Chapter 6

An exemple: the 1-D case

6.1 The setting

We consider here a system in space dimension $d = 1$

$$\partial_t u + A\partial_x u = Bu + f \tag{6.1.1}$$

on the half line $\{x > 0\}$. This equation has to be supplemented by an initial condition

$$u_{|t=0} = h, \tag{6.1.2}$$

and; possibly, with boundary conditions on $\{x = 0\}$:

$$Mu_{|x=0} = g, \tag{6.1.3}$$

where M is a constant coefficient matrix.

We assume that the system has constant coefficients and is hyperbolic in the time direction, which means that A has only real eigenvalues. We assume a little more, namely that A can be diagonalized (that is saying that the system is strongly hyperbolic). Working in a basis where A is diagonal, reduces to the case where

$$A = \begin{pmatrix} a_1 & 0 & \\ 0 & \ddots & 0 \\ & 0 & a_N \end{pmatrix} \tag{6.1.4}$$

so that (6.1.1) is a system of coupled transport equations

$$\partial_t u_j + a_j \partial_x u_j = \sum_{k=1}^{N} b_{j,k} u_k + f_j, \qquad 1 \leq j \leq N. \tag{6.1.5}$$

6.2 The principle

Consider the case where $B = 0$. Then the equations are decoupled and can be solved explicitely. There are two cases:

1. $a_j \leq 0$. Then u_j is uniquely determined on $\{x > 0\}$ by

$$u_j(t,x) = h_j(x - a_j t) + \int_0^t f_j(t', x - a_j(t-t'))dt'. \tag{6.2.1}$$

2. $a_j > 0$. Then the formula above determines u_j only on the domain $\{x > a_j t\}$. On the domain $\{x < a_j t\}$, one has

$$\begin{aligned} u_j(t,x) &= u_j(t - x/a_j, 0) + \int_0^x f(t - (x-x')/a_j, x')dx'/a_j \\ &= u_j(t - x/a_j, 0) + \int_{t-x/a_j}^t f(t', x - a_j(t-t'))dx'. \end{aligned} \tag{6.2.2}$$

The case $a_j \leq 0$ contains two subcases: $a_j < 0$ and $a_j = 0$ which behave differently:

1. When $a_j < 0$, then the trace of u_j on the boundary $\{x = 0\}$ is given by

$$u_j(t,0) = h_j(-a_j t) + \int_0^t f(t', -a_j(t-t'))dt'. \tag{6.2.3}$$

2. When $a_j = 0$, the trace is given by the transport equation $\partial_t u_j = f_j$ on the boundary and

$$u_j(t,0) = h_j(0,0) + \int_0^t f(t', 0)dt'. \tag{6.2.4}$$

Definition 6.2.1. *The vector field $\partial_t + a\partial_x$ is said to be outgoing if $a < 0$, incoming if $a > 0$, tangent if $a = 0$.*

According to this classification, we can group the components of u corresponding to the different categories of vector fields $\partial_t + a_j\partial_x$ and split u intro

$$u = (u_{in}, u_{tg}, u_{out}). \tag{6.2.5}$$

Principle : *the boundary conditions* (6.1.3) *must determine uniquely the traces $u_{j|x=0}$ for the indices j such that $a_j > 0$, that is $u_{in|x=0}$.*

In particular, we need N_{in} boundary conditions, where N_{in} is the number of eigenvalues a_j of A which are positive. To avoid technical complications, we therefore assume that

$$M \text{ is a } N_{in} \times N \text{ matrix} \tag{6.2.6}$$

and according to the splitting (6.2.5) we write

$$M = (M_{in}, M_{tg}, M_{out}), \quad Mu = M_{in}u_{in} + M_{tg}u_{tg} + M_{out}u_{out}. \tag{6.2.7}$$

With these notations, the boundary condition (6.1.3) reads

$$M_{in}u_{in|x=0} = g - M_{tg}u_{tg|x=0} - M_{out}u_{out|x=0}. \tag{6.2.8}$$

The analysis shows that $u_{tg|x=0}$ and $u_{out|x=0}$ are determined by f and h, therefore, to determine uniquely $u_{in|x=0}$, the following condition is necessary:

Assumption 6.2.2. *The $N_{in} \times N_{in}$ matrix M_{in} is invertible.*

In the remaining part of the chapter, assume that this condition is satisfied.

