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A RESIDUAL DISTRIBUTIVE APPROACH FOR ONE-DIMENSIONAL TWO-FLUID MODELS AND ITS RELATION TO GODUNOV FINITE VOLUME SCHEMES

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ABSTRACT

The class of numerical schemes known as Fluctuation Splitting or Residual Distribution (RD) has now become an attractive alternative to more traditional Finite Volume (FV) and Finite Element (FE) approaches for the numerical solution of hyperbolic systems of conservation laws.
The application of the RD approach to complex systems of equations is now making its first steps through the introduction of consistent formulations for unsteady systems, non-homogeneous systems and of a more general conservative formulation of the method.
In this paper we review the results obtained in the last years by applying this new conservative RD approach to a one-dimensional Two-Fluid model. In particular, we will first highlight the equivalence of the one-dimensional formulation of the method with fully upwind conservative FV schemes proposed in literature for the solution of homogeneous and non-homogeneous systems, then we briefly explain how the non-conservative terms can be incorporated in the method and finally show some results obtained by solving the Two-Fluid model proposed by Städtke and his collaborators.
The paper is ended by some remarks on the extension of the approach to the multidimensional case.

1. INTRODUCTION

The class of numerical schemes known as Fluctuation Splitting or Residual Distribution (RD) has become in the last years an attractive alternative to standard Finite Volume (FV) and Finite Elements (FE) approaches for the solution of hyperbolic systems of conservation laws. The compactness of the method (second order of accuracy and monotonicity can be obtained on a minimal nearest neighbors stencil) and a reduced crosswind numerical dissipation with respect to standard FV schemes make its use very appealing. Results obtained in the past for perfect gas steady flow simulations have confirmed the potential of these schemes see (van der Weide, 1999), (Abgrall, 2001) and (Sermeus, 2003) for an overview.

The application of the RD method to general unsteady and non-homogeneous systems is still a topic of intense research. Nevertheless, some important first steps have been made in this direction: consistent second order accurate formulations for unsteady simulations have been introduced based either on a Finite Element interpretation of the method in space and time , as in (Csík, 2002), (Abgrall,

2003) or on a Lax-Wendroff procedure, as in (Hubbard, 2000), (Csík, 2000) and (De Palma, 2001); preliminary results on the design of consistent monotone second order accurate discretizations of relaxation type source terms have been obtained (Ricchiuto, 2002); more general formulations of the method allowing to retain conservation without the need of any conservative (Roe type) linearization of the jacobians of the system have been developed, (Csík, 2002) and (Abgrall, 2002).

In this paper we review the results obtained in the last few years by applying the new conservative Residual Distributive approach to a one-dimensional Two-Fluid model. In particular, after introducing the basics of the RD approach in one space dimension, we show how the method reduces to well-known conservative upwind Flux Vector Splitting and Finite Volume schemes. The equivalence of the standard RD method with Roe's Flux Difference Splitter in one space dimension has been already presented in literature by Wood (1997); here we repeat the analysis for the new conservative formulation of Csík *et al.* (2002) and for the case of non-homogeneous systems of equations. In particular, we show how upwind treatments of the source terms proposed in the past by different authors in the FV community can be naturally obtained within the RD framework. We also consider the issue of the discretization of systems containing non-conservative terms, appearing in most of the Two-Fluid models. Several alternatives are presented and discussed. Finally, we show some results obtained solving a specific Two-Fluid model on some two-phase test cases. In this work we have chosen to solve the model proposed by Städtke and his collaborators (Städtke, 1991) and (Städtke, 1997). The reason of this choice is mainly related to the full hyperbolicity of this model and especially to the possibility of computing analytically once and for all the complete eigenstructure of the system, heavily used in the RD method.

The structure of the paper is the following: in section 2 we present the basic concepts of the Residual Distribution approach as applied to one dimensional homogeneous conservation laws. We introduce most of the definitions for a scalar problem and then extend them to systems in section 2.2; in section 2.3 we highlight the differences between the conservative approach of Csík *et al.* (2002) with respect to the original formulation based on a conservative linearization and in section 2.4 we show how algebraic source terms are handled in this approach. Section 3 is devoted to the equivalence with Godunov Finite Volume schemes and in section 4 we discuss the introduction into the discretization of non-conservative products. Finally, in section 5, we present some sample results. In particular, we first give some generalities on the Two-Fluid model proposed by Städtke and his collaborators and then show results on a few numerical benchmarks. In section 6 we summarize the main results contained in the paper and give some remarks on the extension of the method to the multidimensional case.

2. RESIDUAL DISTRIBUTION SCHEMES: BASIC CONCEPTS

In this section we introduce the main ideas at the basis of the RD approach in one space dimension. Far from being a review on these schemes, this part of the paper is intended just to give a feel of the philosophy behind the method. For more details, the reader is referred to the extensive literature on Residual Distribution schemes given in the references.

2.1 RD schemes for scalar advection

We consider here the solution of the following linear problem:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{1}$$

on the one dimensional spatial domain Ω with the appropriate initial conditions and boundary conditions on ∂Ω. In equation (1) u is an unknown scalar field and a is a known advection speed, independent of u. Although we could be more general, we consider a discretization of Ω composed of segments of equal length h, our numerical unknown being some continuous interpolant of the values of u in the nodes of the grid.

At the very basis of the RD method there is the general idea, due to the work of P.L. Roe (1982), that the nodal values of the numerical unknown evolve in time according to numerical signals coming from the elements containing the node. These signals turn out to be dependent on (most of the times proportional to) the net amount of u which is advected through each element at a given time (net flux balance).

To put this in equations, consider a generic segment E of the mesh; we then define the so-called *Fluctuation* or *Residual* of element E as:

$$\phi_E = \int_E a \frac{\partial u}{\partial x}\, dx \tag{2}$$

In order to obtain a computable definition of the residual we define our numerical unknown u^h, as the linear interpolant of the nodal values of u, namely:

$$u^h = \sum_i u_i N_i \tag{3}$$

where i is the generic node of the grid and the interpolation functions N_i are the linear tent-shaped functions commonly used in the Finite Elements method (see Figure 1).

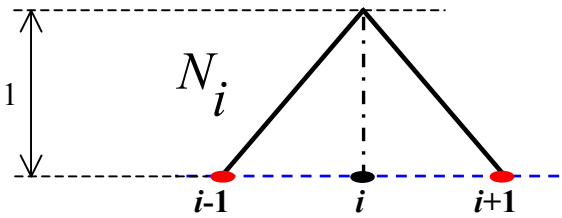

Figure 1. Nodal Shape function N_i

Substituting equation (3) into the definition of the element residual, equation (2), we obtain the following computable expression of the cell fluctuation:

$$\phi_E^h = \int_E a \frac{\partial u^h}{\partial x}\, dx = a_E \Delta u_E \tag{4}$$

where Δu_E is the variation of u on the element and a_E is the average speed, defined as

$$a_E = \frac{1}{h}\int_E a\ dx \tag{5}$$

Each node of the element E will receive a local residual (signal), which is some portion of the element residual. In other words, the fluctuation is distributed to the nodes of the element. In particular, denoting with ϕ_E^i the amount of residual that a generic node i receives from element E, the variation in time of the nodal value of the unknown (total nodal residual) is obtained by assembling the contributions of all the surrounding elements and is given by (see Figure 2):

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} h = -\sum_{E, i \in E} \phi_E^i \tag{6}$$

where Δt is the time step.

