A kinetic model for rain falling on water waves

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Outline

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Rain drops

- Assumptions:
  - large number of drops falling on a unit sea surface;
  - radius $\ll$ any macroscopic length scales (eg. surface wave length)
- a rain drop is a spherical volume of fluid described by:
  - its position $\boldsymbol{x} \in \mathbb{R}^3$;
  - its velocity $\boldsymbol{v} \in \mathbb{R}^3$ (mean velocity of the molecules inside the drop);
  - its radius $r \in]0, +\infty[$.
- density and mass:
  - all drops have a constant density ρ_d;
  - mass of one drop: $m(r) = \rho_d \frac{4}{3}\pi r^3$.

Rain drop distribution

- distribution function: $f(t, \boldsymbol{x}, \boldsymbol{v}, r)$

  number density of drops at time t in the phase space
  $\mathbb{R}^3 \times \mathbb{R}^3 \times]0, +\infty[$

- $f(t, \boldsymbol{x}, \boldsymbol{v}, r) d\boldsymbol{x} d\boldsymbol{v} dr$ is the number of drops that at time t are at position $\boldsymbol{x} \pm d\boldsymbol{x}$ with velocity $\boldsymbol{v} \pm d\boldsymbol{v}$ and with a radius $r \pm dr$

- average quantities:

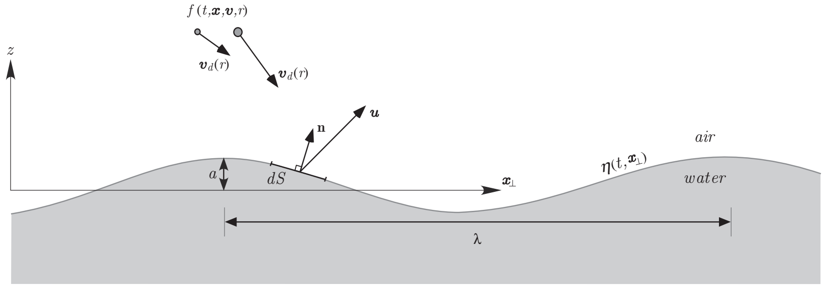
$$n(t, \boldsymbol{x}, r) = \int_{\mathbb{R}^3} f(t, \boldsymbol{x}, \boldsymbol{v}, r)\, d\boldsymbol{v},$$

  number of drops of radius r at time t and position $\boldsymbol{x}$, per unit volume and per radius increment.

Outline

Force exerted by the drops on the free surface

- free surface



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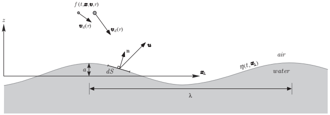
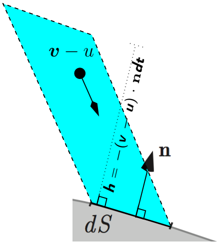


- number of drops of velocity $\boldsymbol{v}$ impacting dS during dt :



(frame of reference attached to the surface)

$$dN = f(t, \boldsymbol{x}, \boldsymbol{v}, r)|\mathcal{V}|drdv$$

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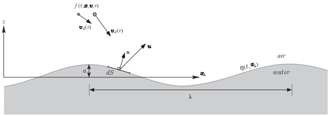

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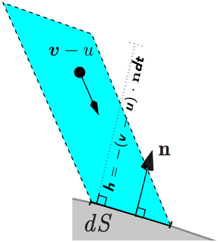

(frame of reference attached to the surface)

$$dN = f(t, \boldsymbol{x}, \boldsymbol{v}, r)[h \times dS]drdv$$

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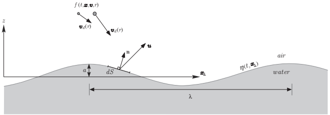

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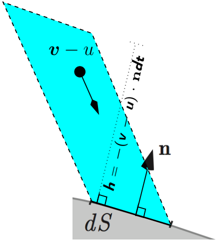

