p-adic dynamical systems of finite order

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Abstract

In this lecture we intend to study the finite subgroups of the group $Aut_R R[[X]]$ of R-automorphisms of the formal power series ring $R[[X]]$.

1 Setting

1.1 Notations

Let R be a discrete valuation ring which is assumed to be complete of unequal characteristic $(0, p)$. Let K be its fraction field and fix (K^{alg}, v) a valued algebraic closure. Let us denote by $R^{alg} := \{z \in K^{alg} | v(z) \geq 0\}$. Let π be a uniformizing parameter for R. The residue field $k := R/\pi$ is assumed to be algebraically closed.

1.2 The p-adic open disc

By using the Weierstrass Preparation theorem we can describe the geometry of the R-scheme $Z := \mathrm{Spec}\, R[\![X]\!]$. Namely, the special fibre $Z \times_R k$ has only one closed point which corresponds to the ideal $(\pi, X)R[\![X]\!]$, and the closed points of the generic fibre $Z \times_R K$, correspond to the irreducible distinguished polynomials of $R[\![X]\!]$. These polynomials have roots in the maximal ideal of R^{alg}. This allows us to identify $Z \times_R K$ with the open disc $\{z \in R^{alg} \mid v(z) > 0\}$ modulo the action by the Galois group of K^{alg}/K. Let $\sigma \in Aut_R R[[X]]$ then $\sigma(X) = \sum_{i \geq 0} a_i X^i$ with $a_0 \in \pi R$ and $a_1 \in R^\times$. In particular σ is a continuous homomorphism. Moreover σ induces an R-automorphism of the open disc Z, which we denote $\tilde{\sigma}$. For a rational point $(X - z) \in Z$ i.e. $z \in \pi R$, one has $\tilde{\sigma}((X - z)) = (X - \tilde{z})$, where $\tilde{z} = \sum_{i \geq 0} a_i z^i$. Identifying $z \in \pi R$ with the rational point $(X - z) \in Z$ we shall identify $\tilde{\sigma}$ with the application $\tilde{\sigma}(z) := \tilde{z}$. A rationnal point $z \in Z$ is a fixed point if and only if $z \in \pi R$ and $z = \tilde{\sigma}(z) = \sum_{i=0}^{\infty} a_i z^i$.

1.3 Prolegomena

In what follows we introduce through examples the questions we list in next section.

- **Iterations of series.** Let $n > 1$ and $\zeta \in R$ be a primitive n-th root of 1. Let $\sigma(X) = \zeta X(1 + a_1 X + a_2 X^2 + ...) \in R[[X]]$. One can calculate with computers for small n the n-th iterate automorphism and it is quite surprising to see that

$$(*) \qquad \sigma^n(X) = X(1 + 0X + 0X^2 + ... + 0X^{n-1} + E_n(a_1, a_2, ..., a_n)X^n + ...)$$

  This makes the use of computers quite limited in order to check if a given series is a torsion series. The equality (*) is a consequence of Cayley-Hamilton theorem. This justifies to look for other methods.

- **Reduction mod π.** Let $\Psi : Aut_R R[[X]] \to Aut_k k[[x]]$ be the reduction homomorphism. The group $Aut_k k[[x]]$ has a lot of finite subgroups. The reason is that finite extensions of the local field $k((t))$ are still local fields. Namely it follows from a theorem due to E. Witt that every finite p-group G can be realized as the Galois group of a finite Galois extension $k((x))/k((t))$ and so as a subgroup of $Aut_k k[[x]]$. Moreover there are infinitely many non conjugated realizations.

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- **Order p automorphisms in char. $p > 0$.** We keep the same notations as in the previous item. Let $G = Z/pZ \simeq < \sigma > \subset Aut_k k[[x]]$. Then $k((x))/k((x))^G$ is a p-cyclic extension and there is an integer m with $(m,p) = 1$ and a parameter t^{-1} of the local field $k((t^{-1})) = k((x))^G$ such that $k((x)) = k((t^{-1}))[w]$ and $\sigma(w) = w + 1$ with $w^p - w = t^m$. Then $x' := w^{-1/m} \in k((t^{-1}))[w]$ and so $\sigma(x') = x'(1+x'^m)^{-1/m}$. It follows that σ is known up to the conjugation by $x = x'(u_0+u_1x'+....)$. Note that $\sigma(x'^m) = x'^m(1+x'^m)^{-1}$ is the homographical transformation on x'^m induced by the transvection 2x2 matrix $B_{2,1}(1)$ with order p.

