Sur quelques questions de géométrie lorentzienne

Soutenance d'Habilitation à Diriger des Recherches

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Lorentzian geometry

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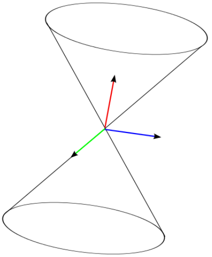

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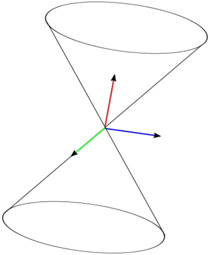

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- A Lorentzian manifold is naturally endowed with a connection, i.e. a way to compute the acceleration, and a volume form (if it is orientable).

Personal themes of research

Space of Lorentzian metrics

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Lorentzian surfaces with a Killing field

- Spacelike Zoll surfaces,
- Conjugate points,
- Extension and classification problems.

A Killing field on a Lorentzian surface (S, g) is a (complete) vector field K such that $\mathcal{L}_K g = 0$ i.e. whose flow is a one parameter group of isometries.

The Clifton-Pohl torus

It is the quotient of $\mathbf{R}^2 \setminus \{(0,0)\}$ endowed with the metric $\frac{2dxdy}{x^2+y^2}$ by an homothety, usually $(x, y) \mapsto 2(x, y)$.
The radial field of $\mathbf{R}^2$ induces a periodic Killing field on the torus.

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We will use these foliations in order to give a representation of the surface.

A picture of the CP-torus

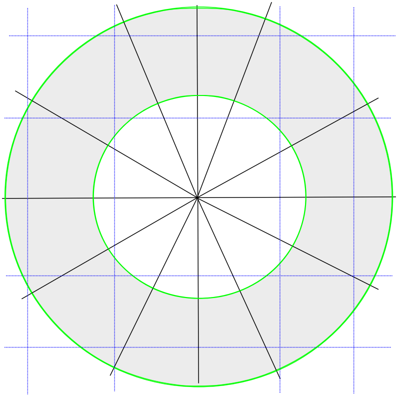

Figure: The three foliations of the Clifton-Pohl plane

Incompleteness of the Clifton-Pohl torus

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Proof: Let F be the following simple change of variables:

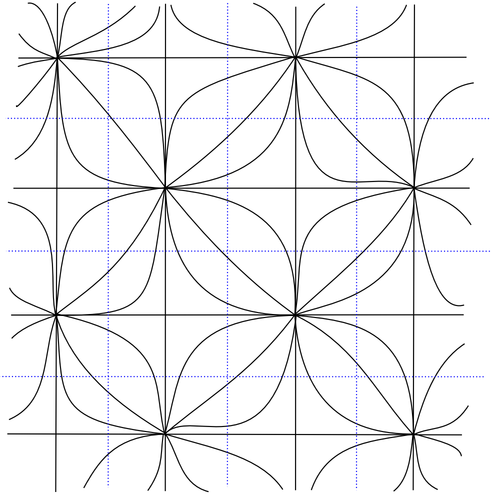
$$\begin{aligned} F : \,]-\pi/2, \pi/2[^2 \setminus \{0\} &\to M \\ (u,v) &\mapsto (\tan(u), \tan(v)). \end{aligned}$$

The metric F^*g reads

$$\frac{2dudv}{\cos^2 u \sin^2 v + \sin^2 u \cos^2 v}.$$

This metric extends to a metric $\hat{g}$ on $\mathbf{R}^2 \setminus \Lambda$. $\square$

The extended CP-plane



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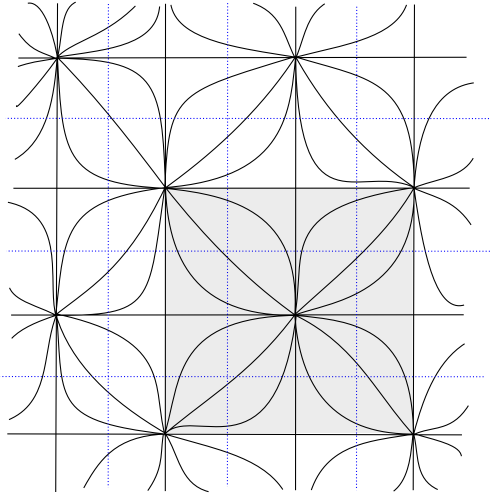

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The CP-torus has no conjugate points.

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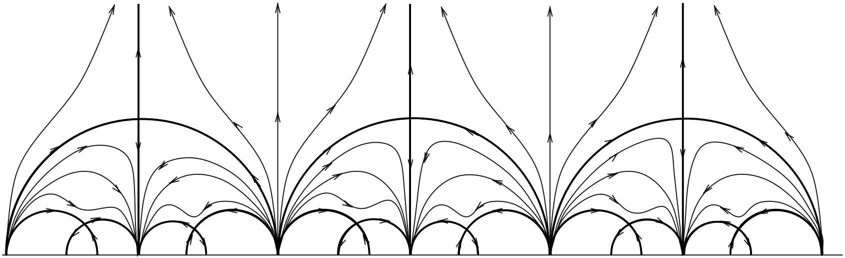
The metric $\hat{g}$ is invariant under an action of $\mathbf{Z}^2$ and therefore gives a metric on a twice punctured torus.
Its fundamental group can be seen as a Fuchsian group.

The universal cover of the extended plane



The properties of the CP-torus we saw are in fact quite general:

Theorem 2 (Bavard, M– (2015))

Let T be smooth (resp. real analytic) Lorentzian 2-torus having a non trivial Killing field K. Then the universal cover of T admits a smooth (resp. real analytic) lightlike complete extension E. In the real analytic case this extension is unique.