(frame of reference attached to the surface)

$$dN = f(t, \boldsymbol{x}, \boldsymbol{v}, r)[(-(\boldsymbol{v} - \boldsymbol{u}) \cdot \mathbf{n})dtdS]drdv$$

Force exerted by the drops on the free surface

- the momentum of an impacting droplet (in the reference frame of the surface) is $m(r)(\boldsymbol{v} - \boldsymbol{u})$

Force exerted by the drops on the free surface

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- momentum of all the droplets impacting dS during dt:

$$\boldsymbol{M} = \int_{(\boldsymbol{v}-\boldsymbol{u})\cdot\mathbf{n}<0} \int_{]0,+\infty[} m(r)(\boldsymbol{v} - \boldsymbol{u}) dN.$$

Force exerted by the drops on the free surface

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$$\boldsymbol{M} = \left(\int_{(\boldsymbol{v}-\boldsymbol{u})\cdot\mathbf{n}<0} \int_{]0,+\infty[} m(r)(\boldsymbol{v} - \boldsymbol{u}) f(t, \boldsymbol{x}, \boldsymbol{v}, r)(-(\boldsymbol{v} - \boldsymbol{u})\cdot\mathbf{n}) \, dr \, d\boldsymbol{v} \right) dS dt.$$

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- assumption: the force exerted by the drops *ejected* from the surface is neglected
- resulting force (Newton):

$$\boldsymbol{F} = -\left(\int_{(\boldsymbol{v}-\boldsymbol{u})\cdot\mathbf{n}<0} \int_{]0,+\infty[} m(r)(\boldsymbol{v} - \boldsymbol{u}) \otimes (\boldsymbol{v} - \boldsymbol{u}) f(t, \boldsymbol{x}, \boldsymbol{v}, r)\, dr\, d\boldsymbol{v} \right) \mathbf{n} dS.$$

Stress

- the force can be written: $\boldsymbol{F} = \boldsymbol{\sigma}_d \mathbf{n} dS$
- where $\boldsymbol{\sigma}_d$ is the stress tensor due to the drops:

$$\boldsymbol{\sigma}_d = - \int_{(\boldsymbol{v}-\boldsymbol{u})\cdot\mathbf{n}<0} \int_{]0,+\infty[} m(r)(\boldsymbol{v}-\boldsymbol{u})\otimes(\boldsymbol{v}-\boldsymbol{u}) f(t,\boldsymbol{x},\boldsymbol{v},r)\, dr d\boldsymbol{v}.$$

Accounting for the rain in a free-surface Navier-Stokes model

- Navier-Stokes: for $t > 0$, $z \geq \eta(t, \boldsymbol{x}_\perp)$

$$\nabla \cdot \boldsymbol{u} = 0,$$
$$\rho \partial_t \boldsymbol{u} + \rho (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} = \nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{g},$$

where $\boldsymbol{u}(t, \boldsymbol{x}_\perp, z) = (\boldsymbol{u}_\perp, w)$;

- kinematic boundary condition: at $z = \eta(t, \boldsymbol{x}_\perp)$

$$w\,(t, \boldsymbol{x}_\perp, z) = \partial_t \eta + \boldsymbol{u}_\perp \cdot \nabla_\perp \eta,$$

- dynamic boundary condition: at $z = \eta(t, \boldsymbol{x}_\perp)$

$$\boldsymbol{\sigma} \mathbf{n} = \boldsymbol{\sigma}_a \mathbf{n} + \Gamma \kappa \mathbf{n} + \boldsymbol{\sigma}_d \mathbf{n}.$$

Euler equations

- neglect viscosity and surface tension
- Euler: for $t > 0$, $z \geq \eta(t, \boldsymbol{x}_\perp)$

$$\nabla \cdot \boldsymbol{u} = 0,$$
$$\rho \partial_t \boldsymbol{u} + \rho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \rho \boldsymbol{g},$$

