

Twists of symmetric bundles

Ph. Cassou-Noguès, B. Erez, M.J. Taylor *

January 12, 2007

Introduction

In 1984 J.-P. Serre, [Se1], proved a formula relating the Hasse-Witt invariant of the quadratic form $x \rightarrow \text{Tr}_{E/K}(x^2)$ associated to a finite separable extension E of a field K of characteristic $\neq 2$ with the second Stiefel-Whitney class of the permutation representation of the Galois group of E . The similarity between this formula and the one he obtained when considering a covering of Riemann surfaces with odd ramification, [Se2], led him to pose the question of the existence of a general result containing the previous ones as particular cases. A first step towards a positive answer to this question was made in 1993 by H. Esnault, B. Kahn and E. Viehweg, [E-K-V], where they gave a formula “à la Serre” for symmetric bundles obtained from certain finite coverings, $\pi : X \rightarrow Y$, either étale or tame, with odd ramification, whenever X and Y are Dedekind schemes. In [CNET1] we generalised this theorem, obtaining, under certain regularity assumptions, the required formula with no restriction on the dimensions of the schemes.

In 1985, Fröhlich, [F], gave his own point of view on Serre’s result, considering it as a particular case of a difference formula between two Hasse-Witt invariants of quadratic forms, namely the form we start with and the twist of this form by an orthogonal representation. The expression he gave for such a difference, [F], Theorems 2 and 3, involved the first and second Stiefel-Whitney classes of the representation and a totally new element, that he called the spinor class. The main goal of this paper is to obtain a theorem “à la Fröhlich” in the geometric situation considered in [CNET1].

In this situation a symmetric bundle (E, f) over a noetherian $\mathbf{Z}[\frac{1}{2}]$ -scheme Y is a vector bundle E over Y equipped with a symmetric isomorphism f

*Royal Society Wolfson Merit Award holder.

between E and its Y -dual E^\vee . A symmetric bundle can also be viewed as a quadratic form on Y and we shall write (E, q) if we adopt this point of view, or, if the form is clear from the context, we shall sometimes simply denote this by E . It is well known how to describe the set of all *twists* of (E, f) , that is the set of symmetric bundles which become isomorphic to (E, f) after an étale base extension. If $\mathbf{O}(E)$ denotes the orthogonal group scheme attached to (E, f) , then this set is $H^1(Y, \mathbf{O}(E))$ (see [Mi], Chapter 3 for a precise definition of this set). For α in $H^1(Y, \mathbf{O}(E))$, let E_α be the twist of E corresponding to α . Every symmetric bundle of rank n is a twist of the standard symmetric bundle $\mathbf{1}_n = (O_Y^n, x_1^2 + \dots + x_n^2)$. Let $\mathbf{O}(n)$ denote the automorphism group of this symmetric bundle. We write α_E for the class of E in $H^1(Y, \mathbf{O}(n))$ ($n = \text{rank}(E)$).

Following Delzant [Dz] and Jardine [J1], for any symmetric bundle E over Y one can define a cohomological invariant, which generalizes the classical invariants of quadratic forms and which is known as the *total Hasse-Witt class*. This is a class $w_t(E)$ in the (graded) étale cohomology group $H^*(Y_{et}, \mathbf{Z}/2\mathbf{Z})$:

$$w_t(E) = 1 + w_1(E)t + w_2(E)t^2 + \dots$$

A brief review of the definitions of the Hasse-Witt invariants can be found in [CNET1], section 1.e. The terms w_1 and w_2 , in degrees one and two, generalise the discriminant and the Hasse-Witt invariant respectively and have the following elementary description: we define δ^1 as the map induced by the determinant map

$$\delta^1 = \delta_E^1 : H^1(Y, \mathbf{O}(E)) \rightarrow H^1(Y_{et}, \mathbf{Z}/2\mathbf{Z})$$

and δ^2 as the boundary map

$$\delta^2 = \delta_E^2 : H^1(Y, \mathbf{O}(E)) \rightarrow H^2(Y_{et}, \mathbf{Z}/2\mathbf{Z})$$

associated to the exact sequence of étale sheaves of groups

$$1 \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow \widetilde{\mathbf{O}}(E) \rightarrow \mathbf{O}(E) \rightarrow 1, \quad (0.1)$$

where $\widetilde{\mathbf{O}}(E)$ is a certain covering constructed with the help of the Clifford algebra of E . Then $w_1(E) = \delta_n^1(\alpha_E)$ and $w_2(E) = \delta_n^2(\alpha_E)$, where, $\delta_n^i = \delta_{\mathbf{1}_n}^i$.

