Stable and convergent residual distribution for time-dependent conservation laws

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1 Generalities

We consider the discretization of the time dependent hyperbolic problem

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}}(\mathbf{u}) = 0 \quad \text{on} \quad \Omega \times [0, t_f] \subset \mathbb{R}^2 \times \mathbb{R}^+ \tag{1}$$

on unstructured grids. We present residual distribution ($\mathcal{RD}$) schemes which (i) give non-oscillatory solutions, (ii) are second order accurate by construction, and (iii) lead to well-posed algebraic problems, that is, they ultimately lead to linear systems $A\,x = y$, with A *invertible*. How to construct nonlinear $\mathcal{RD}$ satisfying (i) and (ii) is known for some time [AM03]. However, it is the satisfaction of (iii) that ensures that a (unique) discrete solution exists, and that second order of accuracy is actually obtained in practice (convergence).

1.1 Residual distribution for time dependent problems

An abstract framework to picture the basics of $\mathcal{RD}$ is the following. Given $\mathcal{T}_h$, unstructured triangulation of Ω, and given $(\mathbf{u}^n, \mathbf{u}^{n-1}, \ldots, \mathbf{u}^0)$, solution in the mesh points at times $(t^n, t^{n-1}, \ldots, 0)$, first discretize the time derivative :

$$\sum_{i=0}^{p} \frac{\alpha_i}{\Delta t_{n+1-i}} \delta \mathbf{u}^{n+1-i} + \sum_{j=0}^{q} \theta_j \nabla \cdot \boldsymbol{\mathcal{F}}^{n+1-j} = 0$$

where $\delta \mathbf{u}^k = \mathbf{u}^k - \mathbf{u}^{k-1}$, $\boldsymbol{\mathcal{F}}^{n+1-j} = \boldsymbol{\mathcal{F}}(\mathbf{u}^{n+1-j})$, and with $\Delta t_k = t_k - t_{k-1}$ the (variable) time step. The α_j and θ_j coefficients may be associated to a multistage method, as well as quadrature weights, in a space-time framework [AAM05, RAD03]. The only unknown being $\mathbf{u}^{n+1}$, we recast the problem as

$$\mathcal{M}(\mathbf{u}^{n+1}) = \mathcal{S}(\mathbf{u}^n, \mathbf{u}^{n-1}, \ldots), \quad \mathcal{M}(\mathbf{u}^{n+1}) = \frac{\alpha_0}{\Delta t_{n+1}} \mathbf{u}^{n+1} + \theta_0 \nabla \cdot \boldsymbol{\mathcal{F}}^{n+1} \tag{2}$$

The nodal values of $\mathbf{u}^{n+1}$ are computed through the following simple steps.

1. $\forall$ triangles $T \in \mathcal{T}_h$ compute the *element residual* $\phi^T = \int_T (\mathcal{M}_h - \mathcal{S}_h)$, where $\mathcal{M}_h$ and $\mathcal{S}_h$ are discrete approximations in space of $\mathcal{M}$ and $\mathcal{S}$
2. define a splitting : $\phi^T = \sum_{j \in T} \phi_j^T$ ($j \in T$ being the nodes of T)

3. compute the values $\mathbf{u}_i^{n+1}$ by solving the (nonlinear) algebraic system
$$\sum_{T|i\in T} \phi_i^T = 0, \quad \forall\, i \in \mathcal{T}_h \tag{3}$$

System (3) is solved by means of an iterative procedure. The issue is how to define the ϕ_j^Ts such that (3) is well posed, *i.e.* it admits a unique solution.

Since we are interested in numerical solutions with a non-oscillatory character, the discretization must have some degree of non linearity. *For smooth problems*, most nonlinear $\mathcal{RD}$ are known to suffer from a lack of iterative convergence. This is the symptom of a subtle instability, and it limits the overall accuracy, due to the poor approximation of (3). For one class of schemes this issue is analyzed in [Abg06]. We proceed along the same lines.

