Trilinear Resonant Interactions of Semilinear Hyperbolic Waves

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1. Introduction.

An important question in the study of nonlinear equations is the understanding of the asymptotic behavior of families of solutions, or approximate solutions. One of the difficulty is known to be the nonlinear interaction of oscillations. This phenomenon has been studied for special classes of solutions, such as oscillatory functions with phases and amplitudes. This is the domain of geometric optics. In some cases, general families of bounded solutions have also been considered. The present paper enters the later category, but is highly inspired by the results of nonlinear geometric optics.

Geometric optics provides a precise description of the asymptotic structure of solutions to both linear (e.g. [Wh]) and nonlinear ([J], [HK], [MR], [HMR], [JMR 1], [JMR 2], [S], ...) hyperbolic PDEs whose initial data $u^\varepsilon(0,x) = u_0(x) + \varepsilon^m u_1(x, \frac{\vec\varphi_0(x)}{\varepsilon})$ have rapid oscillations with prescribed phases $\vec\varphi_0$ and a single prescribed scale $\frac{1}{\varepsilon}$. Such solutions are important in both theory and applications ([L], [KK]). In the lowest approximation, oscillations having different spatial scales interact only when m equals zero, since otherwise the amplitudes of higher-frequency oscillations become negligible relative to those of lower frequencies. Taking $m = 0$ requires L^∞ estimates. Systems that are well-posed in L^∞ occur mostly in one spatial dimension. For such systems, arbitrary uniformly bounded sequences of initial data may be considered, but comparatively little is known about the asymptotic structure of the resulting solutions. Since the oscillations in such solutions may be extremely complex, a natural problem is to determine their non-oscillatory part, which may be identified with the weak limit of the sequence of solutions. The highly oscillatory nature of the solutions manifests itself in the absence of strong convergence in general. Of course, some information about those oscillations will generally be needed in order to determine the weak limit.

The asymptotic structure of uniformly bounded solutions depends strongly on the nature of the nonlinearity: For entropy solutions of a genuinely nonlinear conservation law ([T 1]) or of a pair of such equations in one spatial dimension ([DP]), oscillations of amplitude $O(1)$ do not persist for positive times, so a sequence of solutions converges strongly to the solution whose initial data is the weak limit of the initial data. Although oscillations do persist in linear systems, the weak limit of a sequence of solutions to such systems is just the solution of the same equations whose initial data is the weak limit initially. In contrast to both these cases, the weak limit of solutions to semilinear equations is not determined just by the weak limit of the initial data. For a pair of semilinear PDEs in one space dimension ([T 2], [McLPT]) and for nonresonant systems of three such equations ([JMR 4]) the weak limit of the solution is determined by the weak limits of all continuous functions of single components of the initial data, but even this is no longer true for resonant systems ([JMR 4]). In this paper, we discover what additional information is needed to determine the weak limit for resonant systems of three semilinear equations in one spatial dimension, and obtain equations that characterize that limit. Larger systems with quadratic interactions will also be treated; the precise assumptions are given in §5.

After a linear change of variables, a semilinear hyperbolic system in one spatial dimension takes the form

$$(1.1)\qquad X_k u_{k,n} := \{\partial_t - c_k(t,x)\,\partial_x\}\; u_{k,n} \;=\; F_k(u_{1,n},\dots,\, u_{p,n})\,, \quad \text{for } k \in \{1,\dots p\}\,.$$

The speeds c_k are assumed to be smooth functions, with $c_k(t,x) \neq c_l(t,x)$ when $k \neq l$, and the F_k are assumed to be smooth functions of their arguments. Since the initial-value problem for such systems is well-posed in L^∞ and has a finite propagation speed, initial data that are uniformly bounded in L^∞ on an interval $\omega = [a,b] \subseteq \mathbb{R}$ yield solutions to (1.1) that are uniformly bounded in L^∞ on some region $\Omega \subset \mathbb{R}^2$ contained in the domain of determinacy of ω.

Young measures are a convenient tool for characterizing the weak limits of continuous functions of a uniformly bounded sequence $u_n(y)$ ([T 1]), where in the current context $y := (t,x) \in \Omega$. After restricting to a subsequence, all such weak limits exist and are given by $f(u_n(y)) \rightharpoonup \int f(\lambda)\mu_y(d\lambda)$, where the Young measure $\mu_y(d\lambda)$ is a nonnegative measure of total mass one for each fixed y. Since the sequence u_n converges strongly (in L^p_{loc} for $p < \infty$) iff $\mu_y(d\lambda)$ is a Dirac mass for almost all y ([T 1]), the Young measure does provide some information about the presence of oscillations. However, for any periodic U with period p, any $\varphi \in C^1$ whose gradient almost never vanishes, and any sequence ν_n tending to infinity, $f(U(\nu_n\varphi(y))) \rightharpoonup \frac{1}{p}\int_0^p f(U(z))\,dz$, so the Young measure of the sequence $U(\nu_n\varphi(y))$ contains no information about the particular phase function φ or the scale of oscillation ν_n.

The basic strategy for determining the weak limit of a sequence of solutions is to try to derive an evolution equation for the Young measure of each component. For a single semilinear equation this is easy: Multiplying $Xu_n := \{\partial_t - c(t,x)\partial_x\}u_n = F(u_n)$ by $\Phi'(u_n)$ shows that $X\Phi(u_n) = \Phi'(u_n)F(u_n)$. Describing the weak limit of this equation in the weak sense via the Young measure of u_n, and then integrating by parts in λ in the term on the right side of the result in order to obtain a common factor $\Phi(\lambda)$, yields the evolution equation $X\mu_y(d\lambda) = -\partial_\lambda\,[F(\lambda)\mu_y(d\lambda)]$. This linear PDE is well-posed even for measures, and preserves the total mass of the measure for fixed y since it is in conservation form with respect to the variable λ. However, a serious difficulty arises in applying this strategy to systems: After multiplying the equation for $u_{k,n}$ in (1.1) by $\Phi'(u_n)$ to obtain

$$(1.2)\qquad X_k\Phi(u_{k,n}) = \Phi'(u_{k,n})F(u_{1,n},\dots,u_{p,n}),$$

we see that although the weak limit of the left side of (1.2) can be written in terms of the component Young measure $\mu_{k,y}(d\lambda)$ of the sequence $u_{k,n}$, the weak limit of the right side can in general only be expressed in terms of the joint Young measure $\mu_y(d\lambda_1,\ldots,d\lambda_p)$ such that $G(u_{1,n},\ldots,u_{p,n}) \rightharpoonup \int G(\lambda_1,\ldots,\lambda_p)\mu_y(d\lambda_1,\ldots,d\lambda_p)$. But no evolution equation can be derived for this joint measure μ, because different components $u_{k,n}$ are transported along different characteristics. One therefore tries to obtain a closed set of equations for the component Young measures by determining the joint measure μ_y pointwise in y in terms of the component Young measures $\mu_{k,y}$. For the case of two equations this task has been accomplished by the theory of compensated compactness and its generalizations ([T 1], [M], [T 3], [G]), but it is not possible in general for larger systems because of the phenomenon of resonance, which already plays an important role in the theory of geometric optics for such systems.

Compensated compactness is the study of conditions on f and $u_{k,n}$ weaker than strong convergence which nevertheless ensure that if $u_{k,n} \rightharpoonup \underline{u}_k$ then

$$f(u_{1,n},\ldots,u_{p,n}) \rightharpoonup f(\underline{u}_1,\ldots,\underline{u}_p). \tag{1.3}$$

In particular, the div-curl lemma ([T 1]) shows that for a pair of semilinear equations with distinct constant speeds c_k,

$$u_{1,n}u_{2,n} \rightharpoonup \underline{u}_1\underline{u}_2; \tag{1.4}$$

the results of [T 3], [G] imply that (1.4) remains valid for variable coefficients. This particular result can also be demonstrated directly from the form $u_{k,n}(\psi_k(t,x),t)$ of the solutions, where ψ_k = constant are the integral curves of X_k. The PDEs show that the derivative of $u_{k,n}$ with respect to its second variable is uniformly bounded, which means that the solutions can be approximated uniformly by step functions in the second variable, and this implies that it suffices to prove the result when $u_{k,n} = u_{k,n}(\psi_k(t,x))$. Changing variables from t and x to ψ_1 and ψ_2 in the integral $\iint a(t,x)u_1(\psi_1(t,x))u_2(\psi_2(t,x))\,dx\,dt$ and approximating the resulting test function by a sum of products $\Phi_1(\psi_1)\Phi_2(\psi_2)$ then yields (1.4). Since functions of $u_{k,n}$ satisfy similar equations, and sums of products $\Phi_1(u_{1,n})\Phi_2(u_{2,n})$ are dense in $C^0(\mathbb{R}^2)$, the joint measure $\mu(d\lambda_1,d\lambda_2)$ of two components whose PDEs have different speeds c_k is just the product of the individual Young measures:

$$\mu_y(d\lambda_1,d\lambda_2) = \mu_{1,y}(d\lambda_1)\mu_{2,y}(d\lambda_2). \tag{1.5}$$

Expressing the weak limit of $X_1\Phi_1(u_{1,n}) = \Phi_1'(u_1)F_1(u_{1,n},u_{2,n})$ in terms of the joint measure $\mu(\lambda_1,\lambda_2)$ and integrating over λ_2 therefore yields the equation

$$X_1\mu_1 = -\partial_\lambda\{\mu_1 \int F_1(\lambda,\lambda_2)\mu_{2,y}(d\lambda_2)\}; \tag{1.6}$$

a similar equation is obtained for μ_2. These equations for the Young measures can also be written as equations for "profiles" of those measures, and the latter equations are identical to the equations for the profiles of nonresonant geometric optics with one phase per component: The profiles $U_k(y,\cdot)$, defined as the right-continuous inverse on $]0,1[$ of the distribution function $M_k(Y,\lambda) := \mu_{k,y}(]-\infty,\lambda[)$, satisfy

$$X_1(y,\partial_y)U_1(y,\theta) = \int_{]0,1[^2} F_1(U_1(y,\theta),U_2(y,\theta_2))\,d\theta_2$$

and a similar equation for U_2 (cf. [JMR 4]).

The same argument would yield analogous equations for the Young measures of larger systems whenever the generalization of (1.4) to products of more than two components is valid. The task of determining the evolution of weak limits of solutions to (1.1) therefore reduces to the problem of evaluating such weak limits:

MAIN QUESTION. *Suppose that $v_{k,n}$ are uniformly bounded in $L^\infty(\Omega)$, converge weakly to $\underline{v}_k$, and are such that $X_k v_{k,n}$ is uniformly bounded in $L^\infty(\Omega)$. Is it true that*

$$(1.7)\qquad \prod_{k=1}^{p} v_{k,n} \rightharpoonup \prod_{k=1}^{p} \underline{v}_k\,?$$

If not, what further information is needed to compute the weak limits, and what information about the initial data determine it?

Counter-examples to (1.7) with p at least three were presented already in [T 1]. In fact, the necessary ([T 1]) and sufficient ([M]) conditions of compensated-compactness theory for a polynomial f of degree $p \geq 3$ to satisfy (1.3) when the u_k are solutions of the constant-coefficient linear system $\sum_{i,j} A_{i,j,k}\partial_{x_j} u_i = F_k$ for $k = 1,\dots,p$ are never satisfied by system (1.1) ([JMR 4]). Surprisingly, (1.7) is nevertheless valid for generic variable-coefficient 3×3 systems (1.1) ([JMR 4]). Here we answer the above main question for general 3×3 systems (1.1) by treating the case in which (1.7) fails. For larger systems the last part of the question remains open. Henceforth until §3 we consider only the case of three equations.

What the answer to the main question depends on is whether system (1.1) admits resonances between the oscillations of the components. Recall from [JMR 1] the following definition:

DEFINITION 1.1. *A resonance on $\Omega \subset \mathbb{R}^2$ for the vector fields X_k, $k \in \{1,2,3\}$, is a triplet (ψ_1,ψ_2,ψ_3) of functions in $C^\infty(\Omega)$ such that*

$$(1.8)\qquad X_k\,\psi_k = 0 \quad \textit{for } k \in \{1,2,3\}\,, \quad \textit{and} \quad \sum_{k=1}^{3} \psi_k = 0\,.$$

The resonance is trivial, when all the $d\psi_k$ are identically zero.

We refer to [JMR 1] for a detailed study of the existence of resonances. In particular, when three vector fields are considered as above, the dimension of the space of resonances, modulo constant functions, is zero or one, and the existence of a nontrivial resonance depends upon the vanishing of a geometric invariant, called the curvature of the web defined by the three families of integral curves of the X_k (see [BB], [P]). Thus, for generic vector fields, there is no nontrivial resonance, and it is shown in [JMR 4] that (1.7) is then valid. However, the existence of resonances is an important phenomenon. For example, vector fields $\partial_t - c_k\partial_x$ with constant coefficients are always resonant. The resonant phases are $\psi_k = \alpha_k(x + c_k t)$ with $\sum \alpha_k = 0$ and $\sum \alpha_k c_k = 0$. Other examples of resonant vector fields are given in [JMR 1]. However, when a nontrivial resonance exists for three pairwise independent vector fields, there is a local change of variables such that in the new variables the vector fields are parallel to constant-coefficient fields. Note too that the triplet of phases is uniquely determined up to a constant, since the dimension of the space of resonances is at most one.

When a nontrivial resonance does exist then (1.7) is not always satisfied: For any scale ν_n tending to infinity and any smooth $a_k(y)$, the functions

$$v_{k,n} := a_k(y) e^{i\nu_n \psi_k(y)} \tag{1.9}$$

satisfy the hypotheses of the main question with the weak limits $\underline{v}_k$ equaling zero, yet their product $\prod_k a_k$ is independent of n and can be chosen $\neq$ zero.

This difficulty can only be overcome by refining the analysis using Young measures to account for the resonant interaction of oscillations. We therefore introduce multiscale Young measures, which combine the analysis via Young measures and the multiscale analysis of resonant oscillations. This extends to more general frameworks ideas introduced in the study of homogenization problems and in nonlinear geometric optics (e.g. [A], [N], [E], [ES], [JMR 3]). An important difficulty is that the number of scales which are relevant for a given sequence can be infinite, and furthermore these scales are not known in advance. It is part of the construction to determine them together with the underlying group structure which describes the resonances. The multiscale Young measures are shown to satisfy transport equations, and thus are determined by their Cauchy data. It is noticeable that these equations are again closely related to the equations of semi-linear geometric optics. The main steps in this analysis are the following:

A) The first key observation is that oscillations of the form (1.9), with the phases satisfying (1.8), are the only obstacle to the convergence (1.7):

THEOREM 1.2. *Suppose that $\{v_{k,n}\}$, $k \in \{1,2,3\}$, are three bounded sequence in $L^\infty(\Omega)$, such that the $X_k v_{k,n}$ are bounded in $L^\infty(\Omega)$. Suppose that one of them, $v_{j,n}$, satisfies the condition*

$$\forall \nu \in \mathbf{S} := \mathbb{R}^{\mathbb{N}} , \quad v_{j,n}\, e^{-i\nu_n \psi_j} \rightharpoonup 0 , \tag{1.10}$$

where ψ_j is the unique resonant phase from (1.8). Then, $v_{1,n}v_{2,n}v_{3,n}$ converges weakly to 0 as $n \to +\infty$.

Remark. Theorem 1.2 is the only point in our analysis that does not have an obvious generalization to the case of more than three components.

Note that, for $\nu = 0$, (1.10) implies that $v_{j,n}$ converges weakly to zero. The meaning of (1.10), for unbounded sequences ν, is that $v_{j,n}$ has no oscillations with respect to the phase ψ_j. We denote by $\mathcal{L}^p_{no,\psi_j}$ the space of bounded sequences in $L^p(\Omega)$ which satisfy (1.10).

B) The second step is to extract from a given sequence all the oscillations of the form (1.9). This leads us to consider sums and series of such sequences, with an arbitrary phase function φ. More precisely, when A is a subset of $\mathbf{S} := \mathbb{R}^{\mathbb{N}}$, introduce $\mathcal{P}_\varphi(A)$ the space of sequences of the form

$$w_n(y) \ = \ \sum a_j(y)\ e^{i\,\nu_n^{(j)}\,\varphi(y)}\,,$$

where the sum runs over a finite set of indices, $a_j \in C_0^\infty(\Omega)$, and $\nu^{(j)} \in A$. Introduce next $\mathcal{L}^p_{os,\varphi}(A)$, the asymptotic closure of $\mathcal{P}_\varphi(A)$: $\mathcal{L}^p_{os,\varphi}(A)$ is the space of bounded sequences w_n in $L^p(\Omega)$ such that for all $\delta > 0$, there are a sequence $\tilde w_n$ in $\mathcal{P}_\varphi(A)$ and $n_0 \in \mathbb{N}$, such that

$$\forall n \geq n_0\,, \quad \|\, w_n \ - \ \tilde w_n \,\|_{L^p(\Omega)} \ \leq \ \delta\,.$$

When $A = \mathbf{S}$, we simply note $\mathcal{P}_\varphi = \mathcal{P}_\varphi(\mathbf{S})$ and $\mathcal{L}_{os,\varphi} = \mathcal{L}_{os,\varphi}(\mathbf{S})$.

THEOREM 1.3. *Suppose that v_n is a bounded sequence in $L^2(\Omega)$. Then, there is a subsequence $v_{\ell(n)}$ which belongs to $\mathcal{L}^2_{os,\varphi} \ + \ \mathcal{L}^2_{no,\varphi}$.*

The idea is to perform an asymptotic Fourier analysis of the sequence v_n. For all $\nu \in \mathbf{S}$, one would like to define a Fourier coefficient a_ν as the weak limit of $v_n\ e^{-i\,\nu_n\,\varphi}$. Next, one would introduce $w_n \sim \sum_\nu a_\nu\ e^{i\,\nu_n\,\varphi}$ so that $v_n - w_n \in \mathcal{L}^2_{no,\varphi}$. The main difficulty is that subsequences must be extracted to ensure the weak convergences, while $\mathbf{S}$ is not countable. However, only an at most countable set $A \subset \mathbf{S}$ is expected to contribute and one proves that v_n has a subsequence in a space $\mathcal{L}^2_{os,\varphi}(A) \ + \ \mathcal{L}^2_{no,\varphi}$. The problem is that A is not known in advance, but must be constructed while the subsequence is extracted.

C) Resonance in dispersive PDEs generally involves only a finite number of modes (see e.g. [H]), so such interactions are described by equations for the amplitude of each resonant mode. In contrast, infinitely many modes interact in nonlinear hyperbolic PDEs because any mode can excite all its harmonics. To avoid having an infinity of equations, the theory of geometric optics for such equations combines the modes into a Fourier series

$$\mathcal{V}(y, \nu_n\varphi(y)) := \sum_{k\in\mathbb{Z}} c_k(y) e^{ik\nu_n\varphi(y)}, \tag{1.11}$$

and the entire profile $\mathcal{V}(y,\theta)$ is described by a single equation involving an additional, periodic, independent variable θ. Replacing the index set $\mathbb{Z}$ in (1.11) by an arbitrary countable subset of $\mathbb{R}$ allows consideration of the the more general case of almost-periodic oscillations.

Although both (1.11) and its almost-periodic generalization yield particular cases of sequences in $\mathcal{L}_{os,\varphi}$, such sequences can be more complicated because infinitely many scales may be involved. To avoid having infinitely many equations for the different scales, the modes from all those scales must be combined into one grand sum

$$\sum_{k,j} c_{k,j}(y) e^{ik\nu_n^j \varphi(y)} . \tag{1.12}$$

Just as the sum in (1.11) is the Fourier series on the torus, the sum (1.12) should correspond to the Fourier series on some compact Abelian group G, which has the form

$$\mathcal{V}(y,g) \ = \ \sum_{\alpha} \ a_\alpha(y) \ e_\alpha(g) \, , \tag{1.13}$$

where the index α runs over the dual group $\widehat{G}$ and e_α is the corresponding character (see e.g. [W] or [HR]). Identifying $\tau \in \mathbb{R}$ with the character $t \to e^{i\,\tau\, t}$ on $\mathbb{R}$, there exist dual homomorphisms $\rho_n \in \mathrm{Hom}(\mathbb{R}; G)$ and $\nu_n \in \mathrm{Hom}(\widehat{G}; \mathbb{R})$, such that

$$e_\alpha(\rho_n(t)) \ = \ e^{i\,\nu_n(\alpha)\, t} .$$

Profiles $\mathcal{V}(y, \rho_n(\varphi(y)))$ therefore formally have the form of a sum like (1.12), i.e.

$$\mathcal{V}(y, \rho_n(\varphi(y))) \ = \ \sum_{\alpha} \ a_\alpha(y) \ e^{i\,\nu_n(\alpha)\,\varphi(y)} .$$

This result is exact when the Fourier series (1.13) converges uniformly on $\Omega \times G$. In this case, it defines a sequence v_n in $\mathcal{L}^\infty_{os,\varphi}(H)$, where

$$H \ := \ \{ \ \{\nu_n(\alpha)\}_{n\in\mathbb{N}} \ ; \ \alpha \in \widehat{G} \ \} \ \subset \ \mathbf{S} . \tag{1.14}$$

To avoid redundancy, one requires that

$$\forall \alpha \in \widehat{G}\backslash\{0\} \ : \quad |\,\nu_n(\alpha)\,| \ \to \ +\infty \quad \text{as} \quad n \ \to \ +\infty . \tag{1.15}$$

We say that (G, ρ_n) is admissible when this condition is satisfied.

The next two results show that the study of oscillations in $\mathcal{L}_{os,\varphi}$ can indeed be completely understood in the framework sketched above.

THEOREM 1.4. *For any at most countable $A \subset \mathbf{S}$, there are an increasing map $\ell : \mathbb{N} \to \mathbb{N}$, a compact Abelian group G and a sequence of homomorphisms $\rho_n \in \mathrm{Hom}(\mathbb{R}; G)$ which satisfies condition* (1.15)*, such that, for every sequence v_n in $\mathcal{L}^p_{os,\varphi}(A)$ the subsequence $v_{\ell(n)}$ belongs to $\mathcal{L}^p_{os,\varphi}(H)$, where H is defined in* (1.14).

THEOREM 1.5. *Suppose that G is a compact Abelian group and ρ_n is a sequence of homomorphisms in $\mathrm{Hom}(\mathbb{R}; G)$ which satisfies condition* (1.15)*. For $p \in]1, +\infty[$, the mapping*

$$\mathcal{V} \ \to \ v_n(y) \ = \ \mathcal{V}(y, \rho_n(\varphi(y))) \, ,$$

defined for $\mathcal{V} \in C^0_0(\Omega \times G)$, extends isomorphically from $L^p(\Omega \times G)$ onto $\mathcal{L}^p_{os,\varphi}(H)/\mathcal{L}^p_0$, where $\mathcal{L}^p_0$ denotes the space of sequences in $L^p(\Omega)$ which converge strongly to 0.

The limit cases $p = 1$ and $p = +\infty$ are more delicate (see §4). When (G, ρ_n) is admissible, a sequence v_n in $\mathcal{L}^p_{os,\varphi}(H)$ defines a unique *profile* $\mathcal{V} \in L^p(\Omega \times G)$, and we note

$$v_n(y) \ \sim\ \mathcal{V}(y, \rho_n(\varphi(y))) \quad \text{in} \quad L^p\,.$$

Conversely, for any $\mathcal{V} \in L^p(\Omega \times G)$, there are sequences v_n in $\mathcal{L}^p_{os,\varphi}(H)$ with profile $\mathcal{V}$, and the difference of two such sequences converges to zero strongly in $L^p(\Omega)$.

D) We now answer the main question. Consider bounded families $v_{k,n}$ in $L^2(\Omega)$ such that $X_k v_{k,n}$ is bounded in $L^2(\Omega)$. Extracting a subsequence if necessary, Theorems 1.4 and 1.5 show that one can assume that there are G and $\rho_n \in \mathrm{Hom}(\mathbb{R}; G)$, such that (1.15) is satisfied and $v_{k,n} \in \mathcal{L}^2_{os,\psi_k}(H) + \mathcal{L}^2_{no,\psi_k}$, where H defined in (1.14) and the ψ_k are the unique resonant phases from (1.8). Let $\mathcal{V}_k \in L^2(\Omega \times G)$ be the profile associated to the corresponding oscillations in $\mathcal{L}^2_{os,\psi_k}(H)$.

One can show that $X_k(y, \partial_y)\mathcal{V}_k(y, g) \in L^2(\Omega \times G)$ and that there exist $w_{k,n} \sim \mathcal{V}_k(y, \rho_n(\psi_k(y)))$ in $L^2(\Omega)$ such that $X_k\, w_{k,n}$ is bounded in $L^2(\Omega)$. Using Theorem 1.2 and a direct Fourier analysis for the product of the three oscillations, one obtains :

Theorem 1.6.

$$v_{1,n}\ v_{2,n}\ v_{3,n}\ \rightharpoonup\ \int_{G\times G} \mathcal{V}_1(y, g_1)\ \mathcal{V}_2(y, g_2)\ \mathcal{V}_3(y, -g_1 - g_2)\ dg_1 dg_2\,. \tag{1.16}$$

E) Suppose that $u_{k,n}$ are three bounded sequences in $L^\infty(\Omega)$. Applying Theorem 1.3 successively to the sequences $(u_{k,n})^m$, and using the diagonal process, one finds an at most countable $A \subset \mathbf{S}$ and $\ell : \mathbb{N} \to \mathbb{N}$ such that for all $f \in C^0(\mathbb{R})$, the subsequence $f(u_{k,\ell(n)}) \in \mathcal{L}^2_{os,\psi_k}(A) + \mathcal{L}^2_{no,\psi_k}$. Using Theorem 1.4, one can extract yet another subsequence and find a compact Abelian group G and $\rho_n \in \mathrm{Hom}(\mathbb{R}; G)$ such that (1.15) is satisfied and for all $f \in C^0(\mathbb{R})$, $f(u_{k,\ell(n)}) \in \mathcal{L}^2_{os,\psi_k}(H) + \mathcal{L}^2_{no,\psi_k}$. We summarize this property by saying that (G, ρ_n) is *complete for the sequence* $u_{k,\ell(n)}$ *and the phase* ψ_k.

Replacing the notion of weak limits by profiles, this allows us to define the *multiscale Young measures* associated to the subsequences $u_{k,\ell(n)}$, G, ρ_n and the phase ψ_k.

Theorem 1.7. *For $k \in \{1, 2, 3\}$, there is a unique family of probability measures $\mu_{k,y,g}(d\lambda)$ on $\mathbb{R}$, which depends measurably on $(y, g) \in \Omega \times G$, such that for all $f \in C^0(\mathbb{R})$ and all $\mathcal{A} \in C^0_0(\Omega \times G)$*

$$\int_\Omega f(u_{k,\ell(n)}(y))\ \mathcal{A}(y, \rho_n(\psi_k(y)))\ dy \quad\to\quad \int_{\Omega\times G}\int_{\mathbb{R}} f(\lambda)\ \mathcal{A}(y, g)\ \mu_{k,y,g}(d\lambda)\ dy\ dg\,. \tag{1.17}$$

Moreover, if

$$\mathcal{F}_k(y, g)\ :=\ \int_{\mathbb{R}} f(\lambda)\ \mu_{k,y,g}(d\lambda) \tag{1.18}$$

and $f_{k,n}(y)\ \sim\ \mathcal{F}_k(y, \rho_n(\psi_k(y)))$ in $L^2(\Omega)$, then $f(u_{k,\ell(n)}) - f_{k,n}$ belongs to $\mathcal{L}^2_{no,\psi_k}$.

The multiscale Young measure is the measure μ_k on $\Omega \times G \times \mathbb{R}$, which is defined by the limit integral in (1.17). The classical Young measure is a measure $\tilde{\mu}_k$ on $\Omega \times \mathbb{R}$ which is recovered from μ_k by integration in $g \in G$:

$$\int_{\Omega\times\mathbb{R}} a(y)\, F(\lambda)\, \tilde{\mu}_y(d\lambda)\, dy \;=\; \int_{\Omega\times G\times\mathbb{R}} a(y)\, F(\lambda)\, \mu_{y,g}(d\lambda)\, dg\, dy\,.$$

The second part of Theorem 1.7 shows that the multiscale Young measure contains all the information needed to split $f(u_{k,\ell(n)})$ into oscillations in $\mathcal{L}^2_{os,\psi_k}(H)$ and a remainder in $\mathcal{L}^2_{no,\psi_k}$. Moreover, (1.18) explicitly computes the profile of the oscillations.

