Chapter 4

Symmetric systems. The L^2 linear theory

4.1 Symmetric systems, preliminaries

4.1.1 Definitions

Consider the

$$L = \sum_{j=0}^{d} \widetilde{A}_j(\tilde{x})\partial_{x_j} + \widetilde{B}, \qquad \tilde{x} = (x_0, \ldots, x_d) = (t, x) \tag{4.1.1}$$

Our goal is to solve the Cauchy problem

$$\begin{cases} Lu = f, & t \in [0, T],\ x \in \mathbb{R}^d, \\ u_{|t=0} = h, \end{cases} \tag{4.1.2}$$

assuming that the system is symmetric in the following sense:

Definition 4.1.1. *L is symmetric hyperbolic if the A_j are symmetric and $\widetilde{A}_0$ is positive definite.*

$$\widetilde{A}_0^{-1} = \partial_t + \sum_{j=1}^{d} A_j(\tilde{x})\partial_{x_j} + B, \qquad \tilde{x} = (x_0, \ldots, x_d) = (t, x) \tag{4.1.3}$$

Lemma 4.1.2. *For all $\tilde{x}$, $\widetilde{L}(\widetilde{x}, \widetilde{\xi})$ is strongly hyperbolic in the direction $dt = (1, 0, \ldots 0)$ and the cone of hyperbolic directions $\Gamma_{\tilde{x}}$ is the set of $\tilde{\xi}$ such that $\widetilde{L}(\widetilde{x}, \widetilde{\xi})$ is positive definite.*

Assumption 4.1.3. *The coefficients $\widetilde{A}_j$ are Lipschitz continuous.*

4.1.2 Adjoints and weak solutions

Lemma 4.1.4. *Let $a \in W^{1,\infty}(\Omega)$. For $u \in H^1(\Omega)$ [resp. $L^2(\Omega)$], $a\partial_{x_j} u$ is well defined in $L^2(\Omega)$ [resp. $H^{-1}(\Omega)$]. In particular, for $u \in L^2(\Omega)$ and $v \in H_0^1(\Omega)$,*

$$\langle a\partial_{x_j} u, v\rangle_{H^{-1}\times H_0^1} = -\int u\partial_{x_j}(au)dx.$$

The adjoint of L is

$$L^* = \sum_{j=0}^{d} -\partial_{x_j}\widetilde{A}_j^* + \widetilde{B}^*. \tag{4.1.4}$$

Corollary 4.1.5. *For $u \in H^1(\widetilde{\Omega})$ [resp. $L^2(\widetilde{\Omega})$], Lu is well defined in $L^2(\widetilde{\Omega})$ [resp. $H^{-1}(\widetilde{\Omega})$]. There is a similar result for L^* and for $u \in L^2(\widetilde{\Omega})$ and $v \in H_0^1(\widetilde{\Omega})$,*

$$\langle Lu, v\rangle_{H^{-1}\times H_0^1} = \int u(\tilde{x})\overline{L^* v(\tilde{x})}d\tilde{x}.$$

In particular, for $u \in L^2(\widetilde{\Omega})$ and $f \in L^2(\widetilde{\Omega})$, the equation $Lu = f$ is satisfied *in the weak sense*, that is in $H^{-1}(\widetilde{\Omega})$, if and only if

$$\forall v \in H_0^1(\widetilde{\Omega}), \qquad \int u(\tilde{x})\overline{L^* v}(\tilde{x})d\tilde{x} = \int f(\tilde{x})\overline{v(\tilde{x})}d\tilde{x}. \tag{4.1.5}$$

