Dense Oscillations for the Euler Equations II

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The fact that the resonant interaction of a finite number of oscillatory wave trains might lead, to the creation of an infinity of waves was suggested by Hunter, Majda, and Rosales in §6.2 of [1]. To avoid this possibility they systematically assumed suitable finiteness hypotheses on their phase functions. Joly and Rauch [2] constructed a system of semilinear wave equations in spacetime of dimension 1+2 where waves propagating in a set of directions dense in the unit circle were created. Their intent was to show that nonlinear geometric optics in higher dimensions would not be possible without a finiteness hypothesis. Exactly this unlikey goal was achived in [3, 4]. Schochet [6] then gave an alternate proof in a special case sufficiently general for the examples of this note. Our paper [5] proved that such dense oscillations can be generated by the 1+2 dimensional inviscid compressible Euler equations. In this note we recall and refine that result, answering one of two questions posed at the end of that paper.

The isentropic Euler equations are a 3×3 system of quasilinear equations of the form

$$\partial_t u + \sum_{1\le j\le 2} A_j(u)\,\frac{\partial u}{\partial x_j}\ ,$$

with coefficient matrices $A_j(u)$ defined from

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$$(\partial_t + v_1\partial_1 + v_2\partial_2)v_1 + (p'(\rho)/\rho)\partial_1\rho = 0 \tag{1}$$
$$(\partial_t + v_1\partial_1 + v_2\partial_2)v_2 + (p'(\rho)/\rho)\partial_2\rho = 0 \tag{2}$$
$$(\partial_t + v_1\partial_1 + v_2\partial_2)\rho + \rho(\partial_1 v_1 + \partial_2 v_2) = 0 \tag{3}$$

The same phenomena occur in the nonisentropic case. The present setting is chosen for simplicity. The state vector is denoted $u := (v_1, v_2, \rho)$. Our solutions will be close to a background state $\underline{u} := (0, 0, \underline{\rho})$ which has the fluid at rest and the density constant. In [5] we constructed a family of smooth solutions

$$u^\epsilon = (0, 0, \underline{\rho}) + \epsilon\, U(t,\, t/\epsilon,\, x_1/\epsilon,\, x_2/\epsilon) + O(\epsilon^2) \in C^\infty([0, t_1] \times \mathbf{R}^2)\,. \tag{4}$$

The smooth profile $U(t, T, X)$ is a periodic function of T, X and the solutions u^ϵ are periodic in x. The error is measured in the L^∞ norm with a loss of a factor $1/\epsilon$ for each derivative taken, that is

$$\left|\left|\partial^\beta \left(u^\epsilon - ((0, 0, \underline{\rho}) + \epsilon U(t, t/\epsilon, x_1/\epsilon, x_2/\epsilon))\right)\right|\right|_{L^\infty([0,t_1]\times\mathbf{R}^2)} = O(\epsilon^{2-|\beta|})\,. \tag{5}$$

More general oscillatory solutions would have profiles $U(t, x, T, X)$ depending also on x. For the solutions in 4, the oscillations at time t are essentially the same at all positions x. For this reason we refer to them as homogeneous oscillations in analogy with homogeneous turbulence.

The theory of Nonlinear Geometric Optics [3, 4, 6] provides the following recipe. Given smooth periodic initial data $U(0, 0, X)$ there is a nonlinear evolution equation which is locally solvable determining $U(t, T, X)$ for $0 \le t \le t_1$ for some $t_1 > 0$. Then for ϵ small, the solution of the Euler equation with initial data

$$u(0, x) = \underline{u} + \epsilon\, U(0, 0, x/\epsilon)\,, \tag{6}$$

is smooth on $[0, t_1] \times \mathbf{R}^2$ and the relation 5 holds.

Denote by $L(\partial_T, \partial_X)$ the linearization of the Euler equations at the background state $\underline{u}$. Then L is a constant coefficient strictly hyperbolic partial differential operator. For any $\gamma = (\gamma_0, \gamma_1, \gamma_2) \in \mathbf{R}^3$, $L(\gamma)$ denotes the symbol of L. The characteristic variety of L is given by the equation

$$\det\, L(\gamma) = \gamma_0(\gamma_0^2/c^2 - \gamma_1^2 - \gamma_2^2) = 0\,, \quad c := \sqrt{p'(\underline{\rho})}\,. \tag{7}$$

Here c is the speed of sound and the characteristic variety is the union of the horizontal plane $\gamma_0 = 0$ and the speed c light cone.