In this paper we will be interested in twists of symmetric bundles which arise from certain types of coverings of Y . Before giving the general definition we describe a special case first considered by Fröhlich (see [F] and also

[Es-K-V]). Let G be a finite group and let X be a G -torsor over Y . Consider an orthogonal representation of G given by a symmetric bundle (E, q) over Y together with a group homomorphism $\rho : G \rightarrow \Gamma(Y, \mathbf{O}(E))$. The G -torsor X defines an element $c(X)$ in $H^1(Y, G)$ and so, by push-forward along ρ , it defines an element $\rho(X) = \rho_*(c(X))$ in $H^1(Y, \mathbf{O}(E))$. We refer to this as the twist of E by the representation ρ and we denote it by $E_{\rho, X}$. In comparing the invariants of E and $E_{\rho, X}$, there appear not only the Stiefel-Whitney classes $w_i(\rho)$ of ρ but also the so called spinor classes. Building on the work of Fröhlich [F], Kahn [K] and Snaith [Sn], Jardine showed that, when Y is the spectrum of a field K of characteristic different from 2, there is a class

$$sp_t(\rho) = 1 + sp_1(\rho)t + sp_2(\rho)t^2 + \dots$$

called the *total spinor class*, which satisfies

$$w_t(E_{\rho, X})sp_t(\rho) = w_t(E)w_t(\rho)$$

in $H^*(K, \mathbf{Z}/2\mathbf{Z})$ and whose odd components are all trivial (see [J2]). In fact little else is known about the spinor class except in degree 2.

In [CNET3] we extended the work of Serre [Se1] to bundles over schemes and we obtained as a corollary of our results a proof, different from that of Kahn in [K] Corollary 6.1, of the following theorem which provides a generalisation of Fröhlich's result at least in the étale case.

Theorem 0.2 *Let (E, q) be a symmetric bundle over a regular scheme Y , let X be a G -torsor over Y and let $\rho : G \rightarrow \Gamma(Y, \mathbf{O}(E))$ be an orthogonal representation of G . Let $(E_{\rho, X}, q_{\rho, X})$ be the twist of (E, q) by ρ . Then, we have the equalities:*

- i) $w_1(E_{\rho, X}) = w_1(E) + w_1(\rho)$.
- ii) $w_2(E_{\rho, X}) = w_2(E) + w_1(E)w_1(\rho) + w_2(\rho) + sp_2(\rho)$.

In this paper we shall consider the case of tame coverings. More precisely we let

$$\pi : X \rightarrow Y = X/G$$

be a covering which is tamely ramified along a divisor b with normal crossings and whose ramification indices are all odd. Let ρ be an orthogonal representation of G

$$\rho : G \rightarrow \Gamma(Y, \mathbf{O}(E)) ,$$

where (E, q) is a symmetric bundle over Y . The first problem that we encounter is how to define, in this new situation, the twist of E by ρ . To that end we introduce the symmetric bundle $(\pi_*(\mathcal{D}_{X/Y}^{-1/2}), Tr_{X/Y})$, where $\mathcal{D}_{X/Y}^{-1/2}$ is the locally free sheaf over X whose square is the inverse of the different of X/Y and $Tr_{X/Y}$ is the trace form.

Definition 0.3. The *twist* of (E, q) by ρ is the symmetric bilinear form $(E_{\rho, X}, q_{\rho, X})$ on Y , where

$$E_{\rho, X} = (E \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G$$

is the G -fixed submodule of the G -module $E \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2})$ and where $q_{\rho, X}$ is the form which is the restriction of $|G|^{-1}(q \otimes Tr_{X/Y})$ to the O_Y -module $E_{\rho, X}$.

In the case where $|G|$ is not invertible in Y this abuse of notations will be justified in section 2.b

The appearance of the square root of the different endowed with the trace form can be understood by observing that it is itself the twist of the standard form on a permutation module by a natural orthogonal representation (see Example 2.6).

It is clear that this generalizes the étale case, because in that case the inverse different is just O_X . Of course we have to check that this new object is a symmetric bundle. This is proved in the next theorem.

