Chapter 7

Constant coefficients BVP

7.1 Introduction

We consider a constant coefficient system $L = \sum A_j \partial_{\tilde{x}_j} + B$ and the boundary value problem (in short BVP) on the half space

$$\begin{cases} Lu = f, & x_n > 0, \\ Mu_{|x_n=0} = g. \end{cases} \tag{7.1.1}$$

with $x_n = n \cdot \tilde{x}$ and n is the conormal direction to the boundary. The principal part of L is $L_0 = \sum A_j \partial_j$.

Assumption 7.1.1. *i) L is hyperbolic in a direction $\nu \neq n$.*
ii) $A_n = L_0(n)$ is invertible.

If L is hyperbolic in the normal direction n, the problem is a Cauchy problem and thus already treated, this is why we do not consider this case. The case where the boundary matrix is not invertible could be considered too, but is much more delicate.

The matrices A_j and $L(i\tilde{\xi})$ act from spaces $\mathbb{E}$ to $\mathbb{F}$ and M from $\mathbb{E}$ to $\mathbb{G}$. We assume in this lecture that the boundary is not characteristic, that is that

At the end, we want to solve and initial boundary value problem (in short IBVP) the problem (7.1.1) that is for positive time $t = \nu \cdot \tilde{x}$ with an initial datum at $t = 0$. An intermediate step is to solve the BVP for times t running from $-\infty$ to $+\infty$ (that is in $\mathbb{R}^{1+d}_+ = \{x_n \geq 0\}$), in spaces of functions or distributions which are allowed to have an exponential growth in time at $+\infty$, but still decaying (temperate) at infinity in space. More precisely, we look for solutions of the form

$$e^{\gamma t} u_\flat(\tilde{x}) \tag{7.1.2}$$

with $u_\flat$ tempered. The equations for $u_\flat$ reads

$$\begin{cases} L_\gamma u_\flat = f_\flat & x_n > 0, \\ M u_{\flat|x_n=0} = g_\flat, \end{cases} \tag{7.1.3}$$

where

$$L_\gamma(\partial) = L(\partial) + \gamma L(\nu), \qquad L_\gamma(i\tilde{\xi}) = L(i\tilde{\xi} + \gamma\nu). \tag{7.1.4}$$

So the first goal is to solve (7.1.3) when γ is large enough, say $\gamma \geq \gamma_0$, and next to draw conclusions for (7.1.1) and for the IBVP.

Objectives:

- Introduce the stability condition for (7.1.3), the Lopatinski condition;
- Discuss the causality principle;
- Discuss the finite speed propagation property in relation to the choice of the time direction.

7.2 The basic bvp for o.d.e

Introduce the coordinates

$$\tilde{x} = (t, x', n_n) \quad \text{and the dual variables} \quad \tilde{\xi} = (\tau, \xi', \xi_n). \tag{7.2.1}$$

Apply the tangential Fourier Laplace transform to (7.1.1), that is the Fourier transform to (7.1.3) with respect to the variables (t, x'). To simplify notations, we call u the resulting function. The equations are

$$\begin{cases} \partial_{x_n} u + G(i\zeta)u = f, & x_n > 0, \\ M u_{|x_n=0} = g. \end{cases} \tag{7.2.2}$$

Here $\zeta = (\tau, \xi') \in \mathbb{C} \times \mathbb{R}^{d-1}$, $\operatorname{Im} \tau = -\gamma < 0$ and

$$G(i\zeta) = A_n^{-1} L(i\zeta, 0). \tag{7.2.3}$$

Lemma 7.2.1. *Hyperbolicity implies that there is γ_0 such that for $\operatorname{Im} \tau < -\gamma_0$, $G(i\zeta)$ has no eigenvalue on the imaginary axis.*

Proof. If $G(i\zeta)$ has a purely imaginaryl eigenvalue $i\lambda$, then $\xi_n = -\lambda$ satisfies $\det L(i\zeta, i\xi_n) = \det A_n \det(i\xi_n + G(i\zeta)) = 0$, which requires that $|\operatorname{Im} \tau| \le \gamma_0$ for some γ_0. $\square$