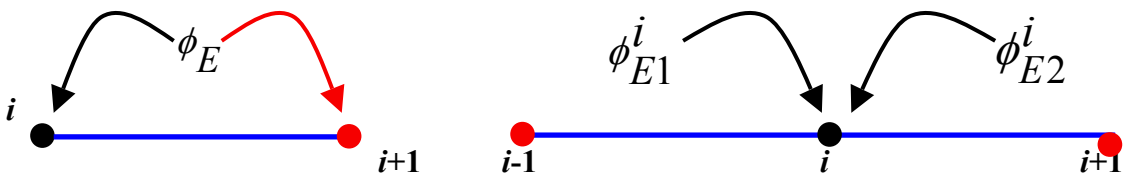

Figure 2. Residual Distribution and Residual Assembly

The properties of the scheme obtained in this way are clearly entirely determined by the definition of the local residuals ϕ_E^i. Particularly important is the following *consistency* requirement:

$$\sum_{v\in E}\phi_E^v = \phi_E \tag{7}$$

Introducing the so-called *distribution coefficients* defined as

$$\beta_E^v = \frac{\phi_E^v}{\phi_E} \tag{8}$$

consistency requires that

$$\sum_{v\in E}\beta_E^v = 1 \tag{9}$$

As an example, with the choice $\beta_E^v = 1/2\ \forall v \in E$ one would end up with central differencing. Also central schemes with artificial dissipation, including Lax-Wendroff schemes, can be recast in this formalism, (Deconinck, 2003). The particular scheme considered in this paper is an upwind scheme. For problem (1), upwinding is obtained by checking the sign of the average speed a_E. In particular, if i and i+1 are the nodes of E, upwind schemes respect the rule:

$$\begin{aligned} a_E > 0 &\Rightarrow \phi_E^i = 0,\ \phi_E^{i+1} = \phi_E \\ a_E < 0 &\Rightarrow \phi_E^i = \phi_E,\ \phi_E^{i+1} = 0 \end{aligned} \tag{10}$$

As a matter of fact constraints (7) and (10) define a unique upwind scheme for which the distribution coefficients can be written as:

$$\beta_E^i = \frac{a_E^-}{a_E},\quad \beta_E^{i+1} = \frac{a_E^+}{a_E} \tag{11}$$

with $a_E^- = \min(a_E, 0)$ and $a_E^+ = \max(a_E, 0)$. Combining equation (11) with the definition of the distribution coefficients (8) and with the expression of the element residual (4), we finally obtain the expression of the local residuals normally implemented:

$$\phi_E^i = a_E^- \Delta u_E,\quad \phi_E^{i+1} = a_E^+ \Delta u_E \tag{12}$$

The scheme defined by (6) and (12) is the starting point of our method. In (Deconinck, 2003), this scheme is further analyzed to show some relations with Finite Elements methods and further additional and general properties. In the same references the reader will find a more general overview on RD schemes, which is out of the scope of this paper.

2.2 Extension to Linear systems

Consider the following hyperbolic system of PDEs:

$$\frac{\partial U}{\partial t} + A\cdot\frac{\partial U}{\partial x} = 0 \tag{13}$$

on a one dimensional spatial domain Ω with the appropriate initial conditions and boundary conditions on ∂Ω. In equation (13) U is an unknown vector field and A is a known constant matrix. Because the system is hyperbolic, A is diagonalizable with all real eigenvalues and real linearly independent eigenvectors. We shall denote with Λ the diagonal matrix containing the eigenvalues of A and with R the matrix whose columns are the eigenvectors of A, so that

$$A = R\Lambda R^{-1} \tag{14}$$

The extension of the scheme presented in the previous section is quite straightforward. In particular, system (13) is solved on a regular discretization of Ω composed of segments of length h; on each segment we compute the vector of the residuals as

$$\Phi_E^h = \int_E A \cdot \frac{\partial U^h}{\partial x}\, dx = A\,\Delta U_E \tag{15}$$

The cell residual is then distributed to the nodes of the element according to the rules of consistency and upwinding. In the case of a system, upwinding is applied along the characteristics. This is easily obtained considering that the element residual can be decomposed in scalar characteristic residuals by multiplying it on the left by the inverse of the matrix of the eigenvectors and by defining the characteristic variables $W=R^{-1}U$. In terms of W variables, system (13) becomes a set of scalar problems of the type

$$W_t^k + \lambda_k \cdot W_x^k = 0 \tag{16}$$

where W^k is tha k-th characteristic variable, and λ_k the corresponding k-the eigenvalue of A. The cell residual along the k-th characteristic becomes

$$\left(\phi_E^h\right)^k = \lambda_k\, \Delta W_E^k \tag{17}$$

Each of the characteristic residual are distributed according to scheme (12), namely

$$\left(\phi_E^i\right)^k = \lambda_k^- \Delta W_E^k,\ \ \left(\phi_E^{i+1}\right)^k = \lambda_k^+ \Delta W_E^k \tag{18}$$

Once the local residuals in characteristic variables have been computed, the residuals in terms of the U variables are obtained simply as

$$\Phi_E^i = R\phi_E^i = R\Lambda^- \Delta W_E,\ \ \Phi_E^{i+1} = R\phi_E^{i+1} = R\Lambda^+ \Delta W_E \tag{19}$$

Inserting into last equation the definition of the characteristic variables finally gives for the local nodal residuals

$$\Phi_E^i = A^- \Delta U_E,\ \ \Phi_E^{i+1} = A^+ \Delta U_E \tag{20}$$

exactly like in the scalar case, with $A^+=R\Lambda^+R^{-1}$ and $A^-=R\Lambda^-R^{-1}$.

We remark that equations (20) and (6) give nothing else that the so-called CIR finite difference scheme (LeVeque, 2002) for system (13). We also note that for the nodal residuals (20) one can define *distribution matrices* gives by

$$\mathrm{B}_E^i = A^- A^{-1} = R\left(\Lambda^- \Lambda^{-1}\right)R^{-1},\ \ \mathrm{B}_E^{i+1} = A^+ A^{-1} = R\left(\Lambda^+ \Lambda^{-1}\right)R^{-1} \tag{21}$$

2.3 Non-linear systems and conservation

In this section we consider the extension of scheme (20) to hyperbolic non-linear systems of conservation laws. Consider then the following differential problem:

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0 \tag{22}$$

on a one dimensional spatial domain Ω with the appropriate initial conditions and boundary conditions on ∂Ω. In equation (22) U is an unknown vector field and $F(U)$ is the vector of the advective fluxes whose components are given by some non-linear function of the components of U.

The form (22) of a system of conservation laws is called the conservative form, since it is obtained directly from the macroscopic conservation laws through the local application of Gauss' theorem. Often (22) is written in quasi-linear form as:

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0 \tag{23}$$

where *A(U)* is the jacobian matrix $A(U)=\partial F/\partial U$. System (22) being hyperbolic, $A(U)$ is diagonalizable with real eigenvalues and real linearly independent eigenvectors.

