
Conservative Multidimensional Upwind Residual Distribution Schemes for Arbitrary Finite Elements

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Abstract. We introduce monotone first order fluctuation splitting schemes for solving hyperbolic systems on arbitrary finite elements, thereby generalizing the N-scheme previously proposed for linear P1 triangles. Conservation is retained by relaxing on strict monotonicity, using a simple method based on contour integration over the element boundaries. Numerical examples are given for the Euler equations solved on Q1 elements for applications ranging from transonic to hypersonic regimes.

1 Introduction

Over recent years, multi-dimensional upwind fluctuation splitting or residual distribution schemes (RDS) have gained some momentum with the development of the *matrix* schemes for the solution of systems of conservation laws [1,2]. These schemes have been mainly constructed for linear P1 elements, although some generalization to Q1 quadrilaterals have already been discussed in [4,6].

Multi-dimensional upwind schemes strongly rely on the quasilinear form of the governing equations, for the computation of the so called *upwind parameters*. For first order schemes, this adds a conservation constraint, which is normally treated by an appropriate linearization of the Jacobians [2]. In the case of the Euler equations on P1 elements a conservative linearization is available, based on the Roe parameter vector [10,8]. This linearization depends on two properties, the linearity of the finite element space (P1 elements) and the requirement to have quadratic fluxes in some variable [10] (LRD approach). Such a set of variables fails to exist for *arbitrary* systems of conservation laws and/or for nonlinear finite elements. Some remedies have been investigated [6,9,11], but the solutions are much too costly or complicated to be of practical use.

In [5], Csik et al. proposed a new formulation of the positive N-scheme, based on a conservative contour flux integration (CRD approach). By relaxing on strict positivity, this formulation is conservative independently from the averaged state used for evaluation of the Jacobians of the system. Due to this property, the conflicting issues of conservation and upwinding along local averaged advection speeds are decoupled, allowing the extension of the schemes to more complex systems and nonlinear elements.

In this paper we address the generalization of the first order monotone N-scheme for P1 elements to *arbitrary* finite elements. This is important since the

existence of a monotone, shock capturing, first order scheme is the basis for the construction of nonlinear higher order schemes, that automatically follow using standard procedures [2,3].

2 Residual distribution schemes

Consider the Euler equations for a perfect gas written as:

$$\mathbf{U}_t + \nabla \cdot \mathbf{F} = \mathbf{0}, \quad \mathbf{U} = (\rho, \rho \mathbf{v}, E)^T, \quad \mathbf{F} = (\rho \mathbf{v}, \rho \mathbf{v} \mathbf{v} + \mathbf{I}p, (E + p)\mathbf{v})^T \quad (1)$$

where \mathbf{v} is the velocity vector, E the total energy density and for closure, the pressure p is computed $p = (\gamma - 1)(E - \frac{1}{2}\rho \mathbf{v} \cdot \mathbf{v})$. Rewriting the system in the quasilinear form (2), where d is the number of dimensions, we obtain:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}_j \frac{\partial \mathbf{U}}{\partial x_j} = \mathbf{0}, \quad j = 1 \dots d \quad (2)$$

Where $\mathbf{A}_j = \partial \mathbf{F}_j / \partial \mathbf{U}$, is the jacobian in terms of \mathbf{U} . In this framework, the solution is approximated in a compact piecewise finite element space over an unstructured mesh composed of elements Ω with m number of nodes.

$$\mathbf{U}^h(\mathbf{x}, t) = \sum_{l=1}^m \mathbf{U}_l(t) N_l(\mathbf{x}), \quad (3)$$

Where $\mathbf{U}_l(t)$ is the time dependent nodal value of the solution at node l and $N_l(\mathbf{x})$ is the shape function with properties of interpolation (4a), constant summation and conservation (4b):

$$N_l(\mathbf{x}_k) = \delta_{kl} : k, l = 1 \dots m \quad (4a)$$

$$\sum N_k(\mathbf{x}) = 1, \quad \sum \nabla N_k(\mathbf{x}) = \mathbf{0} : k = 1 \dots m, \forall \mathbf{x} \in \Omega \quad (4b)$$

Integrating (2) over an element Ω we get the total cell residual ϕ^Ω :

$$\phi^\Omega = \int_{\Omega} \mathbf{A}_j \frac{\partial \mathbf{U}^h}{\partial x_j} d\Omega \quad (5)$$

Making use of (3) and (4a) we can write (5) as:

$$\phi^\Omega = \int_{\Omega} \mathbf{A}_j \frac{\partial N_l}{\partial x_j} \mathbf{U}_l d\Omega = \mathbf{U}_l \int_{\Omega} \mathbf{A}_j \frac{\partial N_l}{\partial x_j} d\Omega = \sum_{l=1}^m \mathbf{K}_l \mathbf{U}_l \quad (6)$$

where the \mathbf{K}_l are defined by

$$\mathbf{K}_l = \int_{\Omega} \mathbf{A}_j \frac{\partial N_l}{\partial x_j} d\Omega \quad (7)$$

