $(V_0^\varepsilon,W_0^\varepsilon)\in\Lambda_0^m$ satisfying*

$$\Pi_0^\flat(x)\,V_0^\varepsilon(x) = \sum_{0\le j\le m} \varepsilon^j\ \boldsymbol{a}_j\big(x,\varphi(0,x)/\varepsilon\big)$$

$$W_0^\varepsilon(x) = \sum_{0\le j\le m} \varepsilon^j\ \mathbf{b}_j\big(x,\varphi(0,x)/\varepsilon\big)$$

and such that the initial data $u_0^\varepsilon := (v_0+\varepsilon\,V_0^\varepsilon, W_0^\varepsilon)$ is compatible up to order m (in the sense of Definition 4.1).

Remark 4.5. In general, there are no approximate WKB solution corresponding to such initial data, except in the case of periodic profiles $\boldsymbol{a}_j(x,\cdot)$ and $\mathbf{b}_j(x,\cdot)$ discussed in section 3. But, for almost periodic profiles, or *jump profiles*, or more general cases, the WKB construction is not available. Hence, Theorem 4.2 proves the existence of solutions containing large amplitude variations, even if a high order asymptotic expansion of the solution is unknown.

Estimations

4.3 Proof of the theorem 4.2

The key ingredient is to prove uniform estimates for the solutions U^ε of the following linear Cauchy problem, where the operator $\mathcal{H}^\varepsilon(\underline{U},\partial)$ was introduced in (2.49)

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$$\mathcal{H}^\varepsilon(\underline{U},\partial)U \;=\; h^\varepsilon\,, \qquad U_{|t=0} = U_0^\varepsilon \tag{4.15}$$

with $U_0^\varepsilon := (a^\varepsilon, b^\varepsilon)$. For all $\varepsilon \neq 0$ *fixed*, the linear symmetric hyperbolic problem (4.15) has obviously a unique solution $U^\varepsilon \in \mathbf{W}^m(0,T_0)$. The corresponding estimates are given by proposition 4.10 (high order L^2 estimates) and proposition 4.14 (L^∞ estimates) of this section, which is mainly concerned with proving them. The last paragraph of subsection 4.3 is devoted to the end of the proof of theorem 4.2.

We recall that, by assumption, $U_0^\varepsilon \in \Lambda_0^m$ and $h^\varepsilon \in \Lambda^m(0,T_0)$ are given data which satisfy the compatibility conditions (4.13) and (4.14).

4.3.1 Norms

For $m \in \mathbb{N}$ and $\varepsilon \neq 0$, introduce the notations

$$|v\|_{m,\varepsilon,T} \;:=\; \sum_{0\le k\le m}\;\sum_{\mathcal{Z}_\varepsilon^k\in\mathbf{Z}_\varepsilon^k}\;\sup_{0\le t\le T}\;\|\mathcal{Z}_\varepsilon^k v(t,\cdot)\|_{L^2(\mathbb{R}^d)}\,, \tag{4.16}$$

and

$$|v|^*_{m,\varepsilon,T} \;:=\; \sum_{0\le k\le m}\;\sum_{\mathcal{Z}_\varepsilon^k\in\mathbf{Z}_\varepsilon^k}\;\sup_{0\le t\le T}\;\|\mathcal{Z}_\varepsilon^k v(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}\,. \tag{4.17}$$

Similarly, we denote by $\|v\|_{\mathbb{A}^m(0,T)}$ and $|v\|_{\mathbb{B}^m(0,T)}$ the supremum with respect to $\varepsilon \in]0,1]$ of $|v\|_{m,\varepsilon,T}$ and $|v|^*_{m,\varepsilon,T}$ respectively.

We also use similar notations for initial datas and denote respectively by $\|v\|_{\mathbb{A}_0^m(0,T)}$ and $\|v\|_{\mathbb{B}_0^m(0,T)}$ the supremum with respect to $\varepsilon \in]0,1]$ of the left hand side of (4.6) and (4.7) respectively.

4.3.2 L^2 estimate

For all $T \in [0,T_0]$, ω_T denotes the strip $]0,T[\times\mathbb{R}^d$.

Lemma 4.6. *Let $R > 0$ and $\underline{U} = (\underline{v},\underline{w})$ such that $\|\,\underline{U}\,\|_{\mathbb{B}^1(0,T_0)} \le R$. Then, there is a constant $C_0(R)$ such that for all $U \in \mathcal{C}_0^\infty(\mathbb{R}^{1+d})$, for all $\varepsilon \in]0,1]$,* *and for all* $T \in [0, T_0]$

labelle

$$|U\|_{0,\varepsilon,T} \le C_0(R) \int_0^T \|\mathcal{H}(\underline{U}, \partial) U(s)\|_{L^2(\mathbb{R}^d)}\, ds + C_0(R)\, |U\|_{0,\varepsilon,0}\,. \tag{4.18}$$

Proof. A Taylor expansion shows that

2.7

$$\mu(v + \varepsilon\, v') = \mu(v) + \varepsilon\, \tilde\mu(\varepsilon, v, v')\,, \qquad \forall (v, v', \varepsilon) \in (\mathbb{R}^{N''})^2 \times]0,1] \tag{4.19}$$

where $\tilde\mu(\cdot,\cdot,\cdot)$ is a $\mathcal{C}^\infty$ function of its arguments. It follows that

$$\begin{aligned} \mathbf{X}_{v_0+\varepsilon\underline{v}} &= \mathbf{X}_{v_0} + \tilde\mu(\varepsilon, v_0, \underline{v}) \cdot \varepsilon\nabla_x \\ &= \mathcal{T}_0 + \sum_{1\le j\le d} \tilde\mu_j(\varepsilon, v_0, \underline{v})\, \varepsilon\partial_j. \end{aligned} \tag{4.20}$$

Therefore, the field $\mathbf{X}_{v_0+\varepsilon\underline{v}}$ is a linear combination of fields in $\mathbf{Z}_\varepsilon$. Using a similar Taylor expansion of the coefficients of the operator $\mathbf{L}(v_0 + \varepsilon\underline{v}, \partial_x)$, we write the equation in the following form:

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$$\sum_{\mathcal{Z}_\varepsilon \in \mathbf{Z}_\varepsilon} \mathbf{S}_{\mathcal{Z}_\varepsilon}(\varepsilon, v_0, \underline{U})\, \mathcal{Z}_\varepsilon U + \mathbf{L}(v_0, \partial_x) U + \mathbf{K}(v_0, \partial v_0, \underline{U}) U = h \tag{4.21}$$

where the matrices $\mathbf{S}_{\mathcal{Z}}(\varepsilon, v, U)$ are $\mathcal{C}^\infty$ functions of their arguments ε, v, U up to $\varepsilon = 0$. Now we proceed in the usual way, taking the product of U and of the equation (4.21) and integrating by parts. Because of the special form (4.21) and because $\|\underline{U}\|_{\mathbb{B}^1(0,T_0)} \le R$, we get a uniform control of the derivatives of the coefficients, and therefore

$$\begin{aligned} |U\|_{0,\varepsilon,T} \le{}& c(R) \int_0^T |U\|_{0,\varepsilon,s}\, ds \\ &+ \int_0^T \|\mathcal{H}(\underline{U}, \partial) U(s,\cdot)\|_{L^2}\, ds + |U\|_{0,\varepsilon,0}\,. \end{aligned} \tag{4.22}$$