It is well known that the CIR scheme (20) can be easily applied to system (22) through introduction of a local conservative linearization. This is done by replacing the matrix A in (20) with the jacobian $A(U)$ evaluated in a local averaged state $\overline{U}$ dependent on the values of U in the nodes forming element E. Such a linearization goes normally under the name of a conservative Roe linearization, since it was introduced by Roe (1981). The most important property of such a linearization is the so-called U-property:

$$A(\overline{U})\Delta U = \Delta F \tag{24}$$

This property guarantees that physical discontinuous solutions of system (13) are correctly computed by the numerical method. In the RD framework this is equivalent to the constraint that the element residual actually distributed to the nodes must be exactly equal to the net flux balance on the element, namely:

$$\Phi_E^h = \int_E \frac{\partial F(U^h)}{\partial x} dx = \Delta F_E \tag{25}$$

The requirement for the cell residual to be equal to the flux balance over the element, combined with the consistency constraint, give the so-called *conservation* property of a residual distribution scheme. We remark that in some way the issue of retaining conservation is in contrast with the upwinding: the first requires the use of the conservative form of the system for the computation of the residual, the second requires the evaluation of the jacobian and of its eigenstructure in some averaged state. In perfect gas inviscid and viscous multidimensional simulations, this property is traditionally guaranteed in the RD approach by the use of a multidimensional generalization of Roe's linearization (Deconinck, 1993). Unfortunately, the existence of such a linearization is not guaranteed for an arbitrary system of conservation laws. In cases in which it does not exist, the use of residual distribution schemes has to find different means to ensure conservation. Lately, two different approaches have been proposed in literature to achieve this goal in more than one space dimension (Csík, 2002) and (Abgrall, 2002). Here we shall not go into technical details, referring the reader to the references for more information. The approach followed here is the one-dimensional version of the one proposed in (Csík, 2002) where the authors present a simple and elegant trick that allows to decouple completely the computation of the element residual from the quasi-linear form of the system still allowing to retain the consistency of the method. This allows to compute the element residual approximating directly the net flux balance over the element through Gauss boundary integration, while evaluating the jacobians of the systems in any arbitrary state. The drawback of this approach is that, giving up in the use of the linearized quasi-linear form of the equations for the computation of the residual, one also makes a lot more difficult the theoretical analysis of the monotonicity of the schemes, which has to be checked numerically. However, the numerical evidence is that the approach of Csik *et al*. does not spoil the monotonicity properties of the underlying schemes. All the multidimensional upwind schemes presented in (Csík, 2002) reduce in on space dimension to a single upwind RD scheme defined by the local residuals (Ricchiuto, 2001):

$$\Phi_E^i = \mathrm{B}_E^i \Phi_E = \overline{A}^- \overline{A}^{-1} \Delta F_E, \quad \Phi_E^{i+1} = \mathrm{B}_E^{i+1} \Phi_E = \overline{A}^+ \overline{A}^{-1} \Delta F_E \tag{26}$$

where $\overline{A}$ is the jacobian *A(U)* evaluated in an arbitrary average state on E and the distribution matrices are basically the same as in the CIR scheme (see also equation (21)).

Finally, scheme (26) is the RD scheme actually used in all the computations.

2.4 Non-homogeneous systems and source terms

We consider now the following non-homogeneous hyperbolic system of conservation laws written in conservative form as:

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = S(U) \tag{27}$$

where $S(U)$ is the vector of the source terms. We consider here the case in which the components of S do not contain derivatives of U, but depend on its components through some algebraic relations.

The extension of scheme (26) to the inhomogeneous case can be easily achieved by modifying the definition of the element residual (25) as follows:

$$\Phi_E^h = \int_E \left[\frac{\partial F(U^h)}{\partial x} - \mathrm{S}(U^h) \right] dx = \Delta F_E - \int_E \mathrm{S}(U^h) dx \tag{28}$$

The source term integral is then evaluated with the second order accurate trapezium quadrature formula. We then distribute the residual defined by (28) according to (26) without any other change.

It must be remarked that in the RD framework, an upwind discretization of the source term is naturally obtained following the philosophy that the variation in time of the nodal value of the unknown is directly linked to the balance of flux gradient and source terms on each element, represented by the residual. Examples of multidimensional generalizations of such a scheme are given in (Ricchiuto, 2002) and (Caraeni, 2000).

3. EQUIVALENCE WITH GODUNOV FINITE VOLUME SCHEMES

In this part of the paper we show the equivalence between the one-dimensional residual distribution schemes presented in the previous section with Finite Volume schemes widely used in literature. We follow the analysis made in (Wood, 1997) where the author has compared the RD schemes based on a conservative linearizarion to Roe's approximate Riemann solver in the solution of the one-dimensional perfect gas Euler equations.

In order to establish the equivalence between the RD method and any FV scheme, we start by observing that while the first leads to algebraic evolution equations for the nodal values of a linear polynomial, the second evolves average values of the unknown (see Figure 3). So one has to keep in mind that whenever we refer to the unknown value U_i, in the RD case we refer to the value of the numerical unknown in node i of the mesh, while the FV unknown is the average value in the computational cell [i-1/2 ; i+1/2].

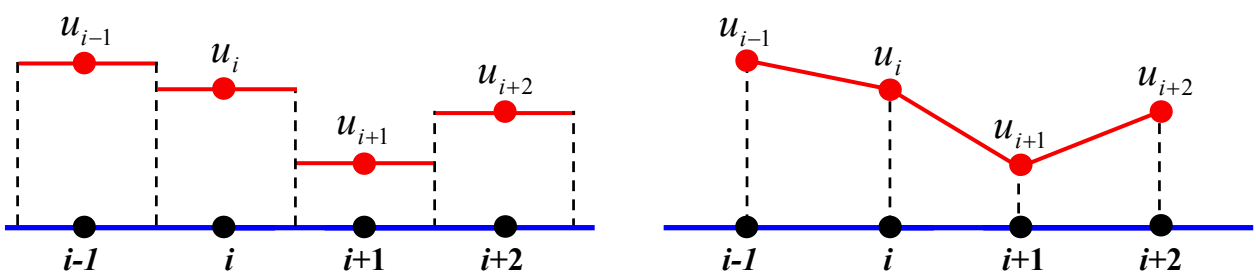

Figure 3. FV Piecewise constant (left) and RD piecewise linear (right) variable representation

Starting from this observation, we can show that the evolution equation obtained for U_i with the RD scheme defined by (6), (25) (or (28)) and (26), can be manipulated such that we get a typical FV scheme for the corresponding average value.

3.1 Conservative RD and FV: homogeneous systems

We consider now the case of the homogeneous system (22). Combining (6), (25) and (26) we obtain:

$$\frac{U_i^{n+1}-U_i^n}{\Delta t}h=-\overline{A}_{i-1/2}^{+}\overline{A}_{i-1/2}^{-1}\left(F_i-F_{i-1}\right)-\overline{A}_{i+1/2}^{-}\overline{A}_{i+1/2}^{-1}\left(F_{i+1}-F_i\right) \tag{29}$$

In itself, (29) represents already a conservative Flux Difference Splitter (Deconinck, 1991) but one can recast it in a more traditional FV notation: remembering that $\overline{A}^{\pm}=\left(\overline{A}\pm\left|\overline{A}\right|\right)/2$, (29) can be rewritten as

$$\begin{aligned}\frac{U_i^{n+1}-U_i^n}{\Delta t}h=&-\frac{1}{2}\left(F_i-F_{i-1}\right)-\frac{1}{2}\left|\overline{A}\right|_{i-1/2}\overline{A}_{i-1/2}^{-1}\left(F_i-F_{i-1}\right)\\&-\frac{1}{2}\left(F_{i+1}-F_i\right)-\frac{1}{2}\left|\overline{A}\right|_{i+1/2}\overline{A}_{i+1/2}^{-1}\left(F_{i+1}-F_i\right)\end{aligned}\quad; \tag{30}$$

denoting with $\overline{D}$ the sign of the jacobian matrix $\overline{A}$, considering that $\left|\overline{A}\right|\overline{A}^{-1}=\mathrm{sign}\left(\overline{A}\right)=\overline{D}$ and adding and subtracting F_i we finally end up with the conservative FV scheme

$$\frac{U_i^{n+1}-U_i^n}{\Delta t}h=-\left(H_{i+1/2}-H_{i-1/2}\right) \tag{31}$$

with the numerical flux functions $H_{i\pm1/2}$ given by

$$H_{i\pm1/2}=\frac{1}{2}\left(F_i+F_{i\pm1}\right)\pm\frac{1}{2}\overline{D}_{i\pm1/2}\left(F_{i\pm1}-F_i\right) \tag{32}$$

Equations (31) and (32) show the equivalence of our conservative upwind RD scheme with the conservative FV scheme of Huang (1981). The flux function (32) can be recast as

$$H_{i\pm1/2}=\left(\overline{A}_{i\pm1/2}^{\pm}\overline{A}_{i\pm1/2}^{-1}\right)F_i+\left(\overline{A}_{i\pm1/2}^{\mp}\overline{A}_{i\pm1/2}^{-1}\right)F_{i\pm1}, \tag{33}$$

which gives the conservative Flux Vector Splitting (FVS) scheme of Städtke *et al.* (1997).