Definition 7.2.2. *For* $\operatorname{Im} \tau < -\gamma_0$*, the incoming space* $\mathbb{E}^{in}(\zeta)$ *[resp. outgoing space* $\mathbb{E}^{out}(\zeta)$ *] is the invariant space of* $G(i\zeta)$ *associated to the eigenvalues in* $\{\operatorname{Re} \lambda > 0\}$ *[resp.* $\{\operatorname{Re} \lambda < 0\}$*]. We denote by* Π^{in} *[resp.* Π^{out}*] the spectral projectors on these spaces.*

The projectors are analytic functions of ζ for $\operatorname{Im} \tau < -\gamma_0$

Lemma 7.2.3. *The dimension of* $\mathbb{E}^{in}$ *is equal to* N_+*, the number of positive eigenvalues of* $L_0(n)^{-1} L_0(\nu)$*.*

Proof. This number is independent of ζ. We compute it for $\zeta = (-i\gamma, 0)$ with $\gamma \to +\infty$. In this case

$$G_\gamma := \gamma^{-1} G(i\zeta) = L_0(n)^{-1} L_0(\nu) + \gamma^{-1} B \to L_0(n)^{-1} L_0(\nu).$$

Thus By Assumption 7.1.1, the eigenvalues of $L_0(n)^{-1} L_0(\nu)$ are real and $\neq 0$. Thus, for γ large, the eigenvalues of G_γ split into two groups. N_+ of them are in $\operatorname{Re} \lambda > 0$ and $N - N_+$ are in $\operatorname{Re} \lambda < 0$. Hence $G(i\zeta)$ has N_+ eigenvalues in $\operatorname{Re} \lambda > 0$ and $N - N_+$ in $\operatorname{Re} \lambda >< 0$. $\square$

We now consider the o.d.e. $(\partial_{x_n} + iG)u = f$ in spaces of temperate (or decaying) functions on $[0, +\infty[$. By Lemma 7.2.1 the solutions of the homogeneous equations $u = e^{-x_n G} a$, split into groups, those which decay exponentially at $+\infty$ when $a \in \mathbb{E}^{in}$ and those which decay exponentially at $-\infty$ when $a \in \mathbb{E}^{out}$. One has the following representation:

$$e^{-x_n G} \Pi^{in} = \frac{1}{2i\pi} \int_{\mathcal{C}^+} e^{-x_n \lambda} \big(G - \lambda \mathrm{Id}\big)^{-1} d\lambda \tag{7.2.4}$$

where $\mathcal{C}^+$ is a contour in $\{\operatorname{Re} \xi_n > 0\}$ surrounding the spectrum of G located in this half space. Similarly

$$e^{-x_n G} \Pi^{out} = \frac{1}{2i\pi} \int_{\mathcal{C}^-} e^{-x_n \lambda} \big(G - \lambda \mathrm{Id}\big)^{-1} d\lambda, \tag{7.2.5}$$

with $\mathcal{C}^- \subset \{\operatorname{Re} \xi_n < 0\}$.

Lemma 7.2.4. $e^{-x_n G} \Pi^{in}$ *[resp.* $e^{-x_n G} \Pi^{out}$ *is exponentially decaying when* $x_n \to +\infty$ *[resp.* $x_n \to -\infty$*]. If* f *is temperate at* $+\infty$*, the temperate solutions of* $(\partial_{x_n} + iG)u = f$ *on* $\mathbb{R}_+$ *are*

$$u(x_n) = e^{-x_n G} a + If(x_n), \qquad a \in \mathbb{E}^{in} \tag{7.2.6}$$

where

$$
(7.2.7)\qquad \begin{aligned} If(x_n) = &\int_0^{x_n} e^{(y_n - x_n)G}\Pi^{in} f(y_n) dy_n \\ &- \int_{x_n}^{\infty} e^{(y_n - x_n)G}\Pi^{out} f(y_n) dy_n. \end{aligned}
$$

Therefore, to solve (7.2.2) is remains to check the boundary condition, that is to solve for $a = \Pi^{in} u_0$

$$
(7.2.8)\qquad a \in \mathbb{E}^{in}(\zeta), \qquad Ma = g - MI(f)_{|x_n=0}
$$

Proposition 7.2.5. *For* $\operatorname{Im}\tau < -\gamma_0$*, the boundary value problem* (7.2.2) *has a unique (temperate) [resp. in the Schwartz class] [resp. in* L^2*] solution for all* f *in the same space and all* $g \in \mathbb{G}$*, if and only if* $M_{|\mathbb{E}^{in}}$ *is an isomorphism from* $\mathbb{E}^{in}$ *to* $\mathbb{G}$*.*

This leads to the natural condition which we assume to be satisfied from now on.