Note that the estimate above strongly relies on the existence of a *good symmetrizor* : the coefficients of $\mathbf{L}(v_0, \partial_x)$ have bounded derivatives because the operator $\mathbf{L}$ does not involve the variable w. Using the Gronwall lemma we deduce

$$|U\|_{0,\varepsilon,T} \le \int_0^T e^{c(R)(T-s)}\, \|\mathcal{H}(\underline{U}, \partial) U(s)\|_{L^2}\, ds + e^{c(R)T}\, |U\|_{0,\varepsilon,0} \tag{4.23}$$

which implies (4.18) with $c_0(R) = e^{c(R)T_0}$. □

4.3.3 Gagliardo-Nirenberg-Moser inequalities

To estimate the derivatives of U we will use Gagliardo-Nirenberg's and Moser's estimates. Let us introduce some notations. For all open subset $\mathcal{O}$ of Ω, all $m \in \mathbb{N}$ and all $p \in [1, \infty]$, we note

$$\|u\|_{\mathbb{L}^{m,p}_\varepsilon(\mathcal{O})} := \sum_{0 \leq k \leq m} \ \sum_{\mathcal{Z}^k_\varepsilon \in \mathbf{Z}^k_\varepsilon} \| \ \mathcal{Z}^k_\varepsilon u \|_{L^p(\mathcal{O})}$$

and

$$\mathbb{L}^{m,p}(\mathcal{O}) := \big\{ \, u^\varepsilon \in L^2(\mathcal{O}) \, ; \ \sup_{0 < \varepsilon \leq 1} \ \|u^\varepsilon\|_{\mathbb{L}^{m,p}_\varepsilon(\mathcal{O})} \, < \, \infty \, \big\} \, .$$

Recall that for all $T > -T_0$, we note $\Omega_T :=] - T_0, T[\times \mathbb{R}^d$. The following Galiardo-Nirenberg estimates hold.

Lemma 4.7. *Let $m \in \mathbb{N}$ with $m \geq 1$. For all $T \in [0, T_0]$ and for all u in the space $L^\infty(\Omega_T) \cap \mathbb{L}^{m,2}(\Omega_T)$*

$$\|\mathcal{Z}^k_\varepsilon u\|_{L^{2m/k}(\Omega_T)} \, \leq \, c_m \, \|u\|^{1-k/m}_{L^\infty(\Omega_T)} \, \|u\|^{k/m}_{\mathbb{L}^{m,2}_\varepsilon(\Omega_T)} \tag{4.24}$$

for all $k \in \{0, \cdots, m\}$, for all $\mathcal{Z}^k_\varepsilon \in \mathbf{Z}^k_\varepsilon$ and for all $\varepsilon \in [0, 1]$.

Proof. It is a special case of the inequality (Ap-II-3), of [10], p. 643. □

The following Moser's estimates follow.

Moser **Lemma 4.8.** *Let $m \in \mathbb{N}$. There is $c_m > 0$ such that for all $T \in [0, T_0]$, for all functions $a_1, \cdots, a_p$ in $H^m(\Omega_T) \cap L^\infty(\Omega_T)$, and for all $\varepsilon \in]0, 1]$*

$$\|\mathcal{Z}^{k_1}_\varepsilon a_1 \cdots \mathcal{Z}^{k_p}_\varepsilon a_p\|_{L^2(\Omega_T)} \, \leq \, c_m \, \sum_k \, \big(\prod_{j \neq k} \|a_j\|_{L^\infty(\Omega_T)} \big) \, \|a_k\|_{\mathbb{L}^{m,2}_\varepsilon(\Omega_T)} \tag{4.25}$$

where $k_1 + \cdots + k_p \leq m$ and $\mathcal{Z}^{k_j}_\varepsilon \in \mathbf{Z}^{k_j}_\varepsilon$ if $j \in \{1, \cdots, p\}$.

4.3.4 Higher order estimates

Next we want estimates for the derivatives $\mathcal{Z}^k_\varepsilon U$. As usual, we apply the L^2 estimate (4.18) to the term $\mathcal{Z}^k_\varepsilon U$ which satisfies the equation

$$\mathcal{H}^\varepsilon(\underline{U}, \partial) \, \mathcal{Z}^k_\varepsilon U \, = \, \mathcal{Z}^k_\varepsilon h \, + \, [\mathcal{H}^\varepsilon(\underline{U}, \partial) ; \mathcal{Z}^k_\varepsilon] \, U \, .$$

Therefore,

$$
\begin{aligned}
|U\|_{m,\varepsilon,T} \;\le\; & C_0(R)\; T\; |h\|_{m,\varepsilon,T} \\
& + C(R)\; T^{1/2}\; \|[\mathcal{H}^\varepsilon(\underline{U},\partial);\mathcal{Z}_\varepsilon^k]U\|_{L^2(\omega_T)} \;+\; C_0(R)\; |U\|_{m,\varepsilon,0}\,.
\end{aligned}
\tag{4.26}
$$

Hence, we are lead to estimate in $L^2(\omega_T)$ the commutator $[\mathcal{H}^\varepsilon(\underline{U},\partial);\mathcal{Z}_\varepsilon^k]\,U$. Here we make use, for the first time, of *the compatibility assumption on the data.* Let us begin with a lifting lemma.

relvt **Proposition 4.9.** *Let $v_0 \in \Lambda_0^m$, $v_1 \in \mathbb{A}_0^{m-1} \cap \mathbb{B}_0^0$, $v_2 \in \mathbb{A}_0^{m-2}, \cdots, v_m \in \mathbb{A}_0^0$. There exists $V \in \Lambda^m(-T_0,T_0)$ such that*

traces

$$
\big((\mathbf{X}_{v_0})^k\, V\big)_{|t=0} = v_k\,, \qquad \forall\, k \in \{0,\cdots,m\}\,.
\tag{4.27}
$$

Proof. We show that there is $V \in \Lambda^m(0,T_0)$ satisfying (4.27). There is a similar lifting to $[-T_0,0]$, and since the traces are equal on $\{t=0\}$, the two functions can be glued together. We follow the proof of classical lifting theorem for $\mathcal{C}^k$ spaces ([14], Corollary 1.3.4, p. 18). Let $m \in \mathbb{N}$ be fixed. We show by induction on $r = 0,\cdots,m$ that there exists $V^r \in \Lambda^m(-T_0,T_0)$ satisfying (4.27) for $0 \le k \le r$. For $r=0$ we take $V^0(t,x) = v_0(x)$. For $1 \le r \le m$, assuming that V^{r-1} is known, we look for V^r of the form $V^r = V^{r-1} + U$ where we want $U \in \Lambda^m(-T_0,T_0)$ to satisfy on $t=0$

$$
\begin{aligned}
&(\mathbf{X}_{v_0})^k\, U = 0\,, \qquad \forall\, k \in \{0,\cdots,r-1\}\,.\\
&(\mathbf{X}_{v_0})^r\, U = v_r - (\mathbf{X}_{v_0})^r\, V^{r-1} := w \in \mathbb{A}_0^{m-r}\,.
\end{aligned}
\tag{4.28}
$$