Integral (7) can be approximated by any particular numerical integration for which we require for consistency $\sum \mathbf{K}_l = \mathbf{0}$. This is automatically satisfied by the

property (4b) if the integration is exact for polynomials of sufficient high order, considering the order of \mathbf{N}_l and the jacobian \mathbf{A}_j . The system being hyperbolic, \mathbf{K}_l have a complete set of real eigenvalues and eigenvectors and considering the positive and negative eigenvalue matrices, we write (8):

$$\mathbf{K}_l = \mathbf{R}_l \mathbf{A}_l \mathbf{L}_l \quad \mathbf{K}_l^+ = \mathbf{R}_l \mathbf{A}_l^+ \mathbf{L}_l, \quad \mathbf{K}_l^- = \mathbf{R}_l \mathbf{A}_l^- \mathbf{L}_l \quad (8)$$

We now distribute the residual in fractions to the nodes that compose the element, using some distribution rule, which defines the numerical scheme. We require for consistency that these fractions sum up to the overall residual:

$$\phi^\Omega = \sum_{l=1}^m \phi_l^\Omega \quad (9)$$

This finally leads to the following semi-discrete scheme for each node l , where S_l is the dual cell area and the summation is performed over the cells surrounding the node.

$$\left(\frac{\partial \mathbf{U}}{\partial t} \right)_l = -\frac{1}{S_l} \sum_{\Omega, l \in \Omega} \phi_l^\Omega \quad (10)$$

The choice of the scheme to compute these fractions ϕ_l^Ω determines the overall properties of the method: multidimensional upwinding, conservativity, monotonicity, k-exactness for polynomials of order k. Refer to [1] for a review on the properties of the matrix schemes.

2.1 The standard N-scheme

For the N-scheme the fractions ϕ_l^Ω are computed as (11). Then we combine (9) and (6) to yield (12) and enforce consistency by defining \mathbf{U}^* as (13). Clearly the scheme is positive and linear, hence first order.

$$\phi_l^N \triangleq \mathbf{K}_l^+ (\mathbf{U}_l - \mathbf{U}^*) \quad (11)$$

$$\phi^\Omega = \sum_{l=1}^m \phi_l^N \triangleq \sum_{l=1}^m \mathbf{K}_l^+ (\mathbf{U}_l - \mathbf{U}^*) \equiv \sum_{l=1}^m \mathbf{K}_l \mathbf{U}_l \quad (12)$$

$$\Rightarrow \mathbf{U}^* = \left(\sum_{l=1}^m \mathbf{K}_l^- \right)^{-1} \sum_{l=1}^m \mathbf{K}_l^- \mathbf{U}_l \quad (13)$$

2.2 Conservation of the N-scheme

The above scheme is not conservative. To gain back conservation we have to ensure (14). This is obtained by redefining \mathbf{U}^* as (15), thereby relaxing on the positive property (CRD approach [5]).

$$\phi^\Omega = \sum_{l=1}^m \phi_l^N \equiv \phi^c = \oint_{\partial\Omega} \mathbf{F}_j \cdot \mathbf{n}_j d\Omega \quad (14)$$

$$\mathbf{U}^* = \left(\sum_{l=1}^m \mathbf{K}_l^+ \right)^{-1} \left(\left(\sum_{l=1}^m \mathbf{K}_l^+ \mathbf{U}_l \right) - \phi^c \right) \quad (15)$$

Here ϕ^c is the conservative residual computed with any integral approximation that satisfies the order of the fluxes involved. In theory any method of integration for the integrals in (14) can be chosen as long as the *telescopic* property is retained. Consequently, a contour integration is chosen with an Simpson rule. Using this approach, the variables chosen for the averaged state to compute the \mathbf{K}_l do not affect conservation.

2.3 Application to particular Finite Elements

The P1 elements: Assuming in (7) linear triangular elements and the jacobian \mathbf{A}_j constant per cell and computed in an averaged state of $\bar{\mathbf{U}}_l$, the general upwind parameters have the form (16).

$$\mathbf{K}_l = \frac{1}{2} \bar{\mathbf{A}}_j \mathbf{n}_{jl}, \quad \bar{\mathbf{A}}_j = \mathbf{A}_j(\bar{\mathbf{U}}) \quad (16)$$

Where \mathbf{n}_{jl} are the inward pointing scaled normals of the triangle, as shown in figure (1). The choice of this averaged state, as mentioned, is at the source of the conservation property. With $\mathbb{L}\mathbb{R}\mathbb{D}$ approach, a linearization with Roe parameter vector is imposed, but $\mathbb{C}\mathbb{R}\mathbb{D}$ allows any choice of variables to linearize. **The Q1 elements:** For quadrilateral elements, with bi-linear shape-functions, and the jacobian evaluated again at an averaged state, we get the same expression (16), but where the normals \mathbf{n}_{jl} are defined as in figure (1). In this case, the $\mathbb{L}\mathbb{R}\mathbb{D}$ with Roe linearization is not accurate, due to the non linearity of the shape-functions. In the computation of (14), the shape-function variation along the edges of arbitrary quadrilaterals is no longer linear. Tests have shown that this error is rather small for roughly regular elements.

