An upper bound for the theta function

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Abstract: This is an appendix to a paper by Parent [5]. I give a new upper bound for the norm of the classical theta function on any complex abelian variety.

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1 Result

Let g be a positive integer. Write $\mathbb{H}_g$ for the Siegel space of symmetric matrices $Z \in \mathrm{M}_g(\mathbb{C})$ such that $\mathrm{Im}Z$ is positive definite. To every $Z \in \mathbb{H}_g$ is associated the theta function defined by

$$\forall z \in \mathbb{C}^g \quad \theta_Z(z) = \sum_{m \in \mathbb{Z}^g} \exp(i\pi {}^{\mathrm{t}}mZm + 2i\pi {}^{\mathrm{t}}mz) \quad ,$$

and its norm defined by

$$\forall z = x + iy \in \mathbb{C}^g \quad \|\theta_Z(z)\| = \sqrt[4]{\det Y} \exp(-\pi {}^{\mathrm{t}}yY^{-1}y)|\theta_Z(z)| \quad ,$$

where $Y = \mathrm{Im}Z$.

My contribution here is the following:

Proposition 1.1: *Let $Z \in \mathbb{H}_g$ and assume that Z is Siegel-reduced. Put $c_g = \dfrac{g+2}{2}$ if $g \leq 3$ and $c_g = \dfrac{g+2}{2}\Big(\dfrac{g+2}{\pi\sqrt{3}}\Big)^{g/2}$ if $g \geq 4$. The upper bound $\|\theta_Z(z)\| \leq c_g (\det \mathrm{Im}Z)^{1/4}$ holds for every $z \in \mathbb{C}^g$.*

This result is used by Parent [5] to bound the height of quadratic points on modular curves. Let us remark that $c_g \leq g^{g/2}$ for every $g \geq 2$. In comparison, Edixhoven and de Jong (see [1] page 231) obtained statement 1.1 with c_g replaced by 2^{3g^3+5g}.

2 Proof

Fix a positive integer g. Denote by $\mathbb{S}_g$ the set of symmetric matrices $Y \in \mathrm{M}_g(\mathbb{R})$ that are positive definite. Let us recall a special case of the functional equation for the theta function (see equation (5.6) of [4] page 195): for every $Y \in \mathbb{S}_g$ and every $z \in \mathbb{C}^g$, one has

$$\theta_{iY^{-1}}(-iY^{-1}z) = \sqrt{\det Y} \exp(\pi{}^{\mathrm{t}}zY^{-1}z)\theta_{iY}(z) \quad . \qquad (1)$$

Lemma 2.1: *Let $Z \in \mathbb{H}_g$ and $z \in \mathbb{C}^g$. Putting $Y = \mathrm{Im}Z$, one has the inequality $\|\theta_Z(z)\| \leq \|\theta_{iY}(0)\| = \theta_{iY}(0)\sqrt[4]{\det Y}$.*

Proof: Put $y = \mathrm{Im}z$. One has

$$|\theta_Z(z)| = \Big|\sum_{m\in\mathbb{Z}^g} \exp(i\pi{}^{\mathrm{t}}mZm + 2i\pi{}^{\mathrm{t}}mz)\Big| \leq \sum_{m\in\mathbb{Z}^g} \Big|\exp(i\pi{}^{\mathrm{t}}mZm + 2i\pi{}^{\mathrm{t}}mz)\Big| = \theta_{iY}(iy) \quad ,$$

that is, $\|\theta_Z(z)\| \leq \|\theta_{iY}(iy)\|$.

The functional equation (1) gives $\|\theta_{iY^{-1}}(Y^{-1}y)\| = \|\theta_{iY}(iy)\|$, and one deduces

$$\|\theta_Z(z)\| \leq \|\theta_{iY^{-1}}(Y^{-1}y)\| \quad . \qquad (2)$$

Applying again (2) with Z replaced by iY^{-1} and z by $Y^{-1}y$, one gets $\|\theta_{iY^{-1}}(Y^{-1}y)\| \leq \|\theta_{iY}(0)\|$. Whence the result. $\square$

Let $Y \in \mathbb{S}_g$. Define $\lambda(Y) = \min_{m\in\mathbb{Z}^g-\{0\}} {}^{\mathrm{t}}mYm$. For every $t \in \mathbb{R}_+^*$, put

$$f_Y(t) = \theta_{itY}(0) = \sum_{m\in\mathbb{Z}^g} \exp(-\pi t{}^{\mathrm{t}}mYm) \quad .$$

Lemma 2.2: *Let $Y \in \mathbb{S}_g$ and put $\lambda = \lambda(Y)$. The following properties hold.*
(α) The function $\mathbb{R}_+^* \to \mathbb{R}$ that maps t to $t^{g/2}f_Y(t)$ is increasing.
(β) One has the estimate $f_Y\Big(\dfrac{g+2}{2\pi\lambda}\Big) \leq \dfrac{g+2}{2}$.

Proof: (α) The functional equation (1) implies $\sqrt{\det Y}t^{g/2}f_Y(t) = f_{Y^{-1}}(1/t)$ for every $t \in \mathbb{R}_+^*$; conclude by remarking that $f_{Y^{-1}}$ is decreasing.

(β) Part α gives $\dfrac{\mathrm{d}}{\mathrm{d}t}[t^{g/2}f_Y(t)] \geq 0$, that is, $\dfrac{g}{2t}f_Y(t) \geq -f_Y'(t)$ for every $t > 0$. On the other hand,

$$-\frac{1}{\pi}f_Y'(t) = \sum_{m\in\mathbb{Z}^g} {}^{\mathrm{t}}mYm \exp(-\pi t{}^{\mathrm{t}}mYm) \geq \sum_{m\in\mathbb{Z}^g-\{0\}} \lambda \exp(-\pi t{}^{\mathrm{t}}mYm) = \lambda[f_Y(t) - 1] \quad .$$

One infers $\dfrac{g}{2t}f_Y(t) \geq \pi\lambda[f_Y(t) - 1]$. Choosing $t = \dfrac{g+2}{2\pi\lambda}$, one obtains the result. $\square$

Proposition 2.3: *Let $Y \in \mathbb{S}_g$. Putting $\lambda = \lambda(Y)$, one has the upper bound*

$$\theta_{iY}(0) \leq \frac{g+2}{2} \max\Big[\Big(\frac{g+2}{2\pi\lambda}\Big)^{g/2}, 1\Big] \quad .$$

Proof: Put $t = \dfrac{g+2}{2\pi\lambda}$. If $t \geq 1$, then lemma 2.2.α implies the inequality $f_Y(1) \leq t^{g/2} f_Y(t)$. If $t \leq 1$, then $f_Y(1) \leq f_Y(t)$ since f_Y is decreasing. In any case, one obtains $\theta_{iY}(0) = f_Y(1) \leq \max(t^{g/2}, 1) f_Y(t)$. Conclude by applying lemma 2.2.β. $\square$

Now, to prove proposition 1.1 from lemma 2.1 and proposition 2.3, it suffices to observe that if $Z \in \mathbb{H}_g$ is Siegel-reduced, then $\lambda(\mathrm{Im} Z) \geq \dfrac{\sqrt{3}}{2}$ (see lemma 15 of [2] page 195).

References

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