Let us introduce a function $j \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $\int j(x)\,dx = 1$, and note $j_\gamma(x) := \gamma^{-d}\, j(x/\gamma)$ for all $\gamma \in \mathbb{R}_*$. We are going to show that if δ is fixed small enough in the interval $]0,1]$, the function

$$
U(t,x) \;:=\; \frac{t^r}{r!}\; j_{t^\delta} * w
$$

satisfies the desired conditions. Denote by $\widehat{\ }$ the Fourier transformation with respect to the variables $x \in \mathbb{R}^d$. We have

$$
\begin{aligned}
\widehat{U}(t,\xi) \;&=\; \frac{t^r}{r\,!}\; \widehat{j}(t^\delta\xi)\; \widehat{w}(\xi)\\
\partial_t\widehat{U}(t,\xi) \;&=\; \frac{t^{r-1}}{(r-1)\,!}\; \widehat{j}(t^\delta\xi)\; \widehat{w}(\xi) \;+\; \delta\;\; t^\delta\; \frac{t^{r-1}}{r\,!}\; (\xi\cdot\nabla_\xi\widehat{j})(t^\delta\xi)\; \widehat{w}(\xi)
\end{aligned}
$$

which implies

$$
|\,\partial_t\widehat{U}(t,\xi)\,| \;\le\; c_1\; t^{r-1}\; j_{(1)}(t^\delta\xi)\; |\,\widehat{w}(\xi)\,|
$$

where $c_1 \in \mathbb{R}_*^+$ and $j_{(1)} \in \mathcal{S}(\mathbb{R}^d;\mathbb{R}^+)$. By induction we also get that for all $p \in \mathbb{N}$ such that $p \leq r$

$$|\,\partial_t^p \widehat{U}(t,\xi)\,| \,\leq\, c_p\; t^{r-p}\; j_{(p)}(t^\delta \xi)\; |\,\widehat{w}(\xi)\,|$$

for some constant $c_p \in \mathbb{R}_*^+$ and $j_{(p)} \in \mathcal{S}(\mathbb{R}^d;\mathbb{R}^+)$. It follows that

$$|\,\xi^\alpha\;\partial_t^p \widehat{U}(t,\xi)\,| \,\leq\, c'_p\; t^{r-p-\delta|\alpha|}\;|\,\widehat{w}(\xi)\,|\,, \qquad \forall\,\alpha \in \mathbb{N}^d$$

where $c'_p = c_p\;\sup_\xi\,\big(|\xi|^{|\alpha|}\; j_{(p)}(\xi)\,\big)$. Using the theorem of dominated convergence, we deduce that $\partial_t^p U$ belongs to

$$\mathcal{C}\big([-T_0,T_0];H^{(r-p)/\delta}\big)$$

and is bounded uniformly with respect to $\varepsilon \in]0,1]$ in this space (remember that U as w depend on ε). We select $\delta \in]0,1/m[$. It is then obvious that U belongs also to $\mathbb{A}^m(-T_0,T_0)$. Moreover, choosing δ small enough in order that $(r-1)/\delta > d/2$, the Sobolev embedding theorem implies that U is contained in $\mathbb{B}^1(-T_0,T_0)$. It follows that U is indeed in $\Lambda^m(-T_0,T_0)$ and that the traces satisfy the expected relations. □

An important consequence of this proposition and of the compatibility conditions is that the functions $\underline{U}$ and U extend to $t<0$ as functions $\underline{\widetilde{U}}$ and $\widetilde{U}$ which both belong to $\mathbb{L}^{m,2}(\Omega_T) \cap \mathbb{L}^{1,\infty}(\Omega_T)$ and such that

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$$\underline{\widetilde{U}}_{|t<0} \,=\, \widetilde{U}_{|t<0} \,=\, \mathcal{V} \tag{4.29}$$

where $\mathcal{V}$ is given by the proposition 4.9 and depends only on the Cauchy data U^0 (and on the choice of the lifting operator in proposition 4.9). Since ω_T is contained in Ω_T we have

$$\|\,[\mathcal{H}^\varepsilon(\underline{U},\partial);\mathcal{Z}_\varepsilon^k]\,U\,\|_{L^2(\omega_T)} \,\leq\, \|\,[\mathcal{H}^\varepsilon(\underline{\widetilde{U}},\partial);\mathcal{Z}_\varepsilon^k]\,\widetilde{U}\|_{L^2(\Omega_T)}\,. \tag{4.30}$$

The commutator $[\mathcal{H}^\varepsilon(\underline{\widetilde{U}},\partial);\mathcal{Z}_\varepsilon^k]\,\widetilde{U}$ writes

$$\begin{aligned}\sum_{\mathcal{Z}\in\mathbf{Z}} [\mathbf{S}_\mathcal{Z}(\varepsilon,v_0,\underline{\widetilde{U}})\mathcal{Z};\mathcal{Z}^k]\,\widetilde{U} \,+\, [\mathbf{L}(v_0,\partial_x);\mathcal{Z}_\varepsilon^k]\,\widetilde{U}& \\ +\, [\mathbf{K}(v_0,\partial v_0,\underline{\widetilde{U}});\mathcal{Z}_\varepsilon^k]\,\widetilde{U}\,.&\end{aligned} \tag{4.31}$$

Each term of the form $[\mathbf{S}_\mathcal{Z}(\varepsilon,v_0,\underline{\widetilde{U}})\mathcal{Z};\mathcal{Z}^k]\widetilde{U}$ is a sum of terms like

$$\Phi(\varepsilon,v_0,\underline{\widetilde{U}})\;\mathcal{Z}^{k_1}\underline{\widetilde{U}}_{i_1}\;\cdots\;\mathcal{Z}^{k_p}\underline{\widetilde{U}}_{i_p}\;\mathcal{Z}^{k_{p+1}}\widetilde{U}_{i_{p+1}}$$

where $\mathcal{Z}^{k_j} \in \mathbf{Z}^{k_j}$ for $1 \le j \le p+1$, with $k_1 + \cdots + k_{p+1} \le k+1$, $k_{p+1} \le k$, $k_1 \ge 1, \dots, k_{p+1} \ge 1$. We write this term as

$$\Phi(\varepsilon, v_0, \underline{\widetilde{U}})\ \mathcal{Z}^{k'_1}(\mathcal{Z}\underline{\widetilde{U}}_{i_1}) \cdots \mathcal{Z}^{k'_p}(\mathcal{Z}\underline{\widetilde{U}}_{i_p})\ \mathcal{Z}^{k'_{p+1}}(\mathcal{Z}\widetilde{U}_{i_{p+1}})$$

and with the lemma [Moser] 4.8, we get

$$\begin{aligned} &\| [\mathbf{S}_{\mathcal{Z}}(\varepsilon, v_0, \underline{\widetilde{U}})\mathcal{Z}; \mathcal{Z}^k]\, \widetilde{U} \,\|_{L^2(\Omega_T)} \le \\ &\qquad C(R)\, \big(\|\widetilde{U}\|_{\mathbb{L}^{m,2}(\Omega_T)} + (1 + \|\widetilde{U}\|_{\mathbb{L}^{1,\infty}(\Omega_T)})\ \|\underline{\widetilde{U}}\|_{\mathbb{L}^{m,2}(\Omega_T)} \,\big)\,. \end{aligned}$$