- kinematic boundary condition: at $z = \eta(t, \boldsymbol{x}_\perp)$

$$w\,(t, \boldsymbol{x}_\perp, z) = \partial_t \eta + \boldsymbol{u}_\perp \cdot \nabla_\perp \eta,$$

- dynamic boundary condition: at $z = \eta(t, \boldsymbol{x}_\perp)$

$$p = p_a + p_d,$$

where

$$p_d = \int_{(\boldsymbol{v}-\boldsymbol{u})\cdot\mathbf{n}<0} \int_{]0,+\infty[} m(r) |(\boldsymbol{v} - \boldsymbol{u})\cdot\mathbf{n}|^2 f(t, \boldsymbol{x}, \boldsymbol{v}, r)\, dr\, d\boldsymbol{v}.$$

Outline

Linear deep water surface gravity waves without rain

- velocity potential: $\boldsymbol{u} = \nabla\Phi$
- surface wave mode: $\eta = a\mathrm{e}^{\mathrm{i}\,(kx-\omega t)}$
- assumption 1: ak is small
- continuity equation and linearized kinematic BC give: $\Phi = -\mathrm{i}\,\frac{a\omega}{k}e^{kz}\mathrm{e}^{\mathrm{i}\,(kx-\omega t)}$
- dispersion relation: linearized momentum equation and dynamics BC $p = p_a$ gives

$$\omega = \sqrt{kg}$$

- real frequency: no attenuation, no amplification
- our goal:
  - extend this analysis with rain induced pressure
  - show that the frequency is complex

Linear deep water surface gravity waves with falling rain

- velocity potential: $\boldsymbol{u} = \nabla \Phi$
- surface wave mode: $\eta = a\mathrm{e}^{\mathrm{i}\,(kx-\omega t)}$
- assumption 1: ak is small
- linear theory gives: $\Phi = -\mathrm{i}\,\frac{a\omega}{k} e^{kz} \mathrm{e}^{\mathrm{i}\,(kx-\omega t)}$
- dispersion relation: we need the dynamic condition

$$p = p_a + p_d\,,$$

with the rain induced pressure

$$p_d = \int_{(\boldsymbol{v}-\boldsymbol{u})\cdot\mathbf{n}<0} \int_{]0,+\infty[} m(r) |(\boldsymbol{v} - \boldsymbol{u})\cdot\mathbf{n}|^2 f(t, \boldsymbol{x}, \boldsymbol{v}, r)\, dr\, d\boldsymbol{v}.$$

Linearized rain induced pressure

- assumption 2: homogeneous in space and monokinetic rain drop distribution

$$f(t, \boldsymbol{x}, \boldsymbol{v}, r) = n(r)\delta_{\boldsymbol{v}-\boldsymbol{v}_{\boldsymbol{d}}(r)}$$

- use this to compute the rain induced pressure :

$$p_d = \int_{(\boldsymbol{v}-\boldsymbol{u})\cdot\mathbf{n}<0} \int_{]0,+\infty[} m(r)|(\boldsymbol{v} - \boldsymbol{u})\cdot\mathbf{n}|^2 f(t, \boldsymbol{x}, \boldsymbol{v}, r)\, dr d\boldsymbol{v}.$$

- assumption 3: $(\boldsymbol{v}_d(r) - \boldsymbol{u})\cdot\mathbf{n} < 0$ for every r (add impact of non-impacting slow drops)

$$p_d = \int_{]0,+\infty[} m(r)n(r)|(\boldsymbol{v}_d(r) - \boldsymbol{u})\cdot\mathbf{n}|^2\, dr.$$