Assumption 7.2.6. *The number of boundary conditions is* N_+*, i.e. the boundary operator acts from* $\mathbb{E}$ *to* $\mathbb{G}$ *where* $\dim \mathbb{G} = N_+$*.*

The analysis above also legitimates the following condition:

Definition 7.2.7. *We say that the* (7.1.1) *satisfies the Lopatinski condition (in the time direction dt) if there is* γ_0 *such that for all* $\zeta = (\tau, \xi')$ *with* $\operatorname{Im}\tau < -\gamma_0$*,* $\mathbb{E}^{in}(\zeta) \cap \ker M = \{0\}$*.*

7.3 Fourier synthesis

To get solutions for (7.1.3), we must be able to perform the inverse Fourier transform, that is we need estimates. For simplicity, we give details in L^2 spaces.

We use the representation (7.2.6) of the solution

$$
(7.3.1)\qquad \hat{u}(x_n, \tau, \xi') = e^{-x_n G(\zeta)}\hat{a}(\zeta) + I(\zeta, \hat{f}(\,\cdot\,, \tau, \xi'))
$$

where $I(\zeta, \hat{f})$ is given by (7.3.3) and $\zeta = (\tau, \xi')$ with $\operatorname{Im}\tau < -\gamma_0$ for some γ_0.

Lemma 7.3.1. *There are* $m_0 \geq 1$*,* $\gamma_0 \geq 0$ *and* C *such that for all real* ξ_n *and all* ζ *with* $\gamma = -\operatorname{Im}\tau \geq \gamma_0$

$$
(7.3.2)\qquad \gamma^{m_0}\big|(i\xi_n \mathrm{Id} + G(i\zeta)^{-1}\big| \leq \langle\zeta\rangle^{m_0 - 1}.
$$

Proof. Recall that the determinant p of $L(i\zeta, i\xi_n)^{-1}$ satisfies for $\operatorname{Im}\tau < -\gamma_0$

$$|p(i\zeta, i\xi_n)| \geq c(\gamma - \gamma_0)^{-N}.$$

Since L^{-1} is the transposed matrix of of the cofactors of L, divided by $\det L$, this implies the resolvent estimate

$$(\gamma - \gamma_0)^{-N} \big|(L(i\zeta, i\xi_n)^{-1}\big| \leq C(|\xi_n| + \langle\zeta\rangle)^{N-1}$$

(see also (2.5.2)). Thus, for $\gamma \geq 2\gamma_0$

$$\gamma^N \big|(i\xi_n \mathrm{Id} + G(i\zeta)^{-1}\big| \leq C(|\xi_n| + \langle\zeta\rangle)^{N-1}$$

This implies (7.3.2) when $|\xi_n| \leq C'\langle\zeta\rangle$. When $|\xi_n|/\langle\zeta\rangle$ is large, $(i\xi_n \mathrm{Id} + G(i\zeta)^{-1} = O(|\xi_n|^{-1})$ which implies (7.3.2). □

Lemma 7.3.2. *$f \in L^2(\mathbb{R}_+)$ then $I(f)$ is the restriction to $\mathbb{R}_+$ of the solution in L^2 of $(\partial_{x_n} + iG)\tilde{u} = \tilde{f}$ where $\tilde{f}$ is the extension of f by 0 on the negative axis.*

Proof. $\tilde{u}$ is given by the formula

$$\tilde{u}(x_n) = \int_{-\infty}^{x_n} e^{i(y_n - x_n)G}\Pi^{in}\tilde{f}dy_n - \int_{x_n}^{\infty} e^{i(y_n - x_n)G}\Pi^{out}\tilde{f}dy_n. \tag{7.3.3}$$

□

Corollary 7.3.3. *There are C and γ_0 such that when* $\operatorname{Im}\tau < -\gamma_0$

$$\gamma^{m_0}\big\|I(f)\big\|_{L^2} \leq C\langle\zeta\rangle^{m_0 - 1}\big\|f\big\|_{L^2}, \tag{7.3.4}$$

$$\gamma^{m_0}\big|I(f)_{|x_n=0}\big| \leq C\langle\zeta\rangle^{m_0 - \frac{1}{2}}\big\|f\big\|_{L^2}. \tag{7.3.5}$$