4.1.3 Weak and strong solutions of the Cauchy problem

Lemma 4.1.6. *If $u \in L^2([0,T]\times\mathbb{R}^d)$ and $\partial_t u \in L^2([0,T]; H^{-1}\mathbb{R}^d)$, then $u \in C^0([0,T]; H^{-\frac{1}{2}}(\mathbb{R}^d))$ and for all $v \in H^1([0,T]\times\mathbb{R}^d)$,*

$$\begin{aligned} -\int u(\tilde{x})\,\overline{\partial_t v(\tilde{x})}d\tilde{x} = & \int_0^T \langle \partial_t u(t), \overline{v}(t)\rangle_{H^{-1}\times H^1} dt \\ & + \langle u(0), \overline{v}(0)\rangle_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}} - \langle u(T), \overline{v}(T)\rangle_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}} \end{aligned} \tag{4.1.6}$$

Also recall that $H^1([0,T]\times\mathbb{R}^d) \subset C^0([0,T]; H^{\frac{1}{2}}(\mathbb{R}^d))$.

Corollary 4.1.7. *If $u \in L^2([0,T]\times\mathbb{R}^d)$ and $Lu \in L^2([0,T]\times\mathbb{R}^d)$, then $u \in C^0([0,T]; H^{-\frac{1}{2}}(\mathbb{R}^d))$ and for all $v \in H^1([0,T]\times\mathbb{R}^d)$,*

$$\begin{aligned} \int u(\tilde{x})\overline{L^* v}(\tilde{x})d\tilde{x} = & \int f(\tilde{x})\overline{v(\tilde{x})}d\tilde{x} \\ & + \langle u(0), \overline{v}(0)\rangle_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}} - \langle u(T), \overline{v}(T)\rangle_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}} \end{aligned} \tag{4.1.7}$$

Definition 4.1.8 (Weak L^2 solutions of the Cauchy problem)**.** *It makes sense*

Corollary 4.1.9. *For $f \in L^2([0,T] \times \mathbb{R}^n)$ and $h \in L^2(\mathbb{R}^n)$, $u \in L^2([0,T] \times \mathbb{R}^n)$ is a weak solution of* (4.1.2) *if and only il, for all $v \in \mathcal{H}^1$ such that $v_{|t=T} = 0$, one has*

$$\int_{[0,T]\times\mathbb{R}^n} f \cdot \overline{v}\; dtdx + \int_{\mathbb{R}^n} h \overline{\cdot v_{|t=0}}\, dx \;= \int_{[0,T]\times\mathbb{R}^n} u \overline{\cdot L^* v}\, dtdx. \tag{4.1.8}$$

Definition 4.1.10 (Strong L^2 solutions of the Cauchy problem)**.** *For $f \in L^2([0,T] \times \mathbb{R}^n)$ and $h \in L^2(\mathbb{R}^n)$, $u \in L^2([0,T] \times \mathbb{R}^n)$ said to be a strong solution of* (4.1.2) *if there is sequences $u_k \in H^1([0,T] \times \mathbb{R}^d)$ such that in the limit $k \to +\infty$:*

i) $\|u - u_k\|_{L^2([0,T]\times\mathbb{R}^n)} \to 0$,
ii) $\|h - u_{k|t=0}\|_{L^2(\mathbb{R}^n)} \to 0$,
iii) $\|f - Lu_k\|_{L^2([0,T]\times\mathbb{R}^n)} \to 0$.

Lemma 4.1.11. *Strong solutions are weak solutions*

4.2 The L^2 energy estimate.

4.2.1 The energy balance

Lemma 4.2.1. *If the matrices A_j are symmetric, and $u \in H^1(\widetilde{\Omega})$ then*

$$2\mathrm{Re}\, Lu.\overline{u} = \sum_{j=0}^{d} \partial_{x_j}(A_j u.\overline{u}) + Ku.\overline{u} \in L^1(\widetilde{\Omega}).$$

with $K = 2\mathrm{Re}\, B - \sum_{j=0}^{d} \partial_{x_j} A_j$.