The FV and FVS schemes (32) and (33) are widely used in the Two-Phase flow community (Toumi, 1999) and, as a matter of fact, they are the same scheme. This was pointed out also by Ghidaglia (1998), who introduced the more general class of Flux Schemes, to which (32) belongs. In this section we have then shown that, in one space dimension, also the conservative RD method of Csík *et al.* (2002) is a Flux Scheme, and it reduces to scheme (32). Of course, in the application of these schemes to two-phase flow simulations some differences arise, which are largely due to modeling issues and, numerically speaking, to the treatment of source and non-conservative terms.

3.2 Conservative RD and FV: inhomogeneous systems

In the case the non-homogeneous system (27), the discrete evolution equation for U_i is obtained combining equations (6), (26) and (28):

$$\begin{aligned}\frac{U_i^{n+1}-U_i^n}{\Delta t}h=&-\overline{A}_{i-1/2}^{+}\overline{A}_{i-1/2}^{-1}\left(F_i-F_{i-1}\right)+\overline{A}_{i-1/2}^{+}\overline{A}_{i-1/2}^{-1}\int_{x_{i-1}}^{x_i}S\left(U^h\right)dx\\&-\overline{A}_{i+1/2}^{-}\overline{A}_{i+1/2}^{-1}\left(F_{i+1}-F_i\right)+\overline{A}_{i+1/2}^{-}\overline{A}_{i+1/2}^{-1}\int_{x_i}^{x_{i+1}}S\left(U^h\right)dx\end{aligned} \tag{34}$$

Proceeding as before we get

$$\frac{U_i^{n+1}-U_i^n}{\Delta t}h=-\left(H_{i+1/2}-H_{i-1/2}\right)+\overline{A}_{i-1/2}^{+}\overline{A}_{i-1/2}^{-1}\int_{x_{i-1}}^{x_i}S\left(U^h\right)dx+\overline{A}_{i+1/2}^{-}\overline{A}_{i+1/2}^{-1}\int_{x_i}^{x_{i+1}}S\left(U^h\right)dx \tag{35}$$

Approximating the integrals of the source terms with the trapezium rule we obtain

$$\frac{U_i^{n+1}-U_i^n}{\Delta t}h=-\left(H_{i+1/2}-H_{i-1/2}\right)+\frac{h}{2}\overline{A}_{i-1/2}^{+}\overline{A}_{i-1/2}^{-1}\left(S_{i-1}+S_i\right)+\frac{h}{2}\overline{A}_{i+1/2}^{-}\overline{A}_{i+1/2}^{-1}\left(S_i+S_{i+1}\right) \tag{36}$$

Using again the relations $\overline{A}^{\pm}=\left(\overline{A}\pm\left|\overline{A}\right|\right)/2$ and $\left|\overline{A}\right|\overline{A}^{-1}=\overline{D}$, we obtain the update scheme

$$\frac{U_i^{n+1}-U_i^n}{\Delta t}h=-\left(H_{i+1/2}-H_{i-1/2}\right)+\sigma_{i-1/2}+\sigma_{i+1/2} \tag{37}$$

where the upwind interface source term contributions $\sigma_{i\pm 1/2}$ are given by

$$\sigma_{i\pm 1/2}=\frac{h}{2}\left(S_i+S_{i\pm 1}\right)\mp\frac{h}{2}\overline{D}_{i\pm 1/2}\left(S_i+S_{i\pm 1}\right) \tag{38}$$

which is the first order upwind source term interface flux proposed, among others, by Hubbard and Garcia-Navarro (2000). Note that equation (36) can be also rewritten as

$$\frac{U_i^{n+1}-U_i^n}{\Delta t}h=-\left(H_{i+1/2}-H_{i-1/2}\right)+\Sigma_i\,, \tag{39}$$

with Σ_i given by

$$\Sigma_i=h\frac{I+\overline{D}_{i-1/2}}{2}S_{i-1}+h\frac{2I+\overline{D}_{i-1/2}-\overline{D}_{i+1/2}}{2}S_i+h\frac{I-\overline{D}_{i+1/2}}{2}S_{i+1}\,, \tag{40}$$

which is the upwind scheme (37) in the form proposed in (Alouges, 1999) and (Ghidaglia, 1999).

As a final remark to this section we underline that this equivalence is peculiar to the one-dimensional case. In multiple space dimensions the upwind treatments of source terms proposed in the FV literature up to now, see (Hubbard, 2000) or (LeVeque, 1999) for example, are considerably different from the consistent treatments proposed in the residual distribution framework, which are largely based on the philosophy typical of residual schemes. The reader can refer to (Ricchiuto, 2002) or (Deconinck, 2003) for an overview on these latter aspects.

4. NON-CONSERVATIVE TERMS

Most of the Two-Fluid models currently used for two-phase flow simulations are differential models containing non-conservative products (Toumi, 1999). These non-conservative terms arise in the modeling of non-viscous interactions at the interface between the phases. In this section we will briefly discuss how these terms can be introduced into the residual distribution discretization.

We start by considering the prototype non-linear homogeneous system containing non-conservative products:

$$\frac{\partial U}{\partial t}+\frac{\partial F(U)}{\partial x}+B(U)\cdot\frac{\partial U}{\partial x}=0 \tag{41}$$

where $B(U)$ is some known matrix function of the components of U. The problem with differential systems like (41) is that, differently from the case of conservative systems like (22), physics gives us no guidelines for the design of the numerics. In particular, we know that problems in conservative form admit weak discontinuous solutions, which satisfy the original macroscopic integral conservation law. This leads directly to the concept of discrete conservation, *i.e.* the idea that our scheme should always guarantee that the integral conservation laws are satisfied at the discrete level. This, as seen in section 2.3, leads to the design of conservative methods. In the case of system (41),

we lack such a more general physical formulation of the problem, hence we do not know how to define weak solutions and if they are admissible. Here we do not discuss this problem, which has been and still is a very interesting research topic (Le Floch, 1988), (Dal Maso, 1995). We limit ourselves to present a few options allowing to include non-conservative products into the RD discretization.