Remark 6.2.3. *The case $N_{in} = 0$ is not excluded. In this case, no boundary condition is required.*

6.3 The case $B = 0$

6.3.1 Continuous solutions

Suppose that f is continuous on $\{t \geq 0, x \geq 0\}$ and h is continuous on $\{x \geq 0\}$. Then the components u_{out} and u_{tg} are continuous on $\{t \geq 0, x \geq 0\}$, as well as their trace on $\{x = 0\}$. Therefore if g is continuous on $\{t \geq 0\}$, the trace $u_{in|x=0}$ is determined and continuous on $\{t \geq 0\}$. This implies that the components of u_j of u_{in} are determined by (6.2.1) when $x > a_j t$ and by (6.2.2) when $x < a_j t$. However, these two formulas do not define a continuous function on $\{t \geq 0, x \geq 0\}$, unless they agree on the line $\{x = a_j t\}$. The limits of u_j on this line from above and from below are the solution of the same transport equation along the line $\{x = a_j t\}$. They coincide if and only if they have the same initial value at the origin, that is

$$\lim_{t \to 0} u_{in}(t, 0) = \lim_{x \to 0} u_{in}(0, x).$$

The limit in the left hand side is

$$M_{in}^{-1}\Big(g(0) - M_{tg}u_{tg}(0,0) - M_{out}u_{out}(0,0)\Big) \\ = M_{in}^{-1}\Big(g(0) - M_{tg}h_{tg}(0) - M_{out}h_{out}(0)\Big).$$

The limit in the right hand side is $h_{in}(0)$. Therefore, a necessary and sufficient conditions in order to get a continuous solution u_{in} is the following *compatibility condition*

$$Mh(0) = g(0). \tag{6.3.1}$$

Proposition 6.3.1. *Suppose that f is continuous on $\{t \geq 0, x \geq 0\}$, h is continuous on $\{x \geq 0\}$, g is continuous on $\{t \geq 0\}$ and satisfy the compatibility condition* (6.3.1). *Then, the boundary value problem* (6.1.1) (6.1.3) *has a unique solution u which is continuous on $\{t \geq 0, x \geq 0\}$. Moreover, there is a constant C such that if the functions are bounded,*

$$\|u(t)\|_{L^\infty} \leq \|h\|_{L^\infty} + C\|g\|_{L^\infty([0,t])} + C\int_0^t \|f(t'\|_{L^\infty} dt' \tag{6.3.2}$$

6.3.2 C^k solutions

For C^k functions, the analysis is similar. The explicit integrations yield C^k functions. However, the C^k regularity of u_j at the interface $x = a_j t$ is more involved and require further compatibility conditions.

For instance, for C^1 solutions, one has the necessary condition

$$\partial_t g(0) = M\partial_t u(0,0)$$

and using the equation, this is equivalent to

$$\partial_t g(0) = Mf(0,0) - MA\partial_x h(0). \tag{6.3.3}$$

Proposition 6.3.2. *Suppose that f is C^1 on $\{t \geq 0, x \geq 0\}$, h is C^1 on $\{x \geq 0\}$, g is C^1 on $\{t \geq 0\}$ and satisfy the compatibility conditions* (6.3.1) (6.3.3). *Then, the boundary value problem* (6.1.1) (6.1.3) *has a unique solution u which is of class C^1 on $\{t \geq 0, x \geq 0\}$.*

For $k \geq 2$, one obtains higher order compatibility conditions, writing

$$\partial_t^k g(0) = M\partial_t^k u(0,0).$$

From the equation,

$$\partial_t^k u = \sum_{j=0}^{k-1} (-A)^j \partial_t^{k-j} \partial_x^j f + (-A)^k \partial_x^k u.$$

Thus the k-the compatibility condition reads

$$\partial_t^k g(0) = \sum_{j=0}^{k-1} M(-A)^j \partial_t^{k-j} \partial_x^j f(0,0) + M(-A)^k \partial_x^k h(0). \tag{6.3.4}$$

Proposition 6.3.3. *Suppose that f is C^k on $\{t \geq 0, x \geq 0\}$, h is C^k on $\{x \geq 0\}$, g is C^k on $\{t \geq 0\}$ and satisfy the compatibility conditions* (6.3.4) *from order* 0 *up to order k. Then, the boundary value problem* (6.1.1) (6.1.3) *has a unique solution u which is of class C^k on $\{t \geq 0, x \geq 0\}$.*

6.3.3 L^p solutions, $p < +\infty$

On the one hand, it is simpler because discontinuities along the lines $\{x = a_j t\}$ are permitted in L^p, and in $C^0([0,T]; L^p(\mathbb{R}))$. On the other hand, for general f_{tg} and h_{tg} in L^p, the trace of u_{tg} on $\{x = 0\}$ is not defined in general, and the boundary condition does not make sense, unless $M_{tg} = 0$. The intrinsic way to express this condition is the following.