Corollary 4.2.2. *If the matrices A_j are symmetric, and $u \in H^1([0,T]\times\mathbb{R}^d)$*

$$\begin{aligned} 2\mathrm{Re} \int_{[0,T]\times\mathbb{R}^d} Lu.\overline{u}\, d\tilde{x} &= \int_{[0,T]\times\mathbb{R}^d} Ku.\overline{u}\, d\tilde{x} \\ &\quad + \int_{\mathbb{R}^d} A_0 u\, \overline{u}(T,x) dx - \int_{\mathbb{R}^d} A_0 u\, \overline{u}(0,x) dx. \end{aligned} \tag{4.2.1}$$

Proposition 4.2.3. *If L is hyperbolic symmetric with Lipschitz coefficients, then there are constants C and γ such that for all $u \in H^1([0,T] \times \mathbb{R}^d)$*

$$\|u(t)\|_{L^2} \le C e^{\gamma t} \|u(0)\|_{L^2} + C \int_0^t e^{\gamma(t-t')} \|Lu(t')\|_{L^2} dt'. \tag{4.2.2}$$

Remark 4.2.4. *On C and γ.*

4.2.2 Uniqueness of strong solutions

Theorem 4.2.5. *Il the system is hyperbolic symmetric, then any strong solution belongs to $C^0([0,T];L^2)$ and satisfies the energy estimate* (4.2.2).

In particular, strong solutions are unique.

Proof. Let u be s strong solution and u_k an approximating sequence. The estimate (4.2.2) can be applied to u_k and also to $u_k - u_l$, proving that the u_k are bounded and form a Cauchy sequence in $C^0([0,T];L^2)$. Therefore the limit u is also in this space, and passing to the limit in the estimates for the u_k we get the estimate for u. □

4.3 Existence of weak solution

4.3.1 The duality method

The system L^* is hyperbolic symmetric. Therefore there are energy estimates for L^* and changing t to $T-t$, we obtain that for $v \in H^1([0,T]\times\mathbb{R}^d)$ et $t \in [0,T]$ on a

$$\|v(t)\|_{L^2} \le C\int_t^T \|L^*v(')\|_{L^2}dt' + C|v(T)\|_{L^2}.$$

Introduce the space $\mathcal{H}^1$ of functions $v \in H^1([0,T]\times\mathbb{R}^n)$ such that $v_{|t=T} = 0$. The estimate above implies the following lemma.

Lemma 4.3.1. *There is a constant C such that for all $v \in \mathcal{H}^1$ on a :*

$$\|v(0)\|_{L^2(\mathbb{R}^d)} + \|v\|_{L^2([0,T]\times\mathbb{R}^d)} \le C\|L^*v\|_{L^2([0,T]\times\mathbb{R}^d)}. \tag{4.3.1}$$

Theorem 4.3.2. *For all $f \in L^2([0,T]\times\mathbb{R}^d)$ and $h \in L^2(\mathbb{R}^d)$, the problem* (4.1.2) *has a solution $u \in L^2([0,T]\times\mathbb{R}^d)$.*

Proof. Consider the space $\mathcal{F} = \{L^*v\,;\, v \in \mathcal{H}^1\}$ which is a subspace of $L^2([0,T]\times\mathbb{R}^d)$. The mapping $\widetilde{L}$ from $\mathcal{H}^1$ L^2 is injective by (4.3.1). Thus there is a linear inverse mapping . $J : \mathcal{F} \mapsto \mathcal{H}^1$. For all $g \in \mathcal{F}$ one has $L^*Jg = g$ and by (4.3.1)

$$\|Jg_{|t=0}\|_{L^2(\mathbb{R}^d)} + \|Jg\|_{L^2([0,T]\times\mathbb{R}^d)} \le C\|g\|_{L^2([0,T]\times\mathbb{R}^d)}. \tag{4.3.2}$$