F) Suppose in addition that the $u_{k,n}$ satisfy the equations (1.1). Then the multiscale Young measures satisfy transport equations, which are derived similarly to (1.6) but using a multiscale oscillatory test function instead of an ordinary one and using Theorems 1.6 and 1.7 in place of (1.4) and (1.5). Thus, multiplying (1.2) by $\mathcal{A}(y, \rho_n(\psi_k(y)))$, integrating over Ω, and passing to the limit by using the definition of the measure μ_k on the left side and Theorems 1.6 and 1.7 on the right side yields the following theorem:

THEOREM 1.8. *With notation as in part E), the multiscale Young measures μ_k associated to the subsequences $u_{k,\ell(n)}$ satisfy*

$$X_k\mu_k \;+\; \partial_\lambda(A_k(y,g,\lambda)\,\mu_k) \;=\; 0\,, \tag{1.19}$$

with

$$A_1(y,g,\lambda) \;:=\; \int_{G\times\mathbb{R}\times\mathbb{R}} f_1(\lambda,\lambda_2,\lambda_3)\; \mu_{2,y,g_2}(d\lambda_2)\; \mu_{3,y,-g-g_2}(d\lambda_3)\; dg_2\,, \tag{1.20}$$

and similar definitions for A_2 and A_3.

Equations (1.19) (1.20) uniquely determine the measures μ_k from their initial value. The multiscale Young measures of the Cauchy data therefore determine the multiscale Young measures of the solutions, and hence also their usual Young measures and their weak limits.

THEOREM 1.9. *Suppose that $(u_{1,n}, u_{2,n}, u_{3,n})$ is a bounded sequence of solutions of (1.1) in $L^\infty(\Omega)$. Suppose that G is a compact Abelian group, and $\rho_n \in \mathrm{Hom}(\mathbb{R}; G)$ satisfy* (1.15). *Suppose that for $k \in \{1,2,3\}$, (G,ρ_n) is complete for the initial data $u^0_{k,n}(x) := u_{k,n}(0,x)$ and the initial phases $\psi^0_k(x) := \psi_k(0,x)$.*

Then (G,ρ_n) is complete for $u_{k,n}$ and the phase ψ_k. The multiscale Young measures μ_k of $u_{k,n}$ are the unique solutions of (1.19) (1.20) *with initial data*

$$\mu_{k\,|t=0} \;=\; \mu_{k,0}\,,$$

where the μ^0_k are the multiscale Young measures associated to the initial data $u^0_{k,n}$.

Analogously to the rewriting of (1.6) as the equations of nonresonant geometric optics, equations (1.19) (1.20) for the multiscale Young measures can be expressed in a form similar to the equations of semilinear resonant geometric optics (see [MR] [J] [HK] [JMR 1]), extended to more general groups G. As in [JMR 4], introduce the functions $U_k(y,g,\theta)$, on $\Omega\times G\times]0,1[$, such that $U_k(y,g,\,.\,)$ is the right continuous inverse of the distribution function $\mu_{k,y,g}(]-\infty,\lambda[)$. Then (1.19) (1.20) are equivalent to

$$(1.21)\qquad \begin{aligned} X_1(y,\partial_y)\,U_1(y,g,\theta) \;&=\\ \int_{G\times]0,1[\times]0,1[} &f_1\big(U_1(y,g,\theta),U_2(y,g_2,\theta_2),U_3(y,-g-g_2,\theta_3)\big)\; dg_2\, d\theta_2\, d\theta_3\,, \end{aligned}$$

and similar equations for U_2 and U_3. Note that while oscillations with the resonant phase are described via the possibly infinite-dimensional group G, a single variable θ per component suffices to describe how all nonresonant oscillations influence the weak limit.

2. Trilinear compensated compactness.

In this section we prove Theorem 1.2, which states that resonance is the only obstruction to trilinear compensated compactness. Let $\Omega\subset\mathbb{R}^2$ be a bounded open set, and let X_k for $k\in\{1,2,3\}$ be three pairwise independent C^∞ vector fields on Ω. We suppose that there exists a nontrivial resonance, i.e., functions $\psi_k\in C^\infty(\Omega;\mathbb{R})$, for $k\in\{1,2,3\}$, such that $d\psi_k\neq 0$, and

$$X_k\,\psi_k \;=\; 0 \quad \text{for } k\in\{1,2,3\} \qquad \text{and} \qquad \psi_1+\psi_2+\psi_3 \;=\; 0\,.$$

Although Theorem 1.2 is stated in terms of the L^∞ norm, it suffices to consider the L^2 norm. We therefore introduce the spaces $W_k(\Omega):=\{u\in L^2(\Omega)\mid X_k u\in L^2(\Omega)\}$. It is proved in [JMR 3] that the product $(u_1,u_2,u_3)\ \to\ u_1u_2u_3$ is defined and continuous for the strong topologies, from $W_1(\Omega)\times W_2(\Omega)\times W_3(\Omega)$ into $L^1_{\rm loc}(\Omega)$. Because of the existence of a nontrivial resonance, this product is not continuous for the weak topologies. Our goal is to study in detail this lack of weak continuity.

A sequence whose terms are u_n will sometimes be denoted u_*. Recall from the introduction that $\mathcal{L}^p_{no,\varphi}$ denotes the space of sequences u_* whose terms are bounded in $L^p(\Omega)$ and satisfy $u_n\,e^{-i\,\nu_n\,\varphi}\ \rightharpoonup\ 0$ for all real sequences ν_*.

Theorem 2.1. *Suppose that for $k\in\{1,2,3\}$, u^k_* are bounded sequences in $W_k(\omega)$. Then $u^1_nu^2_nu^3_n$ is bounded in $L^1_{\rm loc}(\Omega)$. Moreover, if in addition $u^1_*\in\mathcal{L}^2_{no,\psi_1}$ then $u^1_nu^2_nu^3_n$ converges weakly to 0 in the sense of distributions on Ω.*

We must prove that for all $a\in C^\infty_0(\Omega)$,

$$(2.1)\qquad \int a(y)\,u^1_n(y)\,u^2_n(y)\,u^3_n(y)\,dy \quad\to\quad 0\,.$$

This is a local statement, and it is sufficient to prove that for all $\underline{y}\in\Omega$ there is a neighborhood $\omega\subset\Omega$ of $\underline{y}$ such that the convergence (2.1) holds for all $a\in C^\infty_0(\omega)$. Introducing

$\chi \in C_0^\infty(\omega)$ such that $\chi = 1$ on the support of a, and $v_n^1 := a\, u_n^1$, $v_n^2 := \chi\, u_n^2$, $v_n^3 := \chi\, u_n^3$, (2.1) reads

$$\int v_1^n(y)\, v_2^n(y)\, v_3^n(y)\, dy \quad \to \quad 0\,.$$

The v_k^n are bounded sequences in $W_k(\Omega)$, supported in ω, and v_*^1 is in $\mathcal{L}^2_{no,\psi_1}$.

Fix $\underline{y} \in \Omega$. The differentials $d\psi_k$ of the resonant phases ψ_k do not vanish and are pairwise linearly independent (see [JMR 1]). Thus, one can choose local coordinates (t,x) on a neighborhood ω of $\underline{y}$ such that the resonant phases are

$$\psi_1(t,x) = -t - x \quad , \quad \psi_2(t,x) = x \quad , \quad \psi_3(t,x) = t\,.$$

The vector fields X_k are therefore parallel on ω to the fields

$$X_1 = \partial_t - \partial_x \quad , \quad X_2 = \partial_t \quad , \quad X_3 = \partial_x\,. \tag{2.2}$$

Hence, Theorem 2.1 follows from:

Proposition 2.2. *Consider the fields X_k in (2.2). Suppose that for $k \in \{1,2,3\}$, v_*^k are bounded sequences in $L^2(\mathbb{R}^2)$ supported in a fixed ball, such that $X_k v_*^k$ is bounded in $L^2(\mathbb{R}^2)$. Then the product $v_1^n\, v_2^n\, v_3^n$ is bounded in $L^1(\mathbb{R}^2)$.*

In addition, if $v_*^1 \in \mathcal{L}^2_{no,\psi_1}$, then

$$I_n := \int v_n^1(y)\, v_n^2(y)\, v_n^3(y)\, dy \quad \to \quad 0\,. \tag{2.3}$$

Proof. **a)** Introduce $w_n := v_n^2\, v_n^3$. Its Fourier transform is

$$\hat{w}_n(\tau,\xi) = \int \hat{v}^2(\sigma, \xi - \eta)\hat{v}^3(\tau - \sigma, \eta)\, d\sigma\, d\eta.$$

Thus

$$|\, \hat{w}_n(\tau,\xi)\,| \ \leq \int_{\mathbb{R}^2} \langle\sigma\rangle^{-1}\langle\eta\rangle^{-1}\ V_n^2(\sigma, \xi-\eta)\ V_n^3(\tau-\sigma,\eta)\, d\sigma\, d\eta\,, \tag{2.4}$$

where $\langle\cdot\rangle := 1 + |\cdot|$ and the V_n^k belong to $L^2(\mathbb{R}^2)$ and satisfy

$$\|\, V_n^k\,\|_{L^2(\mathbb{R}^2)} \ \leq \ C\ \big(\|\, v_n^k\,\|_{L^2(\mathbb{R}^2)} \ + \ \|\, X_k v_n^k\,\|_{L^2(\mathbb{R}^2)}\big) \tag{2.5}$$

with C independent of n and k.

b) (2.4) implies that

$$|\, \hat{w}_n(\tau,\xi)\,|^2 \ \leq \ C\ \int_{\mathbb{R}^2} |V_n^2(\sigma,\xi-\eta)|^2\ |V_n^3(\tau-\sigma,\eta)|^2 d\sigma\, d\eta\,.$$

Together with (2.5), this implies

$$\| w_n \|_{L^2(\mathbb{R}^2)} \ \le\ C \, \| v_n^2 \|_{W_2(\mathbb{R}^2)} \ \ \| v_n^3 \|_{W_3(\mathbb{R}^2)} \,. \tag{2.6}$$

In particular, this implies that $v_n^1 v_n^2 v_n^3 = v_n^1 \, w_n$ is bounded in $L^1(\mathbb{R}^2)$.

c) (2.4) also implies that

$$\int_{\mathbb{R}} | \, \hat{w}_n(\tau, \xi + \tau) \, | \, d\tau \ \le \int_{\mathbb{R}^2} \langle \sigma \rangle^{-1} \langle \eta \rangle^{-1} \ V_n(\xi, \sigma, \eta) \ d\sigma \, d\eta \,, \tag{2.7}$$

with

$$V_n(\xi, \sigma, \eta) \ := \ \int_{\mathbb{R}} | \, V_n^2(\sigma, \xi + \tau - \eta) \, | \ | \, V_n^3(\tau - \sigma, \eta) \, | \, d\tau \,.$$

Therefore

$$| \, V_n(\xi, \sigma, \eta) \, |^2 \ \le \ \int_{\mathbb{R}} | \, V_n^2(\sigma, \tau) \, |^2 \ d\tau \ \int_{\mathbb{R}} | \, V_n^3(\tau, \eta) \, |^2 \ d\tau \,,$$

and

$$\int_{\mathbb{R}^2} | \, V_n(\xi, \sigma, \eta) \, |^2 \ d\sigma \, d\eta \le \int_{\mathbb{R}^2} | \, V_n^2(\sigma, \tau) \, |^2 \ d\tau \, d\sigma \ \int_{\mathbb{R}^2} | \, V_n^3(\tau, \eta) \, |^2 \ d\tau \, d\eta \,.$$

With (2.7) and (2.5), this implies that

$$\int_{\mathbb{R}} | \, \hat{w}_n(\tau, \xi + \tau) \, | \, d\tau \ \le \ C \, \| v_n^2 \|_{W_2(\mathbb{R}^2)} \ \ \| v_n^3 \|_{W_3(\mathbb{R}^2)} \,, \tag{2.8}$$

where C is independent of ξ and n.

d) The integral I_n in (2.3) is equal to

$$I_n \ = \ (2\pi)^{-2} \int \hat{v}_n^1(\tau, \xi) \ \hat{w}_n(-\tau, -\xi)) \ d\tau \, d\xi \ = \ (2\pi)^{-2} \int \hat{v}_n^1(\tau, \xi{+}\tau) \ \hat{w}_n(-\tau, -\xi{-}\tau)) \ d\tau \, d\xi \,.$$

For $R > 0$, use (2.6) when $|\xi| \ge R$ and (2.8) when $|\xi| \le R$ to estimate this integral. This proves that

$$| \, I_n \, | \ \le \ C \ \big(A_n(R) \ + \ R \, B_n(R) \ \big) \ \| v_n^2 \|_{W_2(\mathbb{R}^2)} \ \ \| v_n^3 \|_{W_3(\mathbb{R}^2)} \,,$$

where

$$A_n(R) \ := \ \Big(\int_{|\xi| \ge R} | \, \hat{v}_n^1(\tau, \xi + \tau) \, |^2 \ d\tau \, d\xi \ \Big)^{1/2} \,,$$

$$B_n(R) \ := \ \sup_{|\xi| \le R} \ \sup_{\tau \in \mathbb{R}} \ | \, \hat{v}_n^1(\tau, \xi + \tau) \, | \,.$$

Now

$$A_n(R) \ \le \ C \ R^{-1} \ \| v_n^1 \|_{W_1(\mathbb{R}^2)} \ \le \tilde{C} \ R^{-1} \,,$$

and so tends to zero as $R \to \infty$, uniformly in n. Therefore, in order to prove that I_n converges to 0, it is sufficient to show that for all fixed R, $B_n(R)$ converges to 0 as n tends to $+\infty$. Upon making the change of variables $(T, X) = (t + x, x)$, this is a consequence of the following result.

Proposition 2.3. *Suppose that v_* is a bounded sequence in $L^2(\mathbb{R}^2)$ supported in a fixed ball. Then, v_* belongs to the space $\mathcal{L}^2_{no,\psi}$ associated to the phase $\psi = T$, if and only if, for all $R > 0$, the Fourier transforms $\hat{v}_n(\tau, \xi)$ converge to 0, uniformly on the slab $\{(\tau, \xi) \in \mathbb{R}^2 \mid \ |\xi| \le R\}$.*

Proof. Suppose that w_n is a bounded sequence in L^2 of functions supported in a fixed ball. Then, their Fourier transforms $\hat{w}_n$ are C^∞, bounded in L^∞, and their derivatives are also bounded in L^∞. Hence, the weak convergence $w_n \rightharpoonup 0$ is equivalent to

$$\hat{v}_n(\tau, \xi) \ \rightarrow \ 0 \ ,$$

uniformly on compacts sets in $\mathbb{R}^2$

Consider a bounded sequence in L^2 of functions u_n supported in a fixed ball. We apply the argument above to $w_n = v_n \ e^{-i\,\tau_n \ T}$. Therefore, v_* belongs to the space $\mathcal{L}^2_{no,\psi}$ if and only if, for every sequence $\tau_* \in \mathbf{S}$,

$$\hat{v}_n(\tau + \tau_n \, , \, \xi) \rightarrow 0$$

uniformly for (τ, ξ) in bounded domains.

This is equivalent to saying that the convergence $\hat{v}_n(\tau, \xi) \rightarrow 0$ is uniform with respect to $\tau \in \mathbb{R}$ and ξ in bounded intervals. □

Theorem 2.1 will be used in §5 in the calculation of the weak limit of the product $u_n^1 u_n^2 u_n^3$ of general bounded sequences, after we have shown that such sequences can be divided into resonant plus nonresonant oscillations and developed the group structure needed to describe the limit of the resonant part.

3. The analysis of oscillations with respect to a given set of phases.

This section is devoted to the proof of Theorem 1.3, which says that arbitrary bounded oscillations can be separated into resonant and nonresonant parts. However, we extend our analysis to any space-time dimension and to a finite number of phases, as we will need it in §5. Throughout this section, Ω is a bounded open subset of $\mathbb{R}^d$ and $\varphi \in C^1(\Omega; \mathbb{R}^m)$ satisfies

$$\forall \xi \in \mathbb{R}^m \backslash \{0\} \, , \quad d(\xi \cdot \varphi) \neq 0 \ \text{ a.e. on } \Omega \, . \tag{3.0.1}$$

In §3.1, we give definitions and preliminary results about spaces of sequences which oscillate with respect to the phase φ. §3.2 contains the main result, i.e. the splitting of suitable subsequences of a given sequence into oscillations with phase φ and a remainder term which has no oscillations with respect to φ.

3.1. Spaces of oscillations.

Introduce first several notations. For $p \in [1, +\infty]$, $\mathcal{L}^p$ denotes the space of bounded sequences in $L^p(\Omega)$ and $\mathcal{L}^p_0$ the subspace of sequences which converge strongly to 0 in

$L^p(\Omega)$. Our goal is to perform an "asymptotic Fourier analysis" of the spaces $\mathcal{L}^p/\mathcal{L}^p_0$. We denote by $\{u_n\}$ or u_* the sequence whose terms are u_n. For $u_* \in \mathcal{L}^p$ we let $\tilde{u}_*$ denote its class in $\mathcal{L}^p/\mathcal{L}^p_0$. For sequences u_* and v_* in $\mathcal{L}^p$, we say that

$$u_n \ \sim\ v_n \quad \text{in} \ \ L^p \tag{3.1.1}$$

when $u_* - v_* \in \mathcal{L}^p_0$, that is when $u_n - v_n$ converges strongly to 0 in $L^p(\Omega)$.

Remark that $\mathcal{L}^p/\mathcal{L}^p_0$ is a Banach space, when equipped with the norm

$$\| \, \tilde{u}_* \, \| \ := \ \limsup_{n\to+\infty} \ \| \, u_n \, \|_{L^p(\Omega)} \, . \tag{3.1.2}$$

$\mathbf{S} := (\mathbb{R}^m)^{\mathbb{N}}$ denotes the set of sequences in $\mathbb{R}^m$.

DEFINITION 3.1.1 *For $A \subset \mathbf{S}$,*

i) $\mathcal{P}_\varphi(A)$ denotes the space of sequences of the form

$$v_n(y) := \sum_j a_j(y) \ e^{i\,\xi^j_n \cdot \varphi(y)} \, , \tag{3.1.3}$$

where the sum is taken over a finite set of indices j, the sequences ξ^j_* are taken in A, and $a_j \in C^\infty_0(\Omega)$.

ii) $\mathcal{L}^p_{os,\varphi}(A)$ is the asymptotic closure of $\mathcal{P}(A)$ in $\mathcal{L}^p$, that is, $v_* \in \mathcal{L}^p_{os,\varphi}(A)$ if and only if, for all $\delta > 0$ there is a sequence $w_* \in \mathcal{P}_\varphi(A)$ such that

$$\limsup_{n\to+\infty} \ || \, v_n - w_n \, ||_{L^p(\Omega)} \ \leq \ \delta \, . \tag{3.1.4}$$

iii) $\mathcal{L}^p_{no,\varphi}$ is the set of sequences $u_* \in \mathcal{L}^p$ such that for all $\xi \in \mathbf{S}$, $u_n \, e^{-i\,\xi_n\cdot\varphi}$ converges to zero in the weak topology, i.e. in the sense of distributions.

$\mathcal{L}^p_{os,\varphi} := \mathcal{L}^p_{os,\varphi}(\mathbf{S})$ is the space of sequences which only have oscillations with respect to φ, while $\mathcal{L}^p_{no,\varphi}$ is the the space of sequences which converge weakly to 0 and have no oscillation with respect to φ. The introduction of subspaces subspaces $\mathcal{L}^p_{os,\varphi}(A)$ of $\mathcal{L}^p_{os,\varphi}$ is motivated by two remarks. The first, is that $\mathbf{S}$ is not countable and our analysis uses countability at several places. The second, is that $e^{i\,\xi_n\,\varphi}$ and $e^{i\,\eta_n\,\varphi}$ are not asymptotically independent for all $\xi \neq \eta$. In particular, expansions like (3.1.3) are not unique if A is too large.

PROPOSITION 3.1.2. *i) If $A \subset B$, then $\mathcal{L}^p_0 \ \subset \ \mathcal{L}^p_{os,\varphi}(A) \ \subset \ \mathcal{L}^p_{os,\varphi}(B)$.*

ii) Suppose that $A \subset \mathbf{S}$ and ℓ is an increasing map from $\mathbb{N}$ to $\mathbb{N}$. Let B the set of all sequences $n \to \xi_{\ell(n)}$ when ξ runs in A. If $u_* \in \mathcal{L}^p_{os,\varphi}(A)$, then $v_* := u_{\ell(*)} \in \mathcal{L}^p_{os,\varphi}(B)$.

Similarly, if $u_* \in \mathcal{L}^p_{no,\varphi}$, then $v_* := u_{\ell(*)} \in \mathcal{L}^p_{no,\varphi}$

iii) Suppose that $A \subset \mathbf{S}$ and $B \subset \mathbf{S}$ satisfy

$$\forall \xi \in A \ , \exists \eta \in B \ , \exists l \in \mathbb{R}^m \ : \quad \xi_n - \eta_n \to l \ \text{as} \ \ n \to +\infty \, . \tag{3.1.5}$$

Then, $\mathcal{L}^p_{os,\varphi}(A) \subset \mathcal{L}^p_{os,\varphi}(B)$.

Proof. The first two points follow directly from the definitions. To prove the third part, it is sufficient to show that $\mathcal{P}_\varphi(A) \subset \mathcal{L}^p_{os,\varphi}(B)$. Thus, it is sufficient to prove that for $a \in C_0^\infty(\Omega)$ and $\xi \in A$, the sequence $u_n := a\, e^{i\,\xi_n\cdot\varphi}$ belongs to $\mathcal{L}^p_{os,\varphi}(B)$. For such a and ξ, let $\eta \in B$ and $l \in \mathbb{R}^m$ such that $\zeta_n := \xi_n - \eta_n - l \to 0$. Therefore

$$u_n = b\, e^{i\,\eta_n\cdot\varphi} \;+\; b\,(\; e^{i\,\zeta_n\cdot\varphi} - 1)\, e^{i\,\eta_n\cdot\varphi}\,,$$

with $b := a\, e^{i\,l\cdot\varphi} \in C_0^1(\Omega)$. Since $b\,(e^{i\,\zeta_n\cdot\varphi} - 1)$ converges strongly to zero in $L^p(\Omega)$, it remains to show that $b\, e^{i\,\eta_n\,\varphi}$ belongs to $\mathcal{L}^p_{os,\varphi}(B)$, which is clear by smoothing b. □

The third part in the proposition above gives an example of possible redundancy in A. If ξ and η in A are such that $\xi_n - \eta_n$ converges to a finite limit as n tends to infinity, then $\mathcal{L}^p_{os,\varphi}(A) = \mathcal{L}^p_{os,\varphi}(A\backslash\{\xi\})$. In the opposite direction, the following result is essential.

LEMMA 3.1.3. *Suppose that $\xi \in \mathbf{S}$ satisfies $|\xi_n| \to +\infty$ as $n \to +\infty$. Then $e^{i\,\xi_n\cdot\varphi}$ converges weakly to zero.*

Proof. We show that for all $a \in C_0^1(\Omega)$.

$$\int a(y)\, e^{i\,\xi_n\cdot\varphi(y)}\; dy \quad\to\quad 0\,,\quad \text{as}\quad n \to +\infty\,. \tag{3.1.6}$$

Taking a subsequence, we can assume that $\omega_n := \xi_n/|\xi_n|$ converges to $\omega \in \mathbb{R}^m$, and $|\omega| = 1$. Since $d(\omega\cdot\varphi) \neq 0$ almost everywhere, it is sufficient to prove that for all $\delta > 0$, (3.1.6) is satisfied when a is supported in a compact set K where $|d(\omega\cdot\varphi)| \geq \delta$. Then, for n large enough, $|d(\omega_n\cdot\varphi)| \geq \delta/2$ on K and the result follows by integration by parts. □

THEOREM 3.1.4. *Suppose that $A := \{\xi^j\}_{j\in J} \subset \mathbf{S}$ satisfies the condition*

$$\forall (j,k) \in J \times J\,,\; j \neq k\,:\quad |\,\xi^j_n \;-\; \xi^k_n\,| \;\to\; +\infty \quad \text{when}\quad n \to +\infty\,. \tag{3.1.7}$$

Then for all $u_* \in \mathcal{L}^2_{os,\varphi}(A)$ and for all $j \in J$, $u_n\, e^{-i\,\xi^j_n\cdot\varphi}$ converges weakly in $L^2(\Omega)$ to a limit $a_j \in L^2(\Omega)$. Moreover

$$\sum_{j\in J} ||\, a_j||^2_{L^2(\Omega)} \;=\; \lim_n ||\, u_n\,||^2_{L^2(\Omega)} \;<\; +\infty\,. \tag{3.1.8}$$

This defines a mapping $u_* \to \{a_j\}_{j\in J}$ from $\mathcal{L}^2_{os,\varphi}(A)$ into $\ell^2(J; L^2(\Omega))$. This operator is surjective, and its kernel is $\mathcal{L}^2_0$.

Proof. Consider

$$v_n := \sum_{k\in K} b_k\, e^{i\,\xi_n^k\cdot\varphi} \in \mathcal{P}_\varphi(A) \tag{3.1.9}$$

where K is a finite subset of A. Then, Lemma 3.1.3 and (3.1.7) imply that

$$\forall j \in J\,, \quad v_n\; e^{-i\,\xi_n^j\cdot\varphi} \;\rightharpoonup\; \begin{cases} b_j & \text{if} \quad j\in K \\ 0 & \text{if} \quad j\notin K \end{cases} \quad \text{as} \;\; n\to+\infty\,, \tag{3.1.10}$$

and

$$||\, v_n\,||^2_{L^2(\Omega)} \quad\to\quad \sum_{j\in K} ||\, b_j||^2_{L^2(\Omega)} \qquad \text{as} \;\; n\to\infty\,. \tag{3.1.11}$$

(3.1.10) and the definition of $\mathcal{L}^2_{os,\varphi}(A)$ implies that for all $u_*\in\mathcal{L}^2_{os,\varphi}(A)$ and all $a\in C_0^\infty(\Omega)$, $\int a(y)\, u_n(y)\, e^{-i\,\xi_n^j\cdot\varphi(y)}\, dy$ is a Cauchy sequence in $\mathbb{C}$, and thus converges. Hence, for all $u_*\in\mathcal{L}^2_{os,\varphi}(A)$ and all $j\in J$, the weak limit

$$C_j(u_*) \;:=\; \text{weak}\lim_{n\to+\infty} \; u_n\, e^{-i\,\xi_n^j\cdot\varphi} \tag{3.1.12}$$

exists and belong to $L^2(\Omega)$. Moreover,

$$||\, C_j(u_*)\,||_{L^2(\Omega)} \;\le\; \limsup_{n\to+\infty} \; ||\, u_n\,||_{L^2(\Omega)}\,. \tag{3.1.13}$$

Let π denotes the canonical mapping $\mathcal{L}^2\to\mathcal{L}^2/\mathcal{L}^2_0$. Introduce $\widetilde{\mathcal{P}}_\varphi(A)=\pi\mathcal{P}_\varphi(A)$ and $\widetilde{\mathcal{L}}^p_{os,\varphi}(A)$ its closure in $\mathcal{L}^2/\mathcal{L}^2_0$. Then, Definition 3.1.1 implies that $\mathcal{L}^p_{os,\varphi}(A)$ is equal to $\pi^{-1}(\widetilde{\mathcal{L}}^p_{os,\varphi}(A))$ and $\pi\big(\mathcal{L}^p_{os,\varphi}(A)\big)=\widetilde{\mathcal{L}}^p_{os,\varphi}(A)$. (3.1.13) implies that $C_j=\widetilde{C}_j\circ\pi$ where $\widetilde{C}_j$ is bounded from $\widetilde{\mathcal{L}}^p_{os,\varphi}(A)$ into $L^2(\Omega)$. Introduce the mapping

$$\tilde{v}_* \;\to\; \widetilde{C}(\tilde{v}_*) \;:=\; \{\widetilde{C}_j(\tilde{v}_*)\}_{j\in J}\,. \tag{3.1.14}$$

Then, (3.1.10) shows that $\widetilde{C}$ maps $\widetilde{\mathcal{P}}_\varphi(A)$ onto $\mathcal{C}$, the space of families $\{b_j\}_{j\in J}$ in $\big(C_0^\infty(\Omega)\big)^J$ with finite support. In addition, (3.1.11) shows that $\widetilde{C}$ is isometric from $\widetilde{\mathcal{P}}_\varphi(A)$ equipped with the norm of $\mathcal{L}^2/\mathcal{L}^2_0$, onto $\mathcal{C}$ equipped with the norm of $\ell^2(J;L^2(\Omega))$. Since $\mathcal{C}$ is dense in $\ell^2(J;L^2(\Omega))$, this implies that $\widetilde{C}$ uniquely extends as an isometry from $\widetilde{\mathcal{L}}^p_{os,\varphi}(A)$ onto $\ell^2(J;L^2(\Omega))$.