Theorem 0.4 *i) The twist $(E_{\rho, X}, q_{\rho, X})$ of (E, q) by ρ is a symmetric bundle over Y .*

ii) Let $\phi : Z \rightarrow Y$ be a scheme flat over Y and let $T' = Z \times_Y X$. For any orthogonal representation ρ of G in $\mathbf{O}(E)$ we have

$$(\phi^*(E))_{\rho, T'} = \phi^*(E_{\rho, X}) .$$

We now wish to compute the difference $w_k(E_{\rho, X}) - w_k(E)$. We do not know how to do this directly based on a cocycle computation as for Theorem 0.2 in [CNET3], although we suspect it might be possible to do so. Hence instead we proceed as in [CNET1] by reducing to the étale case.

Let G_2 be a 2-Sylow subgroup of G . We write $Z = X/G_2$ and we let T be the normalisation of the fiber product $T' = Z \times_Y X$, so that we have the diagram

$$\begin{array}{ccccc} T & \longrightarrow & T' = Z \times_Y X & \longrightarrow & X \\ & & \downarrow & & \downarrow \pi \\ & \searrow \pi_Z & & & \\ & & Z & \xrightarrow{\phi} & Y \end{array}$$

It follows from [Es-K-V], Sect. 3.4, that the action of G on X induces a G -action on T and that π_Z identifies Z with T/G . Moreover we know from [CNET1], Theorem 2.2, that $\pi_Z : T \rightarrow Z$ is étale and hence a G -torsor (see [CEPT1], p. 291). Since the degree of the cover Z/Y is odd, the pull-back map $\phi^* : H^*(Y_{et}, \mathbf{Z}/2\mathbf{Z}) \rightarrow H^*(Z_{et}, \mathbf{Z}/2\mathbf{Z})$ is injective. Therefore we lose no information concerning the difference $w_k(E_{\rho, X}) - w_k(E)$ by instead considering the pull-back $\phi^*(w_k(E_{\rho, X}) - w_k(E))$. We now observe that firstly, we obtain a symmetric bundle over Z by considering the pull-back $(\phi^*(E), \phi^*(q))$ of (E, q) and secondly, that ρ induces an orthogonal representation $\phi^*(\rho) : G \rightarrow \Gamma(Z, \mathbf{O}(\phi^*(E)))$. Hence we are in a situation where we can use our construction of a twist and associate to $(\phi^*(E), \phi^*(q))$ the symmetric bundles $(\phi^*(E), \phi^*(q))_{\phi^*(\rho), T'}$ and $(\phi^*(E), \phi^*(q))_{\phi^*(\rho), T}$. Using the good functorial properties of the Hasse-Witt invariants and of the process of twisting, the previous difference can be written

$$w_k(\phi^*(E)_{\phi^*(\rho), T'}) - w_k(\phi^*(E)) .$$

Our strategy will be to express this difference as a sum of two terms, namely:

$$\left(w_k(\phi^*(E)_{\phi^*(\rho), T}) - w_k(\phi^*(E)) \right) + \left(w_k(\phi^*(E)_{\phi^*(\rho), T'}) - w_k(\phi^*(E)_{\phi^*(\rho), T}) \right) .$$

Since the cover T/Z is étale the first term is known by Theorem 0.2. Therefore the heart of the problem will be to compute the second term. For w_1 the formula is as simple as one could wish, but for w_2 a new class appears. Before stating the result we introduce this new class $R(\rho, X)$.

We start by introducing a divisor on Y which depends on the decomposition of ρ when restricted to the inertia groups of the generic points of the irreducible components of the branch locus b of π . We number the irreducible components b_h of b by $1 \leq h \leq m$, we denote by ξ_h the generic point of b_h and by B_h an irreducible component of the ramification locus of π such that

$\pi(B_h) = b_h$. We let I_h be the inertia group of the generic point of B_h . It follows from our hypotheses that I_h is cyclic of odd order denoted e_h . The action of I_h on the cotangent space at the generic point of B_h is given by a character denoted by χ_h . For $0 \leq k < e_h/2$ we let $d_k^{(h)}(E)$ denote the O_{Y, ξ_h} rank of the χ_h^k -component of E_{ξ_h} considered as an $O_{Y, \xi_h}[I_h]$ -module, (see section 4.a for the details). We then consider the divisor