2 Remarks on the iterative convergence of nonlinear $\mathcal{RD}$

First we recall a procedure allowing to define the ϕ_j^Ts in a way ensuring *by construction* that the scheme is formally second order accurate and monotonicity preserving. At the basis of the construction are the following conditions[3] :

Accuracy For a r-th degree polynomial approximation of the unknown (and of the fluxes) a $r+1$ order accurate scheme is obtained if [Abg06, RAD06]
$$\phi_j = \beta_j \phi\,, \qquad \beta_j \quad \text{uniformly bounded}$$

Monotonicity Given a linear monotonicity preserving first order scheme defined by split residuals ϕ_j^{M}, we look for splittings verifying
$$\phi_j = \lambda_j \phi_j^{\mathrm{M}}\,, \qquad \lambda_j \geq 0$$

We need to satisfy both conditions. One way to do this is to construct split residuals ϕ_j^* by applying to the ϕ_j^{M}s a *uniformly bounded* and *sign preserving* nonlinear mapping, such as (in the scalar case, see [AM03] for systems) :
$$\phi_i^* = \beta_i^* \phi \quad \text{with} \quad \beta_i^* = \frac{\max(0, \beta_i^{\mathrm{M}})}{\sum\limits_{j\in T} \max(0, \beta_j^{\mathrm{M}})}, \quad \text{and} \quad \beta_j^{\mathrm{M}} = \frac{\phi_j^{\mathrm{M}}}{\phi} \tag{4}$$

Due to boundedness of the β_j^*s defined by (4), and due to the properties of the mapping, the resulting scheme is *formally* second order accurate, and it does have a strong monotonicity preserving (*viz.* L^∞-stable) character [Abg06, AM03]. The whole procedure leads to a nonlinear algebraic system of type (3).

Let us suppose to be approximating a smooth solution, such that we can linearize both the nonlinear part of (1) (or equivalently (2)), and also (3). To this linearized version of (3) we associate the matrix engendered by the component of the algebraic problem corresponding to the (linearized) operator $\mathcal{M}(\cdot)$ in (2). Let us denote by M_h^* this matrix. Our (linearized) system reads
$$\mathrm{M}_h^* \mathbf{u}^{n+1} = B_h^* \tag{5}$$

[3] for clarity, we drop the super-script ^{T}

where $\mathbf{u}^{n+1}$ contains the unknown nodal values, and in B_h^* we have dumped whatever does not depend on $\mathbf{u}^{n+1}$.

We make the following remark. Denote by M_h the matrix obtained from the first order monotone scheme, without the application of the mapping. Most monotone schemes used as a basis for our construction are L^∞-stable and also also L^2-stable : they can be shown to be dissipative (in an energy sense). As a consequence, M_h is an irreducibly diagonally dominant invertible $\mathcal{M}$-matrix [BP79]. Thus, the existence of a unique numerical solution is guaranteed.

What can we say of M_h^* ? By construction we know that for some $\lambda_{ij} \geq 0$

$$\left[\mathrm{M}_h^*\right]_{ij} = \lambda_{ij}\left[\mathrm{M}_h\right]_{ij} \tag{6}$$

While ensuring that $\left[\mathrm{M}_h^*\right]_{ii} \geq 0 \ \ \forall\, i$ and $\left[\mathrm{M}_h^*\right]_{ij} \leq 0 \ \ \forall\, i,j$ with $i \neq j$, this does not guarantee the invertiblility of M_h^*, unless we have some more information on of its diagonal entries. This is where the trouble comes from. Even though $|\left[\mathrm{M}_h\right]_{ii}| - \sum_j |\left[\mathrm{M}_h\right]_{ij}| \geq 0 \; \forall\, i$, we have no guarantee at all that

$$\lambda_{ii}|\left[\mathrm{M}_h\right]_{ii}| - \lambda_{ij}\sum_j |\left[\mathrm{M}_h\right]_{ij}| = \lambda_{ii}\left[\mathrm{M}_h\right]_{ii} + \lambda_{ij}\sum_j \left[\mathrm{M}_h\right]_{ij} \geq 0$$

It is even possible that $\lambda_{ii} = 0$, for some i : the mapping, in general, weakens the diagonally dominant character of M_h, eventually leading to a ill-conditioned system matrix M_h^*.

Another way to see it is the following. The whole construction is based on the constraint $\phi_j^{\mathrm{M}} \times \beta_j^* \phi \geq 0$. Upwinding is not included in the process : it is likely that the application of the mapping might lead, locally in an element, to a *down-wind* discretization, known to have poor stability.

Even so, the code never blows-up due to the $L^\infty-$stable character of the scheme. However, if a numerical output is obtained, the iterative convergence in the solution of (3) is often poor, and the result might be affected by spurious modes not identified/dumped (lack of dissipation/lack of uniqueness).

Let us now go back to the full nonlinear case. As mentioned before, the convergence problems are relevant mainly when approximating smooth solutions. When dealing with discontinuities the method has enough numerical viscosity to converge relatively well, both in the inner iterations and with mesh refinement. An heuristic justification of this fact is the following [Abg06].