Consider the anti-linear form on $\mathcal{H}^1$:

$$\Phi(v) = \int_{[0,T]\times\mathbb{R}^d} f\cdot\overline{v}\;dtdx + \int_{\mathbb{R}^d} h\cdot\overline{v_{|t=0}}\,dx \tag{4.3.3}$$

and the antilinear form Ψ on $\mathcal{F}$

$$\Psi(g) = \Phi(Jg). \tag{4.3.4}$$

By (4.3.2) que

$$\big|\Psi(g)\big| \leq M\big\|g\big\|_{L^2([0,T]\times\mathbb{R}^d)} \tag{4.3.5}$$

with $M = C(\|f\|_{L^2} + \|h\|_{L^2})$. Hence Ψ can be continuously extended to the closure of $\mathcal{F}$ in $(L^2([0,T] \times \mathbb{R}^d)$, and next on $(L^2([0,T] \times \mathbb{R}^d)$ as an antilinear form with norm less than or equal to M. By Riesz Theorem, there is $u \in L^2([0,T] \times \mathbb{R}^n)$ such that for all $g \in L^2$:

$$\Psi(g) = \int_{[0,T]\times\mathbb{R}^d} u \cdot \overline{g}\; dtdx.$$

Therefore, for all $v \in \mathcal{H}^1$,

$$\Phi(v) = \int_{[0,T]\times\mathbb{R}^d} u \cdot \overline{L^* v}\; dtdx.$$

This is precisely (4.1.8) and thus u is a solution of (4.1.2). □

4.3.2 The approximation method

Let us explain the principle first. The idea is to replace the spatial derivatives ∂_{x_j} by *approximations* ∂_j^ε such that for all $\varepsilon > 0$ the ∂_j^ε are bounded operators in $L^2(\mathbb{R}^d)$. Of course, their norm in L^2 tends to $+\infty$ as ε goes to 0, but we assume that they are uniformly bounded from L^2 to H^{-1} : there is a constant C such that

$$\big\|\partial_j^\varepsilon u\big\|_{H^{-1}} \leq C\big\|u\big\|_{L^2}, \tag{4.3.6}$$

The adjoint operators in L^2, $\partial_j^{\varepsilon *}$, which need not be exactly $-\partial_j^\varepsilon$, are bounded from H^1 to L^2:

$$\big\|\partial_j^\varepsilon v\big\|_{L^2} \leq C\big\|v\big\|_{H^1}. \tag{4.3.7}$$

Moreover, ∂_j^ε approximates ∂_{x_j} in the distribution sense, that is

$$\begin{aligned} &\forall u \in L^2(\mathbb{R}^d), \quad \partial_j^\varepsilon u \to \partial_{x_j} u \textit{ in } H^{-1}, \\ &\forall v \in H^1(\mathbb{R}^d) \quad \partial_j^{\varepsilon *} v \to -\partial_{x_j} v \textit{ in } L^2. \end{aligned} \tag{4.3.8}$$

Consider

$$L^\varepsilon = A_0\partial_t + \sum_{j=1}^d A_j\partial_j^\varepsilon + B = A_0(\partial_t + K^\varepsilon). \tag{4.3.9}$$

For all $\varepsilon > 0$, K^ε is bounded in L^2 and thus the Cauchy Lipschitz theorem implies that

Lemma 4.3.3. *For all $\varepsilon \in]0,1]$, $h \in L^2(\mathbb{R}^d)$, $f \in L^1([0,T]; L^2(\mathbb{R}^d))$ the problem*

$$L^\varepsilon u^\varepsilon = f, \quad u^\varepsilon_{|t=0} = h \tag{4.3.10}$$

has a unique solution $u^\varepsilon \in C^0([0,T]; L^2(\mathbb{R}^d))$.