First of all we rewrite the problem in the following quasi-linear form:

$$\frac{\partial U}{\partial t} + A(U) \cdot \frac{\partial U}{\partial x} = 0 \tag{42}$$

where the matrix of the system $A(U)$ is now given by

$$A(U) = \frac{\partial F(U)}{\partial U} + B(U) \tag{43}$$

We suppose that the system is hyperbolic and so that A admits always a set of real eigenvalues with real linearly independent eigenvectors. The knowledge of the eigenstructure of (43) allows us to compute the distribution matrices according to (26) without any problem. The remaining ingredient is the computation of the element residual. We define the fluctuation of an element E of our one-dimensional mesh as

$$\Phi_E^h = \int_E \left[\frac{\partial F(U^h)}{\partial x} + \mathrm{B}(U^h) \frac{\partial U_x^h}{\partial x} \right] dx \tag{44}$$

where U^h is the RD continuous linear interpolant. To compute the integral in (44) we have several options. The simplest idea, but also the most expensive, is to use Gauss' integration. Selecting a quadrature formula accurate enough, integral (44) could be computed exactly with respect to the linear variation of U^h. This approach, although correct in principle, would lead to the evaluation of the matrix B in several quadrature points. In the case of conservative systems, similar approaches can be shown to actually give correct results in terms of convergence to the correct weak solution, even in multiple space dimensions (Abgrall, 2002). In the case of a non-conservative problem, the use of Gauss' integration makes the scheme needlessly expensive, since no definition of physically correct weak solution is available. A different route would be to seek for a conservative linearization of the matrix $A(U)$. A lot of work in this sense has been done in (Toumi, 1992) and (Toumi, 1996) where design criteria for weak approximate Riemann solvers have been given. In our view, the problem of this method is how to define the correct jump relations, which eventually lead to the definition of the conservative average state. In (Toumi, 1996) the authors propose the use of a system of equations equivalent to (41), but in conservation form. Unfortunately, in principle there are more than one of such systems. Choosing among them means basically choosing the correct weak solutions of (41) which is our original problem. We have chosen to actually forget about the definition of conservation relations for system (41), remembering that, in two-phase flow models, the additional terms in the phasic momentum and energy equations are related to exchange processes in which one can physically claim conservation only at the global level of the mixture, since each phase is actually loosing or acquiring momentum or energy. As a consequence, we have chosen to treat the last part of integral (44) in a non-conservative way. In particular, we rewrite the element fluctuation as

$$\Phi_E^h = \int_E \left[I + \mathrm{M}(U^h) \right] \frac{\partial F(U^h)}{\partial x} dx = \int_E H(U^h) \frac{\partial F(U^h)}{\partial x} dx \tag{45}$$

where I is the identity matrix and $M(U)$ is defined as

$$M(U) = B(U) \left[\frac{\partial F(U)}{\partial U} \right]^{-1} \tag{46}$$

We then locally linearize in a non conservative fashion the matrix H(U), obtaining for the residual

$$\Phi_E^h = \overline{H}_E \Delta F_E \tag{47}$$

where the average matrix $\overline{H}_E$ is evaluated in an arbitrary state on E.

We remark that this approach is still conservative whenever the non-conservative part of the equations vanishes and, in the case of Two-Fluid models, for the mixture equations which are obtained by the sum of the phasic conservation equations, sum in which all the non-conservative terms cancel out. Additionally, the extra cost of the scheme is kept at the minimum: only an extra matrix evaluation and an extra matrix vector product are necessary.

5. APPLICATION TO A TWO-FLUID MODEL

In this section we show how the scheme described in the previous paragraphs has been applied to a Two-Fluid model. In particular, we have chosen to solve the model proposed by Städtke and his collaborators (1991, 1997). The reason of this choice is that this model is fully hyperbolic and that the eigenstructure of its jacobian can be analytically computed which makes easier the application of the scheme in which the eigenvalue decomposition of the jacobian is heavily used. After introducing briefly the governing equations, we present the full discrete numerical model obtained through the RD upwind scheme described earlier, including source and non-conservative terms. Finally we show results on two benchmark problems, namely a Riemann problem and a steady dispersed flow through a nozzle.

5.1 The Two-Fluid model of Städtke

The general form of a single pressure, inviscid Two-Fluid model for a two-phase gas-liquid flow in absence of phase change, inter-phase heat exchange, external heat sources and neglecting the effect of the surface tension, is the following (Toumi, 1999), (Ishii, 1975):

Conservation of mass

$$\frac{\partial(\alpha_k \rho_k)}{\partial t} + \frac{\partial(\alpha_k \rho_k u_k)}{\partial x} = 0; \quad k = g, l \tag{48}$$

Conservation of momentum

$$\frac{\partial(\alpha_k \rho_k u_k)}{\partial t} + \frac{\partial(\alpha_k \rho_k u_k^2)}{\partial x} + \alpha_k \frac{\partial p}{\partial x} = F_k^{\text{int}} + F_k^{ext}; \quad k = g,l \tag{49}$$

Conservation of energy

$$\frac{\partial(\alpha_k \rho_k E_k)}{\partial t} + \frac{\partial(\alpha_k \rho_k u_k H_k)}{\partial x} + p\frac{\partial \alpha_k}{\partial t} = F_k^{\text{int}} u_k^{\text{int}} + F_k^{ext} u_k^{\text{int}}; \quad k = g,l \tag{50}$$

where α_k is the volume fraction of phase k, ρ_k its density, u_k its velocity, E_k its specific total energy density, H_k its specific total enthalpy density, p is the pressure (common to both phases), F_k^{ext} represents the action of external forces on phase k (gravity, for example), u_k^{int} is an interfacial speed and F_k^{int} is a term modeling the interactions involving momentum exchange between the phases at the interface. The model is closed by the equations of state (EOS) of the two fluids, by the expressions of the external and interfacial forces and by the constitutive relations

$$\begin{aligned} &\alpha_l + \alpha_g = 1; \\ &F_l^{\text{int}} + F_g^{\text{int}} = 0; \\ &E_k = e_k + \frac{u_k^2}{2}; \\ &H_k = E_k + \frac{p}{\rho_k}. \end{aligned} \tag{51}$$

Concerning the EOS, in the computations presented here we have used both analytical and tabulated relations. In the first case, we have used the perfect gas EOS for the gas phase, representative of air,

while for the liquid phase we have chosen a stiffened gas law as in (Paillere, 2003), representative of water. The results obtained with the analytical EOS have also been compared to the ones obtained using the properties computed by the FORTRAN 77 library *STMF*, developed by Franchello.

For the closure of the system, particularly important is the model chosen for the interfacial momentum exchange terms F_k^{int}. These terms take into account both viscous and inviscid interactions. In particular, the viscous momentum exchange between the phases is modeled through a drag algebraic source term, which mimics the effect of the shear stresses at the interface where the phasic velocities must be the same. In the computations of this paper in which this term has been included, it has been modeled as:

$$\left(F_g^{\text{int}}\right)_{drag} = -\left(F_l^{\text{int}}\right)_{drag} = -\frac{3C_D}{8r_{sm}}\alpha_l \rho_g |u_r| u_r \tag{52}$$

with C_D=0.44, r_{sm}=0.125×10^{-3} m and the relative velocity u_r=u_g-u_l. The inviscid momentum exchange, instead, has to take into account different physical phenomena as, for example, virtual mass effects and lift. This term typically contains derivative of the dependent variables and, as a consequence, it affects the mathematical character of the problem. In particular, it has quite an important effect on the hyperbolicity of the system of equations. The system obtained by neglecting this term, known as the Wallis' model, is known to be non-hyperbolic and represents an ill-posed initial value problem. The Wallis' model is often regularized modeling only some of the physics of the interface through the introduction of *ad hoc* differential terms, which allow to show at least the conditional hyperbolicity of the model (Pokharna, 1997) and (Toumi, 1999). A discussion of this matter being out of our scope, we refer the reader to the bibliography for a more detailed overview (Toumi, 1999). The model chosen here includes in an original way most of the inviscid interactions and it includes at the same time differential terms proposed by different authors with the purpose of regularizing the system. This interaction term has been proposed by Städtke and his collaborators (1991, 1997) and has the remarkable property of rendering the system unconditionally hyperbolic. Moreover, it allows to compute analytically the complete eigenstructure of the jacobian of the system, which turns out to be quite handy from the point of view of the study of the model and also of the implementation of upwind schemes. The differential term proposed by Städtke has the following form:

$$\begin{aligned}\left(F_g^{\text{int}}\right)_{inv} = -\left(F_l^{\text{int}}\right)_{inv} = -\alpha_g \alpha_l \rho \left[C_{VM}\left(\frac{d^l u_g}{dt} - \frac{d^g u_l}{dt}\right) - \frac{\alpha_g \rho_l - \alpha_l \rho_g}{\rho} u_u \frac{\partial u_r}{\partial x}\right] \\ -\left(\rho_g + \rho_l\right)\alpha_g \alpha_l u_r^2 \frac{\partial \alpha_g}{\partial x} - \left(\rho_g + \rho_l\right)\alpha_g \alpha_l u_r^2 \left(\frac{\alpha_g}{\rho_g}\frac{d^g \rho_g}{dt} + \frac{\alpha_l}{\rho_l}\frac{d^l \rho_l}{dt}\right)\end{aligned} \tag{53}$$

where the coefficient C_{VM} has been set to 0.2 in all the computations presented and with the following definition of total derivative with respect to phase k

$$\frac{d^k}{dt} = \frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x} \tag{54}$$

As remarked in (Städtke, 1991), the first term in (52) represents a virtual mass force as proposed by Drew (1979). The second term, proportional to the gradient of the void fraction, is exactly the expression normally used to model the so-called interface pressure effect included in the model implemented in the CATHARE code and proposed by many authors in literature (Toumi, 1999). The last term, instead, takes into account the difference in compressibility of the two phases, which also might play an important role in the interfacial momentum exchange. As already mentioned, this model for the inviscid interactions between the phases has the remarkable property of rendering the system of equations unconditionally hyperbolic. Moreover the eigenstructure of the jacobian of the system can be computed analytically.

5.2 Full Discretization of the Two-Fluid model

We introduce the following vectors of conserved quantities U and conservative fluxes F :

$$U = \left[\alpha_g \rho_g \quad \alpha_l \rho_l \quad \alpha_g \rho_g u_g \quad \alpha_l \rho_l u_l \quad \alpha_g \rho_g E_g \quad \alpha_g \rho_g E_g\right]^t$$
$$F = \left[\alpha_g \rho_g u_g \quad \alpha_l \rho_l u_l \quad \alpha_g \rho_g u_g^2 + \alpha_g p \quad \alpha_l \rho_l u_l^2 + \alpha_l p \quad \alpha_g \rho_g H_g \quad \alpha_g \rho_g H_g\right]^t \tag{55}$$

The system of equations (48)-(50) with closure relations (51) and (53) can be analytically rewritten in the following quasi-conservative form:

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} + H^{nc}(U) \cdot \frac{\partial F(U)}{\partial x} = S(U) \tag{56}$$

We dicretize (56) on a one-dimensional grid using the RD scheme described in section 2. In particular, on a generic element E, we define and compute the element residual as

$$\Phi_E^h = \int_E \left\{ \left[I + \mathrm{H}^{nc}\left(U^h\right)\right] \frac{\partial F\left(U^h\right)}{\partial x} - S\left(U^h\right) \right\} dx = \left(I + \overline{H}_E^{nc}\right) \Delta F_E - \int_E S\left(U^h\right) dx \tag{57}$$

with $\overline{H}_E^{nc}$ the value of $H^{nc}(U)$ in an arbitrary average state over E. The last integral is approximated as in section 3.2, using the trapezium integration formula. We then distribute the residual to the two nodes of E. To do this in an upwind fashion, we first rewrite system (56) as

$$\frac{\partial U}{\partial t} + A(U) \frac{\partial U}{\partial x} = S(U) \tag{58}$$

where the matrix $A(U)$ is computed as

$$A(U) = \frac{\partial F(U)}{\partial U} + H^{nc}(U) \frac{\partial F(U)}{\partial U} \tag{59}$$

As anticipated in the previous paragraph, the system is hyperbolic, hence $A(U)$ admits a complete set of real eigenvalues and linearly independent eigenvectors, which can be computed analytically. For brevity, we do not report here the expression of these quantities, which can be found in (Städtke, 1991), (Rubino, 2002) and (Witteveen, 2003). The knowledge of the eigenstructure of A enables us to compute the distribution matrices defined in (26) obtaining the following discrete equation for node i :

$$\begin{aligned} \frac{U_i^{n+1} - U_i^n}{\Delta t} h = & -\overline{A}_{i-1/2}^{+} \overline{A}_{i-1/2}^{-1} \left(I + \overline{H}_{i-1/2}^{nc}\right) \left[\left(F_i - F_{i-1}\right) - \frac{h}{2}\left(S_i + S_{i-1}\right)\right] \\ & - \overline{A}_{i+1/2}^{-} \overline{A}_{i+1/2}^{-1} \left(I + \overline{H}_{i+1/2}^{nc}\right) \left[\left(F_{i+1} - F_i\right) - \frac{h}{2}\left(S_i + S_{i+1}\right)\right] \end{aligned} \tag{60}$$

5.3 Results

In this section we present the results of two two-phase flow benchmarks, obtained with the numerical model represented by equation (60), with the definitions given earlier.

Toumi's water-air shock tube problem

The first test case is a two-fluid shock tube (Toumi, 1999) and (Paillere, 2003). The nature of the time dependent solution of this problem depends heavily on the eigenstructure of the model chosen, since it is entirely governed by wave propagation phenomena. The set up of the problem is the following: a one-dimensional pipe of length 10 m is initially filled with two mixtures with different properties; in particular, we impose the initial conditions

$$\begin{bmatrix} \alpha_g \\ u_l \\ u_g \\ p \\ T_l \\ T_g \end{bmatrix}_L = \begin{bmatrix} 0.25 \\ 0 \\ 0 \\ 2\times 10^7\, Pa \\ 308.15\, K \\ 308.15\, K \end{bmatrix} \quad \text{if } x < 5\,\text{m}; \quad \begin{bmatrix} \alpha_g \\ u_l \\ u_g \\ p \\ T_l \\ T_g \end{bmatrix}_R = \begin{bmatrix} 0.10 \\ 0 \\ 0 \\ 10^7\, Pa \\ 308.15\, K \\ 308.15\, K \end{bmatrix} \quad \text{otherwise.} \tag{61}$$

At time t=0, the two states meet in the middle of the pipe. The structure of the solution is very well described in (Paillere, 2003): the interaction of the two states produces a left-traveling expansion wave, a right-traveling shock wave and two contact waves in the middle across which the pressure stays roughly constant (see Figure 4).

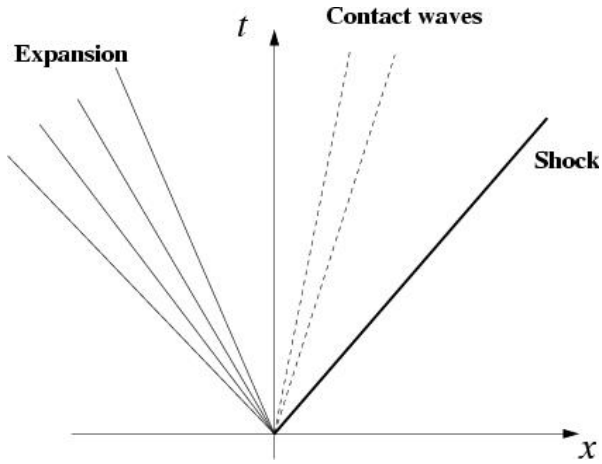

Figure 4. Wave pattern for Toumi's water-air shock tube problem

We present the results obtained at time t=0.006s, on a one-dimensional mesh containing 100 nodes. In the results we show the influence of the use of the analytical EOS with respect to the library developed by Franchello and also the influence of the presence of the interfacial drag for this problem. In figures 5 and 6 the results obtained without drag force are reported.