Therefore $C=\widetilde{C}\circ\pi=\{C_j\}_{j\in J}$ maps $\mathcal{L}^p_{os,\varphi}(A)$ onto $\ell^2(J;L^2(\Omega))$, (3.1.8) is satisfied and the kernel of C is $\mathcal{L}^2_0$. □

This Theorem justifies the following definition.

Definition 3.1.5. *Suppose that* $A=\{\xi^j\}_{j\in J}\subset\mathbf{S}$ *satisfies* (3.1.7). *We note*

$$u_n \;\sim\; \sum_j a_j\, e^{i\,\xi_n^j\cdot\varphi} \tag{3.1.15}$$

when $u_*\in\mathcal{L}^2_{os,\varphi}(A)$ *and* a_j *is the weak limit of* $u_n\, e^{-i\,\xi_n^j\cdot\varphi}$.

Theorem 3.1.4 implies that if (3.1.15) holds then (3.1.8) is satisfied. Conversely, if the series in (3.1.8) converges, then there exists $u_* \in \mathcal{L}^2_{os,\varphi}(A)$ which satisfies (3.1.15). Note, however, that in general we cannot simply take u_* to be $\sum_{j\in J} a_j\, e^{i\,\xi^j_n\cdot\varphi}$ since this sum may not converge in L^2 because its terms are only asymptotically orthogonal. Although we will not make use of this fact, it is not hard to see that we could take u_* to be $\sum_{j\in J(n)} a_j\, e^{i\,\xi^j_n\cdot\varphi}$, where $J(n)$ is a finite subset of J chosen so that the L^2 norm of u_n is uniformly bounded while $\sum_{j\in J(n)} \|a_j\|^2_{L^2}$ tends to $\sum_{j\in J} \|a_j\|^2_{L^2}$ as $n\to\infty$.

Moreover, if u_* and v_* both satisfy (3.1.15), then $u_n - v_n$ belongs to the kernel of the mapping introduced in Theorem 3.1.4 and so converges strongly to 0 in $L^2(\Omega)$. In this same vein, we note the following corollary of Theorem 3.1.4:

Proposition 3.1.6. $\mathcal{L}^2_{os,\varphi} \cap \mathcal{L}^2_{no,\varphi} = \mathcal{L}^2_0$.

Proof. The definitions of $\mathcal{L}^p_{os,\varphi}$ and $\mathcal{L}^p_{no,\varphi}$ imply that both contain $\mathcal{L}^p_0$. If u_* lies in $\mathcal{L}^2_{os,\varphi} \cap \mathcal{L}^2_{no,\varphi}$ then for $\delta > 0$, choose $v_* \in \mathcal{P}_\varphi$ such that

$$\limsup_n \| u_n - v_n \|_{L^2(\Omega)} \;\le\; \delta\,. \tag{3.1.16}$$

Since $u_* \in \mathcal{L}^2_{no,\varphi}$ and $v_* \in \mathcal{P}_\varphi$,

$$\int_\Omega u_n(y)\, \overline{v_n(y)}\; dy \quad\to\quad 0 \quad \text{as} \quad n\to+\infty\,. \tag{3.1.17}$$

Therefore, (3.1.16) implies that

$$\limsup_n \| u_n \|^2_{L^2(\Omega)} \;+\; \| v_n \|^2_{L^2(\Omega)} \;\le\; \delta^2\,. \tag{3.1.18}$$

Since this is true for all $\delta > 0$, the proposition follows. ☐

3.2. The main splitting theorem.

Theorem 1.3 is a consequence of the next result.

Theorem 3.2.1. *Suppose that u_* is a bounded sequence in $L^2(\Omega)$. There exist a subsequence $u_{\ell(*)}$ and an at most countable set $A = \{\xi^j\}_{j\in J} \subset \mathbf{S}$ satisfying (3.1.7), such that*

$$u_{\ell(*)} \in \mathcal{L}^2_{os,\varphi}(A) \;+\; \mathcal{L}^2_{no,\varphi}\,. \tag{3.2.1}$$

The next lemma is used to prove that sequences ξ^j satisfy (3.1.7). We note $\mathbf{S}_\infty$ the set of sequences $\xi\in\mathbf{S}$ such that $|\xi_n| \to +\infty$ as $n\to+\infty$.

Lemma 3.2.2. *Suppose that $\xi\in\mathbf{S}$ and $\mu\in\mathbf{S}$ are two sequences such that $u_n\, e^{-i\,\xi_n\,\varphi}$ converges weakly to 0 and $u_n\, e^{-i\,\eta_n\,\varphi}$ converges weakly to $a\neq 0$. Then $\xi-\eta\in\mathbf{S}_\infty$.*

Proof. If $\xi - \eta \notin \mathbf{S}_\infty$, there is a subsequence $\xi_{\ell(n)} - \eta_{\ell(n)}$ which has a finite limit $l \in \mathbb{R}^m$ as $n \to \infty$. Thus $e^{i\,(\xi_{\ell(n)} - \eta_{\ell(n)})\cdot\varphi} \to e^{i\,l\cdot\varphi}$ uniformly on compacts. This implies that $u_{\ell(n)}\, e^{-i\,\eta_{\ell(n)}\cdot\varphi} = u_{\ell(n)}\, e^{-i\,\xi_{\ell(n)}\cdot\varphi}\, e^{i\,(\xi_{n(k)} - \eta_{n(k)})\cdot\varphi}$ converges weakly to $0 \times e^{i\,l\cdot\varphi} = 0$, contradicting the assumption $a \neq 0$. □

Next we introduce a notation. For any bounded sequence u_* in $L^2(\Omega)$, introduce $K(u_*)$ the set of functions $a \in L^2(\Omega)$ such that there is a subsequence $u_{\ell(n)}$ and $\xi \in \mathbf{S}$ such that $u_{\ell(n)}\, e^{-i\,\xi_n\cdot\varphi}$ converges weakly in $L^2(\Omega)$ to a. This set is bounded and non-empty. Define

$$\delta(u_*) := \sup_{a\in K(u_*)} \|\, a\|_{L^2(\Omega)}\,. \tag{3.2.2}$$

LEMMA 3.2.3. *For all u_* and v_* in $\mathcal{L}^2$,*

$$\delta(u_*) \leq \limsup_{n\to\infty} \|\, u_n\,\|_{L^2(\Omega)}\,. \tag{3.2.3}$$

$$\delta(u_* + v_*) \leq \delta(u_*) + \delta(v_*)\,. \tag{3.2.4}$$

Moreover, $\delta(u_*) = 0$ if and only if $u_* \in \mathcal{L}^2_{no,\varphi}$.

Proof. (3.2.3) is clear. If a subsequence of $(u_n + v_n)e^{-i\,\xi_n\cdot\varphi}$ converges weakly to a limit a, one can extract further another subsequence such that both $u_n\, e^{-i\,\xi_n\cdot\varphi}$ and $v_n\, e^{-i\,\xi_n\cdot\varphi}$ converge weakly to b and c respectively. Thus $a = b + c$, showing that $K(u_* + v_*) \subset K(u_*) + K(v_*)$ and (3.2.4) follows.

$\delta(u_*) = 0$ if and only if, for all $\xi \in \mathbf{S}$ all the limits of weakly convergent subsequences of $u_n\, e^{-i\,\xi_n\cdot\varphi}$ are equal to 0. This means for all $\xi \in \mathbf{S}$, the full sequence $u_n\, e^{-i\,\xi_n\cdot\varphi}$ converges weakly to 0. By definition, this is equivalent to $u_* \in \mathcal{L}^2_{no,\varphi}$. □

Proof of Theorem 3.2.1. **a)** Consider $u_* \in \mathcal{L}^2$. If $\delta(u_*) = 0$, $u_* \in \mathcal{L}^2_{no,\varphi}$ and (3.2.1) is clear. If $\delta(u_*) > 0$, choose $a_1 \in K(u_*)$ such that its L^2 norm $\|a_1\|$ is at least $\delta(u_*)/2$. There are $\xi^1 \in \mathbf{S}$ and a subsequence $u_{\ell^1(n)}$ such that $u_{\ell^1(n)}\, e^{-i\,\xi^1_n\cdot\varphi} \rightharpoonup a_1$. Introduce $u_{2,n} := u_{\ell^1(n)} - a_1\, e^{i\,\xi^1_n\cdot\varphi}$.

If $\delta(u_{2,*}) = 0$, the construction stops. If not, we repeat the construction for the sequence $u_{2,*}$. Following this procedure, one obtains by induction on $j \in \mathbb{N}$, increasing mappings $\ell^j : \mathbb{N} \to \mathbb{N}$, sequences $u_{j,*} \in \mathcal{L}^2$, sequences $\xi^j \in \mathbf{S}$ and functions $a_j \in L^2(\Omega)$, such that

$$\|a_j\|_{L^2(\Omega)} \geq \delta(u_{j,*})\,/2 > 0\,, \tag{3.2.5}$$

$$u_{j,\ell^j(n)}\, e^{-i\,\xi^j_n\cdot\varphi} \rightharpoonup a_j\,, \tag{3.2.6}$$

and the $j+1$-th sequence $u_{j+1,*}$ is defined by

$$u_{j+1,n} \;:=\; u_{j,\ell^j(n)} \;-\; a_j\, e^{i\,\xi^j_n\cdot\varphi}\,. \tag{3.2.7}$$

The construction stops at the j-th step if $\delta(u_{j+1,*}) = 0$. Otherwise, it continues.

b) Introduce u^j_* the j-th extracted subsequence from the initial u_*. It is defined by $u^1_n := u_{\ell^1(n)}$ and $u^j_n := u^{j-1}_{\ell^j(n)}$. Similarly, define $\xi^{j,j} := \xi^j$ and by induction on $k \geq j$, $\xi^{k,j}_n := \xi^{k-1,j}_{\ell^k(n)}$. This is the $(k-j)$-th extracted subsequence from ξ^j. With these notations, one has

$$u^k_n = u_{k+1,n} \;+\; \sum_{j\leq k} a_j\; e^{i\,\xi^{k,j}_n\cdot\varphi}\,. \tag{3.2.8}$$

If the constructions stops at the k-th step, then $\delta(u_{k+1,*}) = 0$ and $u_{k+1,*} \in \mathcal{L}^2_{no,\varphi}$. In this case, (3.2.8) implies (3.2.1). We finish the proof assuming that the construction above runs for all $k \in \mathbb{N}$.

c) We prove by induction on k that

$$\text{for}\;\; 1 \leq j' < j \leq k\;, \quad \xi^{k,j} - \xi^{k,j'} \in \mathbf{S}_\infty\,, \tag{3.2.9}$$

$$\forall j \in \{1,\cdots,k\}\;, \quad u_{k+1,n}\; e^{-i\,\xi^{k,j}_n\cdot\varphi} \;\rightharpoonup 0 \quad \text{as } n \to \infty\,. \tag{3.2.10}$$

When $k=1$, (3.2.9) is void, and (3.2.10) follows directly from the definition of $u_{2,n}$.

Suppose that (3.2.9) and (3.2.10) are proved up to $k-1$. For $j<k$, the sequences $\xi^{k,j}$ are subsequences of $\xi^{k-1,j}$. Therefore (3.2.9) is satisfied for $j' < j < k$. Similarly, taking the subsequence $\ell^k(*)$ in the induction hypothesis (3.2.10) at the order $k-1$ implies that

$$\text{for}\;\; j<k\;, \quad u_{k,\ell^k(n)}\, e^{-i\,\xi^{k,j}_n\cdot\varphi} \;\rightharpoonup\; 0\,. \tag{3.2.11}$$

By definition of $\xi^{k,k} = \xi^k$ and a_k, one has

$$u_{k,\ell^k(n)}\, e^{-i\,\xi^{k,k}_n\cdot\varphi} \rightharpoonup a_k\,. \tag{3.2.12}$$

Since $a_k \neq 0$, (3.2.11), (3.2.12) and Lemma 3.2.2 imply that

$$\text{for}\;\; j<k\;, \quad \xi^{k,k} - \xi^{k,j} \in \mathbf{S}_\infty\,. \tag{3.2.13}$$

This finishes the proof of (3.2.9) at the order k.

By definition of $u_{k+1,n}$, one has

$$u_{k+1,n}\, e^{-i\,\xi^{k,j}_n\cdot\varphi} \;=\; u_{k,\ell^k(n)}\, e^{-i\,\xi^{k,j}_n\cdot\varphi} \;-\; a_k\, e^{i\,(\xi^{k,k}_n - \xi^{k,j}_n)\cdot\varphi}\,. \tag{3.2.14}$$

Using (3.2.11) and (3.2.13) when $j < k$ and (3.2.12) when $j = k$, one obtains that (3.2.10) is satisfied at the order k.

d) (3.2.9) implies that for different indices j, the sequences $a_j\, e^{i\,\xi_n^{k,j}\cdot\varphi}$ are asymptotically orthogonal. (3.2.10) implies that they are asymptotically orthogonal to $u_{k+1,*}$. Therefore, (3.2.8) implies that

$$\limsup_n \| u_n^k \|^2_{L^2(\Omega)} \;=\; \limsup_n \| u_{k+1,n} \|^2_{L^2(\Omega)} \;+\; \sum_{j\le k} \| a_j \|^2_{L^2(\Omega)}\,. \tag{3.2.15}$$

Since u_*^k is a subsequence of u_*, the left hand side is bounded uniformly in k, and

$$\sum_j \| a_j \|^2_{L^2(\Omega)} \;\le\; \limsup_n \| u_n \|^2_{L^2(\Omega)} \;<+\infty\,. \tag{3.2.16}$$

Together with (3.2.5), this implies that

$$\delta(u_{j,*}) \to 0 \quad \text{as} \quad j\to\infty\,. \tag{3.2.17}$$

e) Consider the diagonal subsequence $\tilde u_n := u_n^n$ and for all $j\in\mathbb{N}$, the sequence $\tilde\xi^j$ defined by $\tilde\xi^j_n := \xi_n^{n,j}$ for $n \ge j$ and $\tilde\xi^j_n := 0$ for $n < j$. Let A be the set of sequences $\{\tilde\xi^j\}$.

For all k, $\{\tilde u_n\}$ is for $n\ge k$ a subsequence of $\{u_n^k\}$, and for $k\ge j$, $\{\tilde\xi^j_n\}$ is for $n\ge k$ a subsequence of $\{\xi_n^{k,j}\}$. Therefore, (3.2.9) implies that

$$\text{for} \quad 1\le j' < j\ , \quad \tilde\xi^j - \tilde\xi^{j'} \in \mathbf{S}_\infty\,. \tag{3.2.18}$$

This shows that A satisfies condition (3.1.7). Moreover,

$$r_{k,n} := \tilde u_n - \sum_{j<k} a_j\, e^{i\,\tilde\xi^j_n\cdot\varphi} \tag{3.2.19}$$

is a subsequence of $u_{k,n}$. Therefore,

$$\delta(r_{k,*}) \;\le\; \delta(u_{k,*})\,. \tag{3.2.20}$$

Thanks to (3.2.16) and (3.2.18), Theorem 3.1.4 implies that there exists $v_*\in\mathcal{L}^2_{os,\varphi}(A)$ such that

$$v_n \sim \sum a_j\, e^{i\,\tilde\xi^j_n\cdot\varphi}\,.$$

Fix $\delta>0$. (3.2.17) and (2.1.16) imply that there is $k\in\mathbb{N}$ such that

$$\delta(r_{k,*}) \;\le\; \delta \quad \text{and} \quad \sum_{j\ge k} || a_j ||^2_{L^2(\Omega)} \;\le\; \delta^2\,. \tag{3.2.21}$$

Introduce the partial sum $s_{k,n} := \sum_{j<k} a_j\, e^{i\,\tilde\xi^j_n\cdot\varphi}$. Theorem 3.1.4 implies that

$$v_n - s_{k,n} \sim \sum_{j\ge k} a_j\, e^{i\,\tilde\xi^j_n\;\varphi}\,,$$

$$\limsup_n ||v_n - s_{k,n}||^2_{L^2(\Omega)} \;\le \sum_{j\ge k} || a_j ||^2_{L^2(\Omega)} \;\le\; \delta^2\,.$$

Thus, by Lemma 3.2.3, one has $\delta(v_* - s_{k,*}) \;\le\; \delta$ and, since $\tilde u_* - v_* = r_{k,*} - (v_* - s_{k,*})$,

$$\delta(\tilde u_* - v_*) \;\le\; \delta(r_{k,*}) + \delta(v_* - s_{k,*}) \;\le\; 2\delta\,.$$

Since this estimate holds for all $\delta>0$, one has $\delta(\tilde u_* - v_*) = 0$, hence $u_* - v_* \in \mathcal{L}^2_{no,\varphi}$. $\square$

Theorem 3.2.1 can be extended to countable families of sequences.

THEOREM 3.2.4. *Suppose that* $\{u_*^m\}_{m\in\mathbb{N}}$ *is a countable family of sequences in* $\mathcal{L}^2$. *Then there exist an increasing map* $\ell : \mathbb{N} \to \mathbb{N}$, *and an at most countable subset* A *of* $\mathbf{S}$, *such that for all* $m \in \mathbb{N}$

$$u^m_{\ell(*)} \in \mathcal{L}^2_{os,\varphi}(A) + \mathcal{L}^2_{no,\varphi} . \tag{3.2.22}$$

Proof. Theorem 3.2.1 implies that there is an at most countable $A_1 \subset \mathbf{S}$ and ℓ^1 such that $u^1_{\ell^1(*)} \in \mathcal{L}^2_{os,\varphi}(A_1) + \mathcal{L}^2_{no,\varphi}$. Next, apply Theorem 3.2.1 to the subsequence $u^2_{\ell^1(*)}$. By induction, there are mappings ℓ^k and at most countable $A_k \subset \mathbf{S}$ such that

$$u^k_{\rho^k(*)} \in \mathcal{L}^2_{os,\varphi}(A_k) + \mathcal{L}^2_{no,\varphi} , \tag{3.2.23}$$

where $\rho^k := \ell^1 \circ \cdots \circ \ell^k$.

Consider the diagonal sequence $\sigma(n) := \rho^n(n)$. Similarly, introduce for $k > j$, $\rho^{k,j} = \ell^{j+1} \circ \cdots \circ \ell^k$. Let $\sigma^j(n) := \rho^{n,j}(n)$ for $n \geq j$ and $\sigma^j(n) := 0$ for $n < j$. For all $k > n$, $\sigma(n) = \rho^k(\sigma^k(n))$. Thus, for $n \geq j$, $u^k_{\sigma(n)}$ is a subsequence of $u^k_{\rho^k(*)}$. Introduce $\widetilde{A}_k$ the set of extracted subsequences $\xi_{\sigma^k(*)}$ of sequences $\xi_* \in A_k$. Then, (3.2.23) and the second part of Proposition 3.1.2 imply that $u^k_{\sigma(*)} \in \mathcal{L}^2_{os,\varphi}(\widetilde{A}_k) + \mathcal{L}^2_{no,\varphi} \subset \mathcal{L}^2_{os,\varphi}(A) + \mathcal{L}^2_{no,\varphi}$ if $A := \bigcup \widetilde{A}_k$. □

4. The group structure of frequencies.

In §4.1 we associate profiles with oscillations whose frequencies have an appropriate group structure, and prove Theorem 1.5. Then in §4.2 we show that, at least for appropriate subsequences, all the oscillations introduced in §3.1 have such a group structure, thereby proving Theorem 1.4. In §4.3 we extend this result to account for the oscillations of all continuous functions of a sequence u_n, and introduce the additional structure that will be needed to account for resonances.

4.1. Oscillations and profiles.

Nonlinear geometric optics deal with particular oscillations of the form

$$u^\varepsilon(y) = \mathcal{U}(y, \varphi(y)/\varepsilon) , \tag{4.1.1}$$

where $\mathcal{U}(y,\theta)$ is periodic or almost periodic. In this section, we first generalize (4.1.1) to the case where θ vary in a general compact Abelian group. For example, this allows the superposition of oscillations with wavelength of many different scales. Moreover, in this framework, the L^2 analysis of Theorem 3.1.4 can be extended to L^p.

Our starting point is the following remark.

LEMMA 4.1.1. *Consider a compact abelian group* G *and a sequence* ρ_* *of continuous homomorphisms* $\rho_n \in \mathrm{Hom}(\mathbb{R}^m, G)$. *For* $\mathcal{U} \in C^0_0(\Omega \times G)$, *the sequence*

$$u_n(y) := \mathcal{U}(y, \rho_n(\varphi(y))) , \tag{4.1.2}$$

is bounded in $C^0_0(\Omega)$ *and belongs to* $\mathcal{L}^\infty_{os,\varphi}$.

Proof. u_n is clearly bounded in $C_0^0(\Omega)$. To prove that u_* belongs to $\mathcal{L}^\infty_{os,\varphi}$, we perform a Fourier analysis on G.

Consider a discrete Abelian group $\widehat{G}$, which is isomorphic to the dual group of G. Thus every $\alpha \in \widehat{G}$ corresponds to a unique character e_α on G. For all n and all $\alpha \in \widehat{G}$, the mapping $t \to e_\alpha(\rho_n(t))$ is a character on $\mathbb{R}^m$. Thus there is a unique $\nu_n(\alpha) \in \mathbb{R}^m$ such that

$$\forall \alpha \in \widehat{G}\ ,\ \forall t \in \mathbb{R}^m\ , \quad e_\alpha(\rho_n(t)) \ = \ e^{i\,\nu_n(\alpha)\cdot t}\,. \tag{4.1.3}$$

$\nu_n \in \mathrm{Hom}(\widehat{G};\mathbb{R}^m)$ is the dual of $\rho_n \in \mathrm{Hom}(\mathbb{R}^m;G)$. In particular, every $\alpha \in \widehat{G}$ defines a sequence $\nu_*(\alpha) \in \mathbf{S}$. Introduce

$$H \ := \ \{\nu_*(\alpha) \mid \alpha \in \widehat{G}\ \} \ \subset \ \mathbf{S}\,. \tag{4.1.4}$$

(4.1.3) implies that the mapping $\mathcal{U} \to u_*$ defined by (4.1.2), is a bijection between the space of finite linear combinations

$$\mathcal{U}(y,g) = \sum_\alpha a_\alpha(y)\ e_\alpha(g)\,, \tag{4.1.5}$$

with coefficients $a_\alpha \in C_0^\infty(\Omega)$, and the space $\mathcal{P}_\varphi(H)$.

The finite linear combinations of e_α are dense in $C^0(G)$ (see e.g. [W]). Thus finite linear combinations like (4.1.5), which we call trigonometric polynomials, are dense in $C_0^0(\Omega \times G)$. Therefore, approximating $\mathcal{U} \in C_0^0(\Omega \times G)$ by trigonometric polynomials, provides uniform approximations of u_* by sequences in $\mathcal{P}_\varphi(H)$. In particular,

$$u_* \ \in \ \mathcal{L}^\infty_{os,\varphi}(H)\,. \tag{4.1.6}$$

Our goal is to extend the link between functions on $\Omega \times G$ and oscillations to other spaces. To avoid redundancy and a lack of injectivity, introduce the following condition.

Definition 4.1.2. *We say that (G,ρ_*) is admissible, when*

$$\forall \alpha \in \widehat{G}\backslash\{0\} \ : \quad |\nu_n(\alpha)| \to +\infty \quad \text{as} \ \ n \to +\infty\,. \tag{4.1.7}$$

Lemma 4.1.3. *Suppose that (G,ρ_*) is admissible. Then, for all $\mathcal{U} \in C_0^0(\Omega \times G)$*

$$\int_\Omega \mathcal{U}(y,\rho_n(\varphi(y)))\ dy \ \to \ \int_{\Omega\times G} \mathcal{U}(y,g)\,dy\,dg\,, \tag{4.1.8}$$

and for all $p \in [1,+\infty]$

$$\Big(\int_\Omega |\,\mathcal{U}(y,\rho_n(\varphi(y)))\,|^p\,dy\Big)^{1/p} \ \to \ \Big(\int_{\Omega\times G} |\,\mathcal{U}(y,g)\,|^p\,dy\,dg\Big)^{1/p}\,, \tag{4.1.9}$$

where dg is the normalized Haar measure on G.

Proof. The characters e_α form an orthonormal basis in $L^2(G)$ and $\int e_\alpha(g)\, dg = 0$ when $\alpha \neq 0$ and $\int e_0(g)\, dg = 1$ (see e.g. [W]). Therefore, Lemma 3.1.3 and (4.1.7) imply (4.1.8) for trigonometric polynomials. By density, (4.1.8) extends to $\mathcal{U} \in C^0_0(\Omega \times G)$. Applying (4.1.8) to the function $|\mathcal{U}|^p$ yields (4.1.9) when $p < +\infty$.

Thus, $\| \mathcal{U} \|_{L^p} \leq (\text{meas}(\Omega))^{1/p} \liminf \| u_n \|_{L^\infty}$, where $u_n(y) := \mathcal{U}(y, \rho_n(\varphi(y)))$. On the other hand, $\| u_n \|_{L^\infty} \leq \| \mathcal{U} \|_{L^\infty}$. Since $\| \mathcal{U} \|_{L^p} \to \| \mathcal{U} \|_{L^\infty}$ as $p \to +\infty$, this implies that (4.1.9) is also satisfied when $p = +\infty$.

Theorem 4.1.4. *Suppose that (G, ρ_*) is admissible. Let H be given by (4.1.4). For all $u_* \in \mathcal{L}^p_{os,\varphi}(H)$, $p \in [1, +\infty]$, there is a unique $\mathcal{U} \in L^p(\Omega \times G)$ such that for all $\mathcal{A} \in C^0_0(\Omega \times G)$*

$$\int_\Omega u_n(y)\; \mathcal{A}(y, \rho_n(\varphi(y)))\; dy \;\to\; \int_{\Omega\times G} \mathcal{U}(y,g)\; \mathcal{A}(y,g)\; dy\, dg\,. \tag{4.1.10}$$

It satisfies

$$\| \mathcal{U} \|_{L^p(\Omega\times G)} \;=\; \lim_{n\to+\infty} \| u_n \|_{L^p(\Omega)}\,. \tag{4.1.11}$$

This defines a mapping $\Sigma : u_* \to \mathcal{U}$ from $\mathcal{L}^p_{os,\varphi}(H)$ into $L^p(\Omega \times G)$. This mapping is surjective when $p < +\infty$, and it kernel is $\mathcal{L}^p_0$.

When $\mathcal{U} \in C^0_0(\Omega \times G)$ and $u_n(y) = \mathcal{U}(y, \rho_n(\varphi(y)))$, $\Sigma(u_*)$ is equal to $\mathcal{U}$.

Proof. When, $u_* \in \mathcal{P}_\varphi(H)$, there is a trigonometric polynomial $\mathcal{U}$ such that $u_n(y) = \mathcal{U}(y, \rho_n(\varphi(y)))$ and Lemma 4.1.3 implies that (4.1.11) is satisfied. In particular, $\mathcal{U}$ is uniquely determined. This defines Σ acting from $\mathcal{P}_\varphi(H)$ onto the space of trigonometric polynomials.

For $u_* \in \mathcal{L}^p_{os,\varphi}(H)$, there are $u^k_* \in \mathcal{P}_\varphi(H)$ such that

$$\limsup_{n\to+\infty} \| u_n \;-\; u^k_n \|_{L^p(\Omega\times G)} \;\leq\; 2^{-k}\,. \tag{4.1.12}$$

Then (4.1.11) implies that $\mathcal{U}^k := \Sigma(u^k_*)$ is a Cauchy sequence in $L^p(\Omega \times G)$, and thus converge in this space. (4.1.11) also implies that the limit does not depend on the choice of u^k_* which satisfies (4.1.12). This defines Σ on $\mathcal{L}^p_{os,\varphi}(H)$ and (4.1.11) extends to this space.

The range of Σ contains the space of trigonometric polynomials, and is closed by (4.1.11). Thus the range of Σ always contains $C^0_0(\Omega \times G)$ and is equal to $L^p(\Omega \times G)$ when $p < +\infty$.