$$R(\rho, X) = \sum_{h=1}^m d^{(h)}(E) b_h ,$$

where for $1 \leq h \leq m$ we have put

$$d^{(h)}(E) = \sum_{k=0}^{e_h/2} k d_k^{(h)}(E) .$$

We shall denote by the same symbol this divisor, the class in $\text{Pic}(Y)$ of the line bundle $O_Y(R(\rho, X))$ and also its image in $H^2(Y_{et}, \mathbf{Z}/2\mathbf{Z})$ under the boundary map

$$H^1(Y_{et}, \mathbf{G}_m) \rightarrow H^2(Y_{et}, \mathbf{Z}/2\mathbf{Z})$$

associated to the Kummer sequence

$$0 \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow \mathbf{G}_m \rightarrow \mathbf{G}_m \rightarrow 0$$

(Note that under our assumptions we may identify the group schemes μ_2 and $\mathbf{Z}/2\mathbf{Z}$).

Remarks.

1) One recovers the ramification invariant in [CNET1] by considering the regular representation (see the Example at the end of Section 4).

2) As pointed out to us by Serre, the ramification invariant may be viewed as a ‘‘half of a Woods-Hole element’’ (our terminology). Namely, when considering the Lefschetz-Riemann-Roch theorem one encounters expressions involving terms like $(1 - \zeta)^{-1}$, where ζ is a root of unity, which arise from the inverse of the term $\lambda_{-1}(\mathcal{N})$ (see for instance [E], section 2.a, p. 126). The key-point is the identity

$$\frac{1}{1 - \zeta} = -\frac{1}{e} \sum_{k=1}^{e-1} k \zeta^k$$

where ζ denotes a non-trivial root e -th root of unity.

We are now in a position to state our next main result.

Theorem 0.5 *We have the following equalities:*

$$i) \quad w_1(\phi^*(E_{\rho,X})) = w_1(\phi^*(E)) + w_1(\phi^*(\rho)) .$$

ii)

$$\begin{aligned} w_2(\phi^*(E_{\rho,X})) &= w_2(\phi^*(E)) + w_1(\phi^*(E))w_1(\phi^*(\rho)) + w_2(\phi^*(\rho)) + \\ &\quad + sp_2(\phi^*(\rho)) + \phi^*(R(\rho, X)) . \end{aligned}$$

As in [CNET1] the proof of this result relies on two main ingredients: on the one hand a formula which expresses the difference between the total Hasse-Witt class of the two symmetric bundles

$$\Upsilon^{(0)} = \phi^*(E, q)_{\rho, X} \quad \text{and} \quad \Upsilon^{(m)} = (\phi^*((E), \phi^*(q)))_{\rho, T}$$

and on the other hand an explicit determinant computation. For the sake of simplicity, when there is no risk of ambiguity, we shall just denote by $-E$ the symmetric bundle $(E, -q)$.

The main problem that we encounter in our situation is that $\Upsilon^{(0)} + \Upsilon^{(m)}$ is not always a locally free $\mathcal{O}_Z[G]$ -module and thus $\Upsilon^{(0)} \perp -\Upsilon^{(m)}$ is not in general a symmetric metabolic bundle. This fact leads us to decompose the normalisation map $T \rightarrow T'$ into a sequence of normalisations, where we add in one irreducible component at a time. More precisely we consider the sequence of Z -morphisms

$$T = T^{(m)} \rightarrow T^{(m-1)} \rightarrow \dots \rightarrow T^{(0)} = T' ,$$

numbered by the m components of the branch locus b of the covering and obtained by normalisation along a component of b as described in [CNET1], Section 3. In Section 4 we will prove that, for $0 \leq h \leq m-1$, we obtain an exact sequence of locally free \mathcal{O}_Z -modules

$$0 \rightarrow (\phi^*(E) \otimes_{\mathcal{O}_Z} \mathcal{I}^{(h)})^G \rightarrow \phi^*(E)_{\rho, T^{(h)}} \oplus \phi^*(E)_{\rho, T^{(h+1)}} \rightarrow (\phi^*(E) \otimes_{\mathcal{O}_Z} \mathcal{G}^{(h)})^G \rightarrow 0$$

and that for each h , $0 \leq h \leq m-1$, the symmetric bundle defined as the orthogonal sum $(\phi^*(E), (-1)^h \phi^*(q))_{\rho, T^{(h)}} \perp (\phi^*(E), (-1)^{h+1} \phi^*(q))_{\rho, T^{(h+1)}}$ is metabolic with lagrangian $(\phi^*(E) \otimes_{\mathcal{O}_Z} \mathcal{I}^{(h)})^G$.