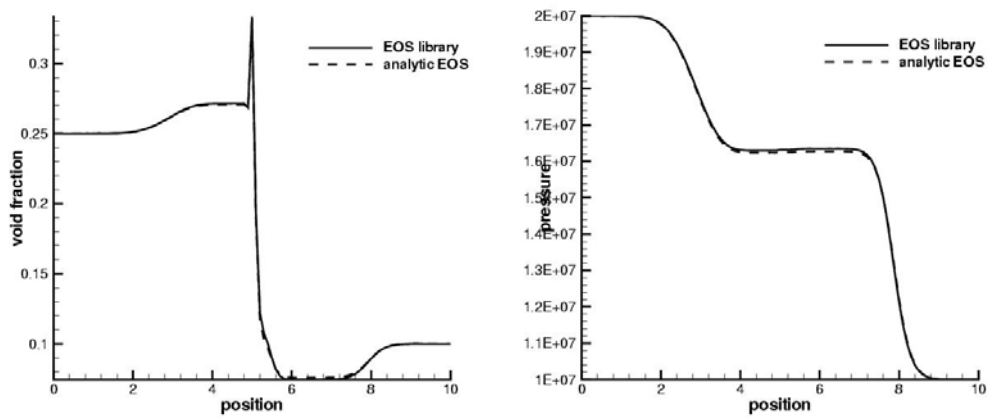

Figure 5. Toumi's water-air shock tube problem. Results at time t =0.006 s without drag force. Left: void fraction. Right: pressure

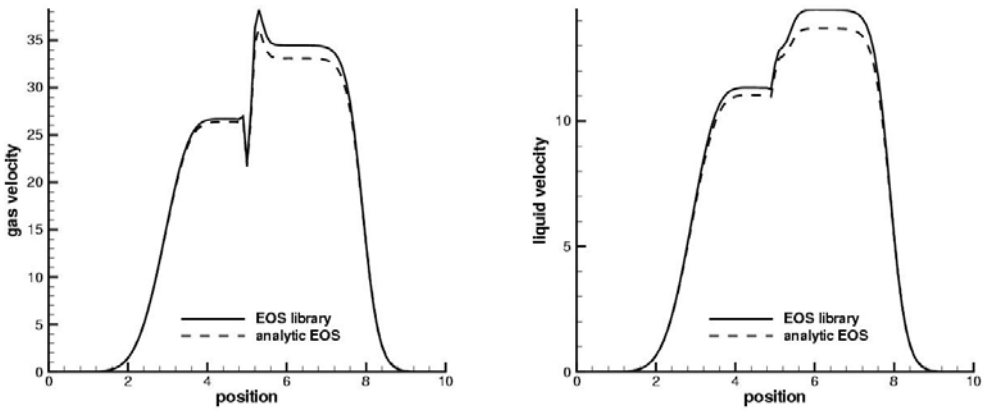

Figure 6. Toumi’s water-air shock tube problem. Results at time *t* =0.006 s without drag force. Left: gas velocity. Right: liquid velocity

The results of figures 5 and 6 show firstly that, for this range of pressures and temperatures, the difference in the use of analytical EOS with respect to the tabulated ones is small. Secondly, we see that the overall structure of the solution is the same of (Paillere, 2003). The main differences, in our opinion mainly due to the different models used, are in the speed of the shock, that is slower in our case, and in the fact that the weaker contact discontinuity is barely visible here, probably due also to the coarseness of the grid used. We also see that the values of the gas velocity are much smaller in our case. This could be due to the extra coupling between the phases, introduced in the model used here by the virtual mass force term in (53). In figures 7 and 8 we report the results obtained including in the model the drag force (52). Again, we see that little difference is observed between the solution obtained using the analytical EOS and the tables. Small differences are observed, with respect to the case without drag, in all the profiles in correspondence of the shock, which seem to have a smaller speed in this case. The spike in the void fraction is much smaller and the weaker contact seems to be completely disappeared. Major differences are of course visible in the velocities. In particular, the contact discontinuity has become much weaker, especially when using analytical EOS, and the values of gas and liquid velocity are much closer as expected.

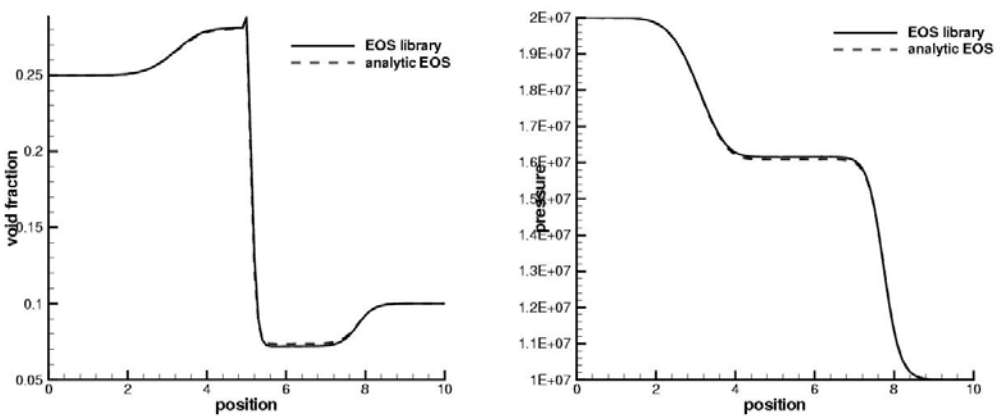

Figure 7. Toumi’s water-air shock tube problem. Results at time *t* =0.006 s with drag force. Left: void fraction. Right: pressure

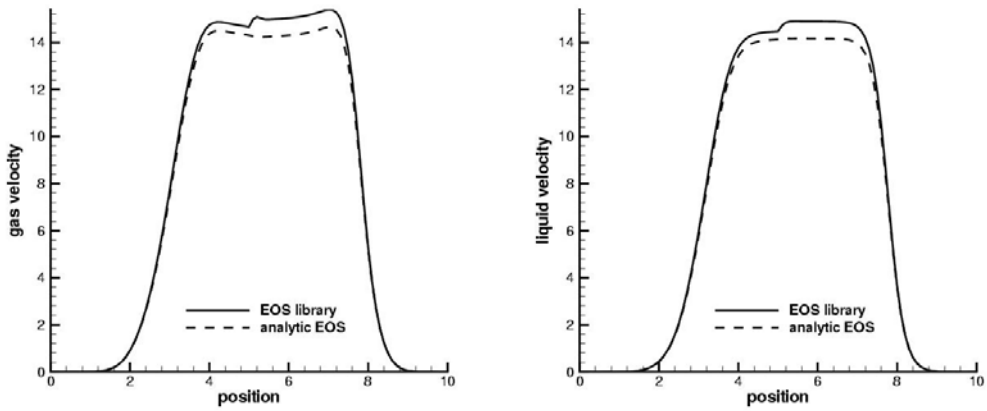

Figure 8. Toumi’s water-air shock tube problem. Results at time *t* =0.006 s with drag force. Left: gas velocity. Right: liquid velocity

ASTAR Benchmark Test Case III: Dispersed two-phase flow in a smooth nozzle

This test case involves the computation of a steady dispersed two-phase flow through a quasi one-dimensional smooth nozzle. We used the quasi-one-dimensional formulation of the model, which includes the variation of the section traversed by the flow (Städtke, 1991). The variation of the section is the one prescribed in the ASTAR benchmark problems specifications (ASTAR, draft). The boundary conditions used in the computations are the following:

Inlet:

Pressure	p =10 bar ;
Liquid temperature	T_l = 570 K ;
Gas temperature	T_g = 570 K ;
Gas volume fraction	α_g = 0.996 ;
Outlet:	
Pressure	p =6 bar, p =0.2 bar .