By using ([coincidencedanslepasse] 4.29), we deduce the following inequality where ρ is some > 0 constant depending only on $\|\mathcal{V}\|_{\mathbb{L}^{m,2}(\Omega_0)}$ and $\|\mathcal{V}\|_{\mathbb{L}^{1,\infty}(\Omega_T)}$, or in other words only on the norm of the initial data U_0^ε in Λ_0^m

5.19 (4.32)
$$\begin{aligned} \|[\mathbf{S}_{\mathcal{Z}}(\varepsilon, v_0, \underline{\widetilde{U}})\, \mathcal{Z}, \mathcal{Z}^k]\, \widetilde{U}\|_{L^2(\Omega_T)} \le\ & C(R)\, \big(\|U\|_{\mathbb{L}^{m,2}(\omega_T)} \\ & + \rho + (\rho + \|U\|_{\mathbb{L}^{1,\infty}(\omega_T)})\ (\rho + \|\underline{U}\|_{\mathbb{L}^{m,2}(\omega_T)}) \big)\,. \end{aligned}$$

Let us also quote the obvious estimate

CS (4.33)
$$\|v\|_{\mathbb{L}_\varepsilon^{m,2}(\omega_T)} \le T^{1/2}\ |v\|_{m,\varepsilon,T}\,, \qquad \forall\, v \in H^m(\Omega_T)\,, \qquad \forall\, \varepsilon \in]0,1]\,.$$

We deduce from ([5.19] 4.32) and ([CS] 4.33) the following control, which is uniform with respect to $\varepsilon \in]0,1]$, where ρ is again a constant depending only on U_0^ε

$$\begin{aligned} \| [\mathbf{S}_{\mathcal{Z}}(\varepsilon, v_0, \underline{\widetilde{U}})\, \mathcal{Z}; \mathcal{Z}^k]\, \widetilde{U} \,\|_{L^2(\Omega_T)} \le\ & C(R)\, \big(T^{1/2}\ |U\|_{m,\varepsilon,T} + \rho \big) \\ & + C(R)\, \big(\rho + |\underline{U}|^*_{1,\varepsilon,T} \|U\|_{\mathbb{L}^{1,\infty}(\omega_T)} \big)\ (\rho + T^{1/2}\ |U\|_{m,\varepsilon,T})\,. \end{aligned} \tag{4.34}$$

The control of $[\mathbf{L}(v_0, \partial_x); \mathcal{Z}_\varepsilon^k]U$ is less easy. Let us note

$$\mathcal{N}(t,x,\partial_x) := |\nabla_x \varphi|^{-1} \sum_{j=1}^d (\partial_j \varphi) \cdot \partial_j\,, \qquad (t,x) \in \Omega\,.$$

The vector field $\mathcal{N}(t,x,\partial_x)$ is transverse to the characteristic foliation $\{\varphi = cte\}$. It can also be seen as a vector field in $\mathbb{R}^d$, parametrized by t, and normal to the hypersurfaces $\{\varphi(t,\cdot) = cte\}$ of $\mathbb{R}^d$. We recall that $\partial_x \varphi(t,x) \neq 0$ for all $(t,x) \in \Omega$, and that the coefficients of $\mathcal{N}$ belong to $\mathcal{C}_b^\infty(\Omega)$. For all $v \in \mathbb{R}^{N''}$, the operator $\mathbf{L}(v, \partial_x)$ writes

$$\mathbf{L}(v, \partial_x) = \mathbf{L}\big(v, \frac{\partial_x \varphi}{|\partial_x \varphi|}\big)\ \mathcal{N}(t,x,\partial_x) + \sum_{1 \le j \le d} M_j(v)\ \mathcal{T}_j(t,x,\partial_t,\partial_x)$$

where the $M_j(v)$ are symmetric with $\mathcal{C}^\infty$ coefficients. The matrix $\mathbf{L}(v,\xi)$ is symmetric and has a constant rank p with $p \le N-1$, on $\mathbb{R}^{N''} \times (\mathbb{R}^d \setminus \{0\})$. Hence there exists a $\mathcal{C}^\infty$ invertible matrix $\mathbf{\Phi}(t,x)$ such that

$$ {}^t\Phi(t,x)\ \mathbf{L}\big(v_0, \frac{\partial_x\varphi}{|\partial_x\varphi|}\big)\ \Phi(t,x) \ = \ \Gamma $$

where Γ is the constant $N \times N$ matrix

MatriceGamma (4.35)
$$ \Gamma \ = \ \begin{pmatrix} \Gamma^\flat & 0 \\ 0 & 0 \end{pmatrix} . \tag{4.35} $$

Introduce the unknown $U'(t,x) = \Phi(t,x)^{-1}\, U(t,x)$. Then express through the equation (4.15) [pblineaire] the quantity $\mathcal{N}(\partial)\,\Gamma\, U'$ in term of the $\mathcal{Z}_\varepsilon U'$ and of the right hand side h. Finally, we get the following result.

prop 2.3 **Proposition 4.10.** *Let $m \in \mathbb{N}$ with $m > d/2+1$. For all $R > 0$, there is a constant $C_m(R)$ satisfying what follows. Suppose that $\underline{U} \in \mathbf{W}^m(0,T_0)$ satisfies the relations*

$$ (\mathbf{X}^k_{v_0}\, \underline{U})_{|t=0} \ = \ \mathcal{F}_k\big(\varepsilon, 0, x, \, (\partial_x^\alpha V_0^\varepsilon)_{|\alpha|\le k}\, , \, (\partial^\beta_{t,x} h^\varepsilon_{|t=0})_{|\beta \le k}\,\big) $$

for $0 \le k \le m$, together with $|\underline{U}|^*_{1,\varepsilon,T_0} < R$. Then, the solution U of the linear problem (4.15) [pblineaire] satisfies the following estimate

$$ \begin{aligned} |U\|_{m,\varepsilon,T} \ \le & C_m(R)\ T\ \big(\, |h\|_{m,\varepsilon,T} \ + \ (1 + |U|^*_{1,\varepsilon,T})\ |U\|_{m,\varepsilon,T}\,\big) \\ & + \ C_m(R)\ T^{1/2}\ (1 + |U|^*_{1,\varepsilon,T}) \ + \ C_m(R)\ |U\|_{m,\varepsilon,0} \end{aligned} \tag{4.36} $$

for all $T \in [0,T_0]$ and for all $\varepsilon \in]0,1]$.