Assumption 6.3.4. $\ker A \subset \ker M$

Lemma 6.3.5. *Suppose that $f \in L^1([0,T]; L^p(\mathbb{R}_+)$ and $h \in L^p(\mathbb{R})$. Then the formulas* (6.2.1) *defines functions u_{out} and u_{tg} in $C^0([0,T]; L^p)$. Moreover, u_{out} which admits a trace $u_{out|x=0}$ in $L^p([0,T])$ such that*

$$\big\| u_{out|x=0} \big\|_{L^p([0,T])} \leq C \big\| h \big\|_{L^p(\mathbb{R}_+)} + C \int_{0^T} \| f_j(t') \|_{L^p} dt'. \tag{6.3.5}$$

Proof. There are two terms. The first is

$$h_j(x - a_j t).$$

For $p < \infty$ and all $h \in L^p$, $\tau_\varepsilon h(x) = h(x - \varepsilon)$ converges to h in L^p as $\varepsilon \to 0$. Thus the first term belongs to $C^0([0,T]; L^p)$ and if $a_j < 0$ and is trace is $h(-a_j t)$ which belongs to $L^p([0,T])$.

The second term is the integral in (6.2.1) which is clearly in $C^0([0,T]; L^p)$. When $a_j < 0$, its trace is

$$v_j(t) = \int_0^t f_j(t', -a_j(t - t'))dt' = \int_0^T \phi(t', t) dt' \tag{6.3.6}$$

where

$$\phi(t', t) = f_j(t', -a_j(t-t'))1_{[t>t']}.$$

Thus

$$\|v_j\|_{L^p} \le \int_0^T \|\phi(t',\cdot)\|_{L^p} dt' \le C \int_{0^T} \|f_j(t')\|_{L^p} dt'$$

and the lemma is proved. □

Thus the natural space for the boundary condition g is $L^p([0,T])$.

Lemma 6.3.6. *Suppose that $a > 0$ and consider the initial-boundary value problem*

$$\partial_t + a\partial_x u = f, \qquad u_{|t=0} = h, \quad u_{|x=0} = g. \tag{6.3.7}$$

If $f \in L^1([0,T]; L^p(\mathbb{R}_+))$, $h \in L^p(\mathbb{R}_+)$, $g \in L^p([0,T])$, then there is a unique solution $u \in C^0([0,T]; L^p(\mathbb{R}_+))$ which satisfies

$$\big\|u(t)\big\|_{L^p} \le C\big\|h\big\|_{L^p} + C\big\|g\big\|_{L^p([0,t])} + C\int_0^t \big\|f(t')\big\|_{L^p} dt'. \tag{6.3.8}$$

Proof. The solution is the sum of three terms. The initial data contributes to

$$h(x-at)1_{[x>at]} = \tilde h(x-at) \tag{6.3.9}$$

where $\tilde h$ is the extension of h by 0 for $x<0$. Il belongs to $C^0([0,T]; L^p)$.

The contribution of f can be written in a unified way, as

$$\int_0^t \tilde f(t', x - a(t-t'))dt'$$

where $\tilde f$ is the extension of f by 0 for $x<0$. This term belongs to $C^0([0,T]; L^p)$. It remains the contribution

$$g(t-x/a)1_{[x<at]} = \tilde g(t-x/a)$$

where $\tilde g$ is the extension of g by 0 for $t<0$. It also belongs to $C^0([0,T]; L^p)$ with L^p norm at time t estimated by $C\|g\|_{L^p([0,t])}$. □

Summing up, we have proved:

Theorem 6.3.7. *Assume that the boundary conditions satisfy the Assumptions* 6.2.2 *and* 6.3.4. *Then, for* $f \in L^1([0,T]; L^p(\mathbb{R}_+))$, $h \in L^p(\mathbb{R}_+)$, $g \in L^p([0,T])$, *the initial boundary value problem* (6.1.1) (6.1.3) *has a unique solution* $u \in C^0([0,T]; L^p(\mathbb{R}_+))$. *Moreover, there is a constant* C *such that*

$$
\begin{aligned}
&\|u(t)\|_{L^p} + \|u_{|x=0}\|_{L^p([0,t])} \leq \\
&\qquad C\|h\|_{L^p} + C\|g\|_{L^p([0,t])} + C\int_0^t \|f(t')\|_{L^p} dt'.
\end{aligned} \tag{6.3.10}
$$

6.3.4 H^s solutions

The question is the following : given $h \in H^s(\mathbb{R}_+)$, $g \in H^s([0,T])$ and $f \in L^1([0,T]; H^s)$, is the solution given by Theorem 6.3.7 in $C^0([0,T]; H^s(\mathbb{R}_+))$?