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It is possible to obtain uniqueness in the smooth setting by asking the extension to have additional symmetries.
When K has no lightlike orbits, T is geodesically complete and the extension is trivial.

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Theorem 3 (Bavard, M– (2015))

If T is real analytic and nonelementary then the exact sequence:

$$0 \to \mathbf{R} \simeq \mathrm{Isom}^0(E) \to \mathrm{Isom(E)} \to \Pi \to 1$$

is split.
Moreover, the action of Π (through the choice of a section) on E is conjugate to the action of a Fuchsian group on the hyperbolic space.

Universal extensions

Let $f : \mathbf{R} \to \mathbf{R}$ be a smooth function that vanishes somewhere. We will assume to simplify things that the connected components of $\mathbf{R} \setminus f^{-1}(0)$ do not cluster. We will say that f is of *finite type*.

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We start with the Lorentzian surface $\mathrm{Rib}_f = (\mathbf{R}^2, 2dxdy + f(x)dy^2)$. It has a Killing field ∂_y but is not lightlike complete (because f vanishes).

Remark

For any connected component C of $\mathbf{R} \smallsetminus f^{-1}(0)$ there exists a symmetry σ_C of $C \times \mathbf{R}$ fixing a geodesic orthogonal to K and permuting the lightlike foliations.

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On $C \times \mathbf{R}$ there exist coordinates (u, v) such that the metric reads $\pm du^2 + h(u)dv^2$ and $\sigma_C(u, v) = (u, -v)$.

The surface X_f

We take 2 copies of Rib_f that we glue together thanks to the symmetries σ_C (we choose one symmetry on each connected component). We obtain a (Hausdorff) Lorentzian surface X_f which also has a Killing field K.

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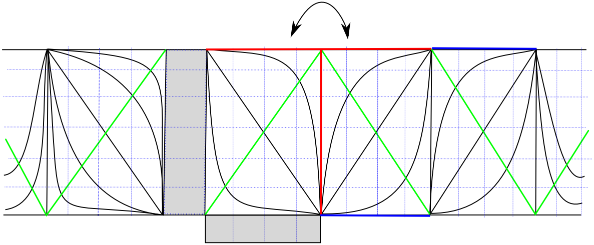

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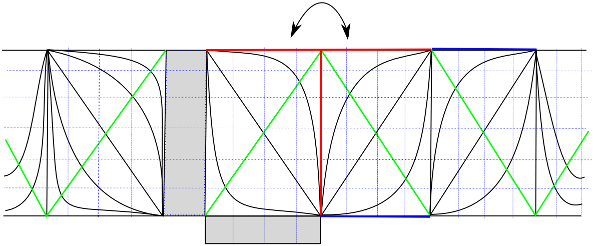
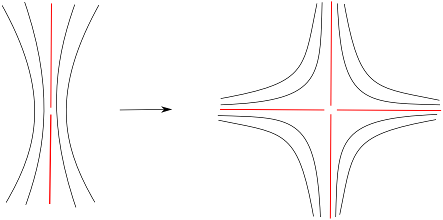


Figure: The surface X_f

The surface X_f is almost lightlike complete. The lightlike (i.e. the vertical and horizontal) orbits of K may carry half complete geodesics. It is the case of the geodesics forming the red T.

Addition of saddle points

To conclude the construction we take a 2-cover Y_f of X_f in order to transform each "half saddle" (like the red T) into a punctured saddle.



For a good choice of gluing symmetries σ_C the punctured saddles can be completed into saddles (i.e. we can add the blue point).
The surface obtained is now lightlike complete, we take its universal cover and call it the universal extension associated to f and denote it E_f^u.

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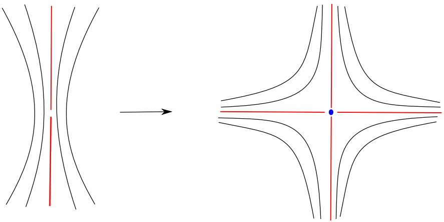

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Link between E_f^u and $\widetilde{T}$

Theorem 2 is obtained by proving that there exists a function f (not necessarily of finite type) such that T is locally modeled on E_f^u and that the associated developing map in injective.
Theorem 3 is a special case of the following result about $\mathrm{Isom}(E_f^u)$.

Theorem 4

Let $f : I \to \mathbf{R}$ be a smooth function and E_f^u the associated extension. If $f''' \neq 0$ then the exact sequence:

$$0 \to \mathbf{R} \simeq \mathrm{Isom}^0(E) \to \mathrm{Isom}(E_f^u) \to \Pi \to 1$$

is split.
Moreover, the action of Π (through the choice of a section) on E_f^u is proper (and therefore conjugate to the action of a Fuchsian group on the hyperbolic space) if and only f is of finite type.

Some consequences

The spaces E_f^u give a better understanding of the Lorentzian surfaces with Killing fields:

- By looking at the space of tori modeled on a given space E_f^u, we were able to give a complete classification of Lorentzian tori and Klein bottles with a Killing field.
- We know the universal cover of any non compact real analytic Lorentzian surface with a Killing field that satisfies a rather weak completeness condition.
- Maybe more anecdotally, we found a Lorentzian surface S such that $\mathrm{Isom}(S)$ is a 2-torsion group acting non properly on S.
- Thanks to this understanding, with Stefan Suhr, we gave three families of explicit examples of spacelike Zoll surfaces.