Theorem 4.3.4. *Suppose that the family u^ε is bounded in $C^0([0,T]; L^2)$. Then the Cauchy problem* (4.1.2) *has a weak solution $u \in L^2([0,T] \times \mathbb{R}^d)$.*

Proof. Using (4.3.6), and the we see that ∂_t^ε is bounded in $L^\infty([0,T]; H^{-1})$ and more precisely that there is C such that for all $\varepsilon \in]0,1]$:

$$\|u^\varepsilon(t) - u^\varepsilon(t')\|_{H^{-1}} \leq C|t - t'|.$$

Hence, by Ascoli's theorem there is a subsequence, still denoted by u^ε, which converges in $C^0([0,T]; L^2_{weak})$ where L^2_{weak} is the L^2 space equipped with the weak topology. The convergence in $C^0([0,T]; L^2_{weak})$ means that for all $\varphi \in L^2(\mathbb{R}^d)$, the function $(u^\varepsilon(t), \varphi)_{L^2}$ converges to $(u(t), \varphi)_{L^2}$ uniformly in time. In particular, $u \in L^2([0,T] \times \mathbb{R}^d)$.

For $v \in H^1([0,T] \times \mathbb{R}^d)$ with $v(T) = 0$, one has

$$\int f \cdot \overline{v} dtdx + \int h \cdot \overline{v_{|t=0}}\, dx = \int u^\varepsilon \cdot \overline{L^{\varepsilon *} v}\, dtdx$$

where

$$L^{\varepsilon *} = -\partial_t A_0 - \sum \partial_j^\varepsilon A_j + B^*$$

Passing to the limit in ε implies that u is a weak solution of (4.1.2). □

Example 1.

Let $J_\varepsilon = (1 - \varepsilon\Delta_x)^{-\frac{1}{2}}$ and $\partial_j^\varepsilon = \partial_{x_j} J_\varepsilon$.

Proposition 4.3.5. *With this choice, the assumption of Theorem* 4.3.4 *is satisfied.*

Sketch of the proof. We repeat the proof of the energy estimate for L^ε. Because of the boundedness in L^2, we can write

$$2\mathrm{Re}\left(A_j\partial_j^\varepsilon u^\varepsilon, u^\varepsilon\right)_{L^2} = \left((A_j\partial_j^\varepsilon - \partial_j^\varepsilon A_j)u^\varepsilon, u^\varepsilon\right)_{L^2}.$$

Using a result of Coiffman and Meyer, one can show that the $(A_j\partial_j^\varepsilon - \partial_j^\varepsilon A_j)$ are uniformly bounded in L^2. From here the proof continues as for Proposition 4.2.3. $\square$

Example 2. We use finite differences: jor $j = 1, \ldots, d$, and $\varepsilon \in]0, 1]$, let

$$\partial_j^\varepsilon u(x) = \frac{1}{2\varepsilon}\big(u(x + \varepsilon e_j) - u(x - \varepsilon e_j)\big) \tag{4.3.11}$$

where $\{e_1, \ldots, e_d\}$ is the canonical basis of $\mathbb{R}^d$.

Proposition 4.3.6. *With this choice, the assumption of Theorem* 4.3.4 *is satisfied.*

We start with a preliminary estimate.

Lemma 4.3.7. *Suppose that $A(x)$ is symmetric and Lipschitz, and $u \in L^2(\mathbb{R}^d)$. Then*

$$\left|\mathrm{Re}\int A_j(x)\partial_j^\varepsilon u(x)\,\overline{u(x)}dx\right| \le \big\|\partial_j A\big\|_{L^\infty}\,\big\|u\big\|_{L^2}^2. \tag{4.3.12}$$

Proof. Let

$$\begin{aligned} w(x,y) := 2\mathrm{Re}\, &A(x)(u(x+y) - u(x-y))\overline{u}(x) \\ =&A(x)u(x+y).\overline{u}(x) - A(x)u(x-y).\overline{u}(x) \\ &+ A(x)u(x).\overline{u}(x+y) - A(x)u(x).\overline{u}(x-y). \end{aligned}$$