4.3.5 L^∞ estimates

For $v \in \mathbb{R}^{N''}$ and $\xi \in \mathbb{R}^d \setminus \{0\}$, we denote by $\mathbf{Q}(v,\xi)$ the pseudo inverse of $\mathbf{L}(v,\xi)$, that is the matrix such that

$$ \mathbf{Q}(v,\xi)\,\mathbf{L}(v,\xi) \ = \ \mathbf{L}(v,\xi)\,\mathbf{Q}(v,\xi) \ = \ \mathrm{Id} - \mathbf{P}(v,\xi)\,. \tag{4.37} $$

To simplify notations, set

$$ \mathbf{P}_0 \ := \ \mathbf{P}\big(v_0(t,x), \partial_x\varphi(t,x)\big)\,, \qquad \mathbf{Q}_0 \ := \ \mathbf{Q}\big(v_0(t,x), \partial_x\varphi(t,x)\big)\,. $$

Lemma 4.11.

$$ \mathbf{P}(v,\xi)\,\mathbf{L}(v,\eta)\,\mathbf{P}(v,\xi) \ = \ 0\,, \qquad \forall\,(v,\xi,\eta) \in \mathbb{R}^{N"} \times (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d\,. $$

Proof. The matrices satisfy

$$\mathbf{P}(v,\xi)\,\mathbf{L}(v,\xi)=0\,,\qquad \mathbf{L}(v,\xi)\,\mathbf{P}(v,\xi)=0\,,\qquad \forall\,(v,\xi)\in\mathbb{R}^{N"}\times\mathbb{R}^d\,.$$

Differentiating the relation $\mathbf{P}\,\mathbf{L}\,\mathbf{P}=0$ with respect to ξ_j gives

$$(\partial_{\xi_j}\mathbf{P})\,\mathbf{L}\,\mathbf{P}+\mathbf{P}\,(\partial_{\xi_j}\mathbf{L})\,\mathbf{P}+\mathbf{P}\mathbf{L}\,(\partial_{\xi_j}\mathbf{P})\;=\;0\,.$$

Since the first term and the third term in this sum are nul, it only remains the second term. And since the matrix $\mathbf{L}$ has the form

$$\mathbf{L}(v,\xi)\;=\;\xi_1\,B_1(v)\;+\;\cdots\;+\;\xi_d\,B_d(v)\,,$$

we get

$$\mathbf{P}(v,\xi)\,B_j(v)\,\mathbf{P}(v,\xi)\;=\;0\,,\qquad \forall\,(v,\xi)\in\mathbb{R}^{N"}\times\mathbb{R}^d\,.$$

This relation being true for $j=1,\cdots,d$, the lemma is proved. □

Using a Taylor expansion of the matrix $\mathbf{L}(v_0+\varepsilon\,\underline{v},\partial_x\varphi/|\partial_x\varphi|)$ and multiplying the equation by $\mathbf{Q}_0$ on the left, we find that U satisfies

$$\begin{aligned}\mathcal{N}(t,x,\partial_x)\big(\mathrm{Id}-\mathbf{P}_0\big)U\;&=\mathbf{Q}_0h+\sum_{\mathcal{Z}_\varepsilon\in\mathbf{Z}_\varepsilon}\mathbf{M}_{\mathcal{Z}}(\varepsilon,t,x,\underline{U})\;\mathcal{Z}_\varepsilon U\\&\quad+\;N(\varepsilon,t,x,\underline{U})\,U\end{aligned}\tag{4.38}$$

where the matrices $\mathbf{M}_{\mathcal{Z}}$ and N are $\mathcal{C}^\infty$ functions.

2.4 **Lemma 4.12.** *Let $m\in\mathbb{N}$ such that $m_0>d/2$. There is $c>0$ such that for all $T\in]0,T_0]$, all $u\in\mathbb{A}^{m_0}(0,T)$ the following inequality holds*

$$|u|^*_{0,\varepsilon,T}\;\leq\;c\;(|u\|_{m_0,\varepsilon,T}+|\mathcal{N}(t,x,\partial_x)u|_{m_0-1,\varepsilon,T})\,,\qquad \forall\,\varepsilon\in]0,1]\,.$$

It follows from lemma 2.4 4.12 that

$$\begin{aligned}\|(\mathrm{Id}-\mathbf{P}_0)U(t,.)\|_{L^\infty}\;&\leq\;c(R)\;|U\|_{m_0,\varepsilon,T}\\&\quad+\;c(R)\;\big(\,|h\|_{m_0,\varepsilon,T}\;+\;|\underline{U}\|_{m_0,\varepsilon,T}\;\|U(t,\cdot)\|_{L^\infty}\,\big)\,.\end{aligned}$$

Repeating the arguments for the ε-tangent (or ε-conormal) derivatives gives:

lem2.6 **Lemma 4.13.** *Let $m_0>d/2+2$. For all $\mathcal{Z}^k_\varepsilon\in\mathbf{Z}^k_\varepsilon$ and for all $k\in\{0,1,2\}$, the following holds*

$$\begin{aligned}\|\mathcal{Z}^k_\varepsilon(\mathrm{Id}-\mathbf{P}_0)U(t)\|_{L^\infty}\;&\leq c_q(R)\;|U\|_{m,\varepsilon,T}\\&\quad+\;c_q(R)\;\big(\,|h\|_{m,\varepsilon,T}\;+\;|\underline{U}\|_{m,\varepsilon,T}\;\|U(t)\|_{L^\infty}\,\big)\,.\end{aligned}$$

Let us write $U = U_I + U_{II}$ with

$$(4.39)\qquad U_I := \mathbf{P}\big(v_0+\varepsilon\,\underline{v}, \frac{\partial_x\varphi}{|\partial_x\varphi|}\big)\,U\,, \qquad U_{II} := \Big(\mathrm{Id}-\mathbf{P}\big(v_0+\varepsilon\,\underline{v}, \frac{\partial_x\varphi}{|\partial_x\varphi|}\big)\Big)\,U\,.$$

Multiplying on the left the equation (5.1 2.47) by the matrix

$$\mathbf{P}\big(v_0+\varepsilon\,\underline{v}, \frac{\partial_x\varphi}{|\partial_x\varphi|}\big)$$

leads to the following equation for U_I:

$$(4.40)\qquad \begin{aligned} S_I(v_0+\varepsilon\,\underline{v},\underline{w})\,\mathbf{X}_{v_0+\varepsilon\underline{v}}\,U_I \;=\; &\mathbf{P}\big(v_0+\varepsilon\,\underline{v}, \frac{\partial_x\varphi}{|\partial_x\varphi|}\big)\,h \\ &-\sum_{1\le j\le d}\mathbf{P}\big(v, \frac{\partial_x\varphi}{|\partial_x\varphi|}\big)\;M_j(v_0+\varepsilon\,\underline{v})\;\mathcal{T}_j U_{II} \\ &-\mathbf{X}_{v_0}U_{II}\;-\;\tilde\mu(\varepsilon,\underline{v})\cdot\varepsilon\partial_x U_{II} \end{aligned}$$

where $\tilde\mu$ is defined in (2.7 4.19) and we have used the notations for $U=(v,w)$

$$(4.41)\qquad S_I(U) := \mathbf{P}\big(v, \frac{\partial_x\varphi}{|\partial_x\varphi|}\big)\;S(U)\;\mathbf{P}\big(v, \frac{\partial_x\varphi}{|\partial_x\varphi|}\big)\,.$$