For a bundle V of rank n over Y we denote by $c_i(V)$ the i -th Chern class of V as an element of $H^{2i}(Y_{et}, \mathbf{Z}/2\mathbf{Z})$, [Gr]. If E is a metabolic form on Y

with lagrangian V , then (see the Main Lemma in 1.c) the total Hasse-Witt class $w_t(E)$ is given by the element of $H^*(Y_{et}, \mathbf{Z}/2\mathbf{Z})$

$$d_t(V) = \sum_{i=0}^n (1 + (-1)^i t) c_i(V) t^{2i} .$$

By applying the Main Lemma to each metabolic bundle

$$(\phi^*(E), (-1)^h \phi^*(q))_{\rho, T^{(h)}} \perp (\phi^*(E), (-1)^{h+1} \phi^*(q))_{\rho, T^{(h+1)}}$$

we obtain the following result.

Theorem 0.6 *In $H^*(Z_{et}, \mathbf{Z}/2\mathbf{Z})$ the class $w_t(\phi^*(E, q)_{\rho, X})$ equals*

$$(w_t(\phi^*(E), (-1)^m \phi^*(q))_{\rho, T})^{(-1)^m} \prod_{0 \leq h \leq m-1} d_t((\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h)})^G)^{(-1)^h} .$$

Note that in principle this formula provides a formula for Hasse-Witt invariants of twists in *all* dimensions. It is quite remarkable to see how the ramification invariant comes out from this. Let us point out that firstly this theorem is new even for curves and secondly it provides a substantial strengthening of the main result of [CNET1], since we no longer need to impose such stringent regularity conditions (see once again the Example at the end of Section 4 for further details).

Theorem 0.5 is a consequence of the previous equalities by making the degree two terms explicit. More precisely for any h , $0 \leq h \leq m-1$ we can consider the metabolic symmetric bundles

$$(\phi^*(E), \phi^*(q))_{\rho, T^{(h)}} \perp (\phi^*(E), -\phi^*(q))_{\rho, T^{(h+1)}}$$

and

$$(\phi^*(E), \phi^*(q))_{\rho, T^{(h+1)}} \perp (\phi^*(E), -\phi^*(q))_{\rho, T^{(h+1)}} .$$

By applying the Main Lemma to each of these bundles we obtain that

$$\frac{w_t((\phi^*(E), \phi^*(q))_{\rho, T^{(h)}})}{w_t((\phi^*(E), \phi^*(q))_{\rho, T^{(h+1)}})} = \frac{d_t((\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h)})^G)}{d_t((\phi^*(E))_{\rho, T^{(h+1)}})}$$

which implies the equality of the degree two terms of each side:

$$w_2((\phi^*(E))_{T^{(h)}}) - w_2((\phi^*(E))_{T^{(h+1)}}) = c_1(\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h)})^G - c_1(\phi^*(E))_{\rho, T^{(h+1)}} .$$

It suffices to add these equalities to obtain $w_2(\phi^*(E)_{\rho, T'}) - w_2(\phi^*(E)_{\rho, T})$ as a sum of images in $H^2(Y_{et}, \mathbf{Z}/2\mathbf{Z})$ of divisors of Z whose supports are contained in the inverse image of the branch locus. The computation of these divisors is achieved in Section 4.

To understand fully Theorem 0.6 it is, in fact, necessary to work in the derived category of bounded complexes of vector bundles over a scheme. The reader is referred to [Ba1] and [Ba2] for the basic theory and to [CNET2] for a detailed discussion in the algebraic geometric context. The key point is that while $\Upsilon^{(0)} \perp -\Upsilon^{(m)}$ is not a metabolic symmetric bundle, it is, however, a metabolic complex in the derived category if we view $\Upsilon^{(0)} \perp -\Upsilon^{(m)}$ as a symmetric complex concentrated in degree 0. Furthermore it has as lagrangian the complex M_{\bullet} which is concentrated in degrees 1 and 0 with $M_0 = \bigoplus_{h=0}^{h=m-1} (\phi^*(E) \otimes \mathcal{I}^{(h)})^G$ and $M_1 = \bigoplus_{h=1}^{h=m} \phi^*(E)_{\rho, T^{(h)}}$.