The convergence (4.1.10) follows from Lemma 4.1.3 when $u_* \in \mathcal{P}_\varphi(H)$. It extends to all $u_* \in \mathcal{L}^p_{os,\varphi}(H)$

Definition 4.1.5. *Suppose that $\mathcal{U} \in L^p(\Omega\times G)$, $1 \leq p \leq +\infty$, and $\{u_n\}$ is a bounded sequence in $L^p(\Omega)$. We say that*

$$u_n(y) \;\sim\; \mathcal{U}(y, \nu_n(\varphi(y))) \quad \text{in } L^p \tag{4.1.13}$$

when $u_* \in \mathcal{L}^p_{os,\varphi}(H)$ and $\Sigma(u_*) = \mathcal{U}$.

Theorem 4.1.4 implies that when $p < +\infty$, for all $\mathcal{U} \in L^p(\Omega \times G)$ there exist $u_* \in \mathcal{L}^p_{os,\varphi}(H)$ which satisfy (4.1.13). For all p, if u_* and v_* satisfy (4.1.13), $u_n - v_n$ converges strongly to 0 in L^p. In addition, when $\mathcal{U} \in C^0_0(\Omega \times G)$, u_* satisfies (4.1.13) if and only if $u_n - \mathcal{U}(y, \rho_n(\varphi(y)))$ converges strongly to 0 in L^p.

REMARK 4.1.6 . Since the characters e_α form an orthonormal basis of $L^2(G)$, the Fourier expansion

$$\mathcal{U}(y,g) = \sum_{\alpha \in \widehat{G}} U_\alpha(y)\; e_\alpha(g)\,, \tag{4.1.14}$$

extends to $\mathcal{U} \in L^2(\Omega \times G)$. It is isometric from $L^2(\Omega \times G)$ to $\ell^2(\widehat{G}; L^2(\Omega))$:

$$||\,\mathcal{U}\,||^2_{L^2(\Omega\times G)} \;=\; \sum_{\alpha \in \widehat{G}} ||\,U_\alpha\,||^2_{L^2(\Omega)}\,. \tag{4.1.15}$$

Therefore, when $p = 2$, Theorem 4.1.4 is a particular case of Theorem 3.1.4. Moreover, $u_n \sim \mathcal{U}(y, \nu_n(y))$ in the sense of Definition 4.1.5 if and only if $u_n \sim \sum U_\alpha\, e^{i\,\nu_n(\alpha)\,\varphi}$ in the sense of Definition 3.1.5.

PROPOSITION 4.1.7. *Suppose that G is a compact Abelian group. Consider $\rho_n \in \mathrm{Hom}(\mathbb{R}^m; G)$ and $\tilde\rho_n \in \mathrm{Hom}(\mathbb{R}^m; G)$. Suppose that (G, ρ_*) is admissible and the dual homomorphisms $\nu_n \in \mathrm{Hom}(\widehat{G}; \mathbb{R}^m)$ and $\tilde\nu_n \in \mathrm{Hom}(\widehat{G}; \mathbb{R}^m)$ satisfy*

$$\forall \alpha \in \widehat{G}\ :\quad \tilde\nu_n(\alpha)\ -\ \nu_n(\alpha)\ \to\ l(\alpha) \in \mathbb{R}^m \quad \text{as} \quad n \to +\infty\,. \tag{4.1.16}$$

Introduce $H := \{\nu_*(\alpha)\ :\ \alpha \in \widehat{G}\}$ and $\tilde H := \{\tilde\nu_*(\alpha)\ :\ \alpha \in \widehat{G}\}$. Then $(G, \tilde\rho_*)$ is admissible and for all $p \in [1, +\infty]$, $\mathcal{L}^p_{os,\varphi}(H) = \mathcal{L}^p_{os,\varphi}(\tilde H)$. Moreover, $\mathcal{U}(y, \rho_n(\varphi(y))) \sim \tilde{\mathcal{U}}(y, \tilde\rho_n(\varphi(y)))$ in L^p, if and only if

$$\mathcal{U}(y,g) \;=\; \tilde{\mathcal{U}}(y, g \,+\, \hat l(\varphi(y)))\,, \tag{4.1.17}$$

where $\hat l \in \mathrm{Hom}(\mathbb{R}^m; G)$ is the dual homomorphism of l.

Proof. **a)** Since ν_* satisfies (4.1.7), (4.1.16) implies that $\tilde\nu_*$ also satisfies (4.1.7), thus $(G, \tilde\rho)$ is admissible.

b) Proposition 3.1.2 implies that for all $p \in [1, +\infty]$, $\mathcal{L}^p_{os,\varphi}(H) = \mathcal{L}^p_{os,\varphi}(\tilde H)$.

c) Suppose that $u_* \in \mathcal{L}^p_{os,\varphi}(H) = \mathcal{L}^p_{os,\varphi}(\tilde H)$. The Fourier coefficients of the associated profiles $\mathcal{U}$ and $\tilde{\mathcal{U}}$, are given by (4.1.10). They are the weak limit of respectively $u_n\; e^{-\,i\,\nu_n(\alpha)\cdot\varphi}$ and $u_n\; e^{-\,i\,\tilde\nu_n(\alpha)\cdot\varphi}$. Thus, (4.1.16) implies that $U_\alpha = e^{\,i\,l(\alpha)\cdot\varphi}\ \tilde U_\alpha$. If $\hat l$ is the dual homomorphism of l, one has $e^{i\,l(\alpha)\cdot\varphi} = e_\alpha(\hat l(\varphi))$ and therefore $U_\alpha(y)\, e_\alpha(g) = \tilde U_\alpha(y)\, e_\alpha(g \,+\, \hat l(\varphi(y)))$. (4.1.17) follows.

4.2. Group structures for the frequencies.

The basic idea for providing an at most countable set of scales with a group structure is to use the group of finite linear combinations of an independent set of scales and restrict to a subsequence for which all other scales converge to linear combinations of those in the independent set. The complication is that the independence of scales depends on the choice of subsequence, so that both must be constructed simultaneously. Thus, our first step is to show that, after extracting a subsequence, every at most countable subset of $\mathbf{S} := \mathbb{R}^{\mathbb{N}}$ is contained in the set of finite linear combinations of some at most countable set of sequences that tend to infinity at independent rates. As long as the coefficients are taken to lie in some field, the set of such linear combinations forms a vector space, which simplifies the construction and also provides the group that will be used to describe the oscillations. In order to explicitly exhibit the resulting set of oscillations as being at most countable, the dual group should be countable, so we will use a countable subfield F of $\mathbb{R}$ rather than $\mathbb{R}$ itself. Since we will later desire that F should contain the coefficients of all the resonance relations among a given set of phase functions, the choice of F will be postponed to §4.3.

Definition 4.2.1. *Let F be a subfield of $\mathbb{R}$. An independent family of scales over F is a family of real sequences $\{\nu_*^j\}_{j\in J}$, such that for all finite $K \subset J$ and all $\lambda \in F^K\backslash\{0\}$*

$$(4.2.1) \qquad \Big| \sum_{j\in K} \lambda_j \, \nu_n^j \Big| \ \to\ +\infty \quad \text{as} \quad n \to +\infty\,.$$

An independent family of scales can be used to define dual homomorphisms ρ_n. To do this, let $\Phi^{(J)}$ denote the space of sequences $\{\alpha_j\}_{j\in J}$ of elements of any set Φ with all the α_j equal to zero except for a finite number of indices. Given a family of scales $\{\nu_*^j\}_{j\in J}$, a subset Φ of some $\mathbb{R}^m$, and $\alpha \in \Phi^{(J)}$, introduce

$$(4.2.2) \qquad \nu_n(\alpha) \ :=\ \sum_j \nu_n^j \, \alpha_j\,.$$

Clearly (4.2.1) is equivalent to

$$(4.2.3) \qquad \forall \alpha \in F^{(J)}\backslash\{0\} \ :\quad |\nu_n(\alpha)| \to +\infty \quad \text{as} \quad n \to +\infty\,.$$

Proposition 4.2.2. *Suppose that A is an at most countable subset of $S := \mathbb{R}^{\mathbb{N}}$. Then for any subfield F of $\mathbb{R}$, there are ℓ, strictly increasing from $\mathbb{N}$ into $\mathbb{N}$, and $\{\nu_*^j\}_{j\in J}$, an at most countable independent family of scales over F, such that*

$$(4.2.4) \qquad \forall \xi_* \in A\,,\quad \exists \alpha \in F^{(J)}\,,\quad \exists l \in \mathbb{R} \ :\quad \xi_{\ell(n)} - \nu_n(\alpha) \to l \quad \text{as} \quad n \to +\infty\,.$$

Proof. **a)** Let $A = \{\xi_*^j\}$. By induction on k, we construct finite dimensional spaces Z_k over F, linear mappings ν_n^k from Z_k into $\mathbb{R}$ and strictly increasing mappings ℓ^k from $\mathbb{N}$ into itself, such that

$$\forall \alpha \in Z_k \backslash \{0\} \ : \quad |\nu_n^k(\alpha)| \to +\infty \quad \text{as} \quad n \to +\infty\,, \tag{4.2.5}$$

$$\forall j \leq k\ , \quad \exists \alpha \in Z_k\ , \quad \exists l \in \mathbb{R}\ : \quad \xi^j_{\ell^k(n)} - \nu_n^k(\alpha) \to l \quad \text{as} \quad n \to +\infty\,. \tag{4.2.6}$$

Adding this sequence to A if necessary, we can assume that $\xi_n^0 = 0$. Therefore we can start from $Z_0 = \{0\}$. Suppose that Z_k, ν_*^k and ℓ^k are constructed. Consider the sequence $\eta_n := \xi^{k+1}_{\ell^k(n)}$. Either,

$$\forall \alpha \in Z_k\ \forall \tau \in F \ : \quad |\tau\eta_n - \nu_n^k(\alpha)| \to +\infty \quad \text{as} \quad n \to +\infty\,, \tag{4.2.7}$$

or there is a subsequence $\eta_{\sigma(n)}$, $\underline{\tau} \in F$, and $\underline{\alpha} \in Z_k$ such that

$$\underline{\tau}\eta_{\sigma(n)} - \nu^k_{\sigma(n)}(\underline{\alpha}) \ \to \ l \in \mathbb{R}^m \quad \text{as} \quad n \to +\infty\,. \tag{4.2.8}$$

In the first case, define $Z_{k+1} := Z_k \times F$, $\nu_n^{k+1}(\alpha,\tau) := \nu_n^k(\alpha) + \tau\eta_n$ and $\ell^{k+1} := \ell^k$. In the second case, since F is a field and Z_k is a vector space over F, it is possible to take $\underline{\tau}$ in (4.2.8) to be 1, so define $Z_{k+1} := Z_k$, $\nu_n^{k+1} := \nu^k_{\sigma(n)}$ and $\ell^{k+1} := \ell^k \circ \sigma$. In either case, properties (4.2.5) and (4.2.6) are satisfied for $Z_{k+1}, \nu_*^{k+1}, \ell^{k+1}$. In the first case, we identify Z_k with the subspace $Z_k \times \{0\}$ of Z_{k+1}. With $\sigma^k = id$ in the first case and $\sigma^k = \sigma$ in the second case, note that

$$Z_k \subset Z_{k+1}\ , \quad \nu_n^{k+1}{}_{|Z_k} = \nu^k_{\sigma^k(n)} \quad \text{and} \quad \ell^{k+1} = \ell^k \circ \sigma^k\,. \tag{4.2.9}$$

b) Consider $Z := \cup Z_k$, which by (4.2.9) is a vector space. It is isomorphic to $F^{(J)}$ where J is at most countable. Z_k is a finite dimensional subspace of Z and there exists a Z'_k such that $Z = Z_k \oplus Z'_k$. We define the linear mapping ν_n, from Z into $\mathbb{R}^m$ by

$$\begin{cases} \nu_n(\alpha) := \nu_n^n(\alpha) & \text{when} \quad \alpha \in Z_n \\ \nu_n(\alpha) = 0 & \text{when} \quad \alpha \in Z'_n \end{cases} \tag{4.2.10}$$

For $k < n$ introduce $\ell^{k,n} := \sigma^k \circ \cdots \circ \sigma^{n-1}$ and for $k = n$, $\ell^{k,n} := id$. (4.2.9) implies that

$$\forall \alpha \in Z_k \ : \quad \nu_n(\alpha) = \nu^k_{\ell^{k,n}(n)}(\alpha) \quad \text{when} \quad n \geq k\,. \tag{4.2.11}$$

Thus, for $\alpha \in Z_k$, $\{\nu_n(\alpha)\}$ is, for $n \geq k$, a extracted subsequence of $\{\nu_n^k(\alpha)\}$. Therefore (4.2.5) implies that

$$\forall \alpha \in Z \backslash \{0\} \ : \quad |\nu_n(\alpha)| \to +\infty \quad \text{as} \quad n \to +\infty\,. \tag{4.2.12}$$

Consider the diagonal sequence $\ell(n) := \ell^n(n)$. Since $\ell^n = \ell^k \circ \ell^{k,n}$, the convergence in (4.2.6) applied to the subsequence $n \to \ell^{k,n}(n)$, implies that

$$\forall j \ , \quad \exists \alpha \in Z \ , \quad \exists l \in \mathbb{R} \ : \quad \xi^j_{\ell(n)} - \nu_n(\alpha) \to l \quad \text{as} \quad n \to +\infty \, . \tag{4.2.13}$$

c) By construction, Z has an at most countable basis, $\{e^j\}_{j\in J}$. For $j \in J$, let $\nu^j_n := \nu_n(e^j)$.Then, (4.2.12) means that $\{\nu^j_*\}_{j\in J}$ is an independent family of scales over F. Then $Z \approx F^{(J)}$ and $\nu_n(\alpha) = \sum \alpha_j \nu^j_n$ for $\alpha = \sum \alpha_j e^j$. Thus, (4.2.12) is equivalent to (4.2.3). □

THEOREM 4.2.3. *Suppose that A is an at most countable subset of $\mathbf{S} := (\mathbb{R}^m)^{\mathbb{N}}$. Then, there are an increasing $\ell : \mathbb{N} \to \mathbb{N}$, a compact Abelian group G and a sequence $\rho_n \in \mathrm{Hom}(\mathbb{R}^m; G)$, such that (G, ρ_*) is admissible, and for all $u_* \in \mathcal{L}^p_{os,\varphi}(A)$, the extracted subsequence $u_{\ell(*)}$ belongs to $\mathcal{L}^p_{os,\varphi}(H)$, where H is given by (4.1.4).*

Furthermore, for any subfield F of $\mathbb{R}$, the dual group $\widehat{G}$ may be taken to be isomorphic to $(F^m)^{(J)}$, where J is the index set of an at most countable independent family of scales $\{\nu^j_*\}_{j\in J}$, and the dual ν_n of ρ_n can be taken to be defined by (4.2.2), where now $\alpha \in (F^m)^{(J)}$.

Proof. The components of the at most countable subset A of $(\mathbb{R}^m)^{\mathbb{N}}$ define an at most countable subset A_1 of $\mathbb{R}^{\mathbb{N}}$. Given a subfield F of $\mathbb{R}$, construct ℓ and $\{\nu^j_*\}_{j\in J}$ for A_1 by Proposition 4.2.2. Let $Z = (F^m)^{(J)}$ and define $\nu_n \in \mathrm{Hom}(Z, \mathbb{R}^m)$ by (4.2.2). Then (4.2.3) implies that

$$\forall \alpha \in Z \backslash \{0\} \ : \quad |\nu_n(\alpha)| \to +\infty \quad \text{as} \quad n \to +\infty \, . \tag{4.2.14}$$

Consider the discrete topology on Z. Then its dual group G is compact and Z is isomorphic to the dual group of G (see e.g.[W]). Denote by e_α the character on G which corresponds to $\alpha \in Z$. By duality, ν_n defines $\rho_n \in \mathrm{Hom}(\mathbb{R}^m; G)$, which is uniquely determined by (4.1.3) :

$$\forall \alpha \in Z \ , \quad \forall t \in \mathbb{R}^m \ : \quad e_\alpha(\rho_n(t)) \ = \ e^{i\,\nu_n(\alpha)\cdot t} \, . \tag{4.2.15}$$

Thus, the set of sequences H associated to (G, ρ_*) by (4.1.4) is

$$H \ := \ \{ \ \nu_*(\alpha) \ ; \ \alpha \in Z \ \} \, . \tag{4.2.16}$$

With (4.2.14), this shows that (G, ρ_*) is admissible in the sense of Definition 4.1.2. Finally, the construction of A_1 together with (4.2.4) ensure that

$$\forall \xi \in A \ , \ \ \exists \alpha \in Z \ , \ \ \exists l \in \mathbb{R}^m \ : \quad \xi_{\ell(n)} \ - \ \nu_n(\alpha) \to l \quad \text{as} \ \ n \to \infty \, . \tag{4.2.17}$$

Proposition 3.1.2 and (4.2.17) imply that for all $u_* \in \mathcal{L}^p_{os,\varphi}(A)$, the subsequence $u_{\ell(*)}$ belongs to $\mathcal{L}^p_{os,\varphi}(H)$. □

Remark 4.2.4. For the purpose of proving the first part of Theorem 4.2.3, it was not necessary to break A down into the set A_1 of scalar sequences, since the procedure of the proof of Proposition 4.2.2 could be applied directly to vector-valued sequences to yield a set of vector-valued independent scales. However, in order to treat resonances it is useful to require that the same scales be present in all components, because resonances among various components yield oscillations in other components.

Remark 4.2.5. Suppose that $\{\nu_*^j\}_{j\in J}$ is an independent family of scales. Let F be a subfield of $\mathbb{R}$, and $\Phi = F^m$. Introduce $Z = \Phi^{(J)}$. We consider it as a discrete group and introduce the dual group G. Then $G = \widehat{\Phi}^J$, where $\widehat{\Phi}$ is the dual group of Φ considered as a discrete group. Recall that there is a natural embedding ι of the dual space $\Theta := \Phi^*$ into $\widehat{\Phi}$. For $\theta \in \Theta$, $\iota(\theta)$ is the character on Φ

$$\iota(\theta)(\xi) \ := \ e^{i\,\xi\cdot\theta}\,. \tag{4.2.18}$$

Recall that continuous functions on $\widehat{\Phi}$ are almost periodic functions on Θ.

The dual of $\nu_n \in \mathrm{Hom}(Z;\Phi)$ is $\rho_n \in \mathrm{Hom}(\Theta;G)$ where

$$\rho_n(\theta) \ = \ \{\iota(\nu_n^j\,\theta)\}_{j\in J}\,. \tag{4.2.19}$$

In particular, if $g = \{g_j\}_{j\in J}$, and $\mathcal{U} \in C^0(G)$ is a function which depends only on a finite number of variables $(g_1,\dots,g_k)$, one can consider $\mathcal{U}$ as an almost periodic function $\tilde{\mathcal{U}}$ on Θ^k and

$$\begin{aligned} u_n(y) \ := \ \mathcal{U}(\rho_n(\varphi(y))) \ &= \ \mathcal{U}(\iota(\nu_n^1\,\varphi(y)),\dots,\iota(\nu_n^k\,\varphi(y))) \\ &= \ \tilde{\mathcal{U}}(\nu_n^1\,\varphi(y),\dots,\nu_n^k\,\varphi(y))\,. \end{aligned} \tag{4.2.20}$$

Remark 4.2.6. In Theorem 4.2.3, the countable set A is arbitrary. One can always add a given set A_0 to it. In particular, one can force the group H to contain a subgroup equivalent in the sense of Proposition 4.1.7 to a given subgroup H_0. This will be used in the study of the Cauchy problem to compare the group structures $(G_0,\rho_{0,*})$ associated to the initial data to the group structures (G,ρ_*) defined by the solutions. More precisely, suppose that $(G_0,\rho_{0,*})$ is admissible and $A \subset \mathbf{S}$ contains $H_0 := \{\nu_{0,*}(\alpha) \,:\, \alpha \in \widehat{G}_0\}$. Let Z, ν_* and ℓ be given by Proposition 2.4.2. Let $G := \hat{Z}$ be the dual group of Z and let $\rho_n \in \mathrm{Hom}(\mathbb{R}^m;G)$ the dual homomorphism of ν_n. We identify Z with $\widehat{G}$. Then (4.2.17) implies that

$$\forall\alpha \in \widehat{G}_0\ ,\quad \exists\beta \in \widehat{G}\ ,\quad \exists l \in \mathbb{R}^m\ \ :\quad \nu_{0,\ell(n)}(\alpha)\ -\ \nu_n(\beta) \to l \quad \text{as}\ \ n \to \infty\,. \tag{4.2.21}$$

(4.2.14) implies that for all $\alpha \in \widehat{G}_0$ there is a unique $\beta \in \widehat{G}$ and therefore a unique $l \in \mathbb{R}^m$ such that (4.2.21) holds with $\beta = \sigma(\alpha)$ and $l = l(\alpha)$.

Since $(G_0,\rho_{0,*})$ is admissible, σ is injective. Therefore its dual homomorphism $\pi \in \mathrm{Hom}(G;G_0)$ is surjective. In particular, G_0 is isomorphic to a quotient group of G. Moreover, if one considers

$$\tilde{\rho}_{0,n} \ := \ \pi\circ\rho_n \in \mathrm{Hom}(\mathbb{R}^m;G_0)\,, \tag{4.2.22}$$

(4.2.21) implies that $\rho_{0,\ell(*)}$ and $\tilde{\rho}_{0,n}$ are equivalent in the sense described in Proposition 4.1.7, that is, $\nu_{0,\ell(*)}$ and $\tilde{\nu}_{0,n}$, the dual of $\rho_{0,n}$ satisfy

$$\forall \alpha \in \widehat{G}_0 \ : \quad \nu_{0,\ell(n)}(\alpha) - \tilde{\nu}_{0,n}(\alpha) \to l(\alpha) \quad \text{as} \quad n \to \infty \,. \tag{4.2.23}$$

4.3. Groups of frequencies adapted to resonances.

Theorems 4.2.3 and 3.2.1 ensure that the oscillations of a scalar sequence with respect to a finite number of phases can be given a group structure. In this subsection we extend this result in two ways, to include the oscillations of all continuous functions of single scalar sequences and to take into account the resonances among the phases for a set of scalar sequences. We begin by defining the additional conditions the group structure should satisfy. Additional properties of resonances that are not needed for the group structure will be described in the next section.

DEFINITION 4.3.1. *Consider a compact Abelian group G and a sequence $\rho_n \in \mathrm{Hom}(\mathbb{R}^m; G)$. Let u_n be a bounded sequence in $L^\infty(\Omega)$. Then (G, ρ_*) is said to be complete for the sequence u_* and the phase φ when (G, ρ_*) is admissible and*

$$\forall f \in C^0(\mathbb{C}) \ : \quad f(u_*) \in \mathcal{L}^2_{os,\varphi}(H) \ + \ \mathcal{L}^2_{no,\varphi} \,, \tag{4.3.1}$$

where H is defined in (4.1.4).

In order to describe resonances we let the space $\Phi := \mathbb{R}^m$ of frequencies be the product of spaces $\Phi_k := \mathbb{R}^{m_k}$, and let Θ_k denote the dual of Φ_k. Also, define $\Phi_k^\sharp := \{0\} \times \ldots \times \{0\} \times \Phi_k \times \{0\} \ldots \times \{0\} \subset \Phi$, with Φ_k in the k-th position.

DEFINITION 4.3.2 *A group R of resonances is a subspace of the vector space Φ over $\mathbb{R}$ such that*

$$\forall\, k, \qquad R \cap \Phi_k^\sharp = \{0\}. \tag{4.3.2}$$

DEFINITION 4.3.3 *A subfield F of $\mathbb{R}$ is consistent with a group $R \subset \mathbb{R}^m$ of resonances if there exists a basis for R whose elements belong to F^m.*

LEMMA 4.3.4. *For any group R of resonances there exists a countable subfield F of $\mathbb{R}$ that is consistent with R.*

Proof. Choose any basis $\{v^i\}$ for R, and let $\{a_j\}_{j=1}^p$ be the set of all components of the v^i. The subfield $F := \mathbb{Q}(a_1, \ldots, a_p)$ of $\mathbb{R}$ obtained by adjoining the a_i to the rationals is countable, as can be seen from the explicit representation of the field obtained by adjoining an element to a subfield (e.g. [Her, p. 210]). By construction, each v^i belongs to F^m. □

Definition 4.3.5. *For $k \in \{1,\ldots,N\}$, consider a compact Abelian group G_k and a sequence $\rho_{k,n} \in \mathrm{Hom}(\Theta_k, G_k)$. Let $\nu_{k,n} \in \mathrm{Hom}(\widehat{G}_k, \Phi_k)$ denote the dual homomorphism of $\rho_{k,n}$. For $\alpha = (\alpha_1, \ldots \alpha_N) \in \widehat{G}_1 \times \ldots \times \widehat{G}_N$, introduce $\nu_n(\alpha) := (\nu_{1,n}(\alpha_1), \ldots, \nu_{N,n}(\alpha_N)) \in \Phi$, and $\tilde{\nu}_n(\alpha)$ its class in $\Phi_R := \Phi/R$. We say that $\big((G_1, \rho_{1,*}), \ldots, (G_N, \rho_{N,*})\big)$ is admissible for R when*

i) for all $k \in \{1,\ldots,N\}$, $(G_k, \rho_{k,*})$ is admissible in the sense of Definition 4.1.2,

ii) there exists a subgroup Z of $\widehat{G}_1 \times \ldots \times \widehat{G}_N$ such that

$$\begin{cases} \forall \alpha \in Z\,, \quad \forall n \in \mathbb{N}\ :\quad \nu_n(\alpha) \in R\,, \\ \forall \alpha \notin Z\ :\ \ |\tilde{\nu}_n(\alpha)| \to +\infty\ , \quad \text{as } n \to +\infty\,. \end{cases} \tag{4.3.3}$$

An important effect of resonance is the creation of new oscillations. In order to account for this in the group structure, the frequencies $\nu_{k,n}(\alpha_k)$ on the k-th mode must include all frequencies that can be created by nonlinear interaction of the other modes. To make this condition precise, define $\widetilde{\Phi}_k$ to be the image of $\Phi_k^\sharp$ in the quotient space Φ/R.

Definition 4.3.6. *For $k \in \{1,\ldots,N\}$, consider a compact Abelian group G_k and a sequence $\rho_{k,n} \in \mathrm{Hom}(\Theta_k, G_k)$. Introduce ν_n and $\tilde{\nu}_n$ as above. Then*

$$\big((G_1, \rho_{1,*}), \ldots, (G_N, \rho_{N,*})\big)$$

is said to be closed for resonances when it satisfies the following condition :

for all $k \in \{1,\ldots,N\}$, if $\tilde{l}_n$ is a bounded sequence in Φ/R and $\alpha = (\alpha_1, \ldots \alpha_N) \in \widehat{G}_1 \times \ldots \times \widehat{G}_N$ are such that for all n in a subsequence, $\tilde{\nu}_n(\alpha) - \tilde{l}_n \in \widetilde{\Phi}_k$, then there exists $\beta \in \widehat{G}_1 \times \ldots \times \widehat{G}_N$ such that $\beta_j = 0$ for all $j \neq k$, and $\tilde{\nu}_n(\beta) = \tilde{\nu}_n(\alpha)$ for all n in the subsequence.

The next theorem states the existence of groups and homomorphisms $(G_k, \rho_{k,*})$ which are admissible for R, closed for resonances and complete for subsequences $u_{k,\ell(*)}$. Morover, they can be chosen so that they extend a given structure $(G_k^0, \rho_{k,*}^0)$. This will be used in section 7.3, with $(G_k^0, \rho_{k,*}^0)$ defined by the initial data.