Hence

$$\begin{aligned} \int w(x,y)dx =& \int (A(x) - A(x+y))u(x+y).\overline{u}(x)dx \\ &+ \int (A(x-y) - A(x))u(x).\overline{u}(x-y).\overline{u}(x).dx \\ &\le 2|y|\big\|\partial A\big\|_{L^\infty}\,\big\|u\big\|_{L^2}^2. \end{aligned}$$

which implies the lemma. $\square$

Proof of Proposition 4.3.6. Consider the energy

$$\mathcal{E}^\varepsilon = \int_{\mathbb{R}^d} A_0 u.\overline{u} dx = \big(A_0 u(t), u(t)\big)_{L^2} \approx \|u^\varepsilon(t)\|^2_{L^2}.$$

We have

$$\frac{d}{dt}\mathcal{E}^\varepsilon = \big(\partial_t A_0 u^\varepsilon(t), u^\varepsilon(t)\big)_{L^2} + 2\mathrm{Re}\,\big(f(t), u^\varepsilon(t)\big)_{L^2} + \sum_{j=1}^d \int w_j(t,x) dx$$

with

$$w_j = 2\mathrm{Re}\ A_j \partial_j^\varepsilon u^\varepsilon\, \overline{u^\varepsilon}.$$

The Lemma implies that

$$\frac{d}{dt}\mathcal{E}^\varepsilon \leq C_0 \|f(t)\|_{L^2}\ \sqrt{\mathcal{E}^\varepsilon} + C_1 \mathcal{E}^\varepsilon +$$

and the proposition follows. □

4.4 Strong solutions of the Cauchy problem

4.4.1 Weak = strong

We are given a weak solution u and we want to exhibit a sequence u_k satisfying the properties listed in the Defintion 4.1.10. The principle of the proof is as follows. We look for *mollifiers* J_ε wich satisfy the following properties:

1. For all $\varepsilon > 0$, J_ε is a bounded operator from $L^2(\mathbb{R}^d)$ to $H^1(\mathbb{R}^d)$ and from to $H^{-1}(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$;

2. The family $\{J_\varepsilon, \varepsilon \in]0,1]\}$ is bounded in the space of operators from L^2 to L^2 and for all $u \in L^2$ [resp. H^1] , $J_\varepsilon u \to u$ in L^2 [resp H^1] as $\varepsilon \to 0$;

3. For all j, the family of operators $\{[A_0^{-1} A_j(t,x)\partial_{x_j}, J_\varepsilon], \varepsilon \in]0,1], t \in [0,T]\}$ is bounded in the space of operators from L^2 to L^2 .

Proposition 4.4.1. *If there exist operators J_ε satisfying the properties above, then for all $f \in L^2([0,T] \times \mathbb{R}^d)$ and $h \in L^2(\mathbb{R}^d)$, any weak solution $u \in L^2([0,T] \times \mathbb{R}^d)$ of the problem* (4.1.2) *is a strong solution.*

Proof. Consider the commutators $C_j^\varepsilon = [A_0^{-1}A_j(t,x)\partial_{x_j}, J_\varepsilon]$ acting in $L^2([0,T]\times\mathbb{R}^d)$. By the property 3, they are uniformly bounded, and by property 2, $C_j^\varepsilon v \to 0$ in L^2 when $v \in H^1([0,T]\times\mathbb{R}^d)$. By density of H^1 in L^2 we conclude that

$$\left\|C_j^\varepsilon u\right\|_{L^2} \ \to \ 0.$$

Write $L = A_0(\partial_t + K)$. What we have proved is that $[K, J_\varepsilon]u \to 0$ in $L^2([0,T]\times\mathbb{R}^d)$.

