Conservative residual distribution schemes for general unsteady systems of conservation laws

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1 Introduction

We present the construction of non-linear monotone schemes for the approximation of weak solutions of time-dependent non-linear systems of conservation laws lacking a Roe linearization [1]. The schemes we consider are of the so-called *Residual Distribution* ($\mathcal{RD}$) or *Fluctuation Splitting* ($\mathcal{FS}$) class [2]. Due to their very compact stencil (only nearest neighbors) and to inherent monotonicity properties (no *tuning* of the numerical dissipation), $\mathcal{RD}$ schemes represent an appealing alternative to both finite volume and standard finite element methods [2, 3]. In this work, we make use of the space-time formulation of residual distribution [4, 5]. In particular, consider the scalar problem

$$\frac{\partial u}{\partial t} + \boldsymbol{\lambda} \cdot \nabla u = 0 \quad \text{on} \;\; \Omega \times [t_0, t_f] \subset \mathbb{R}^2 \times \mathbb{R}^+ \;, \quad \boldsymbol{\lambda} = \text{const} \;. \tag{1}$$

Let $\mathcal{T}_h$ be an unstructured triangulation of Ω and $\{t^1, \ldots, t^M\}$ a sequence of M discrete time levels. Given the nodal values at time t^n, $\{u_i^n\}_{i\in\mathcal{T}_h}$, the unknowns $\{u_i^{n+1}\}_{i\in\mathcal{T}_h}$ are computed by the space-time residual distribution method as the solution of the algebraic system

$$\sum_{T\in\mathcal{D}_i} \phi_i = 0 \quad \forall\, i \in \mathcal{T}_h \tag{2}$$

System (2) is assembled as follows. First a local *space-time residual* is computed in every every triangle $T \in \mathcal{T}_h$:

$$\phi^h = \int_{t^n}^{t^{n+1}} \int_T \left(\frac{\partial u^h}{\partial t} + \boldsymbol{\lambda} \cdot \nabla u^h \right) d\Omega \, dt \;, \tag{3}$$

with u^h a *continuous* numerical approximation of the unknown u. Assuming u^h to be piecewise continuous in space and linear in time one has

$$\phi^h = \sum_{j\in T} \left\{ \frac{|T|}{3} \left(u_j^{n+1} - u_j^n \right) + \frac{\Delta t}{2} k_j u_j^n + \frac{\Delta t}{2} k_j u_j^{n+1} \right\} \tag{4}$$

where the *inflow parameter* k_j is given by

$$k_j = \frac{\boldsymbol{\lambda} \cdot \boldsymbol{n}_j}{2} \tag{5}$$

being $\boldsymbol{n}_j$ is the scaled inward pointing vector normal to the edge of T in front of node j. The inflow parameters (5) allow to easily detect upstream ($k_j < 0$) and downstream nodes ($k_j > 0$). The cell-residual ϕ^h is then distributed to each node of T. The fraction of ϕ^h distributed to a node i (*local nodal residual*) is denoted by ϕ_i. We introduce the *distribution coefficients* $\beta_i = \phi_i/\phi^h$.

The properties of the discretization are determined by the definition of the ϕ_i. For the scope of this paper, we are interested in the following properties:

Consistency: $\sum_{j\in T} \phi_j = \phi^h$ or equivalently $\sum_{j\in T} \beta_j = 1$
Linearity Preservation: a scheme is said to be *linearity preserving* if the β_i coefficients are bounded. In [5, 8] is shown that linearity preserving schemes are second order accurate.
Positivity: rewriting (2) as $A\,U^{n+1} = B\,U^n$, a scheme is *positive* if A is an invertible $\mathcal{M}$-matrix and if B is positive ($B_{ij} \geq 0\ \forall\, i,j$) [5]. Positive schemes exhibit a discrete maximum principle and are essential for a monotone approximation of discontinuous solutions.

It is useful to introduce here the positive linear schemes used in this work. They are space-time extensions of the optimal positive N-scheme [2]. In particular, we will refer to the N1-scheme as to the space-time linear scheme defined by the local nodal residuals [4]:

$$\phi_i^{\mathrm{N1}} = \overline{k}_i^+ (u_i^{n+1} - \overline{u}_{in}); \tag{6}$$