Theorem 4.3.7. *For $k \in \{1,\ldots,N\}$, suppose that $u_{k,n}$ is a bounded sequence in $L^\infty(\Omega)$. Assume that the m_k-dimensional vector-valued phases φ_k satisfy (3.0.1), and with Φ as above, let R be a group of resonances. Then:*

i) There exist a subsequence $\ell : \mathbb{N} \to \mathbb{N}$, groups G_k, and sequences of homomorphisms $\rho_{k,n} \in \mathrm{Hom}(\Theta_k; G_k)$, such that $\big((G_1, \rho_{1,*}), \ldots, (G_N, \rho_{N,*})\big)$ is admissible for R, closed for resonances and, for all $k \in \{1,\ldots,N\}$, $(G_k, \rho_{k,*})$ is complete for $u_{k,\ell(*)}$ and the phase φ_k, in the sense of Definition 4.3.1. Moreover, one can choose the G_k such that their dual groups $\widehat{G}_k$ are countable.

ii) Suppose that $\big((G_1^0,\rho_{1,*}^0),\dots,(G_N^0,\rho_{N,*}^0)\big)$ is admissible for R and the $\widehat{G}_k^0$ are countable. Then, one can choose ℓ, the groups, and the homomorphisms in i) such that for all k there is a surjective $\jmath_k\in\mathrm{Hom}(G_k,G_k^0)$ and there is $l_k\in\mathrm{Hom}(\widehat{G}_k^0;\Phi_k)$ such that

$$\forall\alpha\in\widehat{G}_k^0\ ,\quad \hat\rho_{k,n}(\hat\jmath_k(\alpha))-\hat\rho_{k,\ell(n)}^0(\alpha)\to l_k(\alpha)\quad\text{as}\quad n\to+\infty\,. \tag{4.3.4}$$

Moreover, if Z^0 denotes the subgroup of $\widehat{G}_1^0\times\ldots\times\widehat{G}_N^0$ such the $\nu_{k,*}^0$ satisfy (4.3.3), then $Z^0=\{\alpha\in\widehat{G}_1^0\times\ldots\times\widehat{G}_N^0\ ;\hat\jmath(\alpha)\in Z\}$, where $\hat\jmath:=(\hat\jmath_1,\dots,\hat\jmath_N)$.

Proof. **a)** For each $k\in\{1,\dots,N\}$, consider the sequences $u_n^{(l,p)}:=(u_{k,n})^l\,\overline{(u_n)^p}$ for l,p in $\mathbb{N}$. Since u_n is bounded in L^∞ and Ω is bounded, the sequences $u_*^{(l,p)}$ are bounded in $L^2(\Omega)$. Thus, Theorem 3.2.4 implies that there exist at most countable set $A_k\subset\Phi_k^{\mathbb{N}}$ and ℓ such that

$$f(u_{k,\ell(*)})\in\mathcal{L}^2_{os,\varphi_k}(A_k)\ +\ \mathcal{L}^2_{no,\varphi_k}\,, \tag{4.3.5}$$

for all monomials $f(\lambda)=\lambda^l\,\overline{\lambda^p}$. Taking linear combinations shows that (4.3.5) is satisfied for all polynomial function f of λ and $\overline{\lambda}$.

Suppose that $f\in C^0(\mathbb{C})$. Fix $\rho>\sup||u_{k,n}||_{L^\infty(\Omega)}$. If f_j is a sequence of polynomials which converge uniformly to f on the ball $\{|\lambda|\le\rho\}$, then $f_j(u_{k,n})\to f(u_{k,n})$ in $L^\infty(\Omega)$ as $j\to\infty$, uniformly in n. Since (4.3.5) is satisfied for f_j, and $\mathcal{L}^2_{os,\varphi}(H)\ +\ \mathcal{L}^2_{no,\varphi}$ is asymptotically closed, (4.3.5) is satisfied for all $f\in C^0(\mathbb{C})$. When the $(G_k^0,\rho_{k,*}^0)$ are given, one can choose A_k such that it also contains the sequences $\hat\rho_{k,*}^0(\alpha)$, $\alpha\in\widehat{G}_k^0$. Via the embedding $\Phi_k^\sharp$ of Φ_k into Φ, we can consider A_k to lie in $\Phi^{\mathbb{N}}$. Then define A to be the union of the A_k. Let F be a subfield of $\mathbb{R}$ that is consistent with R, and use Theorem 4.2.3 to construct $\ell:\mathbb{N}\to\mathbb{N}$, $\widehat{G}=(F^m)^{(J)}$, and an at most countable independent set of scales $\{\nu_*^j\}_{j\in J}$ for this A.

b) The group $(F^m)^{(J)}$ factors into the product of groups $H_k:=(F^{m_k})^{(J)}$, and (4.2.2) defines homomorphisms $\nu_{k,n}\in\mathrm{Hom}(H_k,\Phi_k)$. Since $\{\nu_*^j\}_{j\in J}$ is an independent family of scales, the dual groups $G_k:=\widehat{H}_k$ and dual homomorphisms $\rho_{k,n}:=\widehat{\nu}_{k,n}$ are admissible in the sense of definition 4.1.2. Furthermore, the construction of A ensures that the $(G_k,\rho_{k,*})$ are complete for $u_{k,\ell(*)}$. Since F is countable, so are the H_k.

c) Write $\alpha^j\in F^m$ as $\alpha^j=(\alpha_1^j,\dots,\alpha_N^j)$, with $\alpha_k^j\in F^{m_k}$. Then

$$\nu_n(\alpha)\ :=\ \sum_{j\in J}\nu_n^j\,\alpha^j\ =\ \big(\nu_{1,n}(\alpha_1),\dots,\nu_{N,n}(\alpha_N)\big)\ \in\ \Phi \tag{4.3.6}$$

Introduce next $Z:=(R\cap F^m)^{(J)}\subset\prod H_k$. Then Z is a subgroup of $\prod H_k$. When $\alpha\in Z$, then $\alpha=(\alpha^j)$ and all the α^j belong to R. Thus (4.3.6) and the fact that R is a vector space imply that $\nu_n(\alpha)\in R$ for all n.

Let $\tilde{}$ denote the quotient map $\Phi\to\Phi/R$. Since R is closed under scalar multiplication,

$$\forall r\in\mathbb{R}\ \forall v\in\Phi,\qquad \widetilde{rv}=r\tilde v\,. \tag{4.3.7}$$

Hence, by (4.3.6),

$$\tilde{\nu}_n(\alpha) \;=\; \sum_{j\in J} \nu_n^j \,\tilde{\alpha}^j \;\in\; \Phi/R\,. \tag{4.3.8}$$

When $\alpha \notin Z$, at least one of the $\tilde{\alpha}^j$ does not vanish. This means that one of its components in a basis of Φ/R is not equal to zero. Since ν_*^j is an independent family of scales, this implies that the same component of $\tilde{\nu}_n(\alpha)$ tends to infinity. Therefore $|\tilde{\nu}_n(\alpha)| \to \infty$. This proves that the dual groups $G_k := \widehat{H}_k$ and the dual homomorphisms $\rho_{k,n} = \hat{\nu}_{k,n}$ are admissible for R.

d) Suppose that for all n, $\tilde{\nu}_n(\alpha) - \tilde{l}_n \in \widetilde{\Phi}_k$, where $\tilde{l}_n$ is a bounded sequence in $\widetilde{\Phi} := \Phi/R$. Thus the image of $\tilde{\nu}_n(\alpha)$ in the quotient space $\widetilde{\Phi}/\widetilde{\Phi}_k$ is bounded. Take the image of (4.3.8) in this quotient space. Since ν_*^j is an independent family of scales, we conclude that the images of all the $\tilde{\alpha}^j$ are equal to zero, and thus $\tilde{\alpha}^j \in \widetilde{\Phi}_k$ for all $j \in J$. Therefore, there are $\beta^j \in \Phi_k^\sharp$ and $\gamma^j \in R$ such that $\alpha^j = \beta^j + \gamma^j$. We want to show that β^j and γ^j lie in F^m. Now any $\delta \in \Phi$ can be written as $(\delta_1, \ldots, \delta_N) \in \Phi_1 \times \ldots \times \Phi_N$; define $\delta_{\text{non-}k} = (\delta_1, \ldots, \delta_{k-1}, \delta_{k+1}, \ldots, \delta_N)$. Let $\{v^i\}$ be a basis for R with each $v^i \in F^m$; then, since $\beta^j_{\text{non-}k} = 0$,

$$\alpha^j_{\text{non-}k} = \gamma^j_{\text{non-}k} = \sum_i r_i^j v^i_{\text{non-}k}. \tag{4.3.9}$$

Now the $v^i_{\text{non-}k}$ are linearly independent, since if $\sum_i c_i v^i_{\text{non-}k} = 0$ then $\sum_i c_i v^i$ lies in $\Phi_k^\sharp$ and so, by (4.3.2), must equal zero; the independence of the v^i then implies that all the c_i vanish. This implies that the matrix M having elements $v^i_{\text{non-}k} \cdot v^l_{\text{non-}k}$ is nonsingular. Taking the inner product of (4.3.9) with each $v^l_{\text{non-}k}$ therefore yields an invertible linear system for the the r_i^j. Since the coefficients of the matrix and inhomogeneous term all lie in F, so do the r_i^j. This implies that γ^j, and hence also β^j, lie in F^m. Therefore the $\{\gamma^j\}$ and $\{\beta^j\}$ define $\gamma \in Z$ and $\beta = (\beta_1, \ldots, \beta_N) \in H_1 \times \ldots \times H_N$ with $\beta_l = 0$ for $l \neq k$, such that $\alpha = \beta + \gamma$. In particular, $\tilde{\nu}_n(\alpha) = \tilde{\nu}_n(\beta)$. This proves the property of closedness for resonances.

e) Suppose that $(G_k^0, \rho_{k,*}^0)$ were given. Then we chose the A_k in a) so that they contain the sequences $\nu_{k,*}^0(\alpha) := \hat{\rho}_{k,*}^0(\alpha)$, $\alpha \in \widehat{G}_k^0$. Thus

$$\forall \alpha \in \widehat{G}_k^0\ , \quad \exists \beta \in H_k\ , \quad \exists l \in \Phi_k\ :\quad \nu^0_{k,\ell'(n)}(\alpha) - \nu_{k,n}(\beta) \to l \quad \text{as} \quad n \to +\infty\,. \tag{4.3.10}$$

Since $\nu_{k,*}$ is admissible , this defines a group mapping $\sigma_k : \alpha \to \beta$ from $\widehat{G}_k^0$ to $H_k = \widehat{G}_k$. Since $\nu_{k,*}^0$ is admissible, this mapping is injective. Therefore, the dual homomorphism $\jmath_k := \hat{\sigma}_k \in \mathrm{Hom}(G_k; G_k^0)$ is onto. Moreover, (4.3.10) defines $l = l(\alpha)$ with $l \in \mathrm{Hom}(H_k; \Phi_k)$ and (4.3.4) follows. Therefore, with $H^0 := \widehat{G}_1^0 \times \ldots \times \widehat{G}_N^0$, $\hat{\jmath} := (\hat{\jmath}_1, \ldots, \hat{\jmath}_N)$ and $l := (l_1, \ldots, l_N)$, one has

$$\forall \alpha \in H^0\ , \quad \nu_n(\hat{\jmath}(\alpha)) - \nu^0_{\ell(n)}(\alpha) \to l(\alpha) \quad \text{as} \quad n \to +\infty\,. \tag{4.3.11}$$

Passing to the quotient in Φ/R, this implies that

$$(4.3.12) \qquad \forall \alpha \in H^0 \ , \quad \tilde\nu_n(\hat\jmath(\alpha)) - \tilde\nu^0_{\ell(n)}(\alpha) \to \tilde l(\alpha) \quad \text{as} \quad n \to +\infty\,.$$

When $\alpha \in Z^0$, $\tilde\nu^0_{\ell(n)}(\alpha) = 0$ and $\tilde\nu_n(\hat\jmath(\alpha))$ has a finite limit when $n \to +\infty$. The admissibility condition (4.3.3) requires that $\hat\jmath(\alpha) \in Z$. Conversely, if $\hat\jmath(\alpha) \in Z$, then $\tilde\nu_n(\hat\jmath(\alpha)) = 0$ and $\tilde\nu^0_{\ell(n)}(\alpha)$ has a finite limit. The admissibility condition (4.3.3) implies that $\alpha \in Z^0$. □

Remark 4.3.8. If a uniformly bounded sequence u_* lies in $\mathcal{L}^2_{os,\varphi}(H)$ then so does $f(u_*)$ for any continuous f. However, if a uniformly bounded sequence u_* lies in $\mathcal{L}^2_{os,\varphi}(H)+\mathcal{L}^2_{no,\varphi}$ then functions of u_* do not necessarily lie in $\mathcal{L}^2_{os,\varphi}(H) + \mathcal{L}^2_{no,\varphi}$. The reason is that $\mathcal{L}^\infty \cap \mathcal{L}^2_{no,\varphi}$ is not closed under multiplication. For example, if $\varphi = x$ then $e^{inx+in^2x^2}$ and $e^{-in^2x^2}$ both lie in $\mathcal{L}^2_{no,\varphi}$, but their product e^{inx} does oscillate with the phase φ. This is why the construction of the set A in part a) of the proof needed to explicitly include the oscillations of powers.

5. Resonant trilinear interaction.

In this section we first describe the group R of 3-resonances of a finite set of vector fields, and then use Theorem 2.1 and the group structure developed in the previous section to prove Theorem 1.6, which provides a formula for the weak limit of the product of three bounded sequences.

5.1. The equations and resonances.

Consider the semilinear system

$$(5.1.1) \qquad X_k u_k := (\partial_t - c_k(t,x)\,\partial_x)\,u_k \;=\; F_k(t,x,u_1,\cdots,u_N)\,, \quad \text{for} \quad k \in \{1,\cdots,N\}\,.$$

We assume that the speeds c_k are C^∞ and $c_1 < c_2 < \cdots < c_N$. The nonlinearities are of the form

$$(5.1.2) \qquad F_j(y,u) \;=\; \sum_{k,l} F_{j,k,l}(y,u_j,u_k,u_l)\,,$$

where the $F_{j,k,l}$ are C^∞ functions on $\mathbb{R}^2 \times \mathbb{R}^3$. For example, (5.1.2) is satisfied either when $N \le 3$ or when the nonlinearities F_k are quadratic. There is no restriction in assuming that the sum runs over indices (k,l) such that $j \ne k \ne l \ne j$.

Consider an open initial interval $\omega \subset \mathbb{R}$ and a domain

$$(5.1.3) \qquad \Omega \;:=\; \{\ (t,x) \in \mathbb{R}^2\ ;\ 0<t<T \text{ and } \gamma_1(t) < x < \gamma_N(t)\ \}\,,$$

where γ_1 [resp. γ_N] is the integral curve of X_1 [resp. X_N] starting at the left [resp. right] end of ω. T is small enough so that $\gamma_1(t) < \gamma_N(t)$ for $0 \le t \le T$. In particular, Ω is contained in the domain of determinacy of ω.

For any triplet (X_j, X_k, X_l), the existence of non trivial resonances is characterized locally, by the vanishing of a geometric invariant associated to the 3-web generated by these fields, i.e. the set of the three foliations by integral curves. If there exists a non trivial resonance on Ω', then the curvature vanishes on Ω'. Conversely, if the curvature vanishes on a neighborhood of a point, then there exists a nontrivial resonance on a possibly smaller neighborhood ([P], [BB]). Since Ω is contractible and the dimension of the space of resonances on any connected Ω' is not greater than 1, the vanishing of the curvature on Ω is equivalent to the existence of a nontrivial resonance on Ω. For a detailed discussion, we refer the reader to [JMR 1] [JMR 3].

ASSUMPTION 5.1.1. *For all triplets of distinct integers $(j,k,l) \in \{1, \cdots, N\}^3$, either the curvature of the 3-web generated by (X_j, X_k, X_l) vanishes everywhere on Ω, or does not vanish almost everywhere on Ω.*

Let $\mathcal{R}$ denote the set of triplets $(j,k,l) \in \{1, \cdots, N\}^3$, $j \neq k \neq l \neq j$, such that the curvature of the 3-web generated by (X_j, X_k, X_l) vanishes identically on Ω. For $(j,k,l) \in \mathcal{R}$, choose resonant phases $\psi_j^{k,l} \in C^\infty(\Omega; \mathbb{R})$ such that $\psi_j^{k,l} = \psi_j^{l,k}$ and

$$(5.1.4) \qquad X_j\, \psi_j^{k,l} \;=\; 0 \quad \text{and} \quad \psi_j^{k,l} + \psi_k^{l,j} + \psi_l^{j,k} \;=\; 0\,, \quad \text{on } \Omega\,.$$

For $j \in \{1, \ldots, N\}$, consider a finite dimensional space $\Sigma_j \subset C^\infty(\Omega; \mathbb{R})$ of solutions of $X_j \psi = 0$, which contains all the resonant phases $\psi_j^{k,l}$, $(j,k,l) \in \mathcal{R}$. Assume that for all $\psi \in \Sigma_j \backslash \{0\}$, $d\psi \neq 0$ almost everywhere on Ω. We fix a basis $(\varphi_{j,1}, \ldots, \varphi_{j,m_j})$ of Σ_j. This defines a function $\varphi_j \in C^\infty(\Omega; \Theta_j)$, $\Theta_j := \mathbb{R}^{m_j}$, which satisfies

$$(5.1.5) \qquad X_j\, \varphi_j \;=\; 0\,.$$

Denote by Φ_j the dual space of Θ_j. Then, the mapping $\xi \to \xi \cdot \varphi_j$ is an isomorphism from Φ_j to Σ_j, and therefore :

$$(5.1.6) \qquad \forall \xi \in \Phi_j \backslash \{0\} \;:\quad d(\xi \cdot \varphi_j) \neq 0 \quad \text{a.e. on } \Omega\,,$$

$$(5.1.7) \qquad \forall (j,k,l) \in \mathcal{R}\,, \quad \exists\, \eta_j^{k,l} \in \Phi_j \backslash \{0\} \;:\quad \psi_j^{k,l} \;=\; \eta_j^{k,l} \cdot \varphi_j\,.$$

Note that $\eta_j^{k,l} = \eta_j^{l,k}$.

Introduce $\Phi := \Phi_1 \times \ldots \times \Phi_N$. For $(j,k,l) \in \mathcal{R}$, let $R_{j,k,l}$ denote the one dimensional subspace of Φ generated by $\xi = (\xi_1, \ldots, \xi_N)$ such that $\xi_p = 0$ when $p \notin \{j,k,l\}$ and $\xi_p = \eta_p^{q,r}$ when $p \in \{j,k,l\}$, and $\{p,q,r\} = \{j,k,l\}$. The space generated by the sum of the $R_{j,k,l}$, is denoted by R. This is the space of resonances generated by 3-resonances. Note too that (5.1.4) and (5.1.7) imply that

$$(5.1.8) \qquad \forall \xi = (\xi_1, \ldots, \xi_N) \in R \;:\quad \xi \cdot \varphi \;:=\; \xi_1 \cdot \varphi_1 + \;\ldots\; + \xi_N \cdot \varphi_N \;=\; 0\,.$$

Introduce $\Phi_j^\sharp := \{0\} \times \ldots \times \{0\} \times \Phi_j \times \{0\} \ldots \times \{0\}$, with Φ_j in the j-th position. Then (5.1.6) and (5.1.8) imply that

$$\Phi_j^\sharp \cap R \;=\; \{0\}\,. \tag{5.1.9}$$

Hence R satisfies Definition 4.3.2. Similarly, introduce $\Phi_{j,k,l}^\sharp$ the space generated by $\Phi_j^\sharp$, $\Phi_k^\sharp$ and $\Phi_l^\sharp$. Since the dimension of the space of resonances for three vector fields is at most one, one has

$$\Phi_{j,k,l}^\sharp \cap R \;=\; \begin{cases} R_{j,k,l} & \text{when} \quad (j,k,l) \in \mathcal{R}\,, \\ \{0\} & \text{when} \quad (j,k,l) \notin \mathcal{R}\,. \end{cases} \tag{5.1.10}$$

Examples 5.1.2. **a)** When there are no resonances, one can choose $m_1 = \ldots = m_N = 0$. This is the situation studied in [JMR 3]. One can also choose arbitrary spaces Θ_j and $\varphi_j \in C^\infty(\Omega, \Theta_j)$ satisfying (5.1.5) (5.1.6). This corresponds to the situation of nonlinear geometric optics (see [JMR 1]).

b) Suppose that the vector fields $X_k = \partial_t - c_k \partial_x$ have constant coefficients. Then, for all (j,k,l) with $j \neq k \neq l \neq j$ there is a resonance , with

$$\psi_j^{k,l}(t,x) \;:=\; \frac{x + c_j t}{(c_j - c_k)(c_j - c_l)} \;:=\; \eta_j^{k,l}\;(x + c_j t)\,. \tag{5.1.11}$$

One can choose $\Theta_j := \mathbb{R}$ and $\varphi_j := x + c_j t$. Then $\dim \Phi = N$ and $\dim R = N - 2$. Φ/R is isomorphic to the space of linear phases generated by the coordinate functions t and x. Note too that in general the $\eta_j^{k,l}$ for fixed j are not all rational multiples of a fixed real number, so that it is not possible to redefine the phase φ_j to make them all rational. This shows that in general it is not possible to take the subfield F of $\mathbb{R}$ from §4.3 to be the rationals.

Larger spaces Θ_j allow us to incorporate the analysis of the resonant nonlinear geometric optics.

c) When the vector fields have variable coefficients, the analysis of resonances can force us to consider spaces Φ of dimension greater than 1. For example, consider

$$X_1 := \partial_t\,,\; X_2 := \partial_t - \partial_x\,,\; X_3 := \partial_t + \partial_x\,,\; X_4 := \partial_t - \frac{b'(x+t)}{a'(x)}(\partial_t - \partial_x)\,. \tag{5.1.12}$$

where a and b are smooth functions on $\mathbb{R}$ and $a' \neq 0$. In this case, $(1,2,3) \in \mathcal{R}$ and $(1,2,4) \in \mathcal{R}$. Resonant phases are $\psi_1^{2,3} = -2x$, $\psi_2^{3,1} = x+t$, $\psi_3^{1,2} = x-t$ and $\psi_1^{2,4} = -a(x)$, $\psi_2^{4,1} = b(x+t)$, $\psi_4^{1,2} = a(x) - b(x+t)$. Since Σ_1 must contain $\psi_1^{2,3}$ and $\psi_1^{2,4}$, one has necessarily $m_1 \geq 2$ when a is not a linear function. In this case, (5.1.6) is equivalent to saying that the level sets $\{a'(x) = \lambda\}$ have Lebesgue measure equal to zero. This is a very mild assumption, but it is not automatically satisfied.

5.2. Resonant interaction of oscillations.

We now turn to proving a generalization of Theorem 1.6. Although we will need to consider arbitrary triples $\{j,k,l\} \in \{1,\ldots,N\}^3$, for notational simplicity we first consider the case when $N=3$ and a resonance exists. In this subsection we will denote the resonant phases by $\psi_k = \eta_k \cdot \varphi_k$, so that $R = \mathbb{R}\,(\eta_1,\eta_2,\eta_3)$. Introduce

$$\Psi \;:=\; R^{\perp} \;\subset\; \Theta_1 \times \Theta_2 \times \Theta_3\,. \tag{5.2.1}$$

The resonance relation $\psi_1+\psi_2+\psi_3=0$ implies that $\varphi := (\varphi_1,\varphi_2,\varphi_3)$ is valued in Ψ.

For $k \in \{1,2,3\}$, consider a compact abelian group G_k and a sequence $\rho_{k,n}$ in $\mathrm{Hom}(\Theta_k; G)$, such that $(G_k,\rho_{k,*})$ is admissible. Introduce the dual groups $\widehat{G}_k$ and the dual homomorphisms $\nu_{k,n} \in \mathrm{Hom}(\widehat{G}_k;\Phi_k)$. For $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \widehat{G}_1 \times \widehat{G}_2 \times \widehat{G}_3$, introduce $\nu_n(\alpha) := (\nu_{1,n}(\alpha_1),\nu_{2,n}(\alpha_2),\nu_{3,n}(\alpha_3)) \in \Phi$ and $\tilde{\nu}_n(\alpha)$ the class of $\nu_n(\alpha)$ in $\Phi_R := \Phi/R$.

Suppose that there is a subgroup $Z \subset \widehat{G}_1 \times \widehat{G}_2 \times \widehat{G}_3$ such that

$$\forall \alpha \in Z\ ,\ \forall n \in \mathbb{N}\ :\quad \nu_n(\alpha) \in R \tag{5.2.2}$$

$$\forall \alpha \notin Z\ :\quad |\tilde{\nu}_n(\alpha)| \to +\infty \quad \text{in } \Phi_R\ , \quad \text{as } n \to +\infty\,. \tag{5.2.3}$$

Introduce $G \subset G_1 \times G_2 \times G_3$ the group of characters on $\widehat{G}_1 \times \widehat{G}_2 \times \widehat{G}_3$ which are trivial on Z. Denote by $\pi_k \in \mathrm{Hom}(G;G_k)$ the restriction to G of the projection on the k-th factor.

For $k \in \{1,2,3\}$, introduce $H_k := \{\nu_{k,*}(\alpha_k) \ :\ \alpha_k \in \widehat{G}_k\} \subset (\Phi_k)^{\mathbb{N}}$, the set of frequencies (2.3.4) associated to $(G_k,\rho_{k,*})$. Consider bounded sequences $u_{k,*}$ in $W_k(\omega)$ such that,

$$u_{k,*} \in \mathcal{L}^2_{os,\varphi_k}(H_k) \;+\; \mathcal{L}^2_{no,\varphi_k}\,. \tag{5.2.4}$$

Consider $\tilde{u}_{k,*} \in \mathcal{L}^2_{os,\varphi_k}(H_k)$ and $r_{k,*} \in \mathcal{L}^2_{no,\varphi_k}$, such that $u_{k,n} = \tilde{u}_{k,n} + r_{k,n}$. Theorem 4.1.4 implies that there are $\mathcal{U}_k \in L^2(\Omega \times G_k)$ such that

$$\tilde{u}_{k,n}(y) \;\sim\; \mathcal{U}_k(y,\rho_{k,n}(\varphi_k(y)))\,. \tag{5.2.5}$$

Proposition 3.1.6 implies that $\tilde{u}_{k,*}$ and $r_{k,*}$ are uniquely determined, up to sequences which converge strongly to zero in $L^2(\Omega)$. Therefore, $\mathcal{U}_k$ is uniquely determined by the sequence $u_{k,*}$. Our goal is to prove the following result :

THEOREM 5.2.1. *Under the assumptions above, one has :*

i) the sequence $u_{1,n}\, u_{2,n}\, u_{3,n}(y)$ is bounded in $L^1_{loc}(\Omega)$,

ii) $\mathcal{U}_1(y,\pi_1(g))\,\mathcal{U}_2(y,\pi_2(g))\,\mathcal{U}_3(y,\pi_3(g))$ is locally integrable on $\Omega \times G$,

iii) for all $a \in C_0^\infty(\Omega)$,

$$\int_\Omega a(y)\; u_{1,n}(y)\; u_{2,n}(y)\; u_{3,n}(y)\; dy \quad \to \quad \int_{\Omega\times G} a(y)\; \mathcal{U}_1(y,\pi_1(g))\; \mathcal{U}_2(y,\pi_2(g))\; \mathcal{U}_3(y,\pi_3(g))\; dy\, dg\,, \tag{5.2.6}$$

as $n \to +\infty$.

EXAMPLE 5.2.2. Consider the case where $m_1 = m_2 = m_3 = 1$ and $\varphi_k = \psi_k$. Then $\Phi = \mathbb{R}^3$ and R is generated by $(1,1,1)$. Consider a compact group G_0 and $\rho_{0,n} \in \mathrm{Hom}(\mathbb{R}; G_0)$. Suppose that $(G_0, \rho_{0,*})$ is admissible. Consider three copies G_k of G_0 and define $\rho_{k,n} := \rho_{0,n}$. Thus $\nu_{k,n} = \nu_{0,n} \in \mathrm{Hom}(\widehat{G}_0; \mathbb{R})$. Introduce Z the group of the $\alpha := (\alpha_1, \alpha_2, \alpha_3) \in (\widehat{G}_0)^3$ such that $\alpha_1 = \alpha_2 = \alpha_3$. Then (5.2.2) and (5.2.3) are satisfied.

In this case

$$G \;=\; \{\ (g_1, g_2, g_3) \in (G_0)^3 \ :\ g_1 + g_2 + g_3 = 0\ \}\,. \tag{5.2.7}$$

It can be parametrized by (g_1, g_2), and the limit in (5.2.6) is equal to

$$\int_{\Omega \times G_0 \times G_0} a(y)\, \mathcal{U}_1(y, g_1)\, \mathcal{U}_2(y, g_2)\, \mathcal{U}_3(y, -g_1 - g_2)\; dy\, dg_1\, dg_2\,. \tag{5.2.8}$$

The first part of Theorem 5.2.1 is proved in Theorem 2.1. The first step in the proof of the other two parts is to show that the profiles $\mathcal{U}_k$ inherit smoothness from the $u_{k,*}$.