The initial condition used in the computation is also prescribed in (ASTAR, draft) and consists of constant states throughout the nozzle for all the variables. In particular, the pressure is set to 10 bar, liquid and gas velocities are set to 0.0 m/s, liquid and gas temperatures are set to 570 K and the void fraction is set to 0.996. In the computation the drag has been included in the model.

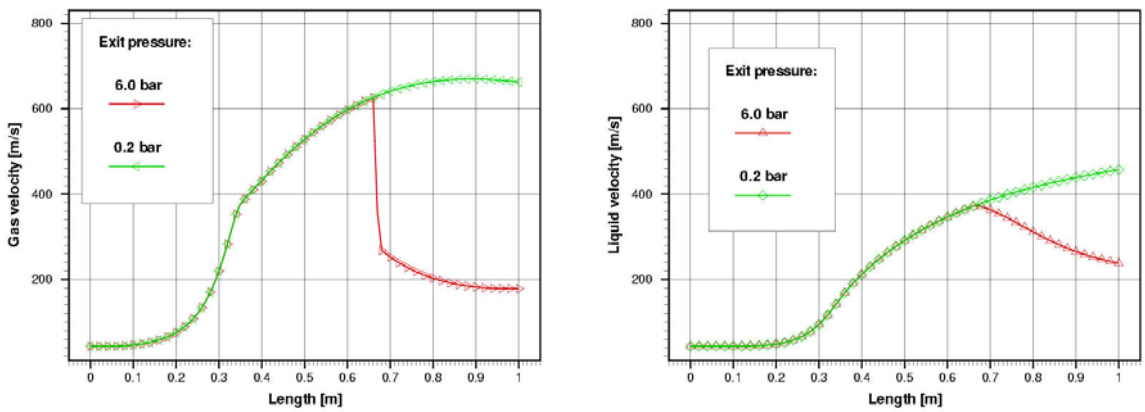

Figure 9. Dispersed flow in a smooth nozzle. Results for outlet pressure of 6 and 0.2 bar. Left: gas velocity. Right: liquid velocity

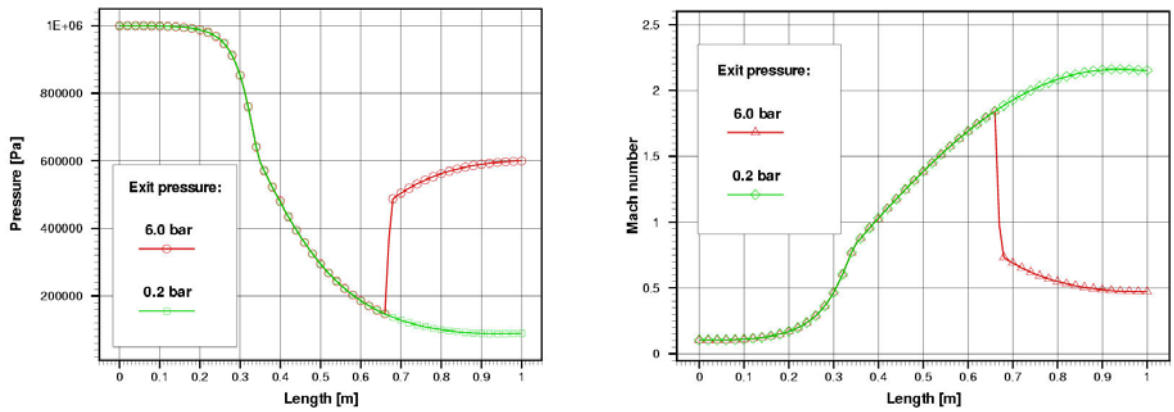

Figure 10. Dispersed flow in a smooth nozzle. Results for outlet pressure of 6 and 0.2 bar. Left: pressure. Right: Mach number

The results obtained for the two values of the outlet pressure are displayed in figures 9 and 10 in terms of distribution along the nozzle of gas and liquid velocities, pressure and Mach number. In the first case (outlet pressure 6 bar), we see the formation of a transonic shock clearly visible in the profiles of pressure, gas velocity and Mach number. We also see that the deceleration of the liquid phase across the shock is practically smooth. This is certainly due to the larger inertia of the water. It must be remarked how the scheme produces a beautiful sharp and monotone shock. As a matter of fact, for steady computations, the scheme presented here can be shown to be second order accurate in space, thanks to the consistent treatment of the source term (Deconicnck, 2003), (Ricchiuto, 2002). We also remark that second order of accuracy is obtained on a stencil of nearest neighbors. In the second case (outlet pressure 0.2 bar), we also see nice and smooth profiles for all the variables. In particular, for the geometry used, the value of 0.2 bar is below the design outlet pressure of the nozzle, which is the value actually reached by the pressure at the end of the nozzle. The flow reaches a maximum Mach number of about 2.2 before the end of the nozzle. The decrease in Mach number at the end of the domain could be due to the deceleration in the gas phase, visible in figure 9, probably due to the fact that at its right end the nozzle has constant area and there the effect of the drag might become dominant slowing down the gas and hence reducing the Mach number.

6. CONCLUSION

In this paper we have described the application of an upwind conservative residual distribution scheme to the one-dimensional Two-Fluid model of Städtke. We have shown that in one space dimension the upwind conservative RD schemes based on boundary Gauss integration of Csík *et al.* (2002) reduce to well-known conservative FV schemes, when applied to homogeneous systems. We have also shown that the consistent inclusion of algebraic source terms in the RD discretization, which is quite straightforward in one space dimension, allows to recover one-dimensional upwind source terms treatments previously proposed in literature. The equivalence between conservative residual distribution and finite volume schemes presented here further extends similar analyses performed in the past by other authors. We have then discussed the extension of the method to models containing non-conservative products and proposed a simple solution. We then applied the method to the solution of the two-fluid model of Städtke showing results on two two-phase benchmarks. Particularly interesting are the results obtained for the dispersed flow in a smooth nozzle, where the accuracy of the consistent treatment of the source term can be seen.

A few remarks have to be added concerning the extension of the approach to the multidimensional case. Most of the ideas introduced here carry on unchanged in multiple space dimensions. This is the case, for example, for the definition of the element fluctuation, which, also in more than one space dimension can be defined as in (57), replacing the space derivative of the flux vector with the divergence of the flux tensor. This definition would certainly allow to perform second order accurate steady simulations in multiple space dimensions. Unfortunately, it would also not guarantee the monotonicity of the solution. Monotone non-linear second order RD schemes in multiple space dimensions can be designed but their extension to complex systems of equations as the one used in this paper is still a topic of research. Particularly important will be, from this point of view, the study of non-linear schemes, which include the source terms in a consistent way. Although some preliminary studies have been performed in (Ricchiuto, 2002) and also in (Degrez 1999), a systematic and consistent study has not been presented up to now. Equally important is a consistent treatment of the time derivative. As shown in (Abgrall, 2003) and (Mezine, 2003), the simple iterative procedure (4) represents an inconsistent discretization, which limits the accuracy in unsteady computations to first order. The use of one of the space-time consistent approaches already present in literature is a short-term feasible solution for this issue.

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