The first of the equations determining U is that, as a function of the fast variables T, X, U is a solution of the linearized equation

$$L(\partial_T, \partial_X)\, U(t,T,X) = 0\,.$$

Thus standard linearization yields the approximate solution $\underline{u}+\epsilon U(0, t/\epsilon, x/\epsilon)$, which is accurate only for times t small compared to ϵ. Expanding U in a Fourier series yields

$$U(t) = \sum_{\gamma}\ U_\gamma(t)\ e^{i\gamma.(T,X)}\,. \tag{8}$$

The sum is over γ from the characteristic variety and the corresponding amplitude U_γ belongs to the kernel of $L(\gamma)$. The solutions 1.4 have as principal contributions the terms

$$\epsilon\, U_\gamma(t)\ e^{i\gamma.(T,X)/\epsilon}$$

which are plane waves of amplitude and wavelength of order ϵ. When $\gamma_0 = 0$, the corresponding waves are stationary and are called entropy or vorticity waves. Otherwise, they are acoustic waves which move with speed c in the direction of the unit vector $(\gamma_1, \gamma_2)/|\gamma_1, \gamma_2|$.

Denote by $\mathbf{E}_\gamma$ the spectral projection of $\mathbf{R}^3$ onto kernel $L(\gamma)$. Then $\mathbf{E}_\gamma = |r_\gamma >< l_\gamma|$ for normalized right and left eigenvectors r_γ and l_γ. Introduce the operator $\mathbf{E}$ on trigonometric series by

$$\mathbf{E}(\sum_{\gamma}\ V_\gamma(t)\ e^{i\gamma.(T,X)}) := \sum_{\gamma}\ \mathbf{E}_\gamma V_\gamma(t)\ e^{i\gamma.(T,X)}\,. \tag{9}$$

The equations determining the profile U from its initial data $U(0,0,X)$ are then

$$\mathbf{E}U = U\,, \quad \text{and} \quad \mathbf{E}\Big(U_t + \sum_j (\mathbf{A}_j\, U)\, \frac{\partial U}{\partial X_j}\,) = 0\,, \tag{10}$$

$$\mathbf{A}_j := D_u A_j(\underline{u})\,.$$

The main result of [JMR] studies the behavior of the solution of 10 with initial data so that

$$U(0,T,X) = r_\alpha\, f(\alpha.(T,X)) \ + r_\beta\, f(\beta.(T,X)) \ + \ r_\nu\, f(\nu.(T,X)) \tag{11}$$

$$\alpha := (c,1,0)\,, \quad \beta := (0,1,0)\,, \quad \text{and} \quad \nu := (0,3,4)\,,$$

consists of three plane waves. The function f is given by

$$f(\theta) := \sum_{n\in\mathbf{Z}} e^{-|n|}\, e^{in\theta}\,. \tag{12}$$

Definiton 1 *Denote by Λ the lattice of integer linear combinations of the vectors α, β, and, ν.*

The characteristic covectors in Λ are precisely the wavenumbers one expects to find in the spectrum of U generated by nonlinear effects. In [5] we analyse the initial value problem 1.10, 1.11. There is a unique local solution $U \in C^\infty([0,t_1] \times \mathbf{R}^3)$ which is $2\pi/c, 2\pi, 2\pi$ periodic in T, X_1, X_2. The spectrum of $U(t)$ is contained in the characteristic points of Λ. Concerning the set of amplitudes which are ignited we prove the following.

Theorem 1 *The set of characteristic points $\gamma := (\tau, \xi)$ in $\Lambda \cap \{\tau = c|\xi|\}$ have directions $\xi/|\xi|$ dense in the unit circle. In addition for any γ in this set which is not proportional to any of the five vectors $(7,-24)$, $(\pm 3, 4)$, $(1,0)$, $(0,1)$ one has*

$$\frac{d^2 U_\gamma(0)}{dt^2} \neq 0\,, \quad \text{and,} \quad \frac{d^j U_\gamma(0)}{dt^j} = 0 \quad \text{for} \quad j = 0, 1\,. \tag{13}$$

This result asserts that waves traveling in directions dense in the unit sphere are generated by the nonlinear interaction of three waves present initially. The above result does not asssert that the $U_{\tau,\xi}(t)$ are all nonzero for a particular t. In this note we provide a simple proof that they are all nonzero except for at most a countable number of t. We leave unanswered the question of whether for some $0 < t_2 < t_1$, they are all nonzero for $0 < t < t_2$.