PROPOSITION 5.2.3. *i) For* $k \in \{1,2,3\}$, $X_k(y, \partial_y)\mathcal{U}_k \in L^2(\Omega \times G_k)$.

ii) $\mathcal{U}_1(y, \pi_1(g))\, \mathcal{U}_2(y, \pi_2(g))\, \mathcal{U}_3(y, \pi_3(g)) \in L^1_{loc}(\Omega \times G)$.

Proof. **a)** Theorem 4.1.4 implies that for all $\mathcal{A} \in C^0_0(\Omega \times G_k)$

$$\int_\Omega u_{k,n}(y)\; \mathcal{A}(y, \rho_n(\varphi_k(y)))\; dy \quad \to \quad \int_{\Omega \times G_k} \mathcal{U}_k(y, g)\; \mathcal{A}(y, g)\; dy\, dg\,. \tag{5.2.9}$$

Consider trigonometric polynomials

$$\mathcal{A}(y, g) \;=\; \sum_\alpha a_\alpha(y)\; e_\alpha(g) \tag{5.2.10}$$

where the sum is carried over a finite subset of $\widehat{G}_k$ and $a_\alpha \in C^\infty_0(\Omega)$. Then, denoting by $X^*_k(y, \partial_y)$ the adjoint operator of $X_k(y, \partial_y)$, one has

$$X^*_k(y, \partial_y)\, \mathcal{A}(y, g) \;=\; \sum_\alpha \big(X^*_k(y, \partial_y)\, a_\alpha(y)\big)\; e_\alpha(g)\,. \tag{5.2.11}$$

Introduce

$$a_{k,n}(y) \;:=\; \mathcal{A}(y, \rho_n(\varphi_k(y))) \;=\; \sum_\alpha a_\alpha(y)\; e^{i\,\nu_n(\alpha)\cdot\varphi_k(y)}\,, \tag{5.2.12}$$

where $\nu_n \in \mathrm{Hom}(\widehat{G}_k; \mathbb{R})$ is the dual homomorphism of ρ_n. Since $X_k \varphi_k = 0$, remark that

$$X^*_k(y, \partial_y)\, a_{k,n}(y) \;=\; \big(X^*_k(y, \partial_y)\, \mathcal{A}\big)\, (y, \rho_n(\varphi_k(y)))\,. \tag{5.2.13}$$

Thus, Theorem 4.1.4 implies that for all trigonometric polynomial $\mathcal{A}$, one has

$$\int_\Omega (X_k\, u_{k,n})(y)\ a_{k,n}(y)\ dy\ \to\ \int_{\Omega\times G_k} \mathcal{U}_k\ (X_k^*\,\mathcal{A})\ dy\, dg\,. \tag{5.2.14}$$

By Theorem 4.1.4, the L^2 norm of $a_{k,n}$ is asymptotically equal to the L^2 norm of $\mathcal{A}$. Therefore, the density of trigonometric polynomials in $L^2(\Omega\times G)$, and the boundedness of $u_{k,*}$ in $W_k(\Omega)$ imply that there is $\mathcal{F}_k \in L^2(\Omega\times G_k)$ such that

$$\int_{\Omega\times G_k} \mathcal{U}_k\ (X_k^*\,\mathcal{A})\ dy\, dg\ =\ \int_{\Omega\times G_k} \mathcal{F}_k\ \mathcal{A}\ dy\, dg\,. \tag{5.2.15}$$

This means that $X_k(y,\partial y)\mathcal{U}_k(y,g) = \mathcal{F}_k(y,g) \in L^2(\Omega\times G_k)$.

b) Expand $\mathcal{U}_k$ and $\mathcal{F}_k$ into Fourier series,

$$\mathcal{U}_k(y,g) \sim \sum_\alpha U_{k,\alpha}(y)\ e_\alpha(g)\ ,\quad \mathcal{F}_k(y,g) \sim \sum_\alpha F_{k,\alpha}(y)\ e_\alpha(g)\,. \tag{5.2.16}$$

Then, (5.2.15) applied to monomials $a(y)e_\alpha(g)$, implies that $X_k U_{k,\alpha} = F_{k,\alpha}$. This proves that $U_{k,\alpha} \in W_k(\Omega)$. Moreover, Plancherel's formula implies that

$$\sum_\alpha \|\, U_{k,\alpha}\,\|^2_{W_k(\Omega)}\ =\ \|\,\mathcal{U}_k\,\|^2_{L^2(\Omega\times G_k)}\ +\|\,\mathcal{F}_k\,\|^2_{L^2(\Omega\times G_k)}\,. \tag{5.2.17}$$

Introduce the space $W_k(\Omega\times G_k)$ of functions $\mathcal{U}_k \in L^2(\Omega\times G_k)$ such that $X_k\mathcal{U}_k \in L^2(\Omega\times G_k)$, equipped with the norm

$$\|\,\mathcal{U}_k\,\|_{W_k(\Omega\times G_k)}\ :=\ \|\,\mathcal{U}_k\,\|_{L^2(\Omega\times G_k)}\ +\ \|\,X_k\,\mathcal{U}_k\,\|_{L^2(\Omega\times G_k)}\,. \tag{5.2.18}$$

Then, (5.2.17) implies that the finite sums

$$\mathcal{V}_k(y,g)\ =\ \sum_\alpha U_{k,\alpha}(y)\ e_\alpha(g)\,, \tag{5.2.19}$$

with $U_\alpha \in W_k(\Omega)$, are dense in $W_k(\Omega\times G_k)$.

c) For $k\in\{1,2,3\}$ consider finite sums of the form (5.2.19). Introduce

$$\mathcal{W}(y,g)\ :=\ \mathcal{V}_1(y,\pi_1(g))\ \mathcal{V}_2(y,\pi_2(g))\ \mathcal{V}_3(y,\pi_3(g))\,. \tag{5.2.20}$$

The first part of Theorem 2.1 implies that for all relatively compact subset ω contained in Ω, there is a constant C such that

$$\|\,\mathcal{W}(\,.\,,g)\,\|_{L^1(\omega)}\ \le\ C\ A_1(\pi_1(g))\ A_2(\pi_2(g))\ A_3(\pi_3(g))\,, \tag{5.2.21}$$

with

$$A_k(g_k)\ :=\ \|\,\mathcal{V}_k(\,.\,,g_k)\,\|_{W_k(\Omega)}\,. \tag{5.2.22}$$

We now use the following result : if G and G' are compact Abelian groups, each equipped with its normalized Haar measure, and if π is a continuous homomorphism from G onto G', then for all integrable function f on G', $f \circ \pi$ is integrable on G and

$$\int_{G'} f(g')\, dg' \;=\; \int_G f(\pi(g))\, dg\,. \tag{5.2.23}$$

Since $\eta_k \neq 0$ and $(G_k, \rho_{k,*})$ is admissible, $(\widehat{G}_1 \times \widehat{G}_2 \times \{0\}) \cap Z = \{0\}$. This implies that (π_1, π_2) maps G onto $G_1 \times G_2$. Thus, $G_1 \times G_2$ is a quotient group of G and for all functions F on $G_1 \times G_2$:

$$\int_G F(\pi_1(g), \pi_2(g))\; dg \;=\; \int_{G_1 \times G_2} F(g_1, g_2)\; dg_1\, dg_2\,. \tag{5.2.24}$$

Similarly, π_3 is surjective from G to G_3 and

$$\int_G F(\pi_3(g))\; dg \;=\; \int_{G_3} F(g_3)\; dg_3\,. \tag{5.2.25}$$

Therefore, (5.2.21) implies that

$$\|\, \mathcal{W}\, \|_{L^1(\omega \times G)} \;\leq\; C\, \|\, A_1\, \|_{L^2(G_1)}\, \|\, A_2\, \|_{L^2(G_2)}\, \|\, A_3\, \|_{L^2(G_3)}\,. \tag{5.2.26}$$

Thus

$$\|\, \mathcal{W}\, \|_{L^1(\omega \times G)} \;\leq\; C\, \|\, \mathcal{V}_1\, \|_{W_1(\Omega \times G_1)}\, \|\, \mathcal{V}_2\, \|_{W_2(\Omega \times G_2)}\, \|\, \mathcal{V}_3\, \|_{W_3(\Omega \times G_3)}\,. \tag{5.2.27}$$

This estimate implies that the mapping $(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3) \;\to\; \mathcal{W}$ extends continuously from $W_1(\Omega \times G_1) \times W_2(\Omega \times G_2) \times W_3(\Omega \times G_3)$ to $L^1_{loc}(\Omega \times G)$. □

Proposition 5.2.4. *Suppose that* $\mathcal{U}_k \in L^2(\Omega \times G_k)$ *is such that* $X_k \mathcal{U}_k \in L^2(\Omega \times G_k)$. *Then, there exists* $v_{k,*} \in \mathcal{L}^2_{os,\varphi_k}(H)$ *such that*

i) $v_{k,n}(y) \;\sim\; \mathcal{U}_k(y, \rho_n(\varphi_k(y)))$ *in* $L^2(\Omega)$,

ii) $v_{k,*}$ *is bounded in* $W_k(\Omega)$.

Proof. Approximate $\mathcal{U}_k$ by partial sums of its Fourier series. Then (5.2.18) implies that for all N, there is a finite sum $\mathcal{V}_k^N$ of the form (5.2.19), such that

$$\|\, \mathcal{U}_k \;-\; \mathcal{V}_k^N\, \|_{W_k(\Omega \times G_k)} \;\leq\; 2^{-N}\,. \tag{5.2.28}$$

Introduce

$$v^N_{k,n}(y) \;:=\; \mathcal{V}_k^N(y, \rho_n(\psi_k(y)))\,. \tag{5.2.29}$$

Then, (5.2.13) and Theorem 4.1.4 imply that

$$\lim_{n\to+\infty} \| v_{k,n}^{N+1} - v_{k,n}^{N} \|_{W_k(\Omega)} = \| \mathcal{V}_k^{N+1} - \mathcal{V}_k^{N} \|_{W_k(\Omega\times G)} \le 2^{-N} . \tag{5.2.30}$$

Thus, there is $n(N)$ such that for $n \ge n(N)$

$$\| v_{k,n}^{N+1} - v_{k,n}^{N} \|_{W_k(\Omega)} \le 2^{1-N} . \tag{5.2.31}$$

One can assume that $n(N)$ increases with N and tends to $+\infty$ as $N \to +\infty$. Introduce

$$v_{k,n} := v_{k,n}^{0} + \sum_{\{N\,;\,n(N)\le n\}} (v_{k,n}^{N+1} - v_{k,n}^{N}) = v_{k,n}^{N(n)+1} , \tag{5.2.32}$$

where $N(n)$ is the largest integer N such that $n(N) \le n$). Then (5.2.31) implies that $v_{k,*}$ is bounded in $W_k(\Omega)$. Moreover, for all N, (5.2.31) implies that for $n \ge n(N)$, one has

$$\| v_{k,n} - v_{k,n}^{N} \|_{L^2(\Omega)} \le \| v_{k,n} - v_{k,n}^{N} \|_{W_k(\Omega)} \le 2^{2-N} . \tag{5.2.33}$$

With (5.2.28) and (5.2.29), this implies that $v_{k,n}(y) \sim \mathcal{U}_k(y, \rho_n(\varphi_k(y)))$ in $L^2(\Omega)$.

LEMMA 5.2.5. *For all* $\xi = (\xi_1, \xi_2, \xi_3) \in \Phi\backslash R$ *:*

$$\xi \cdot d\varphi = \xi_1 \cdot d\varphi_1 + \xi_2 \cdot d\varphi_2 + \xi_3 \cdot d\varphi_3 \ne 0 \quad \text{a.e. on } \Omega . \tag{5.2.34}$$

Proof. The statement is local. Therefore, one can choose local coordinates such that $\psi_1 = t$, $\psi_2 = x$ and $\psi_3 = t + x$. Thus, $\xi_1 \cdot \varphi_1 = a(t)$, $\xi_2 \cdot \varphi_2 = b(x)$ and $\xi_3 \cdot \varphi_3 = c(t + x)$. Then, $\xi \cdot d\varphi$ vanishes on the closed set K of points (t, x) such that

$$a'(t) = b'(x) = - c'(t + x) . \tag{5.2.34}$$

If all ξ_k are parallel to η_k, then a', b' and c' are constant, and $K = \emptyset$ if $\xi \notin R$. Suppose that one of the ξ_k is not parallel to η_k, say ξ_1. Then (5.1.6) implies that $d(a(t) + \gamma t) \ne 0$ a.e. which implies that for all γ, the set of t such that $a'(t) = \gamma$ is negligible. Therefore, for all x, the set of t such that $(t, x) \in K$ is negligible, and the Lebesgue measure of K vanishes.

COROLLARY 5.2.6. *For* $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \widehat{G}_1 \times \widehat{G}_2 \times \widehat{G}_3$, *introduce the phase* $\nu_n(\alpha) \cdot \varphi(y) := \nu_{1,n}(\alpha_1) \cdot \varphi_1(y) + \nu_{2,n}(\alpha_2) \cdot \varphi_2(y) + \nu_{3,n}(\alpha_3) \cdot \varphi_3(y)$. *Then,*

1) $e^{i\,\nu_n(\alpha)\cdot\varphi(y)} = 1$ *when* $\alpha \in Z$,

2) $e^{i\,\nu_n(\alpha)\cdot\varphi(y)}$ *converges weakly to 0, when* $\alpha \notin Z$.

Proof. Since $\varphi(y) \in R^\perp$, one has $\nu_n(\alpha) \cdot \varphi(y) = \tilde\nu_n(\alpha) \odot \varphi(y)$, where $\tilde\nu_n(\alpha)$ is the class of $\nu_n(\alpha)$ in $\Phi_R := \Phi/R$ and $\odot$ denotes the duality between Φ_R and $R^\perp$. Lemma 5.2.5 implies that for all $\tilde\xi \in \Phi_R\backslash\{0\}$, one has $\tilde\xi \odot d\varphi(y) \ne 0$ almost everywhere. Then, (5.2.3) and Lemma 3.1.3 imply that $e^{i\,\nu_n(\alpha)\cdot\varphi(y)}$ converges weakly to 0, when $\alpha \notin Z$. Moreover, (5.2.2) shows that $\tilde\nu_n(\alpha) \odot \varphi(y) = 0$ when $\alpha \in Z$.

Proof of Theorem 5.2.1.

It remains to prove (5.2.6). Propositions 5.2.3 and 5.2.4 show that there exist bounded sequences in $W_k(\Omega)$, $v_{k,n}$, such that $v_{k,n}(y) \sim \mathcal{U}_k(y, \rho_n(\varphi_k(y)))$ in $L^2(\Omega)$. Then, (5.2.4) and Proposition 3.1.6 imply that $r_{k,n} := u_{k,n} - v_{k,n}$ belongs to $\mathcal{L}^2_{no,\varphi_k} \subset \mathcal{L}^2_{no,\psi_k}$. Moreover, $r_{k,n}$ is bounded in $W_k(\Omega)$, since $u_{k,n}$ and $v_{k,n}$ are bounded in this space. Therefore, Theorem 2.1 reduces the proof of (5.2.6) to the case where $u_{k,n} = v_{k,n} \sim \mathcal{U}_k(y, \rho_n(\varphi_k(y)))$. Moreover, Propositions 5.2.3 and 5.2.4 imply that it is sufficient to prove (5.2.6) when the Fourier series of $\mathcal{U}_k$ have only finitely many nonvanishing terms. By linearity, we can reduce further the proof to the case where $\mathcal{U}_k(y, g_k)$ is a monomial, $U_k(y)\ e_{\alpha_k}(g_k)$, with $\alpha_k \in \widehat{G}_k$.

In this case,

$$(5.2.35)\qquad u_{1,n}(y)\, u_{2,n}(y)\, u_{3,n}(y) \;=\; U_1(y)\, U_2(y)\, U_3(y)\ e^{i\ \nu_n(\alpha)\cdot\varphi(y)}\,.$$

Remark that $U_1 U_2 U_3 \in L^1_{loc}(\Omega)$ since $U_k \in W_k(\Omega)$. Thus Corollary 5.2.6 implies that

$$(5.2.36)\qquad u_{1,n}\, u_{2,n}\, u_{3,n} \quad \rightharpoonup \quad \begin{cases} 0, & \text{when} \quad \alpha \notin Z\,, \\ U_1\, U_2\, U_3 & \text{when} \quad \alpha \in Z\,. \end{cases}$$

On the other hand, denoting by e_α the character on $G_1 \times G_2 \times G_3$ associated to $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \widehat{G}_1 \times \widehat{G}_2 \times \widehat{G}_3$, one has

$$(5.2.37)\qquad e_{\alpha_1}(\pi_1(g))\ e_{\alpha_2}(\pi_2(g))\ e_{\alpha_3}(\pi_3(g)) \;=\; e_\alpha(g)\,.$$

By definition of G, this vanishes when $\alpha \in Z$ and $g \in G$. On the other hand, when $\alpha \notin Z$ then e_α is a non trivial character on G, and thus the integral of $e_\alpha(g)$ over G vanishes. Hence,

$$(5.2.38)\qquad \int_G e_{\alpha_1}(\pi_1(g))\ e_{\alpha_2}(\pi_2(g))\ e_{\alpha_3}(\pi_3(g))\ dg \;=\; \begin{cases} 0, & \text{when} \quad \alpha \notin Z\,, \\ U_1\, U_2\, U_3 & \text{when} \quad \alpha \in Z\,. \end{cases}$$

With (5.2.36), this proves that (5.2.6) is satisfied when $u_{k,n}(y) = \mathcal{U}_k(y, \rho_n(\varphi_k(y)))$ and $\mathcal{U}_k(y, g_k)$ is a monomial $U_k(y)\, e_{\alpha_k}(g_k)$ with $U_k \in W_k(\Omega)$. □

5.3. Resonant interaction of triples.

We now return to the general case of N vector-valued phases. For $k \in \{1, \ldots, N\}$, consider a bounded sequence $u_{k,*}$ in $L^\infty(\Omega; \mathbb{R})$ such that $X_k\, u_{k,*}$ is bounded in $L^\infty(\Omega; \mathbb{R})$. Suppose that $(G_1, \rho_{1,*}), \ldots, (G_N, \rho_{N,*})$ are admissible for the resonances R in the sense of Definition 4.3.5 and $(G_k, \rho_{k,*})$ is complete for $u_{k,*}$ and φ_k in the sense of Definition 4.3.1.

To take into account the resonances, introduce the subgroup $Z \subset \widehat{G}_1 \times \ldots \times \widehat{G}_N$ such that (4.3.3) holds and

$$(5.3.1)\qquad G := \{ g \in G_1 \times \ldots \times G_N\ ;\ \forall \alpha \in Z\,,\ e_\alpha(g) = 0 \}\,.$$

For $g = (g_1, \ldots, g_N) \in G_1 \times \ldots \times G_N$ and $\alpha = (\alpha_1, \ldots, \alpha_N) \in \widehat{G}_1 \times \ldots \times \widehat{G}_N$ we have noted $e_\alpha(g) := e_{\alpha_1}(g_1) \ldots e_{\alpha_N}(g_N)$. Note that the duality theorem implies that Z is exactly the set of $\alpha \in \widehat{G}_1 \times \ldots \times \widehat{G}_N$ such that $e_\alpha(g) = 0$ for all $g \in G$.

Consider a triplet (j, k, l), with $j \neq k \neq l \neq j$. As in §5.1, introduce $\Phi^\sharp_{j,k,l} \subset \Phi_1 \times \ldots \times \Phi_N$, which is isomorphic to $\Phi_j \times \Phi_k \times \Phi_l$. Similarly, consider $\widehat{G}_j \times \widehat{G}_k \times \widehat{G}_l$ as a subgroup of $\widehat{G}_1 \times \ldots \times \widehat{G}_N$, identifying $(\alpha_j, \alpha_k, \alpha_l)$ with α whose other components are equal to zero. Introduce $Z' \subset \widehat{G}_j \times \widehat{G}_k \times \widehat{G}_l$ as the intersection of Z with $\widehat{G}_j \times \widehat{G}_k \times \widehat{G}_l$. Let $G' \subset G_j \times G_k \times G_l$ denote the set of characters on $\widehat{G}_j \times \widehat{G}_k \times \widehat{G}_l$, which are trivial on Z'.

Considering $\widehat{G}_j \times \widehat{G}_k \times \widehat{G}_l$ as a subset of $\widehat{G}_1 \times \ldots \times \widehat{G}_N$ as above, for all $\alpha' \in \widehat{G}_j \times \widehat{G}_k \times \widehat{G}_l$, $\nu_n(\alpha')$ belongs to $\Phi^\sharp_{j,k,l} \approx \Phi_j \times \Phi_k \times \Phi_l$. This defines a sequence of homomorphisms ν'_n, from $\widehat{G}_j \times \widehat{G}_k \times \widehat{G}_l$ into $\Phi_j \times \Phi_k \times \Phi_l$.

Lemma 5.3.1 *i) For $j \neq k$ let $\pi_{j,k}$ denote the mapping $g \to (g_j, g_k)$ from G to $G_j \times G_k$. Then $\pi_{j,k}$ is surjective.*

ii) Let $\pi_{j,k,l}$ denote the mapping $g \to (g_j, g_k, g_l)$ from G into $G_j \times G_k \times G_l$, and π'_r the mapping $g' \to g'_r$ from G' to G_r. Then

$$G' \;=\; \pi_{j,k,l}(G) \quad \text{and} \quad \pi_r = \pi'_r \circ \pi_{j,k,l}\,. \tag{5.3.2}$$

iii) If $R_{j,k,l} \neq \{0\}$, then Z' and ν'_n satisfy the condition (4.2.7) (4.2.8).

iv) If $R_{j,k,l} = \{0\}$, then $Z' = \{0\}$ and $G' = G_j, \times G_k \times G_l$.

Proof. i) If $\pi_{j,k}(G)$ is strictly smaller than $G_j \times G_k$, there is a nontrivial character $(\alpha_j, \alpha_k) \in \widehat{G}_j \times \widehat{G}_k$ such that

$$\forall g \in G\,, \quad e_{\alpha_j}(\pi_j(g)) \;+\; e_{\alpha_k}(\pi_k(g)) \;=\; 0\,. \tag{5.3.3}$$

Let $\alpha \in \widehat{G}_1 \times \ldots \times \widehat{G}_N$ with j-th and k-th components equal to α_j and α_k respectively, and others components equal to zero. Then (5.3.3) means that for all $g \in G$, $e_\alpha(g) = 0$. Therefore, $\alpha \in Z$. Thus $\nu_n(\alpha) \in R$ for all n. On the other hand $\nu_n(\alpha) \in \Phi^\sharp_j \oplus \Phi^\sharp_k$. Since the vector-fields X_j and X_k are linearly independent, the space of resonances $R \cap (\Phi^\sharp_j \oplus \Phi^\sharp_k) = \{0\}$. Therefore, for all n, $\nu_n(\alpha) = 0$ which means that $\nu_{j,n}(\alpha_j) = 0$ and $\nu_{k,n}(\alpha_k) = 0$. Since $\alpha_j \neq 0$ or $\alpha_k \neq 0$, this contradicts the admissibility assumption.

ii) For $\alpha' = (\alpha'_j, \alpha'_k, \alpha'_l) \in \widehat{G}_j \times \widehat{G}_k \times \widehat{G}_l$ introduce $\alpha \in \widehat{G}_1 \times \ldots \widehat{G}_N$ such that $\alpha_r = 0$ when $r \notin \{j, k, l\}$ and $\alpha_r = \alpha'_r$ when $r \in \{j, k, l\}$. Then, for all $g = (g_1, \ldots, g_N) \in G_1 \times \ldots \times G_N$,

$$e_\alpha(g) \;= e_{\alpha'}(g_j, g_k, g_l)\,.$$

By definition, α' belongs to Z' if and only if α belongs to Z. This shows that $\pi_{j,k,l}$ maps G into G'. If it is not onto, there is a character, α' which is trivial on $\pi_{j,k,l}(G)$ and nontrivial on G'. Introduce $\alpha \in \widehat{G}_1 \times \dots \widehat{G}_N$ as above. Then $e_\alpha(g) = 0$ for all $g \in G$ and therefore, $\alpha \in Z$. Thus, $\alpha' \in Z'$, and α' is trivial on G', which contradicts the assumption.

iii) Consider $\alpha' \in \widehat{G}_j \times \widehat{G}_k \times \widehat{G}_l$. If $\alpha' \in Z' \subset Z$, then $\nu_n(\alpha') \in R \cap \Phi^\sharp_{j,k,l}$. Thus (5.1.10) implies that $\nu'(\alpha')$ belongs to the space R' generated by the unique resonance in $\Phi' := \Phi_j \times \Phi_k \times \Phi_l$. If $\alpha \notin Z'$, then $\alpha \notin Z$ and the admissibility assumption (4.3.3) implies that $|\nu_n(\alpha')| \to \infty$ in Φ/R. (5.1.10) implies that the image of $\Phi^\sharp_{j,k,l}$ in Φ/R is isomorphic to the quotient Φ'/R'. Therefore, $|\nu'_n(\alpha')| \to \infty$ in Φ'/R'.

iv) If $\alpha' \in Z'$, then $\alpha \in Z$ and the admissibility condition implies that $\nu_n(\alpha)$ is valued in $R_{j,k,l}$. When this space is trivial, this implies that $\nu_n(\alpha)$ vanishes identically. Since each $(G_r, \rho_{r,*})$ is admissible, this is possible only when $\alpha = 0$. Thus $Z' = \{0\}$ and $G' = G_j \times G_k \times G_l$. □

We can now extend Theorem 5.2.1 to the case of triples $\{j,k,l\}$ of distinct sequences. Note that this triple does not necessarily have a resonance and that the group G is now given by (5.3.1) and so in particular is independent of the choice of the triple. We also extend the theorem by using oscillatory rather than just fixed test functions. Let the $u_{k,*}$ satisfy the same assumptions as for Theorem 5.2.1; in particular, $u_{k,n} = \tilde u_{k,n} + r_{k,n}$ with $\tilde u_{k,n} \in \mathcal{L}^2_{os,\varphi_k}(H_k)$ and $r_{k,n} \in \mathcal{L}^2_{no,\varphi_k}$, and $\tilde u_{k,n}(y) \sim \mathcal{U}_k(y, \rho_{k,n}(\varphi_k(y)))$.

Theorem 5.3.2. *For all $\mathcal{A} \in C^0(\overline{\Omega} \times G_j)$ and all triples $\{j,k,l\}$ of distinct numbers,*

(5.3.4)
$$\int_\Omega \mathcal{A}(y, \rho_{j,n}(\varphi_j(y)))\, u_{j,n}(y)\; u_{k,n}(y)\; u_{l,n}(y)\; dy \quad \to$$
$$\int_{\Omega\times G} \mathcal{A}(y, \pi_j(g))\, \mathcal{U}_j(y, \pi_j(g))\, \mathcal{U}_k(y, \pi_k(g))\, \mathcal{U}_l(y, \pi_l(g))\; dy\, dg\,,$$

as $n \to +\infty$, where dg denotes the normalized Haar measure on G.

Proof. **a)** It is sufficient to consider the case where

$$\mathcal{A}(y, g_j) \;:=\; \sum_\alpha \mathcal{A}_\alpha(y)\; e_\alpha(g_j) \tag{5.3.5}$$

is a trigonometric polynomial. The sum runs over a finite subset of $\widehat{G}_j$ and $\mathcal{A}_\alpha \in C^\infty_0(\Omega \cup \omega)$. Then clearly $\mathcal{A}(y, \rho_{j,n}(\varphi_j(y)))\tilde u_{j,n}(y) \;\sim\; \mathcal{A}(y, \rho_{j,n}(\varphi_j(y)))\mathcal{U}_j(y, \rho_{j,n}(\varphi_j(y)))$ lies in $\mathcal{L}^2_{os,\varphi_j}(H_j)$, and $\mathcal{A}(y, \rho_{j,n}(\varphi_j(y)))r_{j,n}$ lies in $\mathcal{L}^2_{no,\varphi_j}$. Note too that $\mathcal{A}(y, \rho_{j,n}(\varphi_j(y)))u_{j,n}(y)$ lies in W_j since $X_j\varphi_j = 0$. Hence define

$$\begin{aligned} \mathcal{V}_j(y, g_j) &= \mathcal{A}(y, g_j)\, \mathcal{U}_j(y, g_j)\,, \\ \mathcal{V}_r(y, g_r) &= \mathcal{U}_r(y, g_r) \quad \text{for} \quad r \in \{k, l\}\,. \end{aligned} \tag{5.3.6}$$

b) We use the notations introduced for Lemma 5.3.1.