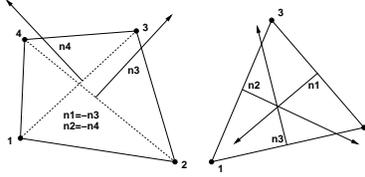


Fig. 1. Quadrilateral and Triangle \mathbf{n}_l normals

General Elements: For general higher-order elements, including three dimensional ones, the upwind parameters (7) are computed by averaging the jacobian in the element Ω and computing (17) to satisfy $\sum \mathbf{K}_l = 0$.

$$\mathbf{K}_l = \bar{\mathbf{A}}_j \cdot \int_{\Omega} \partial_j \mathbf{N}_l \, d\Omega \quad (17)$$

2.4 Higher Order schemes: a boundedness issue

Once a first order monotone scheme is available as discussed above, a high order scheme preserving exact solutions belonging to the Finite Element space can be

obtained. An example for the case of a scalar conservation law follows:

$$\beta_l^N = \frac{\phi_l^N}{\phi_\Omega} \quad \phi_j^\Omega = \frac{\max(0, \beta_l^N)}{\sum_{l=1}^m \max(0, \beta_l^N)} \phi_\Omega \quad (18)$$

3 Results and Discussion

3.1 Transonic flow, $Ma = 0.675$ circular arc bump channel

This transonic test-case, also known as GAMM channel, consists of a 3×1 channel, with an unity chord circular arc in the middle with 10% height. This results in a sonic pocket that culminates in a transonic shock at $\approx 72\%$ of the chord, reaching $Ma \approx 1.32$, [7]. The reference solution was calculated with a finer 300×90 quadrilateral mesh with the CRD N scheme. The triangle schemes perform better in edge aligned flows, where the cross diffusion is minimal. The quadrilateral schemes, although conservative, have shown to be very diffusive, as their cross-diffusion minimizes in diagonal aligned flows.

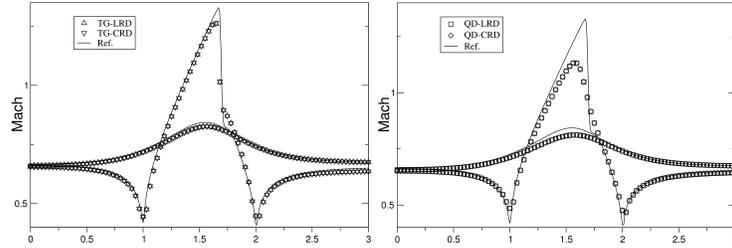


Fig. 2. Mach distribution along the top and bottom walls in the GAMM channel. In the left, the results of the isotropic triangle mesh (TG) ≈ 3400 nodes, in the right the computed results with the quadrilateral (QD) 100×30 mesh.

3.2 Hypersonic flow, $Ma = 6$ bow shock in cylinder

To demonstrate the robustness of the N-scheme for Q1 elements, we present the results of a simulation of a hypersonic bow shock in front of a cylinder of unity radius. The flow Mach number is 6. Plots along the stagnation line are presented, fig.(3). In this case, a strong normal shock is observed in the stagnation line, and the scheme yielded monotone evolutions. The computation grid had 60×70 Q1 elements. The reference solution was computed in a grid with approximately 10 times more elements.

4 Conclusion

The general definition of the upwind parameters \mathbf{K}_l for *arbitrary* finite elements is presented and applied in conjunction with the CRD approach to retain conservation. This separates the conflicting issues of multidimensional upwinding and linearization of the jacobians. It shows potential for application to non-linear or higher order finite elements.

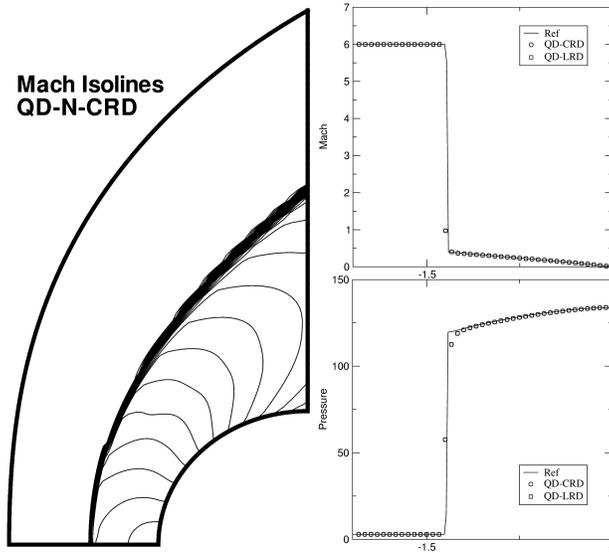


Fig. 3. Left: Q1 flow field results, Mach distribution with conservative N-scheme. Right: Q1 Stagnation line results for N-scheme with both LRD and CRD. Top Right: Mach number evolution. Bottom Right: pressure distribution.

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