Apart from the notation, it is important to note that the *inflow state* $\overline{u}_{in}$ represents the linearly interpolated value of u^h in the most upstream point of the space-time prism $T \times [t^n, t^{n+1}]$, *i.e.* the most upstream point with respect to the space-time characteristic line crossing the prism (left on figure 1). The parameters $\overline{k}_j$ are space-time equivalents of the inflow parameters (5). Scheme (6) represents a truly space-time generalization of the linear N-scheme. It is consistent and positive provided that the residual can be expressed as in (4) and under a time-step constraint [4]. A different scheme has been introduced in [5], defined by the local nodal residual

$$\phi_i^{\mathrm{N2}} = \frac{|T|}{3}(u_i^{n+1} - u_i^n) + \frac{\Delta t}{2} k_i^+ (u_i^n - u_{in}^n) + \frac{\Delta t}{2} k_i^+ (u_i^{n+1} - u_{in}^{n+1}) \tag{7}$$

In this case, the state u_{in} represents the linearly interpolated value of u in the most upstream point in triangle T, *i.e.* the most upstream point on the streamline crossing the element (right on figure 1). Scheme (7) corresponds to the standard N-scheme with Crank-Nicholson time integration. It is consistent and positive provided that the residual can be expressed as in (4) and under a time-step constraint [5]. We will refer to scheme (7) as to the N2-scheme. Note that for both the N1 and the N2 scheme consistency is achieved provided that the residual can be expressed as in (4).

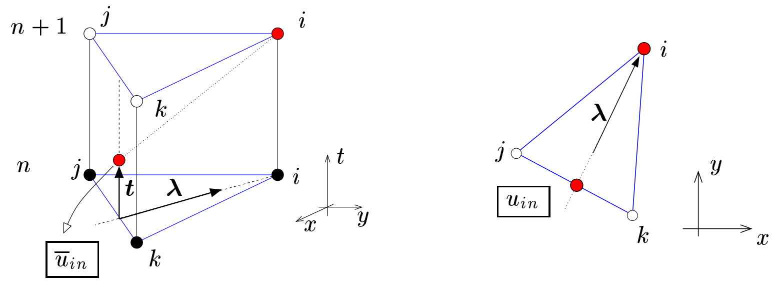

Fig. 1. Space-time inflow state $\overline{u}_{in}$ (left) and space inflow state u_{in} (right)

2 Limited residual distribution schemes

An analog of Godunov's theorem for residual distribution [8, 9] states that linear schemes cannot be at the same time linearity preserving and positive. As a consequence, non-linear schemes must be considered if one is interested in a monotone and sharp approximation of discontinuous solution. Here we follow [5, 8, 10]: given a linear first order positive scheme with local residuals $\phi_i^{\mathcal{P}}$ and distribution coefficients $\beta_i^{\mathcal{P}} = \phi_i^{\mathcal{P}}/\phi^h$, one can introduce the class of nonlinear schemes whose distribution coefficients β_i respect the conditions:

$$\begin{cases} \beta_i\beta_i^{\mathcal{P}} \geq 0 & \text{(for positivity)} & (8.a) \\ |\beta_i| < C < \infty & \text{(for linearity preservation)} & (8.b) \\ \sum\limits_{j\in T} \beta_j = 1 & \text{(for consistency)} & (8.c) \end{cases} \tag{8}$$

Mappings satisfying (8) are presented in [5, 8, 10]. Here we give a condition on the $\phi_i^{\mathcal{P}}$'s guaranteeing the well-posedness of the procedure. The consistency constraint (8.c) requires the existence of at least one positive β_i, hence, due to (8.a), at least one positive $\beta_i^{\mathcal{P}}$ must exist. If the linear scheme is consistent, that is if $\sum_{j\in T}\beta_j^{\mathcal{P}} = 1$, this condition will be met. However, if $\sum_{j\in T}\phi_j^{\mathcal{P}} = \phi^1 \neq \phi^h$, one could run into the unfortunate case $\phi^1\phi^h \leq 0$ and

$$\sum_{j\in T}\beta_j^{\mathcal{P}} = \frac{\sum\limits_{j\in T}\phi_j^{\mathcal{P}}}{\phi^h} = \frac{\phi^1}{\phi^h} = \mathit{\Gamma} \leq 0\,, \quad \operatorname{sign}(\beta_j^{\mathcal{P}}) = -1 \quad \forall j \in T \tag{9}$$

In this case, (8) cannot be satisfied, since either the first or the last relation would have to be violated compromising the positivity or the consistency of the nonlinear scheme. So we have the general compatibility requirement

Proposition 1. $\forall$ *mappings* $\{\beta_j^{\mathcal{P}}\} \to \{\beta_j\}$ *respecting (8), a sufficient condition for the existence of the scheme defined by* $\phi_i = \beta_i\phi^h$ *, is* $\sum_j \phi_j^{\mathcal{P}} = \phi^h$.

Using the argument that the only role of the positive linear scheme is to give the correct sign of the β_is, nonlinear schemes based on inconsistent linear ones have been proposed in [8, 10, 11]. Even though the compatibility requirement is not a necessary condition, and one could find an inconsistent positive linear scheme for which (9) are never met, in [8, 10, 11] fixes compromising positivity are introduced to retain consistency.