The matrix S being positive definite, the matrix $S_I(U)$ is invertible on the range of $\mathbf{P}(v, \frac{\partial_x\varphi}{|\partial_x\varphi|})$. Therefore, writing $\mathbf{P}$ instead of $\mathbf{P}(v_0+\varepsilon\underline{v}, \frac{\partial_x\varphi}{|\partial_x\varphi|})$, there holds:

$$(4.42)\qquad \mathbf{P}\,\mathbf{X}_{v_0+\varepsilon\underline{v}}\,U_I \;=\; T_I(\varepsilon,v_0,\underline{U})\;S_I(v_0+\varepsilon\underline{v},\underline{w})\;\mathbf{X}_{v_0+\varepsilon\underline{v}}\,U_I$$

where T is a pseudo inverse of $S_I(U)$, $\mathcal{C}^\infty$ function of its arguments $(\varepsilon, v_0, \underline{U})$. Moreover, since

$$\mathbf{X}_{v_0+\varepsilon\underline{v}}\,U_I \;=\; \mathbf{P}\;\mathbf{X}_{v_0+\varepsilon\underline{v}}\,U_I + \big(\mathbf{X}_{v_0+\varepsilon\underline{v}}(\mathbf{P})\big)\cdot U_I$$

we deduce that U_I satisfies

2.18
$$(4.43)\qquad \begin{aligned} \mathbf{X}_{v_0+\varepsilon\underline{v}}\,U_I \;=\; &F(\varepsilon,t,x,\underline{U})\;h \\ &+\sum_{\mathcal{Z}\in\mathbf{Z}_\varepsilon} C_{\mathcal{Z}}(\varepsilon,t,x,\underline{U})\;\mathcal{Z}U_{II}\;+\;C_0(\varepsilon,t,x,\underline{U})\,U \end{aligned}$$

where F, $C_{\mathcal{Z}}$ and C_0 are again $\mathcal{C}^\infty$ matrices, bounded with all their derivatives on $[0,1]_\varepsilon\times\Omega\times K$, for all compact subset K of $\mathbb{R}^N$. By integration along the vector field $\mathbf{X}_{v_0+\varepsilon \underline{v}}$, and using the assumption $\|\underline{U}\|_{\mathbb{B}^1(0,T_0)} \leq R$, we deduce the following L^∞ inequality

$$|U_I|^*_{0,\varepsilon,T} \;\leq\; c(R)\; T\; \big(\, |h|^*_{0,\varepsilon,T} \;+\; |U|^*_{0,\varepsilon,T} \;+\; |U_{II}|^*_{1,\varepsilon,T} \,\big) \;+\; |U_I|^*_{0,\varepsilon,0}\,. \tag{4.44}$$

Applying the field $\mathcal{Z}_\varepsilon \in \mathbf{Z}_\varepsilon$ to the equation (4.43) gives

$$\begin{aligned} \mathbf{X}_{v_0+\varepsilon \underline{v}}\; \mathcal{Z}_\varepsilon U_I \;&=\; f \;+\; \mathit{coeff} \cdot \mathcal{Z}_\varepsilon h \\ &\quad+\; \mathit{coeff} \cdot \mathcal{Z}_\varepsilon U \;+\; \sum_{\mathcal{Z} \in \mathbf{Z}_\varepsilon} \;\mathit{coeff} \cdot \mathcal{Z} U \\ &\quad+\; \sum_{\mathcal{Z}^2_\varepsilon \in \mathbf{Z}^2_\varepsilon} \;\mathit{coeff} \cdot \mathcal{Z}^2_\varepsilon U_{II} \end{aligned} \tag{4.45}$$

where the notation *coeff* means a matrix with entries $\mathcal{C}^\infty$ in $\varepsilon, t, x, \underline{U}$, as in the equation (4.43). The following equation follows

$$|U_I|^*_{1,\varepsilon,T} \;\leq\; c(R)\; T\; \big(\, |h|^*_{1,\varepsilon,T} \;+\; |U|^*_{1,\varepsilon,T} \;+\; |U_{II}|^*_{2,\varepsilon,T} \,\big) \;+\; |U_I|^*_{1,\varepsilon,0}\,.$$

In order to estimate $|U_{II}|^*_{2,\varepsilon,T}$, we write

$$U_{II} = \Big(\, \mathbf{P}_0 - \mathbf{P}\big(v_0 + \varepsilon\, \underline{v}, \frac{\partial_x \varphi}{|\partial_x \varphi|}\big) \Big)\, U \;+\; (I - \mathbf{P}_0) U\,. \tag{4.46}$$

The second term of this sum is controled by Lemma 4.13. A Taylor expansion shows that in order to control the derivatives in $\mathcal{Z}^2$ in the first term of the sum, it is sufficient to find a bound in L^∞ of terms of the form, using obvious notations,

2.22
$$\varepsilon\; \underline{U}\; \mathcal{Z}^2 U\,, \qquad \varepsilon\; \mathcal{Z}\underline{U}\; \mathcal{Z} U\,, \qquad \varepsilon\; \mathcal{Z}^2 \underline{U}\; U\,. \tag{4.47}$$

This last point follows from the standard Sobolev embedding theorem which implies that for $m_0 > d/2$ there is a constant $c > 0$ such that for all $T \in [0, T_0]$, and for all $\varepsilon \in]0,1]$:

$$|\varepsilon\, v|^*_{0,\varepsilon,T} \;\leq\; c\; |v\|_{m_0,\varepsilon,T}\,, \qquad \forall\, v \in \mathbb{A}^{m_0}(0,T)\,. \tag{4.48}$$

Hence, if $|\underline{U}|^*_{1,\varepsilon,T} \leq R$, each one of the terms (4.47) is bounded by

$$c(R)\; (|\underline{U}\|_{m_0+2,\varepsilon,T}\; |U|^*_{0,\varepsilon,T} \;+\; |U\|_{m_0+2,\varepsilon,T})\,.$$

Hence we have proved that

$$|U_{II}|^*_{2,\varepsilon,T} \;\leq\; c(R)\; \big(\, |U\|_{m,\varepsilon,T} \;+\; |h\|_{m,\varepsilon,T} \;+\; |\underline{U}\|_{m,\varepsilon,T}\; |U\|^*_{0,\varepsilon,T} \,\big)\,. \tag{4.49}$$

Furthermore, integrating along the characteristics of the field $\mathbf{X}_{v_0}$ gives

$$|U_{II}|^*_{1,\varepsilon,T} \;\leq\; T\; |U_{II}|^*_{2,\varepsilon,T} \;+\; |U_{II}|^*_{1,\varepsilon,0}\,.$$

Summarizing, we have proved the following result.