Suppose first that $R_{j,k,l} \neq \{0\}$. Lemma 5.3.1 shows that the assumptions of Theorem 5.2.1 are satisfied and thus the left hand side of (5.3.4) converges to

$$\int_{\Omega\times G'} \mathcal{V}_1(y,\pi'_j(g'))\ \mathcal{V}_2(y,\pi'_k(g'))\ \mathcal{V}_3(y,\pi'_l(g'))\ dy\, dg'\,, \tag{5.3.7}$$

When $R_{j,k,l} = \{0\}$, Assumption 5.1.1 implies that the curvature of the 3-web generated by (X_j, X_k, X_l) does not vanish almost everywhere on Ω. Then [JMR 3] implies that the left hand side of (5.3.4) converges to the integral (5.3.7). Note that the two cases $R_{j,k,l} \neq \{0\}$ and $R_{j,k,l} = \{0\}$ are deeply different, since the definition of G' changes.

c) Finally, (5.3.2) implies the conditions leading to (5.2.23) are satisfied, so the integral in (5.3.7) can be lifted to G. Since $\pi'_r \circ \pi_{j,k,l} = \pi_r : G \to G_r$, this integral is equal to

$$\int_{\Omega\times G} \mathcal{V}_1(y,\pi_j(g))\ \mathcal{V}_2(y,\pi_k(g))\ \mathcal{V}_3(y,\pi_l(g))\ dy\, dg\,, \tag{5.3.8}$$

which in view of (5.3.6) is the same as (5.3.4). □

Remark 5.3.3. If the nonlinearities $F_{j,k,l}$ from (5.1.2) were linear in each of the u_r separately, say $F_{j,k,l} = c_j(y)u_j u_k u_l$, then Theorem 5.3.2 could be used to obtain evolution equations for the profiles $\mathcal{U}_j(y,g_j)$ analogous to the profile equations of nonlinear geometric optics: For simplicity, consider the case of three equations. The fact that $X_j\varphi_j = 0$ implies that $X_j^*\mathcal{A}(y,\rho_{j,n}(\varphi_j(y))) = X_j^*\mathcal{A}(y,g)_{|g=\rho_{j,n}(\varphi_j(y))}$. Hence by Theorem 5.3.2 with $u_{k,n} \equiv 1 \equiv u_{l,n}$, the limit of $\int_\Omega \mathcal{A}(y,\rho_{j,n}(\varphi_j(y)))X_j u_{j,n}$ is $\int_{\Omega\times G}\mathcal{A}(y,\pi_j(g))X_j\mathcal{U}_j(y,\pi_j(g))$. Since the oscillatory test function $\mathcal{A}$ is arbitrary, and the G can be parametrized by $G_1\times G_2$, this equation together with (5.3.4) show that $X_1\mathcal{U}_1(y,g) = \int_{G_2} c_1(y)\mathcal{U}_1(y,g)\mathcal{U}_2(y,g_2)\mathcal{U}_2(y,-g-g_2)$, which is a special case of (1.21). However, if the F_j are nonlinear in some u_r then we must apply Theorem 5.3.2 to sequences that are powers of the u_r. The profiles of these powers are not determined by those for u_r itself since, as already noted in Remark 4.3.8, $\mathcal{L}^2_{no,\varphi}\cap\mathcal{L}^\infty$ is not closed under multiplication. In order to determine the profiles of the u_r we therefore need to simultaneously determine the profiles of all of their powers. In order to avoid having an infinity of equations, we therefore need to encode all these profiles into a single object: the multiscale Young measure.

6. Multiscale Young measures

In this section, we first review the construction of multiscale Young measures, extending to general compact groups the constructions made for tori in [JMR 2], [E], [ES]. We then prove Theorem 1.7, and use it to extend Theorem 5.3.2 by giving a formula for the weak limit of arbitrary continuous functions of three bounded sequences.

Definition 6.1. *Suppose that G is a compact Abelian group and ρ_n is a sequence in $\mathrm{Hom}(\mathbb{R}^m; G)$ such that (G,ρ_*) is admissible. A bounded family u_n in $L^\infty(\Omega)$ is said to be* *adapted to (G, ρ_*) and the given phase φ, when for all $f \in C^0(\mathbb{C})$ and all $\mathcal{A} \in C_0^0(\Omega \times G)$, the following integrals*

$$\int_\Omega f(u_n(y))\ \mathcal{A}(y, \rho_n(\varphi(y)))\ dy \tag{6.1}$$

have limits when $n \to +\infty$.

Remark 6.2. When $\widehat{G}$ is countable, any bounded sequence u_n in $L^\infty(\Omega)$ contains subsequences $u_{\ell(n)}$ which adapted to $(G, \rho_{\ell(*)})$.

Theorem 6.3. *Suppose that G is a compact Abelian group and ρ_n is a sequence in* $\mathrm{Hom}(\mathbb{R}^m; G)$ *such that (G, ρ_*) is admissible. Suppose that u_n is a bounded sequence in $L^\infty(\Omega)$ which is adapted to (G, ρ_*) and the phase φ. Then there exists a measurable family of probabilities on $\mathbb{C}$, $\mu_{y,g}$ parametrized by $\Omega \times G$, such that for all $f \in C^0(\mathbb{C})$ and all $\mathcal{A} \in C_0^0(\Omega \times G)$,*

$$\int_\Omega f(u_n(y))\ \mathcal{A}(y, \rho_n(\varphi(y)))\ dy \quad \to \quad \int_{\Omega \times G} \int f(\lambda)\ \mathcal{A}(y, g)\ \mu_{y,g}(d\lambda)\ dy\, dg\,. \tag{6.2}$$

Proof. This is a repetition of the construction of Young measures (see e.g. [T 1], [Ev], [E]), extended to a more general setting. The limits of integrals (6.1) define a functional on $C^0(\mathbb{C}) \otimes C_0^0(\Omega \times G)$. Moreover, for all $F \in C_0^0(\Omega \times G \times \mathbb{C})$, the integrals

$$\int_\Omega F(y, \rho_n(y), u_n(y))\, dy$$

are uniformly bounded by the supremum of $|F|$ on $\Omega \times G \times \{|\lambda| \leq \Lambda|\}$, times the measure of Ω. Thus, the limits of these integrals exist for all $F \in C_0^0(\Omega \times G \times \mathbb{C})$, and define a Borel measure μ, on $\Omega \times G \times \mathbb{C}$:

$$\int_\Omega F(y, \rho_n(y), u_n(y))\, dy \quad \to \quad \int_{\Omega \times G \times \mathbb{C}} F(y, g, \lambda)\, \mu(dy\, dg\, d\lambda). \tag{6.3}$$

When $f \equiv 1$, the weak limit of $f(u_n)$ is just the constant function 1 and the limit of (6.1) is

$$\int_{\Omega \times G} \mathcal{A}(y, g)\, dy\, dg\,.$$

This shows that the projection of μ on $\Omega \times G$ is the measure $dy\, dg$.

Because μ is a finite measure, we can consider it to define a probability and apply methods from probability theory: Since $\mathbb{C}$ is a Borel space, by [Br, Theorem 4.34 and

Proposition 4.36] there exists a regular conditional probability $\mu_{y,g}(d\lambda)$ for λ given (y, g) such that

$$E(F(\lambda, y_1, g_1) \mid y_1 = y, g_1 = g) = \int F(\lambda, y, g)\mu_{y,g}(d\lambda). \tag{6.4}$$

Furthermore, by [Br, (4.21)] and the fact that the projection of μ on $\Omega \times G$ equals $dy\, dg$,
(6.5)

$$\int_{\Omega\times G\times\mathbb{C}} F(\lambda, y, g)\mu(dy\, dg\, d\lambda) = E(F(\lambda, y, g)) = E(E(F(\lambda, y_1, g_1) \mid y_1 = y, g_1 = g))$$
$$= \int_{\Omega\times G} E(F(\lambda, y_1, g_1) \mid y_1 = y, g_1 = g)\, dy\, dg.$$

Combining (6.3), (6.4) and (6.5) for the function $F(y, g, \lambda) = f(\lambda)\mathcal{A}(y, g)$ yields (6.2). □

We now apply this construction to groups which are large enough to describe all the oscillations of suitably chosen subsequence. Recall Definition 4.3.1 of the notion of completeness of a group and homomorphism for a sequence.

THEOREM 6.4. *Consider a compact Abelian group G, a sequence $\rho_n \in \mathrm{Hom}(\mathbb{R}^m; g)$ and a bounded sequence u_n in $L^\infty(\Omega)$. Assume that (G, ρ_*) is complete for the sequence u_* and the phase φ. Then u_* is adapted to (G, ρ_*) and the phase φ, and the multiscale Young measure of u_* satisfies the following property:*

for all $f \in C^0(\mathbb{C})$, the function

$$\mathcal{F}(y, g) \ := \int f(\lambda)\ \mu_{y,g}(d\lambda)\,, \tag{6.6}$$

is defined a.e. on $\Omega \times G$ and belongs to $L^\infty(\Omega \times G)$. Moreover, if $p < +\infty$ and if $\widetilde{f}_n(y) \sim \mathcal{F}(y, \rho_n(\varphi(y)))$ in $L^p(\Omega)$, then $\widetilde{f}_* \in \mathcal{L}^p_{os,\varphi}(H)$ and

$$f(u_*) \ - \ \widetilde{f}_* \ \in \mathcal{L}^p_{no,\varphi}\,. \tag{6.7}$$

Proof. Suppose that $f \in C^0(\mathbb{C})$. Then $v_n := f(u_n)$ is bounded in $L^\infty(\Omega)$ and

$$v_n \ = \ f_n \ + \ r_n\ , \quad \text{with} \quad f_* \in \mathcal{L}^2_{os,\varphi}(A)\ , \quad r_* \in \mathcal{L}^2_{no,\varphi}\,.$$

Theorem 2.3.4 implies that there is $\mathcal{F} \in L^2(\Omega \times G)$ such that for all $\mathcal{A} \in C^0_0(\Omega \times G)$,

$$\int_\Omega f_n(y)\ \mathcal{A}(y, \rho_n(\varphi(y)))\ dy \quad \to \quad \int_{\Omega\times G} \mathcal{F}(y, g)\ \mathcal{A}(y, g)\ dy\ dg\,. \tag{6.8}$$

On the other hand, by the definition of $\mathcal{L}^2_{no,\varphi}$, Theorem 2.3.4 implies that

$$\int_\Omega r_n(y)\ \mathcal{A}(y, \rho_n(\varphi(y)))\ dy \quad \to \quad 0\,, \tag{6.9}$$

when $\mathcal{A}$ is a trigonometric polynomial. (6.9) extends to all $\mathcal{A} \in C^0_0(\Omega \times G)$. Adding (6.8) and (6.9) imply that u_* is adapted. Moreover, comparing the limit with (6.2), implies that the profile $\mathcal{F}$ in (6.8) satisfied

$$\int_{\Omega\times G} \mathcal{F}(y,g)\ \mathcal{A}(y,g)\ dy\ dg \quad = \quad \int_{\Omega\times G}\int f(\lambda))\ \mathcal{A}(y,g)\ \mu_{y,g}(d\lambda)\ dy\, dg\,. \tag{6.10}$$

Therefore, $\mathcal{F}$ is given by (6.6) and thus $\mathcal{F} \in L^\infty(\Omega \times G)$.

Fix $p \in [1, +\infty[$ and consider $\tilde{f}_n(y) \sim \mathcal{F}(y, \rho_n(\varphi(y)))$ in L^p. Theorem 2.3.4 implies that such $\tilde{f}_n$ always exist. Introduce $\tilde{r}_n := v_n - \tilde{f}_n = f_n - \tilde{f}_n + r_n$. Remark that $\tilde{r}_n$ is bounded in $L^p(\Omega)$, since v_n is bounded in $L^\infty(\Omega)$ and $\tilde{f}_n$ is bounded in $L^p(\Omega)$. By definition of $\mathcal{F}$, $f_n(y) \sim \mathcal{F}(y, \rho_n(\varphi(y)))$ in L^2. Therefore $f_n - \tilde{f}_n(y) \sim 0$ in L^q, and $f_n - \tilde{f}_n \to 0$ strongly in L^q, for $q := \min(2,p)$. This implies that $f_n - \tilde{f}_n$ satisfies :

$$\forall \xi \in \mathbf{S}\ :\quad (f_n - \tilde{f}_n)\ e^{i\ \xi_n\cdot\varphi} \rightharpoonup 0 \tag{6.11}$$

r_* satisfies the same property, since it belongs to $\mathcal{L}^2_{no,\varphi}$. Therefore, $\tilde{r}_n$, which is bounded in $L^p(\Omega)$, also satisfies (6.11) and thus belongs to $\mathcal{L}^p_{no,\varphi}$. □

Formula (6.6) for the profiles of functions of a bounded sequence, together with formula (5.3.4) for the weak limit of the product of three sequences, enable us to obtain a formula for the weak limit of arbitrary continuous functions of such sequences. Given a function $F_{j,k,l} \in C^0(\overline{\Omega} \times \mathbb{R}^3)$, define F on $\Omega \times \mathbb{R}^N$ by

$$F(y, \lambda_1, \dots, \lambda_N)\ :=\ F_{j,k,l}(y, \lambda_j, \lambda_k, \lambda_l)\,. \tag{6.12}$$

THEOREM 6.5. *Suppose that $(G_1, \rho_{1,*}), \dots, (G_N, \rho_{N,*})$ are admissible in the sense of Definition 4.3.5 and each $(G_k, \rho_{k,*})$ is complete for $u_{k,*}$ and φ_k in the sense of Definition 4.3.1. Assume further that each sequence $u_{k,n}$ lies in $L^\infty \cap W_k$. Then for F a sum of terms of the form (6.12) and all $\mathcal{A} \in C^0(\overline{\Omega} \times G_j)$,*

$$\begin{aligned} &\int_\Omega \mathcal{A}(y, \rho_{j,n}(\varphi_j(y)))\, F(y, u_n(y))\, dy \quad \to \\ &\int_{\Omega\times G} \mathcal{A}(y, \pi_j(g))\, F(y, \lambda_1, \dots, \lambda_N)\, \mu_{1,y,\pi_1(g)}(d\lambda_1) \dots \mu_{N,y,\pi_N(g)}(d\lambda_N)\ dg\, dy \end{aligned} \tag{6.13}$$

as $n \to +\infty$, where dg denotes the normalized Haar measure on the group G defined in (5.3.2).

Proof. $F_{j,k,l}$ can be approximated by polynomials uniformly on compact sets. Since the sequences are uniformly bounded, it is sufficient to consider polynomial $F_{j,k,l}$, and one can further reduce the proof to the case where

$$F = F_{j,k,l}(y, \lambda_j, \lambda_k, \lambda_l)\ =\ \lambda_j^{p_j}\overline{\lambda_j}^{\,q_j}\ \lambda_k^{p_k}\overline{\lambda_k}^{\,q_k}\ \lambda_l^{p_l}\overline{\lambda_l}^{\,q_l}\,. \tag{6.14}$$

Introduce $v_{r,n} \; := \; (u_{r,n})^{p_r} \in L^\infty \cap W_r$. Theorem 6.4 implies that

$$v_{r,n}(y) \;\sim\; \mathcal{V}_r(y, \rho_{r,n}(\varphi_r(y))) \;+\; w_{r,n}, \tag{6.15}$$

where $w_{r,*} \in \mathcal{L}^2_{no,\varphi_r}$ and the profiles $\mathcal{V}_r \in L^\infty(\Omega \times G_k)$ are

$$\mathcal{V}_r(y, g_r) \;=\; \int_{\mathbb{R}} \lambda_r^{p_r} \overline{\lambda_r}^{\,q_r} \; \mu_{r,y,g_r}(d\lambda_r)\,. \tag{6.16}$$

Here μ_k is the multiscale Young measure of $u_{k,*}$ relative to $(G_k, \rho_{k,*})$ given by Theorem 6.3. Given (6.14) and (6.15), Theorem 5.3.2 implies that the left side of (6.13) converges to

$$\int_{\Omega\times G} \mathcal{A}(y, \pi_j(g)) \; \mathcal{V}_j(y, \pi_j(g)) \; \mathcal{V}_k(y, \pi_k(g)) \; \mathcal{V}_l(y, \pi_l(g)) \; dy\, dg\,,$$

and in view of (6.16), this is equal to the right side of (6.13) for the case (6.14). □

7. Propagation of Multiscale Young Measures

In this section we prove Theorems 1.8 and 1.9 that show how the multiscale Young measures evolve in time, and discuss the connection with geometric optics.

7.1. Transport equations for the multiscale Young measures

For $k \in \{1, \ldots, N\}$, consider a sequence of real valued solutions $u_{k,*}$ of (5.1.1), bounded in $L^\infty(\Omega)$, with Cauchy data

$$u_{k,n}(0, x) \;:=\; u^0_{k,n}(x)\,. \tag{7.1.1}$$

Introduce

$$\varphi^0_k(\,.\,) \;:= \varphi_k(0, \,.\,) \in C^\infty(\omega; \Theta_k)\,. \tag{7.1.2}$$

Since φ_k satisfies the equation $X_k \varphi_k = 0$, (5.1.6) implies that

$$\forall \xi \in \Phi_k \backslash \{0\}\,, \quad d(\xi \cdot \varphi^0_k) \neq 0 \quad \text{a.e. on } \omega\,. \tag{7.1.3}$$

Suppose that $(G_1, \rho_{1,*}), \ldots, (G_N, \rho_{N,*})$ are admissible and $(G_k, \rho_{k,*})$ is complete for $u_{k,*}$ and φ_k. Let μ_k denote the multiscale Young measure of $u_{k,*}$ relative to $(G_k, \rho_{k,*})$. Thanks to (7.1.3) one can also analyze the oscillations of the initial data relative to the initial phases. Assume that $(G_k, \rho_{k,*})$ is complete for $u^0_{k,*}$ and φ^0_k and introduce μ^0_k the corresponding multiscale Young measure.

Introduce

$$G'_k \;=\; \{ g \in G \;;\; \pi_k(g) = 0 \}\,. \tag{7.1.4}$$

Lemma 5.3.1 implies that G_k is isomorphic to G/G'_k. This identification allows the following version of Fubini's Theorem (see e.g. [W]) : for all $f \in L^1(G)$,

$$\int_{G'_k} f(g + g') dg' \tag{7.1.5}$$

is invariant by G'_k and defines (almost everywhere) a function $\tilde f$ on G_k. We denote it

$$\tilde f(g_k) \;=\; \int_{\pi_k^{-1}(g_k)} f(g)\;dg'_k\,; \tag{7.1.6}$$

This function is integrable and

$$\int_{G_k} \tilde f(g_k)\,dg_k \;=\; \int_G f(g)\,dg\,. \tag{7.1.7}$$

THEOREM 7.1.1. *The measures μ_k satisfy on $\Omega\times G_k\times\mathbb{C}$*

$$X_k\,\mu_k \;+\; \partial_\lambda(A_k\mu_k) \;=\; 0\,,\quad \mu_{k|t=0} = \mu_k^0\,, \tag{7.1.8}$$

with

$$A_k(y,g_k,\lambda_k) \;:=\; \int_{\pi_k^{-1}(g_k)} \mathcal{F}_k(y,g,\lambda_k)\;dg'_k\,, \tag{7.1.9}$$

$$\mathcal{F}_k(y,g,\lambda_k) \;:=\; \int_{\mathbb{R}^{N-1}} F_k(y,\lambda_1,\ldots,\lambda_N)\,\mu'_{k,y,g}(d\lambda'_k)\,, \tag{7.1.10}$$

where $\mu'_{k,y,g}(d\lambda'_k)$ denotes the product of the $\mu_{j,y,\pi_j(g)}(d\lambda_j)$ for $j\in\{1,\ldots,N\}$, $j\neq k$.

Proof. Consider $h\in C^\infty(\mathbb{R})$ and $v_{k,n} := h(u_{k,n})$. Then

$$X_k v_{k,n} \;=\; f_{k,n} \;:=\; F_k(y,u_n)\,\frac{\partial h}{\partial\lambda}(u_{k,n})\,. \tag{7.1.11}$$

Consider a trigonometric polynomial on $\Omega\times G_k$

$$\mathcal{A}(y,g_k) \;:=\; \sum_\alpha \mathcal{A}_\alpha(y)\;e_\alpha(g_k)\,. \tag{7.1.12}$$

The sum runs over a finite subset of $\widehat{G}_k$ and $\mathcal{A}_\alpha\in C_0^\infty(\Omega\cup(\{0\}\times\omega))$. Introduce

$$a_n(y) \;:=\; \mathcal{A}(y,\rho_{k,n}(\varphi_k(y))) \;:=\; \sum_\alpha \mathcal{A}_\alpha(y)\;e^{i\,\hat\rho_{k,n}(\alpha)\,\varphi_k}\,. \tag{7.1.13}$$

Thus $a_n\in C_0^\infty(\Omega\cup(\{0\}\times\omega))$, and (4.1.11) implies that

$$\int_\Omega v_{k,n}(y)\,{}^tX_k a_n(y)\,dy \;=\; \int_\Omega f_{k,n}\,a_n(y)\,dy \;-\; \int_\omega v_{k,n}(0,x)\,a_n(0,x)\,dx\,, \tag{7.1.14}$$

where ${}^tX_k := -\partial_t + \partial_x c_k$ is the transposed operator of X_k. Since $X_k\varphi_k = 0$, one has

$${}^tX_k\,a_n(y) \;=\; ({}^tX_k(y,\partial_y)\mathcal{A})\,(y,\rho_{k,n}(\varphi_k(y)))\,. \tag{7.1.15}$$

Thus, the first term in (7.1.14) tends to

$$(7.1.16) \qquad \int_{\Omega\times G_k\times\mathbb{R}} ({}^tX_k\mathcal{A})(y,g_k)\,h(\lambda_k)\;\mu_{k,y,g_k}(d\lambda_k)\,dg_k\,dy$$

as n tends to infinity. Similarly, the third term converges to

$$(7.1.17) \qquad \int_{\omega\times G_k\times\mathbb{R}} \mathcal{A}(0,x,g_k)\,h(\lambda_k)\;\mu^0_{k,x,g_k}(d\lambda_k)\,dg_k\,dx\,.$$

The limit of the middle term is computed in Theorem 6.5. It is equal to

$$\int_{\Omega\times G\times\mathbb{R}^N} \mathcal{A}(y,\pi_k(g))\,\partial_{\lambda_k}h(\lambda_k)F(y,\lambda)\,\mu_{1,y,\pi_1(g)}(d\lambda_1)\ldots\mu_{N,y,\pi_N(g)}(d\lambda_N)\;dg\,dy$$

Using the notations (7.1.9) and Fubini's Theorem (7.1.7), these limits are equal to

$$(7.1.18) \qquad \int_{\Omega\times G_k\times\mathbb{R}} \mathcal{A}(y,\pi_k(g))\,\partial_{\lambda_k}h(\lambda_k)A_k(y,g_k,\lambda)\,\mu_{1,y,g_k}(d\lambda_k)dg_k\,dy\,.$$

Adding up, we have proved that

$$(7.1.19) \qquad \begin{aligned} &\int_{\Omega\times G_k\times\mathbb{R}} ({}^tX_k\mathcal{A})(y,g_k)\,h(\lambda_k)\;\mu_{k,y,g_k}(d\lambda_k)\,dg_k\,dy \;= \\ &\int_{\omega\times G_k\times\mathbb{R}} \mathcal{A}(0,x,g_k)\,h(\lambda_k)\;\mu^0_{k,x,g_k}(d\lambda_k)\,dg_k\,dx\;+ \\ &\int_{\Omega\times G_k\times\mathbb{R}} \mathcal{A}(y,g_k)\,\partial_{\lambda_k}h(\lambda_k)A_k(y,g_k,\lambda)\,\mu_{k,y,g_k}(d\lambda_k)dg_k\,dy\,. \end{aligned}$$

This means that μ_k satisfies (7.1.8) in the sense of distributions.

7.2. The link with nonlinear geometric optics

Suppose that the assumptions of §7.1 are satisfied. Then the probability measures $\mu_{k,y,g}(d\lambda)$ can be seen as the distribution law of random variables $U_k(y,g,\,.\,)$. In this section we show that the transport equations (7.1.8) for the μ_k are equivalent to the equations of nonlinear geometric optics for the U_k. A similar argument is developed in [JMR 3]. The only difference comes from the new variables $g\in G$ which are parameters in the equations. However, these new parameters increase the coupling between the equations, and we check below that this new coupling does not affect the argument.

Introduce the normalized distributions functions, $M_k(y,g_k,.)$ of μ_{k,y,g_k} defined by

$$(7.2.1) \qquad M_k(y,g_k,\lambda)\;:=\;\mu_{k,y,g_k}(]-\infty,\lambda[)\,.$$

These functions are non decreasing and left continuous in λ. Moreover, there is Λ such that $|u_{k,n}(y)|\le\Lambda$ for all n and almost all y. Thus μ_{k,y,g_k} is supported in $[-\Lambda,\Lambda]$ for almost all $(y,g_k)\in\Omega\times G_k$ and

$$(7.2.2) \qquad M_k(y,g_k,\lambda)=0\quad\text{for }\lambda<-\Lambda\quad\text{and}\quad M_k(y,g_k,\lambda)=1\quad\text{for }\lambda>\Lambda\,.$$

Furthermore, $\mu_k = \partial_\lambda M_k$ in the sense of distributions.

Similarly, introduce the distribution functions M_k^0 on $\omega \times G_k \times \mathbb{R}$ associated to the probabilities μ_k^0 :

$$M_k^0(x, g_k, \lambda) \;:=\; \mu_{k,x,g_k}(]-\infty, \lambda[)\,. \tag{7.2.3}$$

THEOREM 7.2.1. *Then the M_k satisfy*

$$X_k\, M_k \;+\; A_k\, \partial_\lambda M_k \;=\; 0\,, \qquad M_{k|t=0} = M_k^0\,. \tag{7.2.4}$$

with A_k defined in (7.1.9).

Proof. The idea is to integrate equation (7.1.8) with respect to λ. Suppose that $h \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} h \subset [-\Lambda_1, \Lambda_1]$, where $\lambda_1 > \Lambda$, $\tilde h \in C_0^\infty(\mathbb{R})$ and $\tilde h = 1$ on $[-\Lambda_1, \Lambda_1]$. Introduce

$$h_1(\lambda) \;:=\; \tilde h(\lambda) \int_{-\infty}^{\lambda} h(s)\, ds \tag{7.2.5}$$

so that

$$\partial_\lambda h_1 \;=\; h \;+\; \tilde h_1\,, \quad \operatorname{supp} \tilde h_1 \subset [\Lambda_1, +\infty[\,. \tag{7.2.6}$$

Consider a trigonometric polynomial $\mathcal{A}$ on $\Omega \times G_k$, as in (7.1.12). Using (7.2.6), the identity $\mu_k = \partial_\lambda M_k$ and that $\tilde h = 1$ on a neighborhood of the support of μ_k, one obtains

$$\begin{aligned} &\int_{\Omega\times G_k\times\mathbb{R}} ({}^tX_k\mathcal{A})(y, g_k)\, h_1(\lambda_k)\; \mu_{k,y,g_k}(d\lambda_k)\, dg_k\, dy \;= \\ &\qquad -\int_{\Omega\times G_k\times\mathbb{R}} ({}^tX_k\mathcal{A})(y, g_k)\, h(\lambda_k)\; M_k(y, g_k, \lambda_k)\, d\lambda_k\, dg_k\, dy \\ &\qquad -\int_{\Omega\times G_k\times\mathbb{R}} ({}^tX_k\mathcal{A})(y, g_k)\, \tilde h_1(\lambda_k)\; M_k(y, g_k, \lambda_k)\, d\lambda_k\, dg_k\, dy\,. \end{aligned} \tag{7.2.7}$$

Because $\tilde h_1$ is supported in $[\Lambda_1, +\infty[$ and $M_k = 1$ there, the third integral is equal to

$$\gamma \int_{\Omega\times G_k} ({}^tX_k\mathcal{A})(y, g_k)\, dg_k\, dy \;=\; -\,\gamma \int_{\omega\times G_k} \mathcal{A}(0, x, g_k)\, dg_k\, dx\,. \tag{7.2.8}$$

where

$$\gamma := \int_{\mathbb{R}} \tilde h_1(\lambda_k)\, d\lambda_k\,. \tag{7.2.9}$$

Similarly,

$$
\begin{aligned}
\int_{\omega\times G_k\times\mathbb{R}} \mathcal{A}(0,x,g_k)\, h_1(\lambda_k)\ \mu^0_{k,x,g_k}(d\lambda_k)\, dg_k\, dx \ &= \\
- \int_{\omega\times G_k\times\mathbb{R}} \mathcal{A}(0,x,g_k)\, h(\lambda_k)\ M^0_k(x,g_k,\lambda_k)\, d\lambda_k\, dg_k\, dx & \\
+\ \gamma \int_{\omega\times G_k\times\mathbb{R}} \mathcal{A}(0,x,g_k)\, dg_k\, dx\,. &
\end{aligned}
\tag{7.2.10}
$$

Because μ_k is supported in $[-\Lambda,\Lambda]$, one has

$$
\int_{\Omega\times G_k\times\mathbb{R}} \mathcal{A}(y,g_k)\, \tilde h_1(\lambda_k)\ \mu_{k,y,g_k}(d\lambda_k)\, dg_k\, dy \ = \ 0\,.
\tag{7.2.11}
$$

Applying (7.1.19) with h_1 in place of h, and using (7.2.6), these identities imply that

$$
\begin{aligned}
&\int_{\Omega\times G_k\times\mathbb{R}} ({}^tX_k\mathcal{A})(y,g_k)\, h(\lambda_k)\ M_k(y,g_k,\lambda_k)\, d\lambda_k\, dg_k\, dy \ = \\
&\int_{\omega\times G_k\times\mathbb{R}} \mathcal{A}(0,x,g_k)\, h(\lambda_k)\ M^0_k(y,g_k,\lambda_k)\, d\lambda_k\, dg_k\, dx \ + \\
&\int_{\Omega\times G_k\times\mathbb{R}} \mathcal{A}(y,\pi_k(g))\, h(\lambda_k) A_k(y,g_k,\lambda)\, \mu_{1,y,g_k}(d\lambda_k) dg_k\, dy\,.
\end{aligned}
\tag{7.2.12}
$$

Since $\mu_k = \partial_\lambda M_k$, this implies that M_k satisfies (7.2.4) in the sense of distributions.