Theorem 2 *For the solution $U(t,T,X)$ of the last theorem, each of the functions $U_\gamma(t)$ is real analytic on $[0,t_1]$. In particular, for the τ, ξ for which 13 is proved, the $U_{\tau,\xi}(t)$ vanishes for at most a finite set of $t \in [0,t_1]$.*

Proof. We indicate two different arguments showing that $U(t,T,X)$ is real analytic. The first is an application of the abstract Cauchy-Kowalewski theorem to the initial value problem 11-12. Toward this end introduce a scale of Banach spaces.

Definiton 2 *$\mathbf{B}_s$ is the Hilbert space of triply periodic solutions of the linearized equation,*

$$F(T,X) = \sum_{\lambda \in \Lambda} F_\lambda e^{i\lambda.(T,X)}, \quad \mathbf{E}_\lambda F_\lambda = F_\lambda \tag{14}$$

such that

$$\sum ||F_\lambda||_{\mathbf{C}^3}\, e^{s|\lambda|} < \infty\,. \tag{15}$$

The abstract treatment of the Cauchy-Kowalewski Theorem (see [9] and the references therein) applies virtually without change in the present setting. It shows that if $U(0)$ belongs to $\mathbf{B}_\sigma$ for some $\sigma > 0$, then there is a $t_3 > 0$ and a unique solution U of 10,11 on $t < t_3$ such that for all $\underline{t} < t_3$ U is continuous on $[0, \underline{t}[$ with values in $B_s(\underline{t})$ with $s(t)$ of the form $\sigma - \lambda t$ for some constant λ. Using the equation to express time derivatives in terms of spatial derivatives it follows that U is a real analytic function of t, T, X for $0 \le t < t_3$. Therefore all the Fourier coefficients $U_\lambda(t)$ are real analytic on $[0, t_3[$.

The advantage of this method is that it is quick and easy. The disadvantage is that it applies on a possibly shorter interval $[0, t_3]$. For hyperbolic Cauchy problems the analogous gap was filled by the result of Mizohata [7] in the linear case and by Alinhac and Metivier [8] in the nonlinear case. They show that for real analytic initial data, solutions remained real analytic as long as they are C^1. Their proof extends to the present context without essential modification. The main difference is the presence of the factor $\mathbf{E}$ in the equation for the profile U. However, $\mathbf{E}$ commutes with derivatives and is bounded on $H^2(\mathbf{T}^3)$ which is all that is needed to control it. For the proof one estimates the rate of growth of the derivatives of U as the order of derivation increases. One proves by induction on $|\alpha| > 1$ that there are constants M, λ, σ such that for $t < t_1$

$$||\partial^\alpha_{T,X} U(t,.,.)||_{H^2(\mathbf{T}^3)} \le M \sigma^{1-|\alpha|} \, \frac{(|\alpha|-1)!}{|\alpha|^2} \, e^{(|\alpha|-1)\lambda t} \,. \tag{16}$$

To prove 16, differentiate the equation for U to find

$$\partial_t \, \partial^\alpha U + \mathbf{E}\big(\sum_j (\mathbf{A}_j U) \, \partial_{X_j} \partial^\alpha U \big) = \mathbf{E} G_\alpha \,, \tag{17}$$

$$G_\alpha := \sum_{0 < \beta \le \alpha} \binom{\alpha}{\beta} \mathbf{A}_j \, \partial^\beta U \; \partial_{X_j} \partial^{\alpha - \beta} U \,. \tag{18}$$

The basic energy estimate for the profile equation then shows that

$$||\partial^\alpha U(t)||_{H^2(\mathbf{T}^3)} \le ||\partial^\alpha U(0)||_{H^2(\mathbf{T}^3)} + C \int_0^t ||G_\alpha(t')||_{H^2(\mathbf{T}^3)} \, dt' \tag{19}$$

with a constant C that depends on the C^1 norm of U. For the details of the proof of 16 using 18 and 19 see [8].

This shows that U is real analytic on $[0, t_1] \times \mathbf{R}^3$ which implies the analyticity of the Fourier coefficients $U_\gamma(t)$. $\diamondsuit$

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