Real and Complex Regularity are Equivakent for Hyperbolic Characteristic Varieties

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If $P(\eta)$ is a real homogeneous polynomial one associates real and a complex algebraic varieties

$$\mathbf{V_R} \ := \ \big\{\eta \in \mathbf{R}^n \setminus 0 \ : \ P(\eta) = 0\big\} \quad \text{and} \quad \mathbf{V_C} \ := \ \big\{\eta \in \mathbf{C}^n \setminus 0 \ : \ P(\eta) = 0\big\}.$$

A homogeneous polynomial is hyperbolic with timelike direction $\theta \in \mathbf{R}^n \setminus 0$ iff for all real η the equation

$$P(\eta + s\theta) \ = \ 0$$

has only real roots s.

Then $\mathbf{V_R}$ (resp. $\mathbf{V_C}$) is a variety of real (resp. complex) codimension equal to one. $\mathbf{V_R}$ is called the characteristic variety.

A point $\eta \in \mathbf{V_R}$ (resp. $\mathbf{V_R}$) is a regular point point of $\mathbf{V_R}$ (resp. $\mathbf{V_C}$) iff in an $\mathbf{R}^n$ (resp. $\mathbf{C}^n$) neighborhood of η, $\mathbf{V_R}$ (resp. $\mathbf{V_C}$) is a smooth manifold of real (resp. complex) codimension equal to 1.

In general these two notions are in general distinct as the following example shows.

Example. The polynomial

$$P(\eta) \ = \ (\eta_1 - \eta_2)\big(((\eta_1 - \eta_2)^2 + \eta_3^2\big)$$

has

$$\mathbf{V_R} \ = \ \big\{\eta_1 = \eta_2\big\}\,, \qquad \mathbf{V_C} \ = \ \big\{\eta_1 = \eta_2\big\} \ \cap \ \big\{\eta_3 = i(\eta_1 - \eta_2)\big\} \ \cap \ \big\{\eta_3 = -i(\eta_1 - \eta_2)\big\}\,.$$

The real points are all regular points of $\mathbf{V_R}$, and they are all singular points of $\mathbf{V_C}$.

In this note we prove that for hyperbolic polynomials, the two notions are equivalent. This fact is used by B. TEXIER in his recent and elegant derivation of the algebraic identities of geometric optics.

Theorem. *If P is a homogeneous hyperbolic polynomial, then $\eta \in \mathbf{V_R}$ is a smooth point of $\mathbf{V_R}$ if and only if η is a smooth point of $\mathbf{V_C}$.*

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Proof. To prove the if assertion, suppose that $\underline{\eta}$ is a regular point of $\mathbf{V_C}$. Express

$$P \;=\; \Pi_\alpha \big(P_\alpha\big)^{m(\alpha)}$$

as a product of irreducible factors. Then (see [?]) $\underline{\eta}$ is a root of exactly one of the P_α and for that α, $dP(\underline{\eta}) \neq 0$.

The irreducilble factors of a hyberbolic polynomial are hyperbolic ([H, thm. ?]). Therefore without loss of generality we can suppose that P_α has real coefficients in which case $dP(\underline{\eta})$ is a nonzero real covector.

On a neighborhood of $\underline{\eta}$, $\mathbf{V_R} = \{P_\alpha(\eta) = 0\}$. The implicit function theorem implies that on a real neighborhood of $\underline{\eta}$, $\mathbf{V_R}$ is a real analytic manifold of codimension equal to 1. Thus $\underline{\eta}$ is a regular point of $\mathbf{V_R}$.

For the converse, suppose that $\underline{\eta}$ is a real regular point of $\mathbf{V_R}$. Choose coordinates

$$\eta = (\tau, \xi)\ \in \mathbf{R} \times \mathbf{R}^{n-1}\,, \qquad \underline{\eta} = (\underline{\tau}, \underline{\xi})\,,$$

so that $\theta = (1, 0, \dots, 0)$. In [R] it is proved that at a regular point of $\mathbf{V_R}$ the real tangent plane has equation

$$\ell\Big((\tau, \xi) - (\underline{\tau}, \underline{\xi})\Big) \;=\; 0\,,$$

where ℓ is a real homogeneous hyperbolic polynomial of degree equal to 1 with θ timelike [2]. Thus $\ell(1, 0, \dots, 0) \neq 0$.

The implicit function theorem implies that on a real neighborhood of $(\underline{\tau}, \underline{\xi})$, the real variety is given by a real analytic equation $\tau = \lambda(\xi)$.

Dividing P by $P(1, 0, \dots, 0)$ reduces to the case where this coefficient is equal to 1. Consider

$$P(\tau, \xi) \;=\; \tau^m + p_1(\xi)\tau + p_2(\xi)\tau^2 + \cdots + p_m(\xi)$$

as a polynomial in τ depending on ξ. For $\xi = \underline{\xi}$, denote by k, the multiplicity of the root $\underline{\tau} = \lambda(\underline{\xi})$. For ξ near $\underline{\xi}$, $P(\tau, \xi) = 0$ has exactly k roots near $\lambda(\underline{\xi})$. Hyperbolicity implies that they are all real. Since $\mathbf{V_R}$ is regular at $\underline{\xi}$ there is exactly one point $(\lambda(\xi), \xi)$ near $(\underline{\tau}, \underline{\xi})$ projecting to ξ. Therefore the k roots are all equal to $\lambda(\xi)$ so $\lambda(\xi)$ is a root of multiplicity exactly equal to k.

For ξ near $\underline{\xi}$, divide $P(\tau, \xi)$ by $\big(\tau - \lambda(\xi)\big)^k$,

$$P(\tau, \xi) \;=\; \big(\tau - \lambda(\xi)\big)^k \Big(\tau^{m-k} + a_1(\xi)\tau^{m-k-1} + \cdots + a_{m-k}(\xi)\Big) \tag{1}$$

$$= \Big(\tau^k + C_1^k \tau^{k-1}(-\lambda) + C_2^k \tau^{k-2}(-\lambda)^2 + \cdots + (-\lambda)^k\Big)\Big(\tau^{m-k} + a_1(\xi)\tau^{m-k-1} + \cdots + a_{m-k}(\xi)\Big)\,.$$

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[2] It is proven that the leading term in the Taylor expansion of P at $\underline{\eta}$ is proportional to a power of $\ell(\eta - \underline{\eta})$.

Equating coefficients of powers of τ yields the equations

$$p_1 = C_1^k(-\lambda) + a_1\,,$$
$$p_2 = C_2^k(-\lambda)^2 + C_1^k(-\lambda)a_1 + a_2\,,$$

and so on. Since λ and p_1 are real analytic, the first equality implies that a_1 is a real analytiuc function of ξ. Then, the second equality implies that a_2 is real analytic. By iunduction one finds that all the $a_j(\xi)$ are real analytic functions of ξ, and therefore continuous on a complex neighborhood of $(\underline{\tau}, \underline{\xi})$.

Since the second factor on the right of (1) is nonzero at $(\underline{\tau}, \underline{\xi})$, it is nonzero on a complex neighborhood of this point. Therefore, on a complex neighborhood of $(\underline{\tau}, \underline{\xi})$, the complex variety $\mathbf{V}_\mathbf{C}$ is given by the real analytic equation $\tau = \lambda(\xi)$. Thus, $(\underline{\tau}, \underline{\xi})$ is a regular point of $\mathbf{V}_\mathbf{C}$ and the proof is complete.

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