In elements containing a singularity, the element residual ϕ^T and the first order monotone residuals ϕ_j^{M} scale according to the same power of the mesh size h. In particular, in two dimensions, simple arguments lead to

$$\phi^T \approx h\|\varDelta\boldsymbol{\mathcal{F}}\|_T \quad \text{and} \quad \phi_j^{\mathrm{M}} \approx h\|\varDelta\mathbf{u}\|_T$$

being $\|\varDelta\boldsymbol{\mathcal{F}}\|_T$ and $\|\varDelta\mathbf{u}\|_T$ reference values for the norms of flux and variable differences over T. The two scaling are easily obtained from the definition of ϕ^T, and from the one of positive first order dissipation terms. If T contains a singularity, what we can say is that $\|\varDelta\boldsymbol{\mathcal{F}}\|_T$ and $\|\varDelta\mathbf{u}\|_T$ are bounded, which leads to $\phi_j^{\mathrm{M}} = \mathcal{O}(h)$ and $\phi^T = \mathcal{O}(h)$. Since $\phi_j^* = \mathcal{O}(\phi^T)$, then $\phi_j^*/\phi_j^{\mathrm{M}} = \mathcal{O}(1)$. This means that, across a discontinuity, the mapping is likely to preserve more the algebraic structure of the system obtained with the low order method.

3 Convergent schemes: a possible solution

We want to improve the properties of the matrix M_h^* in (5). An idea comes from the observation that our problem is not far away from the one encountered when discretizing (1) with a pure Galerkin scheme. The simplest way to improve things could be then to add a streamline dissipation (SD) term. For years this has been successfully used to stabilize Galerkin discretizations. A recent analysis focusing on the time dependent case is given in [BGS04].

Forgetting for the moment about shocks, we rewrite our discretization as

$$\phi_i = \phi_i^* + \phi_i^{\mathrm{s}} = \beta_i^* \phi^T + h\,\mathrm{C}_T \int_T \frac{\partial \boldsymbol{\mathcal{F}}(\mathbf{u}_h)}{\partial \mathbf{u}} \cdot \nabla \varphi_i \left(\mathcal{M}_h - \mathcal{S}_h\right) dx \tag{7}$$

where the superscript $^{\mathrm{s}}$ stands for stabilization, C_T is a positive definite matrix, and φ_i is the finite element (FE) shape function of node i. Numerical experiments show that this modification solves the problem, the term added introducing dissipation[4]. Another way to see it is that the SD term introduces some kind of upwind bias. The choice of C_T seems irrelevant from the point of view of the numerical results. This confirms that the algebraic nature of the problem (full L^2/entropy-stability not required, even though still desirable...).

Since the evaluation of ϕ_i^{s} in (7) is quite expensive, and since this term, which destroys the monotonicity preserving character of the discretization, is only needed in smooth areas, we modify the stabilization term as follows :

$$\phi_i^s = \beta_i^{\mathrm{s}}\,\phi^T, \qquad \beta_i^{\mathrm{s}} = \Theta(\mathbf{u}_h, h) K_i, \qquad K_i = \frac{1}{2}\frac{\partial \boldsymbol{\mathcal{F}}(\mathbf{u}^*)}{\partial \mathbf{u}} \cdot \mathbf{n}_i \tag{8}$$

with $\theta(\mathbf{u}_h)$ a solution monitor ensuring that the extra term is only active in smooth regions, and $\mathbf{u}^*$ an arbitrary average of $\mathbf{u}$ over T. The following remarks can be made:

- ϕ_i^{s} is a rough approximation of the SD term (7) (exact for linear problems)
- ϕ_i^{s} still introduces some kind of upwind bias

This qualitatively explains the reason why this fix works : for smooth problems, the upwind bias improves the structure of M_h^* in (5). The mechanism is roughly the same guaranteeing the stability of the SD-Galerkin FE scheme. In the case of nonlinear $\mathcal{RD}$, it is rather difficult to formalize this with energy/entropy estimates, even though an analysis similar to the one made in [BGS04] is possible (see also [Abg06]).

Lastly, different definitions can be used for $\theta(\mathbf{u}_h)$. Here, we take the simplest possible [Abg06] :

$$\Theta = \mathrm{C}'_T \Big[\min\big(1, \frac{|T|\,\|\mathbf{u}\|_T}{\|\phi^h\|}\big)\Big]^m, \quad \mathrm{C}'_T \text{ positive definite} \tag{9}$$

Provided that C'_T is of $\mathcal{O}(h^{-1})$, across singularities, this definition leads to $\phi_i^s = \mathcal{O}(h^m) \times \mathcal{O}(\phi^T)$ [Abg06]. Thanks to this, the amplitude of any oscillations appear in the solution decreases as some power of h.