- **Lifting as a torsion series.** Let us assume that R contains ζ a primitive p-th root of 1. With the same notations as in in the previous item. We can ask for a deformation of $B_{2,1}(1)$ as a 2x2 matrix in $PGL_2(R)$ of order p. Let $A := (a_{ij})$ be the lower triangular matrix with diagonal $\{\zeta^m, 1\}$ and $a_{2,1} = 1$. Its order in $PGL_2(R)$ is p and it is a lifting of $B_{2,1}(1)$. Let $\tilde{\sigma}(X) := \zeta X(1 + X^m)^{-1/m} \in R[[X]]$. Its order is p and lifts the automorphism $\sigma(x) = x(1 + x^m)^{-1/m}$. Let us look at the set $Fix(\tilde{\sigma})$ of fixed points for $\tilde{\sigma}$. We solve the equation $z \in R^{alg}$ with $v(z) > 0$ and $\tilde{\sigma}(z) = z$ i.e. $z = \zeta z(1 + z^m)^{-1/m}$. We get $z = 0$ or $z^m = \zeta^m - 1$ whose p-adic valuation is $v(p)/m(p-1) > 0$. So $Fix(\tilde{\sigma})$ consists in $m+1$ points with equal mutual distance $v(p)/m(p-1) > 0$.

- **Lifting Galois covers.** We come back to item 3. There we consider the p-cyclic cover $w^p - w = t^m$ of the field $k((t^{-1}))$ (Artin-Schreier theory). This equation defines as well a p-cyclic étale cover of the affine line $\mathbb{A}^1_k$ which is totally ramified at $t = \infty$. The equation $w^p - w = t^m$ defines a non singular affine curve with a high singularity at ∞. The search for a parameter x' of the local field $k((t^{-1}))[w]$ geometrically corresponds to a local uniformization and so to a desingularization. The geometric counterpart to item 4 is thus to look for a lifting over R of $w^p - w = t^m$ as a $G = \mathbb{Z}/p\mathbb{Z}$-cover of the projective R-line $\mathbb{P}^1_R$ in a way that the normalization process of the corresponding R-curve induces a smooth R curve. There is a "deformation" of Artin-Schreier theory for étale covers over k to Kummer theory for étale covers over K. Namely, let $\lambda := \zeta - 1 \in R$. We remark that $[(\lambda W + 1)^p - 1]/\lambda^p = \prod_{1 \leq i \leq p}[W - (1 + \zeta + \zeta^2 + ... + \zeta^{p-1})]$ which mod λ reduces to $w^p - w$. Now there is a numerical criterion in order to check that the R-curve defined by $[(\lambda W + 1)^p - 1]/\lambda^p = T^m$ induces after normalization a smooth R-curve $\mathcal{C}$. Moreover $T : \mathcal{C} \to \mathbb{P}^1_R$ is a $G = \mathbb{Z}/p\mathbb{Z}$-cover of the projective R-line $\mathbb{P}^1_R$ which is a lifting of the smooth compactification of the affine curve $w^p - w = t^m$ to $\mathbb{P}^1_k$. The formal fiber $Z := \pi^{-1}(\infty)$ at $t = \infty$ is a G-stable open disc. We recover in this way the results of item 4.

2 Questions we propose to look at and references

- What is the geometry of fixed points for an order p automorphism in $Aut_R R[[X]]$? Necessary conditions and realization.

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- What about order p^2 and more generally order p^n automorphisms?

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- What about elementary abelian p-groups in $Aut_R R[[X]]$?

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- Find obstructions to the lifting of subgroups in $Aut_k k[[x]]$ as subgroups in $Aut_R R[[X]]$?

  References: [Be],[Be-Me 1], [Br-We], [Ch-Gu-Ha 2], [Gr-Mat 1]

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3 The global counterpart

A motivation to look at torsion series is related to global aspects. Namely the relationship with good reduction (or stable reduction) for Galois covers of smooth projective R-curves and lifting of galois covers of smooth or stable k-curves. There is a vast litterature relative to the global aspects but this is another story.

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