Because $u \in L^2([0,T]\times\mathbb{R}^d)$ and $\partial_t u \in L^2([0,T]; H^{-1}(\mathbb{R}^d))$, one easily shows that

$$\begin{aligned}
&1) \quad \partial_t J_\varepsilon u = J_\varepsilon \partial_t u, \qquad in\ L^2([0,T]; H^{-1}(\mathbb{R}^d)),\\
&2) \quad (J_\varepsilon u)_{|t=0} = J_\varepsilon (u_{|t=0} \qquad in\ H^{-\frac{1}{2}}(\mathbb{R}^d).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&1) \quad J_\varepsilon u \to u \qquad in\ L^2([0,T]; H^{-1}(\mathbb{R}^d))\\
&2) \quad LJ_\varepsilon u = A_0\big(J_\varepsilon A_0^{-1} f + [K, J_\varepsilon]u \to f in\ L^2([0,T]; H^{-1}(\mathbb{R}^d))\\
&3) \quad (J_\varepsilon u)_{|t=0} = J_\varepsilon h \to h \qquad in\ L^2(\mathbb{R}^d).
\end{aligned}$$

proving that u is a strong solution. □

4.4.2 Friedrichs Lemma

Consider a function $\jmath \in C_0^\infty(\mathbb{R}^d)$, $\jmath \geq 0$, with

$$\int \jmath(x)dx = 1. \tag{4.4.1}$$

Let

$$\jmath_\varepsilon(x) = \varepsilon^{-d}\jmath(x/\varepsilon), \qquad J_\varepsilon u = \jmath_\varepsilon \star u. \tag{4.4.2}$$

Lemma 4.4.2. *The operators J_ε have the properties 1, 2, 3 listed above.*

Proof. Consider a function Lipschitz function a and $u \in H^1$. Let $K_\varepsilon u = J_\varepsilon(a\partial_{x_j}u) - a\partial_{x_j}J_\varepsilon u$. Then

$$K_\varepsilon u(x) = \int \jmath_\varepsilon(y)(a(x-y) - a(x))\partial_{x_j}u(x-y)dy$$

$$K_\varepsilon u(x) = \int K_\varepsilon(x,y)u(x-y)dy.$$

with

$$K_\varepsilon(x,y) = \partial_{y_j}\big(\jmath_\varepsilon(y)(a(x-\varepsilon y) - a(x))\big)$$

One has

$$|K_\varepsilon(x,y)| \le 2\|\nabla a\|_{L^\infty}\tilde\jmath_\varepsilon(y)$$

with

$$\tilde\jmath_\varepsilon(y) = \varepsilon^{-d}\tilde\jmath(y/\varepsilon), \qquad \tilde\jmath(y) = \jmath(y) + |y||\partial_j\jmath(y)|.$$

Hence

$$|K_\varepsilon u(x)| \le \int C\tilde\jmath_\varepsilon(y)|u(x-y)|dy$$

and

$$\|K_\varepsilon u\|_{L^2} \le C\|\tilde\jmath\|_{L^1}\|u\|_{L^2}. \tag{4.4.3}$$

By density of smooth functions in L^2, the estimate implies the K_ε are uniformly bounded functions from L^2 into L^2. Because $K_\varepsilon u \to 0$ in L^2 when H^1, the uniform bound also implies that

$$\forall u \in L^2, \quad \lim_{\varepsilon\to 0}\|K_\varepsilon u\|_{L^2} = 0.$$

The proof is similar when a and u also depend on t, and for matrices and vectors. □

4.5 The local theory

4.5.1 The cone of hyperbolic directions

Proposition 4.5.1. *The cone $\Gamma(t,x)$ of hyperbolic directions at (t,x) is the set of $\nu = (\nu_0, \nu_1, \dots, \nu_d)$ such that the matrix $\sum \nu_j A_j(t,x)$ is definite positive.*

Proof. □

Lemma 4.5.2. *Let $\lambda_k(t,x,\xi)$ denote the eigenvalues of $\sum_{j=1}^d \xi_j A_0^{-1}A_j(t,x,\xi)$ and*

$$c = \max_{[0,T]\times\mathbb{R}^d\times S^{d-1}} \max_k |\lambda_k(t,x,\xi)| < +\infty. \tag{4.5.1}$$