Consider the case $s = 1$. Functions in $H^1(\mathbb{R}_+)$, $H^1([0,T])$ and in $C^0([0,T]; H^1)$ are continuous and therefore the compatibility condition (6.3.1) is certainly necessary. One can prove

Theorem 6.3.8. *Assume that the boundary conditions satisfy the Assumptions* 6.2.2 *and* 6.3.4. *For* $f \in L^1([0,T]; H^1(\mathbb{R}_+))$, $h \in H^1(\mathbb{R}_+)$, $g \in H^1([0,T])$ *satisfying the compatibility condition* (6.3.1), *the initial boundary value problem* (6.1.1) (6.1.3) *has a unique solution* $u \in C^0([0,T]; H^1(\mathbb{R}_+))$.

The case $s > 1$ is much more delicate. For instance, the compatibility condition (6.3.3) uses the value of f at the origin $(0,0)$ and this leads to require more regularity in time for f. This will be discussed later on.

6.4 The general case, $B \neq 0$.

If $B \neq 0$ the incoming and outgoing components are coupled, so one cannnot solve the equations as easily. However, one can solve the equation using an iterative scheme

$$
\begin{cases}
\partial_t u^n + A\partial_x u^n = Bu^{n-1} + f \\
u^n_{|t=0} = h, \\
Mu^n_{|x=0} = g,
\end{cases} \tag{6.4.1}
$$

for $n \geq 1$, starting with $u^0 = 0$. We state the result in L^2, but it can be extended to the other cases. We can also allow B to depend on the variables (t,x) provided that

$$
B \in L^\infty([0,T] \times \mathbb{R}_+). \tag{6.4.2}
$$

Theorem 6.4.1. *Assume that the boundary conditions satisfy the Assumptions* 6.2.2 *and* 6.3.4. *Then, for* $f \in L^1([0,T]; L^2(\mathbb{R}_+))$, $h \in L^2(\mathbb{R}_+)$, $g \in L^2([0,T])$, *the initial boundary value problem* (6.1.1) (6.1.3) *has a unique solution* $u \in C^0([0,T]; L^p(\mathbb{R}_+))$. *Moreover, there is are constant* C *and* γ *such that*

$$\begin{aligned} &\big\|u(t)\big\|_{L^2} + \big\|u_{|x=0}\big\|_{L^2([0,t])} \leq \\ &\qquad Ce^{\gamma t}\big\|h\big\|_{L^2} + Ce^{\gamma t}\big\|g\big\|_{L^2([0,t])} + C\int_0^t e^{\gamma(t-t')}\big\|f(t')\big\|_{L^p} dt'. \end{aligned} \tag{6.4.3}$$

Proof. By Theorem 6.3.7 the first iterate $u^1 \in C^0[0,T]; L^2)$ and satisfies the estimate (6.3.10). By induction, the same theorem gives the iterates $u^n \in C^0[0,T]; L^2)$. Writing the equation for $w^n = u^{n+1} - u^n$ and using the estimate, we see that there is a constant γ, which depend on $\|B\|_{L^\infty}$, such that for all $n \geq 1$ and $t \in [0,T]$:

$$\big\|w^n(t)\big\|_{L^2} + \big\|w^n_{|x=0}\big\|_{L^2([0,t])} \leq \gamma \int_0^t \big\|w^{n-1}(t')\big\|_{L^2} dt'. \tag{6.4.4}$$

We start from $w^0 = u^1 - u^0$, and by induction on n, the estimate implies that for $n \geq 1$:

$$\big\|w^n(t)\big\|_{L^2} + \big\|w^n_{|x=0}\big\|_{L^2([0,t])} \leq \gamma \int_0^t \frac{(\gamma(t-t'))^{n-1}}{(n-1)!}\big\|u^1(t')\big\|_{L^2} dt'.$$

This shows that the series $\sum w^n$ and $\sum w^n_{|x=0}$, hence the sequences u^n and $u^n_{|x=0}$, converge in $C^0([0,T]; L^2)$ and in $L^2([0,T])$ respectively. The limit $u = \lim u_n$. Then, $u - u^1 = \sum_{n\geq 1} w^n$ satisfies

$$\big\|u(t) - u^1(t)\big\|_{L^2} + \big\|(u-u^1)_{|x=0}\big\|_{L^2([0,t])} \leq \gamma \int_0^t e^{\gamma(t-t')}\big\|u^1(t')\big\|_{L^2} dt'.$$

Using the estimate (6.3.10) for u^1, one obtains (6.4.3). $\square$