lem 2.7 **Proposition 4.14.** *Let $m > d/2 + 2$. The following estimates hold*

$$
\begin{aligned}
|U_I|^*_{1,\varepsilon,T} \;&\leq\; c(R)\; T\; \big(\, |h|^*_{1,T} \,+\, |U|^*_{1,T} \,+\, |U_{II}|^*_{2,\varepsilon,T} \big) \,+\, |U_I|^*_{1,\varepsilon,0}\,,\\
|U_{II}|^*_{1,\varepsilon,T} \;&\leq\; T\; |U_{II}|^*_{2,\varepsilon,T} \,+\, |U_{II}|^*_{1,\varepsilon,0}\,,\\
|U_{II}|^*_{2,\varepsilon,T} \;&\leq\; c(R)\; \big(\, \|U\|_{m,\varepsilon,T} \,+\, |h\|_{m,\varepsilon,T} \,+\, |\underline{U}\|_{m,\varepsilon,T}\; |U|^*_{0,\varepsilon,T} \big)\,.
\end{aligned}
$$

4.3.6 End of the proof of theorem 4.2

Observe that

5.35 (4.50)

$$
\begin{aligned}
\|U\|_{m,\varepsilon,0} \;&\leq\; \kappa \sum_{0\leq k\leq m} \|\big(\mathbf{X}^k_{u_0} U\big)_{|\{t=0\}}\|_{\mathbb{A}^{m-k}_0} + \kappa\; \|U^\varepsilon_0\|_{0,\varepsilon,0}\\
&+ \kappa \sum_{1\leq k\leq m} \|\mathcal{F}_k\big(\varepsilon, 0, x, (\partial^\alpha_x U^\varepsilon_0\,)_{|\alpha|\leq k}, (\partial^\beta_{t,x} h^\varepsilon{}_{t=0})_{|\beta|\leq k-1}\big)\|_{\mathbb{A}^{m-k}_0}\,,
\end{aligned}
$$

and

5.36 (4.51)

$$
\begin{aligned}
\|U\|^*_{1,\varepsilon,0} \;&\leq\; \kappa' \sum_{0\leq k\leq 1} \|\big(\mathbf{X}^k_{u_0} U\big)_{|\{t=0\}}\|_{\mathbb{B}^{1-k}_0}\\
&\leq\; \kappa'\; \big(\|U^\varepsilon_0\|_{\mathbb{B}^1_0} \,+\, \|\mathcal{F}_1\big(\varepsilon, 0, x, (\partial^\alpha_x U^\varepsilon_0\,)_{|\alpha|\leq 1}, h^\varepsilon_{t=0}\big)\|_{\mathbb{B}^0_0}\big)\,.
\end{aligned}
$$

The constants κ and κ' only depend upon the choice of the vector fields $\mathcal{T}_j$ and on m, but not on U nor on ε. It follows then from the estimates (4.50)-(4.51) and from the compatibility conditions satisfied by the Cauchy data, that the quantities $\|U\|_{m,\varepsilon,0}$ and $|U|^*_{1,\varepsilon,0}$ are uniformly bounded with respect to $\varepsilon \in]0,1]$. Now, choosing $T > 0$ small enough, Theorem 4.2 follows as a classical consequence of Propositions 4.10 and 4.14.

4.4 proof of theorem 4.4

Let us call $U^\varepsilon_0 := (V^\varepsilon_0, W^\varepsilon_0)$. We look for U^ε_0 of the form

stracciatela (4.52)

$$
\begin{aligned}
U^\varepsilon_0(x) \;&=\; \mathbf{U}^0_0\big(x, \varphi(x)/\varepsilon\big) \,+\, \varepsilon\; \mathbf{U}^1_0\big(x, \varphi(x)/\varepsilon\big)\\
&\quad+\, \cdots \,+\, \varepsilon^M\; \mathbf{U}^M_0\big(x, \varphi(x)/\varepsilon\big)
\end{aligned}
$$

where the profiles $\mathbf{U}^j_0(x,z)$ belong to the space $\mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}; \mathbb{R}^d)$. Since everything is local, making a smooth change of independent coordinates, we can assume that in the compact $\mathbf{C}$ the phase $\varphi(t,x)$ is x_d. Recall that

$$
\mathbf{L}(v,\xi) \;=\; \mathbf{L}_1(v)\,\xi_1 \,+\, \cdots \,+\, \mathbf{L}_d(v)\,\xi_d\,, \qquad \mathbf{L}_d(v_0) = \mathbf{L}(v_0, \partial_x\varphi)\,.
$$

We work with the unknown $U^\varepsilon = (V^\varepsilon, W^\varepsilon)$. Multiplying on the left the equation (2.49) by $\mathbf{S}^\varepsilon(v_0+\varepsilon V, W)^{-1}$, and using a first order Taylor expansion, the equation (2.49) writes

aghetti frutti di mare

$$\mathbf{X}_{v_0} U = \mathbf{N}(v_0, W)\, \partial_d U + \sum_{\mathcal{Z}\in\mathbf{Z}} \mathbf{A}^\varepsilon_{\mathcal{Z}}(v_0, U)\, \mathcal{Z} U + \mathbf{Q}^\varepsilon(v_0, \partial v_0, U) U \tag{4.53}$$

where the matrices $\mathbf{N}, \mathbf{A}, \mathbf{Q}$ are $\mathcal{C}^\infty$ functions of their arguments up to ε up to $\varepsilon = 0$, and

$$\mathbf{N}(v_0, w) \;=\; \big(\mathbf{S}^0(v_0, w)\big)^{-1}\; \mathbf{L}_d(v_0)\,.$$

Since $\mathbf{L}_d(v_0)$ has a constant rank there exists some $N\times N$ matrix with $\mathcal{C}^\infty_b$ entries such that the matrix $\mathbf{L}_d(v_0)\,\Phi(t,x)$ has a constant kernel in $\mathbb{R}^N$. Replacing U by $\Phi(t,x)\tilde U$ and forgetting the " ˜ ", we can assume that the matrix $\mathbf{N}$ has a constant kernel iand that $\mathbf{N} = \mathbf{M}\,\mathbf{\Gamma}$ where $\mathbf{\Gamma}$ in the constant $N\times N$ matrix

$$\mathbf{\Gamma} \;=\; \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{bmatrix}$$

and $\mathbf{M}$ is some $N\times N$ invertible matrix. It is sufficient to prove the theorem after all these reductions are done. In particular we assume that $\Pi_0(x)$ does not depend on x, implying that

$$(\mathrm{Id} - \mathbf{\Gamma})\, U \;=\; {}^t\big(\Pi_0(x) V, W\big)\,, \qquad \forall\, U = (V, W) \in \mathbb{R}^{N''}\times\mathbb{R}^{N'}\,.$$

Our goal is to find profiles $\mathbf{U}^j_0$ satisfying

gongonzola

$$(\mathrm{Id} - \mathbf{\Gamma})\mathbf{U}^j_0 \;=\; {}^t(\boldsymbol{a}^j, \mathbf{b}^j) \tag{4.54}$$

and such that the local smooth solution $U^\varepsilon(t,x)$ of (4.53) with initial data (4.52) satisfies (when ε goes to 0)

runello di Montalcino

$$\big(\mathbf{X}^j_{v_0} U^\varepsilon\big)_{|t=0} \;=\; O(1)\,, \qquad \forall\, j\in\{1,\cdots,m\}\,. \tag{4.55}$$

Here, $O(1)$ means that the $\mathbb{A}^m_0\cap\mathbb{B}^m_0$ norm is bounded, for all $m\in\mathbb{N}$. Because of (4.54), the problem is reduced to determine

$$\mathbf{\Gamma}\,\mathbf{U}^0_0\,,\quad \cdots\,,\quad \mathbf{\Gamma}\mathbf{U}^M_0\,.$$

In a first step, we solve this problem in the sense of asymptotic series in powers of ε. Next, we check that the conditions (4.55) are satisfied.