Then

$$\Gamma = \{\nu_0 > c|\nu'|\} \subset \cap_{t,x}\Gamma(t,x). \tag{4.5.2}$$

Proof. This is clear when $\nu' = 0$. When $\nu' \neq 0$, we can assume that $|\nu'| = 1$ and the assumption is that thus $\nu_0 > c$. Thus the eigenvalues of $A := \nu_0 \mathrm{Id} + \sum \nu_j A_0^{-1} A_j$ are positive, as well as the eigenvalues of the conjugate matrix

$$A_0^{\frac{1}{2}} A A_0^{-\frac{1}{2}} = A_0^{-\frac{1}{2}} (\nu_0 A_0 + \sum \nu_j A_j) A_0^{-\frac{1}{2}}.$$

Thus this symmetric matrix is definite positive, implying that $\nu_0 A_0 + \sum \nu_j A_j$ is also positive. □

4.5.2 Local energy estimates

Integrate the energy balance on $\Omega \subset [0,T] \times \mathbb{R}^d$:

$$2\mathrm{Re} \int_\Omega (Lu, u) dt dx - \int (Ku, u) dt dx = \sum_{j=0}^{d} \int_{\partial\Omega} \nu_j (A_j u, u) d\sigma$$

where $(\nu_0, \ldots, \nu_d)$ is the outward normal to ∂_Ω.

Consider the polar cone of Γ :

$$\Gamma^\circ = \{(t,x) \in \mathbb{R}^{1+d} : |x| \leq ct\}, \tag{4.5.3}$$

and a backward cone

$$\Omega = \{(t,x), t \in [0, \underline{t}], |x - \underline{x}| \leq c(\underline{t} - t)\}. \tag{4.5.4}$$

The lateral boundary of Ω is

$$\partial_l \Omega = \{(t,x), t \in [0, \underline{t}], |x - \underline{x}| = c(\underline{t} - t)\}. \tag{4.5.5}$$

Lemma 4.5.3. *On $\partial_l \Omega$, the boundary matrix $\sum \nu_j A_j$ is nonnegative.*

Proof. Take for simplicity $\underline{x} = 0$. The outer normal at $(t,x) \in \partial_l \Omega$ is $\delta(c, x/|x|)$ with $\delta = (1+c^2)^{\frac{1}{2}}$. Thus the matrix boundary matrix is $\delta(cA_0 + \sum \nu_j A_j)$ with $\nu_j = x_j/|x|$ for $j = 1, \ldots, d$. By the lemma above, it is non negative. □

Consider $\underline{t} \leq T$ and $\underline{x} \in \mathbb{R}^d$ and Ω as above. For $t \in [0, \underline{t}]$, let $\omega_t = \{x : |x - \underline{x}| \leq c(\underline{t} - t)\}$. One has the local energy estimate

Proposition 4.5.4. *There are constants G and γ, such that for $u \in H^1(\Omega)$,*

$$\|u(t)\|_{\omega_t} \leq C e^{\gamma t} \|u(0)\|_{L^2(\omega_0)} + C \int_0^t e^{\gamma(t-t')} \|Lu(t')\|_{L^2(\omega_{t'})} dt'. \tag{4.5.6}$$

Proof. The energy balance applied on $\Omega_t = \Omega \cap \{t' < t\}$ and the lemma imply that

$$\int_{\omega_t} (A_0 u(t,x), u(t,x))dx \le \int_{\omega_t} (A_0 u(0,x), u(0,x))dx$$
$$+ 2\mathrm{Re} \int_{\Omega_t} (Lu, u)dt'dx + \int_{\Omega_t} |(Ku, u)|dt'dx.$$

We conclude by Gronwall's argument. □

Corollary 4.5.5. *If u is a strong solution of the Cauchy problem with source term which vanishes on Ω and initial data which vanishes on ω_0, then $u = 0$ on Ω.*

Theorem 4.5.6. *For $u_0 \in L^2(\omega_0)$ and $f \in L^2(\Omega)$, the Cauchy problem has a unique strong solution in $L^2(\Omega)$, which in addition is continuous in times with values in L^2 and satisfies* (4.5.6)*.*