An important point is uniqueness for the equations (7.1.8).

THEOREM 7.2.2. *The Cauchy Problem (7.1.8) has at most one solution* $\mu = (\mu_1,\dots,\mu_N)$ *such that each* λ_{k,y,g_k} *is a measurable family of probability measures on* $\mathbb{R}$*, supported in a compact interval* $[-\Lambda,\Lambda]$.

Proof. Suppose that $\mu^{(1)}$ and $\mu^{(2)}$ are two solutions. Introduce the distribution functions $M^{(1)}$ and $M^{(2)}$ and $N := M^{(1)} - M^{(2)}$. Then N satisfies in the sense of distributions

$$
X_k\, N_k \ + \ A^{(2)}_k \partial_\lambda\, N_k \ = \ (A^{(1)}_k - A^{(2)}_k)\mu^{(1)}_k\ , \quad N_{k|t=0} = 0\,.
\tag{7.2.13}
$$

One has

$$
(A^{(1)}_k - A^{(2)}_k)(y,g_k,\lambda_k) \ := \int_{\pi_k^{-1}(g_k)} \big(\mathcal{F}^{(1)}_k - \mathcal{F}^{(2)}_k\big)(y,g,\lambda_k)\ dg'_k\,,
\tag{7.2.14}
$$

where

$$
\big(\mathcal{F}^{(1)}_k - \mathcal{F}^{(2)}_k\big)(y,g,\lambda_k) \ := \int_{\mathbb{R}^{N-1}} F_k(y,\lambda_1,\dots,\lambda_N)\, (\mu'^{(1)}_{k,y,g} - \mu'^{(2)}_{k,y,g})(d\lambda'_k)\,,
$$

and $\mu'^{(r)}_{k,y,g}(d\lambda'_k)$ denotes the product of the $\mu^{(r)}_{j,y,\pi_j(g)}(d\lambda_j)$ for $j \neq k$. Therefore, $\mu'^{(1)}_{k,y,g} - \mu'^{(2)}_{k,y,g}$ is the sum of $N-1$ terms, each of them being the product of $N-2$ factors $\mu^{(r)}_{l,y,\pi_l(g)}$ and one factor $\mu^{(1)}_{j,y,\pi_j(g)} - \mu^{(2)}_{j,y,\pi_j(g)}$. Since $M^{(1)}_j(y,\pi_j(g),\,.\,) - M^{(2)}_j(y,\pi_j(g),\,.\,)$ has compact support by (7.2.2), one can integrate by parts to obtain

$$\int_{\mathbb{R}} F_k(y,\lambda_1,\ldots,\lambda_N)(\mu^{(1)}_{j,y,\pi_j(g)} - \mu^{(2)}_{j,y,\pi_j(g)})(d\lambda_j) \;=\; -\int_{\mathbb{R}} \partial_{\lambda_j} F_k(y,\lambda_1,\ldots,\lambda_N) N_j(y,\pi_j(g),\lambda_j)\, d\lambda_j \,.$$

Using Fubini's theorem, we conclude that there is C such that for all $y \in \Omega$, $g \in G$ and $|\lambda_k| \leq \Lambda$

$$|(\mathcal{F}^{(1)}_k - \mathcal{F}^{(2)}_k)(y,g,\lambda_k)| \;\leq\; C \sup_{j\neq k} \int_{\mathbb{R}} |N_j(y,\pi_j(g),\lambda_j)|\, d\lambda_j \,.$$

Therefore

$$\begin{aligned} &|(A^{(1)}_k - A^{(2)}_k)(y,g_k,\lambda_k)| \;\leq \\ &\qquad C \sup_{j\neq k} \int_{\pi_k^{-1}(g_k)} \int_{\mathbb{R}} |N_j(y,\pi_j(g),\lambda_j)|\, d\lambda_j \; dg'_k \\ &\qquad\qquad = \; C \sup_{j\neq k} \int_{G_j} \int_{\mathbb{R}} |N_j(y,g_j,\lambda_j)|\, d\lambda_j \; dg_j \,, \end{aligned} \tag{7.2.15}$$

where the last equality follows from Lemma 5.3.1 i). This implies that the measurable family of bounded Borel measures $\tilde{\mu}_{k,y,g_k} := (A^{(1)}_k - A^{(2)}_k)\mu^{(1)}_k$, which are supported in $[-\Lambda,\Lambda]$, satisfies

$$\|\tilde{\mu}_{k,y,g_k}\|_{\mathcal{M}} \;\leq\; C \sup_{j\neq k} \int_{G_j} \|N_j(y,g_j),\,.\,)\|_{L^1(\mathbb{R})}\, dg_j \,. \tag{7.2.16}$$

with C independent of y and g_k. The norm in the left hand side is the total variation norm in the space of bounded Borel measures on $\mathbb{R}$. Lemma 4.3.4 of [JMR 3], with g_k as parameters, implies that the solution N_k of (7.2.13) satisfies

$$\int_{\omega_t \times G_k} \|N_j(t,x,g_k,\,.\,)\|_{L^1(\mathbb{R})}\; dx\, dg_k \;\leq\; C \int_{\Omega_t \times G_k} \|\tilde{\mu}_{k,y,g_k}\|_{\mathcal{M}}\; dy\, dg_k \tag{7.2.17}$$

where $\omega_t := \{x \in \mathbb{R}\ ;\ (t,x) \in \Omega\}$ and $\Omega_t := \{(s,x) \in \Omega\ ;\ 0 < s < t\}$.

Using (7.2.16) and Gronwall's lemma, this implies that

$$n(t) \;:=\; \sup_k \int_{\omega_t \times G_k} \|N_k(t,x,g_k,\,.\,)\|_{L^1(\mathbb{R})}\; dx\, dg_k$$

vanishes identically, proving Theorem 7.2.2.

Introduce next $U_k(y, g_k, \,.\,)$ the right continuous inverse function of $M_k(y, g_k, \,.\,)$. It is defined on $]0,1[$ and characterized by the property

$$M_k(y, g_k, \lambda) > \theta \quad \Longleftrightarrow \quad \lambda > U_k(y, g_k, \theta)\,. \tag{7.2.18}$$

In addition, (7.2.2) implies that for almost all y and g_k,

$$|U_k(y, g_k, \theta)| \;\leq\; \Lambda\,. \tag{7.2.19}$$

The relation with μ_k is that for all $f \in C^0(\mathbb{R})$

$$\int_{\mathbb{R}} f(\lambda)\; \mu_{k,y,g_k}(d\lambda) \;=\; \int_0^1 f(U_k(y, g_k, \theta)\; d\theta\,. \tag{7.2.20}$$

(see e.g. §4.4 in [JRM 3]). The profiles $U_k^0(x, g_k, \theta)$ associated to the initial values M_k^0 are defined analogously.

THEOREM 7.2.3. *The profiles U_k satisfy*

$$\begin{aligned} &X_k\, U_k(y, g_k, \theta_k) \;= \\ &\int_{\pi_k^{-1}(g_k)} \int_{]0,1[^{N-1}} F_k(y, U_1(y, \pi_1(g), \theta_1) \ldots, U_N(y, \pi_N(g), \theta_N))\; d\theta_k'\, dg_k'\,, \end{aligned} \tag{7.2.21}$$

$$U_{k\,|t=0} = U_k^0\,. \tag{7.2.22}$$

In (7.2.21), $d\theta_k'$ denotes the products of the Lebesgue measures $d\theta_j$ for $j \neq k$.

Proof. Consider the Cauchy problem (7.2.21) (7.2.22). Picard's iterations provide a solution $\widetilde{U} = (\widetilde{U}_1, \ldots, \widetilde{U}_N)$, defined for $t \leq T'$ where $0 < T' \leq T$, and such that $\widetilde{U}_k \in L^\infty(\Omega' \times G_k \times \mathbb{R})$, $\Omega' := \Omega \cap \{t < T'\}$.

Consider $\tilde{\mu}_k$ the measure defined by (7.2.20) with $\widetilde{U}_k$ in place of U_k. For almost all (y, g_k), μ_{k,y,g_k} is a probability measure supported in $[-\widetilde{\Lambda}, \widetilde{\Lambda}]$ where $\widetilde{\Lambda} := \max \|\widetilde{U}_k\|_{L^\infty}$.

For each $k \in \{1, \ldots, N\}$. Let $\tilde{\mu}'^{(r)}_{k,y,g}(d\lambda_k')$ be the product of the measures $\tilde{\mu}^{(r)}_{j,y,\pi_j(g)}(d\lambda_j)$ for $j \neq k$, and define

$$\widetilde{\mathcal{F}}_k(y, g, \lambda_k) \;:=\; \int_{\mathbb{R}^{N-1}} F_k(y, \lambda_1, \ldots, \lambda_N)\, \tilde{\mu}'^{(1)}_{k,y,g} d\lambda_k')\,.$$

Then (7.2.20) implies that the right hand side of (7.2.21) computed for $\widetilde{U}$ is equal to $\widetilde{A}_k(y, g_k, \widetilde{U}_k(y, g_k, \theta_k))$ where

$$\widetilde{A}_k(y, g_k, \lambda_k) \;:=\; \int_{\pi_k^{-1}(g_k)} \widetilde{\mathcal{F}}_k(y, g, \lambda_k)\; dg_k'\,. \tag{7.2.23}$$

For all $h \in C^\infty(\mathbb{R})$ one has $X_k h(\widetilde{U}_k) = h'(\widetilde{U}_k)\, \widetilde{A}_k(y, g_k, \widetilde{U}_k)$. Therefore, for all trigonometric polynomial $\mathcal{A}$ on $\Omega' \times G_k$, with compact support contained in $\big(\Omega' \cup (\{0\} \times \omega)\big) \times G_k$, one has

$$\begin{aligned}
&\int_{\Omega\times G_k\times\mathbb{R}} ({}^t X_k \mathcal{A})(y, g_k)\, h(\widetilde{U}_k(y, g_k, \theta_k))\, d\theta_k\, dg_k\, dy \;=\\
&\int_{\omega\times G_k\times\mathbb{R}} \mathcal{A}(0, x, g_k)\, h(U_k^0(x, g_k, \theta_k))\, d\theta_k\, dg_k\, dx \;+\\
&\int_{\Omega\times G_k\times\mathbb{R}} \mathcal{A}(y, g_k)\, h'(\widetilde{U}_k(y, g_k, \theta_k))\widetilde{A}_k(y, g_k, \widetilde{U}_k(y, g_k, \theta_k))\, d\theta_k\, dg_k\, dy\,.
\end{aligned} \tag{7.2.24}$$

With (7.2.20), this implies that $\tilde{\mu}_k$ satisfies (7.1.19) and $\tilde{\mu}$ satisfies (7.1.8). Thus, Theorem 7.2.2 implies that $\tilde{\mu} = \mu$ on Ω'.

Next we remark that (7.2.21) is an ordinary differential equation for $\widetilde{U}_k$, along the integral curves of X_k : $X_k \widetilde{U}_k = \widetilde{A}_k(y, g_k, \widetilde{U}_k)$. Therefore since the Cauchy data are nondecreasing function of θ_k, $\widetilde{U}_k$ is nondecreasing in θ_k. One can normalize it to be right continuous. Thus $\widetilde{U}_k = U_k$, the unique right continuous function on $]0, 1[$ such that (7.2.20) holds.

Therefore, U satisfies (7.2.21) on Ω'. Since U is defined an bounded for $y \in \Omega$, a continuation argument shows that U satisfies (7.2.21) on Ω.

7.3. The main result

Consider equations (5.1.1). Suppose that $u_n := (u_{1,n}, \ldots, u_{N,n})$ is a bounded family of solutions in $L^\infty(\Omega)$, with initial data $u_n^0 \in L^\infty(\omega)$, where Ω is given by (5.1.3). The phases φ_j are chosen as indicated in §5.1.

Theorem 7.3.1. *Suppose that* $(G_1, \rho_{1,*}), \ldots, (G_N, \rho_{N,*})$ *are admissible in the sense of Definition* 4.3.5 *and closed for resonances. Suppose that* $(G_k, \rho_{k,*})$ *is complete for* $u_{k,*}^0$ *and* φ_k^0 *in the sense of Definition* 4.3.1. *Then* $(G_k, \rho_{k,*})$ *is complete for* $u_{k,*}$ *and* φ_k *and the corresponding multiscale Young measures are uniquely determined by equations* (7.1.8).

Note that Theorem 4.3.7 implies the existence of admissible and closed for resonance $(G_1, \rho_{1,*}), \ldots, (G_N, \rho_{N,*})$ which are complete for subsequences of $u_{k,*}^0$. Similarly, that theorem implies that there exist analogous structures that are complete for the solution, which can furthermore be chosen to be extensions of those for the initial data. In order to prove that the structures for the initial data actually suffice for the solution as well, we will show that profiles for the solution are essentially independent of the extension. This follows from the facts that the extension is not needed for the initial data and that the evolution equations for the profiles preserve this independence.

Proof. Denote μ_k^0 the multiscale Young measures of the initial data.

a) Theorem 4.3.7 implies that there are strictly increasing mappings $\ell : \mathbb{N} \to \mathbb{N}$, $\big((G_1^e, \rho_{1,*}^e), \ldots, (G_N^e, \rho_{N,*}^e)\big)$ admissible in the sense of Definition 4.3.5 such that $(G_k^e, \rho_{k,*}^e)$

is complete for $u_{k,\ell(*)}$ and φ_k, surjective $\jmath_k \in \mathrm{Hom}(G_k^e, G_k)$ and $l_k \in \mathrm{Hom}(\widehat{G}_k; \Phi_k)$ such that

$$\forall \alpha \in \widehat{G}_k\ , \quad \nu_{k,n}^e(\hat{\jmath}_k(\alpha)) - \nu_{k,\ell(n)}(\alpha) \to l_k(\alpha) \quad \text{as} \quad n \to +\infty\,. \tag{7.3.1}$$

where $\nu_{k,n}^e$ and $\nu_{k,n}$ denote the dual homomorphisms of $\rho_{k,n}^e$ and $\rho_{k,n}$ respectively. In addition, with notations similar to those in Theorem 4.3.7, $\hat{\jmath} := (\hat{\jmath}_1, \ldots, \hat{\jmath}_N)$ maps Z into Z^e and $Z = \hat{\jmath}^{-1}(Z^e)$.

b) Consider a profile $\mathcal{V} \in L^p(\omega \times G_k)$, where $1 \leq p < +\infty$. Introduce

$$\mathcal{V}^e(x, g_k^e) \ :=\ \mathcal{V}(x, \jmath_k(g_k^e) - \hat{l}_k(\varphi_k^0(x)))\,. \tag{7.3.2}$$

where $\hat{l}_k \in \mathrm{Hom}(\Theta_k; G_k)$ is the dual homomorphism of l_k. $\mathcal{V}^e$ is defined on $\omega \times G_k^e$ and $\mathcal{V}^e(x, \rho_{k,n}^e(\varphi^0(x))) = \mathcal{V}(x, \jmath_k \circ \rho_{k,n}(\varphi^0(x)) - \hat{l}_k(\varphi_k^0(x)))$ when $\mathcal{V}$ is continuous. Since $\jmath_k$ is surjective, $\mathcal{V}^e \in L^p(\omega \times G_k^e)$ if and only if $\mathcal{V} \in L^p(\omega \times G_k)$. Therefore, $w_n \sim \mathcal{V}^e(x, \rho_{k,n}^e(\varphi^0(x)))$ in $L^p(\omega)$ if and only if $w_n \sim \mathcal{V}(x, \jmath_k \circ \rho_{k,n}(\varphi^0(x)) - \hat{l}_k(\varphi_k^0(x)))$ in $L^p(\omega)$.

Since $\hat{\rho}_{k,n}^e \circ \hat{\jmath}_k$ is the dual map of $\jmath_k \circ \rho_{k,n}^e$, Proposition 4.1.7 and (7.3.1) imply that, if $v_n \sim \mathcal{V}(x, \rho_{k,n}(\varphi^0(x)))$ in $L^p(\omega)$, then $v_{\ell(n)} \sim \mathcal{V}(x, \jmath_k \circ \rho_{k,n}(\varphi^0(x)) - \hat{l}_k(\varphi_k^0(x)))$ and thus $v_{\ell(n)} \sim \mathcal{V}^e(x, \rho_{k,n}^e(\varphi^0(x)))$ in $L^p(\omega)$.

This implies that $(G_k^e, \rho_{k,*}^e)$ is complete for $u_{k,\ell(*)}^0$ and φ_k^0 and that the corresponding multiscale Young measure $\mu_k^{0,e}$ is equal to

$$\mu_{k,x,g_k^e}^{0,e} \ =\ \mu_{k,x,\jmath_k(g_k^e) - \hat{l}_k(\varphi_k^0(x))}^0\,. \tag{7.3.3}$$

For the profiles U^0 of §7.2, this is equivalent to the identity

$$U_k^{0,e}(x, g_k^e, \theta_k) \ =\ U_k^0(x, \jmath_k(g_k^e) - \hat{l}_k(\varphi_k^0(x)), \theta_k)\,. \tag{7.3.4}$$

In particular, $\mu_k^{0,e}$ and $U_k^{0,e}$ are invariant by translations in G_k^e, parallel to $\ker \jmath_k$.

c) Consider $\jmath := (\jmath_1, \ldots, \jmath_N)$ from $G_1^e \times \ldots \times G_N^e$ to $G_1 \times \ldots \times G_N$. It is surjective, since each $\jmath_k$ is. Since $\alpha \in Z$ if and only if $\hat{\jmath}(\alpha) \in Z^e$, $\jmath$ maps $G^e \subset G_1^e \times \ldots \times G_N^e$, the annihilator of Z^e, onto $G \subset G_1 \times \ldots \times G_N$, the annihilator of Z. We show that

$$\pi_j^e(\ker \jmath \cap G^e) \ =\ \ker \jmath_j \subset G_j^e\,. \tag{7.3.5}$$

where π_j^e denotes he projection on the j-th factor. First, note that $\ker \jmath = \ker \jmath_1 \times \ldots \times \ker \jmath_N$. Thus $\pi_j^e(\ker \jmath \cap G^e)$ is contained in $\ker \jmath_j$. Therefore, to prove (7.3.5), it is sufficient to show that if $\alpha_j^e \in \widehat{G_j^e}$ annihilates $\pi_j^e(\ker \jmath \cap G^e)$, then it annihilates $\ker \jmath_j$. Let $\alpha^e \in \widehat{G_1^e} \times \ldots \times \widehat{G_N^e}$ with all components equal to zero, except the j-th one. If α_j^e annihilates $\pi_j^e(\ker \jmath \cap G^e)$, then α^e annihilates $\ker \jmath \cap G^e$. Therefore, there are $\alpha \in \widehat{G}_1 \times \ldots \times \widehat{G}_N$ and $\gamma^e \in Z^e$, such that $\alpha^e = \hat{\jmath}(\alpha) + \gamma^e$. (7.3.1) implies that

$$\nu_{\ell(n)}(\alpha) - \nu_n^e(\alpha^e) + \nu_n^e(\gamma) \ \to l(\alpha) \in \Phi \quad \text{as} \quad n \to +\infty\,.$$

Passing to the quotient in Φ/R, this implies that

$$\tilde\nu_{\ell(n)}(\alpha) - \tilde\nu^e_n(\alpha^e) \ \to \tilde l(\alpha) \quad \text{as} \quad n \to +\infty \,.$$

Our choice of α^e implies that $\nu^e_n(\alpha^e) \in \Phi^\sharp_j$. Since $(G_1, \rho_{1,*}), \dots, (G_N, \rho_{N,*})$ is closed for resonances, this implies that there is $\beta = (\beta_1, \dots, \beta_N) \in \widehat G_1, \times \dots \times \widehat G_N$, with $\beta_l = 0$ when $l \neq j$, such that $\tilde\nu_{\ell(n)}(\alpha) = \tilde\nu_{\ell(n)}(\beta)$. The admissibility condition implies that $\gamma := \alpha - \beta$ belongs to Z.

Therefore, $\alpha^e - \hat\jmath(\beta) \ = \hat\jmath(\gamma) + \gamma^e \in Z^e$. Since $\nu^e_n(\alpha^e - \hat\jmath(\beta)) \in \Phi^\sharp_j$, and $\nu^e_n(\hat\jmath(\gamma) + \gamma^e) \in R$, (5.1.9) implies that $\nu^e_n(\alpha^e - \hat\jmath(\beta)) = 0$, hence $\nu^e_{j,n}(\alpha^e_j - \hat\jmath_j(\beta_j)) = 0$. Since $\nu^e_{j,*}$ is admissible, this implies that $\alpha^e_j = \hat\jmath_j(\beta_j)$. Thus, α^e_j annihilates $\ker \jmath_j$, and (7.3.5) is proved.

d) Introduce the multiscale Young measures μ^e_k of the subsequences $u_{k,\ell(*)}$, relative to $(G^e_k, \rho^e_{k,*})$. We prove that they are invariant by translations parallel to $\ker \jmath_k$. Introducing the profiles U^e_k of §7.2, we show that

$$\text{(7.3.6)} \qquad \forall \sigma_k \in \ker \jmath_k \,, \quad U^e_k(y, g^e + \sigma_k, \theta_k) \ = \ U^e_k(y, g^e, \theta_k) \quad \text{a.e. on } \Omega \times G^e_k \times]0,1[\,.$$

The U^e_k satisfy (7.2.21) with Cauchy data $U^{0,e}_k$, which satisfy the analogue of (7.3.6) on $\omega \times G^e_k \times]0,1[$. Constructing the solution of (7.2.21) by Picard's iterations, it is sufficient to show that the Picard's iterates satisfy (7.3.6). Thus, it is sufficient to prove that if the U^e_l satisfy (7.3.6), then the integral

$$\text{(7.3.7)} \qquad V^e_k(y, g^e_k, \theta_k) \ := \ \int_{(\pi^e_k)^{-1}(g^e_k)} \int_{]0,1[^{N-1}} F(y, g^e, \theta) \, d\theta'_k \, dg^{e\prime}_k \,,$$

with

$$\text{(7.3.8)} \qquad F(y, g^e, \theta) := F_k(y, U^e_1(y, \pi^e_1(g^e), \theta_1) \dots, U^e_N(y, \pi^e_N(g^e), \theta_N)) \,,$$

satisfies (7.3.6). By definition,

$$\text{(7.3.9)} \qquad V^e_k(y, g^e_k, \theta_k) \ := \ \int_{\ker \pi^e_k \cap G^e} \int_{]0,1[^{N-1}} F(y, g^e + g', \theta) \, d\theta'_k \, dg' \,,$$

for all $g^e \in G^e$, such that $\pi^e_k(g^e) = g^e_k$. Consider $\sigma_k \in G^e_k$ such that $\jmath_k(\sigma_k) = 0$. By (7.3.5), there is $\sigma \in G^e$, such that $\jmath(\sigma) = 0$ and $\pi^e_k(\sigma) = \sigma_k$. Therefore, $\pi^e_k(g^e + \sigma) = g^e_k + \sigma_k$ and thus

$$V^e_k(y, g^e_k + \sigma_k, \theta_k) \ := \ \int_{\ker \pi^e_k \cap G^e} \int_{]0,1[^{N-1}} F(y, g^e + \sigma + g', \theta) \, d\theta'_k \, dg' \,.$$

The assumptions on U^e_k immediately imply that the function F defined by (7.3.8) satisfies $F(y, g^e + \sigma, \theta) = F(y, g^e, \theta)$ for all $g^e \in G^e$ and all $\sigma \in G^e \cap \ker \jmath$. Therefore, $V^e_k(y, g^e_k + \sigma_k, \theta_k) = V^e_k(y, g^e_k, \theta_k)$, and (7.3.6) is proved.

e) Next we show that $(G_k, \rho_{k,\ell(*)})$ is complete for $u_{k,\ell(*)}$ and φ_k. Since $(G^e_k, \rho^e_{k,*})$ is complete, for all $f \in C^0(\mathbb{R})$ one has

$$\text{(7.3.10)} \qquad f(u_{k,\ell(n)}) \ = \ v_n \ + \ w_n \,,$$

with $w_n \in \mathcal{L}^2_{no,\varphi_k}$ and $v_n \sim \mathcal{V}^e(y, \rho^e_{k,n}(\varphi_k(y)))$ in $L^2(\Omega)$, where $\mathcal{V}^e \in L^2(\Omega \times G^e_k)$ is given by

$$(7.3.11) \qquad \mathcal{V}^e(y, g^e_k) = \int_{\mathbb{R}} f(\lambda)\, \mu^e_{k,y,g^e_k}(d\lambda) = \int_0^1 f(U^e_k(y, g^e_k, \theta))\, d\theta\,.$$

From (7.3.6), we deduce that for all $\sigma_k \in \ker \jmath_k$, $\mathcal{V}^e_k(y, g^e + \sigma_k) = \mathcal{V}^e_k(y, g^e)$ a.e. on $\Omega \times G^e_k$. Since $\jmath_k$ is surjective, this implies that there is a unique $\mathcal{V} \in L^2(\Omega \times G_k)$ such that

$$(7.3.12) \qquad \mathcal{V}^e(y, g^e_k) \ = \ \mathcal{V}(y, \jmath_k(g^e_k))\,.$$

Introduce $\mathcal{V}_1(y, g_k) := \mathcal{V}(y, g_k + \hat{l}_k(\varphi_k(y)))$. Thus

$$(7.3.13) \qquad \mathcal{V}^e(y, g^e_k) \ := \ \mathcal{V}_1(y, \jmath_k(g^e_k) - \hat{l}_k(\varphi^0_k(y)))\,.$$

Arguing as in part a) above, one proves that

$$v_n \sim \mathcal{V}^e(y, \rho^e_{k,n}(\varphi_k)) \sim \mathcal{V}_1(y, \rho_{k,\ell(n)}(\varphi_k)) \quad \text{in } L^2(\Omega)\,.$$

With (7.3.10), this proves that $(G_k, \rho_{k,\ell(*)})$ is complete for $u_{k,\ell(*)}$.

f) Since the $(G_k, \rho_{k,\ell(*)})$ are complete for all the subsequences of $u^0_{k,\ell(*)}$, we deduce from the reasoning above from all the subsequences of u_n, one can extract a subsequence $u_{\ell(*)}$ such that $(G_k, \rho_{k,\ell(*)})$ is complete for $u_{k,\ell(*)}$. This implies that $(G_k, \rho_{k,*})$ is complete for the full sequence $u_{k,*}$. Then Theorem 7.1.1 implies that the multiscale Young measures associated to the sequence and the groups satisfy equation (7.1.8). □

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