For $j=1$, the condition reduces to $(\mathbf{\Gamma}\,\partial_d U^\varepsilon)_{|t=0} = O(1)$. Thus, $\mathbf{\Gamma}\,\partial_z U^0_0 = 0$, and

$$\mathbf{\Gamma}\, U^0_0(x,z) \;=\; \mathbf{c}^0(x)$$

is a solution for all $\mathbf{c}^0 \in \mathcal{C}_0^\infty(\mathbb{R}^d;\mathbb{R}^N)$ which is supported in the compact set $\mathbf{C}$.

Assuming that the condition is satisfied for $j = 1, \cdots, k$, we find that

$$\begin{aligned}
\mathbf{X}_{v_0}^{k+1} U^\varepsilon &= \mathbf{X}_{v_0}^{k} \Big(\mathbf{M}(v_0, W^\varepsilon)\, \boldsymbol{\Gamma}\, \partial_d U^\varepsilon + \sum \mathbf{A}_{\mathcal{Z}}(U^\varepsilon)\, \mathcal{Z} U^\varepsilon + \mathbf{Q}(U^\varepsilon)\, U^\varepsilon \Big) \\
\mathbf{X}_{v_0}^{k+1} U^\varepsilon{}_{|t=0} &= \mathbf{X}_{v_0}^{k} \big(\mathbf{M}(v_0, W^\varepsilon)\, \boldsymbol{\Gamma}\, \partial_d U^\varepsilon \big)_{|t=0} + \bigcirc(1) \\
&= \big(\mathbf{M}(v_0, W_0^\varepsilon)\, \boldsymbol{\Gamma}\, \partial_d\, \mathbf{X}_{v_0}^{k} U^\varepsilon \big)_{|t=0} + \bigcirc(1)
\end{aligned}$$

and the condition $\big(\mathbf{X}_{v_0}^{k+1} U^\varepsilon\big)_{|t=0} = O(1)$ reduces to

$$\boldsymbol{\Gamma}\, \partial_d \mathbf{X}_{v_0}^{k} U^\varepsilon{}_{|t=0} \;=\; O(1)\,. \tag{4.56}$$

Now $\mathbf{X}_{v_0}^{k} U^\varepsilon_{|t=0}$ is a function of the form

$$\mathbf{X}_{v_0}^{k} U^\varepsilon_{|t=0} = \sum_{j=-k}^{0} \varepsilon^j\, F_j^k\big(x, \varphi(0,x)/\varepsilon\big) \;+\; O(\varepsilon)$$

and since by induction hypothesis all the terms F_j^k with $j < 0$ are zero, we just have to solve

$$\mathbf{X}_{v_0}^{k} U^\varepsilon_{|t=0} \;=\; F_0^k\big(x, \varphi(0,x)/\varepsilon\big) \;+\; O(\varepsilon)$$

which yields

$$\boldsymbol{\Gamma}\, \partial_d\, \mathbf{X}_{v_0}^{k} U^\varepsilon{}_{|t=0} \;=\; \varepsilon^{-1}\, \boldsymbol{\Gamma}\, \partial_z F_0^k\big(x, \varphi(0,x)/\varepsilon\big) \;+\; O(1)\,.$$

Hence we are lead to solve the equation

izabel
$$\boldsymbol{\Gamma}\, \partial_z F_0^k \;\equiv\; 0\,. \tag{4.57}$$

The function $F_0^k(x,z)$ has the following form

$$F_0^k \;=\; \big(\mathbf{M}(v_0, W_0^0)\, \boldsymbol{\Gamma}\, \partial_z\big)^k\, \mathbf{U}_0^k \;+\; G_0^{k-1}$$

where the term G_0^{k-1} depends only on the profiles $\mathbf{U}_0^j$ with $j \le k-1$

$$\begin{aligned}
G_0^{k-1} \;= \mathcal{G}_0^{k-1}\big(\, &D^\alpha v_{0|t=0}\,,\; \partial_{t,x}^{\beta_j} \partial_z^{p_j} \mathbf{U}_0^j\,; \\
&0 \le j \le k-1\,,\; |\alpha| \le k\,,\; |\beta_j| + p_j \le k\,,\; p_j \le k-1 \big)\,.
\end{aligned}$$

Hence the equation (4.57) can be written as

elabet
$$\boldsymbol{\Gamma}\, \partial_z \big(\mathbf{M}(v_0, W_0^0)\, \boldsymbol{\Gamma}\, \partial_z\big)^k\; \boldsymbol{\Gamma} \mathbf{U}_0^k \;=\; -\,\boldsymbol{\Gamma}\, \partial_z G_0^{k-1} \tag{4.58}$$

where the unknown is $\mathbf{\Gamma U}_0^k$. This equation can be solved through $k+1$ repeated integrations with respect to z, and multiplications by the matrix $\mathbf{M}^{-1}(v_0, W_0^0)$, the solution depending on the choice of $k+1$ arbitrary $\mathcal{C}^\infty$ functions $\mathbf{c}_0^k(x), \cdots, \mathbf{c}_k^k(x)$ (the constants of integration) that we can all choose supported in the compact $\mathbf{C}$. This shows that the problem can be solved from a formal point of view with $\mathbf{\Gamma U}^k \in \mathcal{C}^\infty(\mathbb{R}^d; \mathbb{R}^N)$ supported in $\mathbf{C} \times \mathbb{R}$. But the same induction shows that $|\mathbf{\Gamma U}_0^k(x,z)| \leq cte\,(1+|z|^k)$ and more generally for all $\alpha \in \mathbb{N}^d$ and $j \in \mathbb{N}$

$$(4.59) \qquad |\partial_x^\alpha \partial_z^j \, \mathbf{\Gamma U}_0^k(x,z)| \;\leq\; c_{k,\alpha,j}\; (1+|z|^{k-j})\,, \qquad \forall\,(x,z) \in \mathbb{R}^d \times \mathbb{R}\,.$$

This implies that

$$\varepsilon^k \; \mathbf{\Gamma U}_0^k(x, x_d/\varepsilon) \in \mathbb{A}_0^m \cap \mathbb{B}_0^m\,, \qquad \forall\, m \in \mathbb{N}\,.$$

We already know that the function $(\mathrm{Id} - \mathbf{\Gamma})\mathbf{U}_0^k = {}^t(\boldsymbol{a}_j, \mathbf{b}_j)$ is prescribed in the space $\mathbf{J_C}(\mathbb{R}^d; \mathbb{R}^N)$ which implies that

$$(\mathrm{Id} - \mathbf{\Gamma})\mathbf{U}_0^k(x, x_d/\varepsilon) \in \mathbb{A}_0^m \cap \mathbb{B}_0^m\,, \qquad \forall\, m \in \mathbb{N}\,.$$

Therefore, U_0^ε is actually a compatible initial data, and the proof is complete. □

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