université de BORDEAUX

ANNÉE UNIVERSITAIRE 2019 / 2020
Session 1 d'automne
PARCOURS / ÉTAPE : 4TMA903U
Code UE : 4TTN901S, 4TTN901S
Épreuve : Algebraic number theory
Due date: 2019/10/21

Collège Sciences et technologies

Exercise 1

(1) Find the limit of the sequence of integers $(3, 33, 333, 3333, \ldots)$ (written in base 10) in $\mathbf{Q}_5$.
(2) Let p be a prime integer and $x \in \mathbf{Z}$ such that $\mathsf{gcd}(p, x) = 1$. Show that the sequence $(x^{p^n})_{n \in \mathbf{Z}_{\geqslant 0}}$ converges in $\mathbf{Q}_p$. Show that the limit is a root of unity, that depends only on the image of x in $\mathbf{F}_p^\times$.
(3) Let p be a prime integer. Prove that $X^p - X - 1$ is irreducible in $\mathbf{Q}_p[X]$.

Exercise 2

Let $(K, |.|)$ be a non archimedean valued field such that $\mathsf{char}(K) = \mathsf{char}(\kappa_K)$. Show that if $x, y \in K$ are distinct roots of unity, then $|x - y| = 1$.

Exercise 3

Show that the polynomial $(X^2 - 2)(X^2 - 17)(X^2 - 34)$ has a root in $\mathbf{R}$ and in $\mathbf{Z}_p$ for every prime p, but no root in $\mathbf{Q}$.

Exercise 4

Let $A = \mathbf{Z}\left[\sqrt{-5}\right]$ and $K = \mathsf{Frac}(A) = \mathbf{Q}(\sqrt{-5})$. Explain why $I = 3A + (1 + \sqrt{-5})A \subset K$ is a projective A-module. Show it explicitely as a direct factor of A^2. Show that it is not free.

Exercise 5

Let A be a Dedekind ring, K its fraction field and X an indeterminate.
(1) The *content* of a polynomial $P \in A[X]$ is the ideal $\mathfrak{c}(P)$ generated by the coefficients of P. Show that $\mathfrak{c}(PQ) = \mathfrak{c}(P)\mathfrak{c}(Q)$ for all $P, Q \in A[X]$.
(2) Let $S = \{P \in A[X]\,;\, \mathfrak{c}(P) = A\}$. Show that S is a multiplicative part in $A[X]$: let

$$B = S^{-1}(A[X]) \subset \mathsf{Frac}(A[X])$$

be the associated localization. Show that if $P, Q \in A[X]$ and $Q \neq 0$, then $\frac{P}{Q} \in B$ if and only if $\mathfrak{c}(P) \subset \mathfrak{c}(Q)$.
(3) Show that $K \cap B = A$. Let $J \subset B$ be an ideal: show that $J = IB$ where $I = J \cap A$, and that the map $I \mapsto IB$ is a bijection between the set of ideals of A onto the set of ideals of B.
(4) Prove that B is a PID.

Exercise 6

Show that every non-trivial non archimedean absolute value on $\mathbf{R}$ has divisible value group and algebraically closed residue field.