3 Conservative schemes for multidimensional systems

Consider the hyperbolic system of conservation laws

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot \mathcal{F}(\boldsymbol{u}) = 0 \quad \text{on} \quad \Omega \times [t_0, t_f] \subset \mathbb{R}^2 \times \mathbb{R}^+ \; ; \quad \mathcal{F} = (F, G) \tag{10}$$

Writing (10) in the quasilinear form

$$\frac{\partial \boldsymbol{u}}{\partial t} + A_x \frac{\partial \boldsymbol{u}}{\partial x} + A_y \frac{\partial \boldsymbol{u}}{\partial y} = 0 \, ; \quad A_x = \frac{\partial F}{\partial \boldsymbol{u}} \, , A_y = \frac{\partial G}{\partial \boldsymbol{u}} \tag{11}$$

its hyperbolicity guarantees that $\forall \boldsymbol{\xi} \in \mathbb{R}^2$ the matrix $K_{\boldsymbol{\xi}} = A_x \xi_x + A_y \xi_y$ has real eigenvalues and linearly independent eigenvectors. The *matrix* variant of space-time $\mathcal{RD}$ approximates solutions of linear hyperbolic systems [4, 5] as described in sections 1 and 2 except that the k_j parameters (5) become matrices defined as $K_j = K_{\boldsymbol{n}_j}/2$, the distribution coefficients become distribution matrices and the scalar u^h is replaced by a discrete unknown vector U^h. The N1 and N2 schemes are defined as in (6) and (7) with the proper change of notation. Non-linear schemes are built through a wave decomposition procedure [5, 8]. The case of a non-linear system is more complex. Non-linear systems can evolve discontinuities and it is essential that across these discontinuities the discretization consistently approximates the integral weak form of (10). In the case of the Euler equation for a perfect gas, the existence of a conservative Roe linearization [1, 7] implies an equivalence between the integral and the quasi-linear form of the equations, so that conservation is guaranteed by evaluating the K_j matrices in the Roe averaged state [7]. This allows to compute the element residual using a formula analog to (4) and to use the N1 and N2 schemes. Due to the conservative linearization, these are conservative and consistent, hence the limiting procedure can be applied and is well-posed. In absence of a conservative linearization, the N1-scheme and N2-scheme cannot be conservative since the residual cannot be expressed as in (4). In this case, we propose to compute ϕ^h as

$$\phi^h = \sum_{j \in T} \frac{|T|}{3} \left(\boldsymbol{u}_j^{n+1} - \boldsymbol{u}_j^n \right) + \frac{\Delta t}{2} \oint_{\partial T} \mathcal{F}^n \cdot \hat{n} \; \partial T + \frac{\Delta t}{2} \oint_{\partial T} \mathcal{F}^{n+1} \cdot \hat{n} \; \partial T \; , \tag{12}$$

approximating the boundary integrals with Gauss integration. The problem is now to define consistent monotone linear schemes. Here we follow [6]. Consider the case of the N1-scheme. For a non-linear system, the local residuals obtained with the matrix version of (6) would not yield a consistent scheme. Nevertheless, it is easy to see that the two conditions

$$\phi_i^{\mathrm{N1}} = \overline{K}_i^+ \left(\boldsymbol{u}_i - \overline{\boldsymbol{u}}_{in} \right) ; \quad \sum_{j \in T} \phi_j^{\mathrm{N1}} = \phi^h ,$$

with ϕ^h given by (12), uniquely define $\overline{U}_{in}$. In particular, a unique conservative matrix variant of the N1-scheme (6) is obtained defining $\overline{\boldsymbol{u}}_{in}$ as [4, 6]

$$\overline{\boldsymbol{u}}_{in} = \Big(\sum_{j\in T} \overline{K}_j^+\Big)^{-1} \Big(\sum_{j\in T} \overline{K}_j^+ \boldsymbol{u}_j^{n+1} - \phi^h\Big)\,, \quad \text{with } \phi^h \text{ given by (12)}$$

Proceeding in a similar way, a unique conservative matrix variant of the N2-scheme (7) is obtained, by defining the inflow states as [5, 6, 12]

$$\boldsymbol{u}_{in}^k = \Big(\sum_{j\in T} K_j^+\Big)^{-1} \Big(\sum_{j\in T} K_j^+ \boldsymbol{u}_j^k - \phi^k\Big); \quad \phi^k = \oint_{\partial T} \mathcal{F}^k\, \partial T; \;\; k = n, n+1$$

The schemes obtained in this way are indeed conservative but their monotonicity is to be verified numerically [6]. Starting from the conservative variants of the linear schemes we construct limited schemes, which we will refer to as limited N1-scheme and limited N2-scheme. The well-posedness of the limiting is guaranteed by the conservative formulation of the linear schemes.