Proof. $\tilde{u}$ can be computed using a Fourier transform in x_n: its Fourier transform is

$$\hat{u}(\xi_n) = -i(\xi_n + G)^{-1}\hat{f}$$

where $\hat{f}$ is the Fourier transform of $\tilde{f}$. The L^2 estimate of $\tilde{u}$ follows from (7.3.2). The second estimate follows using the equation and the inequality

$$|\tilde{u}(0)|^2 \leq 2\big\|\tilde{u}\big\|_{L^2}\big\|\partial_{x_n}\tilde{u}\big\|_{L^2} \leq 2\big\|\tilde{u}\big\|_{L^2}\big\|\tilde{f}\big\|_{L^2} + O(\langle\zeta\rangle)\big\|\tilde{u}\big\|_{L^2}^2. \tag{7.3.6}$$

□

For the first term in (7.3.1), we use the following estimate.

Lemma 7.3.4. *There is C such that for $\operatorname{Im}\tau < -\gamma_0$ and $a \in \mathbb{E}^{in}(\zeta)$, $u = e^{-ix_nG}a$ satisfies*

$$\gamma^{m_0}\big\|u\big\|_{L^2(\mathbb{R}_+)} \le \langle\zeta\rangle^{m_0-1}|a|. \tag{7.3.7}$$

Proof. Introduce $L^* = -\partial_x - iG^*$ the adjoint of $L = \partial_x + iG$. Then

$$\big(Lu, v\big)_{L^2(\mathbb{R}_+)} - \big(u, L^*v\big)_{L^2(\mathbb{R}_+)} = -\big(u(0), v(0)\big). \tag{7.3.8}$$

In particular, if $u = e^{-ix_nG}a$ with $a \in \mathbb{E}^{in}$, one has

$$\big(u, L^*v\big)_{L^2(\mathbb{R}_+)} = \big(a, v(0)\big). \tag{7.3.9}$$

For $f \in L^2(\mathbb{R}_+)$, extend it by 0 for negative x_n and consider the solution v of $L^*v = \tilde{f}$. L^* satisfies the same estimate (7.3.2) as L and repeating the proof of the Corollary above, we obtain the estimate

$$\gamma^{m_0}|v(0)| \le C\langle\zeta\rangle^{m_0-\frac{1}{2}}\big\|f\big\|_{L^2}. \tag{7.3.10}$$

With (7.3.8), this implies (7.3.7). □

Next we need estimates for the solutions of the equation (7.2.8). The Lopatinski condition says that there is an inverse mapping $R(\zeta) : \mathbb{G} \mapsto \mathbb{E}^{in}(\zeta)$ such that $MR(\zeta) = \mathrm{Id}_{\mathbb{G}}$.

Lemma 7.3.5. *If the Lopatinski condition is satisfied, there are γ_1, m and C such that for $\operatorname{Im}\tau \le -\gamma_1$*

$$a \in \mathbb{E}^{in}(\zeta) \quad \Rightarrow \quad |\operatorname{Im}\tau|^m\big|u\big| \le C\langle\zeta\rangle^m\ \big|Ma\big|. \tag{7.3.11}$$

Equivalently, this means that

$$\big|R(\zeta)\big| \le C|\operatorname{Im}\tau|^m/\langle\zeta\rangle^m. \tag{7.3.12}$$

Proof. Again, the polynomial bound depends on properties of semi-algebraic functions. □

Summing up, we have proved the following:

Theorem 7.3.6. *Suppose that the system is hyperbolic in the time direction and the Lopatinski condition is satisfied. Then, there are C, m and γ_0 such that, when $\operatorname{Im}\tau < -\gamma_0$, for all $f \in L^2(\mathbb{R}_+)$ and all $g \in \mathbb{C}^{N_+}$, the problem (7.2.2) has a unique solution $u \in H^1(\mathbb{R}_+)$ wich satisfies,*

$$\gamma\big\|u\big\|^2_{L^2} + \big|u(0)\big|^2 \le C\big(\langle\zeta\rangle/\gamma\big)^m\big(\gamma^{-1}\big\|f\big\|^2_{L^2} + \big|g\big|^2\big). \tag{7.3.13}$$

where $\gamma = -\operatorname{Im}\tau$.