[4] even though, to be rigorous, we should evaluate the extra term in entropy variables

4 Computational results

We discuss some results obtained solving the Euler equations with the nonlinear discretization constructed starting from the Lax-Frederich's scheme combined with Crank-Nicholson time integration :

$$\phi_i^{\mathrm{LF}} = \frac{|T|}{3}\frac{\delta \mathbf{u}^{n+1}}{\Delta t_{n+1}} + \frac{\psi_i^n + \psi_i^{n+1}}{2}\,, \quad \psi_i = \frac{1}{3}\int_T \nabla \cdot \boldsymbol{\mathcal{F}}\, dx + \alpha^T \sum_{j \in T} (\mathbf{u}_i - \mathbf{u}_j) \quad (10)$$

We call LLF scheme the one obtained by applying (4) to (10). The scheme obtained by adding the stabilization term (8) is referred to as LLFs.

We first consider a smooth problem, consisting of the advection of a vortex in inviscid flow (see [DD06] for details). On fig. 1 we report, on the top row, the exact solution (pressure contours), the solution of the LLF scheme and of the LLFs scheme on a coarse grid ($h = 1/40$). On the bottom row we report the solution of the LLFs scheme on the finer mesh ($h = 1/80$), and the 1D pressure distributions through the vortex core. The top row pictures clearly show the spurious modes not dumped by the LLF scheme, and the effectiveness of stabilization in suppressing these modes. The contour plot on the bottom row, as well as the 1D line plots, confirm this observation and demonstrate the truly second order of convergence of the stabilized scheme (as shown by the reduction in the L^∞ pressure error).

Then on fig. 2 we report the results on the well known problem of the Mach 3 wind tunnel with a forward facing step [WC84]. The result show the monotonicity preserving character of the scheme, and the effectiveness of the definition of the solution monitor (9).

5 Conclusions

We discussed the construction of non-oscillatory $\mathcal{RD}$ schemes which are stable and give genuine second order of accuracy in practical applications. The approach proposed opens the way to a new class of schemes which need very few matrix operations. Hence, they are more efficient, while retaining the advantages of the $\mathcal{RD}$ approach. In particular, the extension to arbitrary accuracy is quite natural and it is under way.

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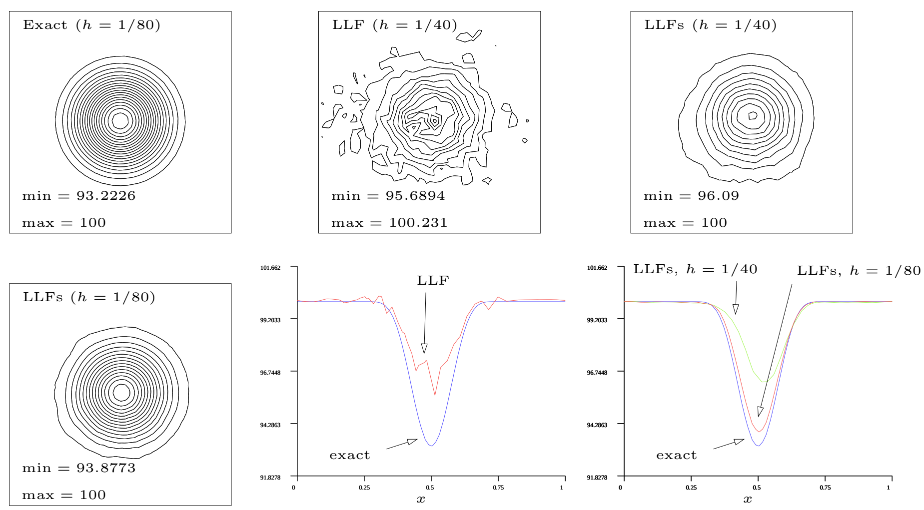

Fig. 1. Vortex advection. Top (left to right): Exact, LLF and LLFs ($h = 1/40$). Bottom (left to right): LLFs ($h = 1/80$), cut along centerline for LLF and LLFs

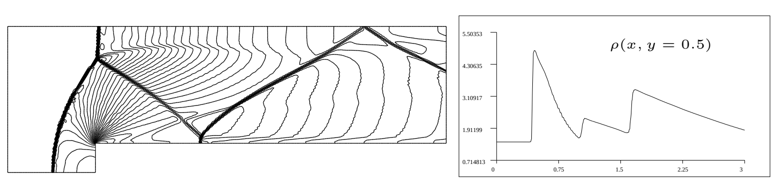

Fig. 2. Mach 3 forward facing step. Density contours (left) and distribution at $y = 0.5$ (right). Results at time $t = 4$ computed with the LLFs scheme