4 Results

We apply the schemes to the hyperbolic two-phase flow model defined by

$$\boldsymbol{u} = \begin{pmatrix} \alpha_g\rho_g \\ \alpha_l\rho_l \\ \rho u \\ \rho v \end{pmatrix}, \; \mathcal{F} = \begin{pmatrix} \alpha_g\rho_g u & \alpha_g\rho_g v \\ \alpha_l\rho_l u & \alpha_l\rho_l v \\ \rho u^2 + p & \rho u v \\ \rho u v & \rho v^2 + p \end{pmatrix}; \; \begin{cases} \rho = \alpha_g\rho_g + \alpha_l\rho_l \\ \alpha_g + \alpha_l = 1 \\ p = \varGamma_g \left(\dfrac{\rho_g}{\rho_{g0}}\right)^{\gamma_g} \\ p = \varGamma_l \left[\left(\dfrac{\rho_l}{\rho_{l0}}\right)^{\gamma_l} - 1\right] + p_{l0} \end{cases}$$

with α_g and α_l gas and liquid *volume fractions*, ρ_g and ρ_l gas and liquid densities, p the pressure, $\boldsymbol{u} = (u, v)$ the flow speed and ρ the mixture density. The constants in the equations of state are taken as in [13]. No conservative linearization is available for this model but, being in strong conservative form, one can compute exact Rankine-Hugoniot relations. On the first row in figure 2 we show the computation of a planar shock moving in a mixture with $\alpha_g = 0.5$ performed on a 2D mesh ($h \approx 1/100$) with periodic boundary conditions in the y direction. The shock speed u_s is defined by the Mach number $M_s = u_s/\sqrt{p_R/\rho_R}$. The results in the picture show the mixture density distribution in the middle of the 2D domain for $M_s = 10$. The shock is correctly captured by all schemes, confirming their conservative character. Very sharp shock capturing is obtained with the non-linear schemes. On the last row of figure 2 we present the computation of the interaction of a $M_s = 3$ shock with a circular discontinuity in the volume fraction. The solutions are obtained with the non-linear schemes (Top: limited N1. Bottom: limited N2) on a 2D mesh ($h \approx 1/200$). Both schemes give a very good resolution of the interaction between the shock and the *bubble* comparing to similar results available in literature [14, 15, 16, 17].

5 Concluding remarks

We propose conservative, non-linear and linearity preserving space-time $\mathcal{RD}$ schemes for the discretization of time-dependent systems lacking a conservative linearization [1, 7]. Making use of the technique proposed in [6] we obtained conservative variants of the linear first order schemes of [4] and [5] to be used as a basis for the construction of non-linear schemes through the limiting procedure of [5, 10, 8]. We have shown that the well-posedness of this procedure is guaranteed by the conservative formulation of the linear schemes. Results involving the solution of a hyperbolic two-phase flow model have shown promising features of the schemes proposed: discrete conservation, generality, monotone and sharp capturing of discontinuities.

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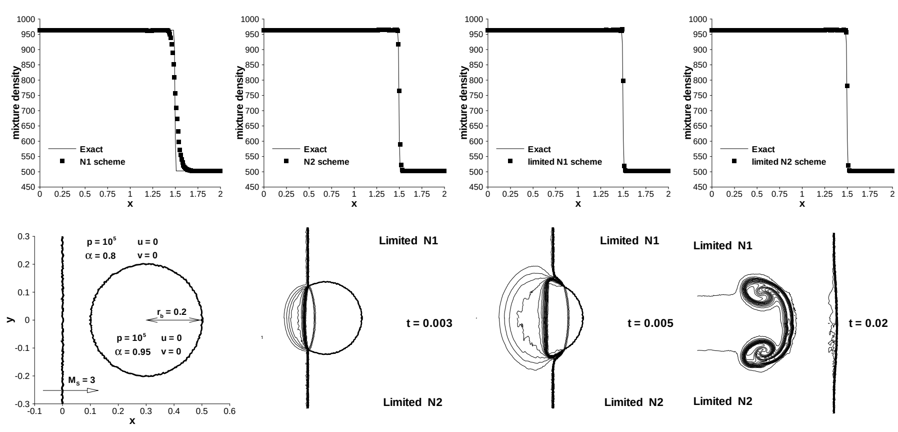

Fig. 2. First row: $M_s = 10$ moving shock. Second row: Shock bubble interaction.