Linearized rain induced pressure

- assumption 4: neglect $(ak)^2$, $a^2 k\omega$ and $ak(a\omega)^2$ terms $\Rightarrow$

$$p_d = p_d^{(0)} + p_d^{(1)},$$

where

$$\begin{aligned} p_d^{(0)} &= \int_{]0,+\infty[} m(r)n(r)w_d(r)^2\, dr, \\ p_d^{(1)} &= (Ik + J\omega)\mathrm{i}\, a\mathrm{e}^{\mathrm{i}\,(kx-\omega t)}. \end{aligned}$$

where I and J are constants depending on moments of the drop distribution.

$$I = -2\int_{]0,+\infty[} m(r)n(r)u_d(r)w_d(r)\, dr, \qquad J = 2\int_{]0,+\infty[} m(r)n(r)w_d(r)\, dr.$$

Dispersion relation

- we find complex frequencies: $\omega = \omega_R + \mathrm{i}\,\omega_I$
- real part:

$$\omega_R = \pm\sqrt{\frac{1}{2}\left(\sqrt{\left(kg - \frac{k^2 J^2}{4\rho^2}\right)^2 + \frac{k^4 I^2}{\rho^2}} + kg - \frac{k^2 J^2}{4\rho^2}\right)}.$$

- imaginary part:

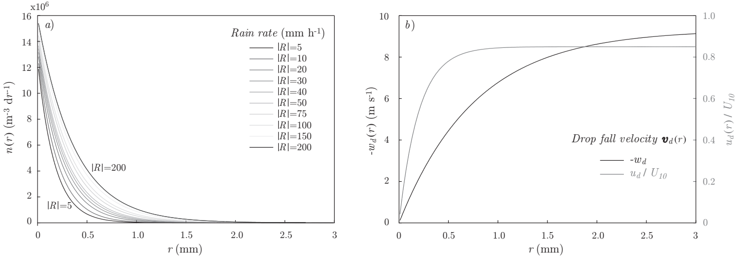
$$\omega_I = \frac{kJ}{2\rho} + sign(I)\sqrt{\frac{1}{2}\left(\sqrt{\left(kg - \frac{k^2 J^2}{4\rho^2}\right)^2 + \frac{k^4 I^2}{\rho^2}} - kg + \frac{k^2 J^2}{4\rho^2}\right)},$$

attenuation if $\omega_I < 0$, amplification if $\omega_I > 0$.

Outline

Rain size and velocity distributions

- Marshall-Palmer size distribution: $n(r) = \mathcal{N} \mathrm{e}^{-344.34|R|^{-0.21} r}$, where R is the *rain rate* (in m s^{-1})
- vertical fall velocity (Best, 1950): $w_d(r) = -\Upsilon_w (1 - \mathrm{e}^{-\alpha_w r})$, where $\Upsilon_w = 9.32$ m s^{-1} is the maximum fall velocity
- rain drop horizontal velocity (Mueller, Veron 2009) $u_d(r) = \Upsilon_u (1 - \mathrm{e}^{-\alpha_u r})$, where $\Upsilon_u = 0.85 U_{10}$ (85% of the 10-m wind speed)



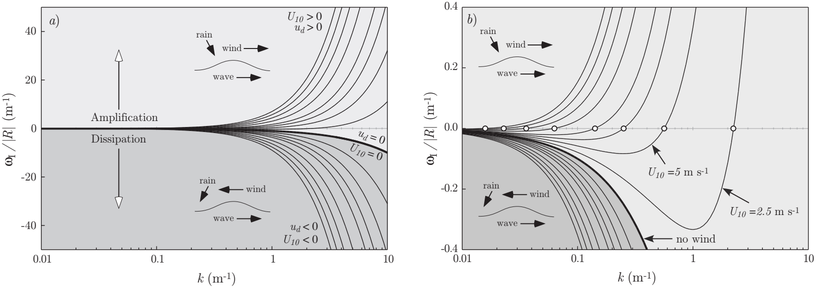
Validity of assumptions

- inviscid irrotational flows: $\lambda > O(1)$m
- Smoothness of the free surface and the validity of the kinetic approach: largest drops ($r \approx 3$mm) generate impact craters of radii $R \approx 30$mm $\Rightarrow$ the surface is smooth if $\lambda > 100R \approx 3$m
- wave slope and velocity: $(ak)^2$, $a^2 k\omega$ and $ak(a\omega)^2$ are negligeable if $\lambda < 250$m (and surface velocity $a\omega < 1$m s^{-1})
- "impact" of non-impacting drops: the error made on the true pressure is at most 10^{-4}%.

summary: analysis valid for $O(1-3)\ \mathrm{m} < \lambda < O(250)\ \mathrm{m}$

Results

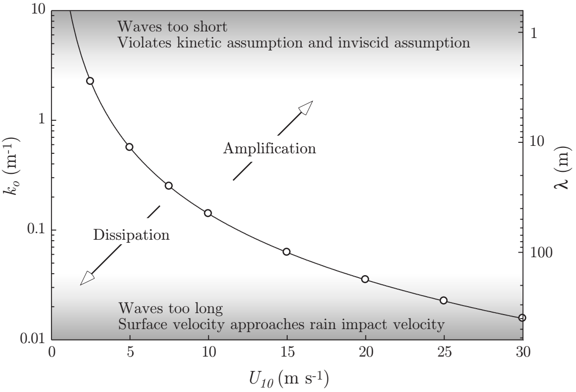
- $\omega_R \approx \sqrt{kg}$
- ω_I as a function of the wave number k, for various wind-speeds:



- for $U_{10} < 0$ (wind and rain opposed to the wave): attenuation
- for $U_{10} > 0$ (wind, rain, and waves travel in the same direction): attenuation or amplification

Transition from dissipation to amplification

- transition at $k_o = gJ^2/I^2$
- k_o as a function of U_{10}:



Example

- $U_{10} = 0$ m s^{-1} (no wind)
- rain rate of $|R| = 50$ mm h^{-1}
- the amplitude of a wave of $\lambda = 5$m is damped
  - at 94% after 1 hour (at 12% in wave energy density)
  - at 88% after 2 hours (22%)
  - at 73% after 5 hours (46%)

Comparison with Le Méhauté and Khangaonkar

- Le Méhauté and Khangaonkar, 1990:

$$\omega_I = \frac{\rho_d R}{2\rho}\left(2k - \frac{2U_d k^2}{\omega_R}\right).$$

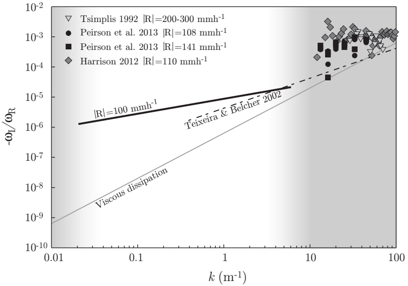
- our approach, with homogeneous and monokinetic rain distribution, can be viewed as an extension
- retains the dependence of the rain-induced momentum flux on the velocity and size of the drops
- possibly much more general f
- possible inclusion of drop specific parameters such as temperature

Comparison with experimental data

- experiments all use short waves generated in laboratory
- fall outside the wavenumber validity range of our theory
- example 1: Peirson et al. 2013 $\lambda \approx 0.5$m and $|R| = 141$ mm h^{-1} attenuates 5 times faster as our theory predicts
- example 2: laboratory observations show weak dependence of the wave attenuation rate on the rain rate, at least at high rain rates. This result is also in contrast with our theory.

Comparison with experimental data

- example 3: attenuation as a function of k (no wind) predicted by our theory (black thick line), a theory taking into account subsurface turbulence (dash line), and experiments



these data show significant scatter...

Perpectives

- experiments up to large k?
- Navier-Stokes simulations with rain stress
- include mass and heat transfer, difference in density (salinity) between rain drops and sea water
- for short laboratory scales: account for subsurface turbulence and wave-turbulence interaction