Using a generalisation of the Main Lemma to derived categories we then get

$$w_t(\Upsilon^{(0)})w_t(-\Upsilon^{(m)}) = d_t(M_0 - M_1) .$$

This formula lies behind our theorem.

To conclude this introduction we indicate some other related developments. The twisting results for forms over fields play an important role in the work of T. Saito on the sign of the functional equation of the L-function of an orthogonal motive (see [S1]). More recently the results of [CNET1] have been used by Glass in [Gl] to relate the Galois invariants to ϵ -factors. In a different vein, Saito has generalized Serre's original formula to the case of (smooth) non-finite morphisms $X \rightarrow Y$ [S2]. The conjunction of these results suggests a beautiful picture, parts of which are still hidden, in which the formulæ obtained in this paper seem to hold a special place.

1 Hasse-Witt class of metabolic bundles

In this section we start by recalling the basic definitions and properties of the objects we will consider throughout this paper as well as some of the results of [CNET1] that will play an important role in this work.

1.a Symmetric bundles on schemes

Let Y be a scheme and assume that 2 is invertible over Y , then the theory of symmetric bilinear forms over Y is equivalent to that of quadratic forms over Y . A *vector bundle* E on Y is a locally free O_Y -module of finite rank. The dual of a vector bundle E is the vector bundle E^\vee such that, for any open subscheme Z of Y

$$E^\vee(Z) = \text{Hom}_{O_Z}(E|_Z, O_Z) .$$

There is a natural evaluation pairing $\langle \cdot, \cdot \rangle$ between E and E^\vee and one can identify E with the double dual $E^{\vee\vee}$ by

$$\kappa : E \cong E^{\vee\vee} ,$$

where $\langle \alpha, \kappa(u) \rangle = \langle u, \alpha \rangle$. A *symmetric bilinear form* on Y is a vector bundle E on Y equipped with a map of sheaves

$$q : E \times_Y E \rightarrow O_Y ,$$

which on sections over an open subscheme restricts to a symmetric bilinear form. This defines an *adjoint* map

$$\varphi = \varphi_q : E \rightarrow E^\vee ,$$

which because of the symmetry assumption equals its transpose:

$$\varphi = \varphi^t : E \xrightarrow{\kappa} E^{\vee\vee} \xrightarrow{\varphi^\vee} E^\vee .$$

We shall say that (E, q) is *non-degenerate* (or unimodular) if the adjoint φ is an isomorphism. From now on we will call a *symmetric bundle* any vector bundle endowed with a non-degenerate quadratic form.

1.b Metabolic bundles

The notion of a metabolic bundle was introduced by Knebusch for his definition of the Witt group of a scheme in [Kne]. Hyperbolic forms are metabolic and the greater generality of the latter notion is the right one in the global situation of non-affine schemes.

Let E be a vector bundle over Y . A sub- O_Y -module V of E is a *sub-bundle* of E if it is *locally* a direct summand, *i.e.* for any y in Y , there is an

open Z containing y such that $V|_Z$ has a direct summand in $E|_Z$. If V is a sub-bundle of E , then V and the quotient E/V are both vector bundles. Suppose now that (E, q) is a vector bundle endowed with a quadratic form. For a sub-module $i : V \subset E$ one defines the orthogonal space, which is the sub- O_Y -module V^\perp , whose sections over the open Z consist of those sections of E which are orthogonal to all sections of V over any open subset of Z . Alternatively:

$$V^\perp = \ker(E \xrightarrow{\varphi_q} E^\vee \xrightarrow{i^\vee} V^\vee) .$$

If furthermore V is a sub-bundle of E , then i^\vee is an epimorphism. Assume now that (E, q) is a symmetric bundle, then by definition φ_q is an isomorphism and therefore we have an isomorphism

$$\alpha : E/V^\perp \cong V^\vee ,$$

E/V^\perp is locally free and V^\perp is also a sub-bundle. There is also an isomorphism

$$\beta : V^\perp \cong (E/V)^\vee .$$

(The form $(E/V)^\vee$ can be identified with the sub- O_Y -module of E whose sections over $Z \subset Y$ are the linear forms $\lambda : E|_Z \rightarrow O_Z$ which vanish on V , so that $V^\perp = \varphi_B^{-1}((E/V)^\vee)$).

A sub-bundle V of a symmetric bundle (E, q) is a *totally isotropic sub-bundle* (also called a *sub-lagrangian*) if $V \subset V^\perp$. If V is a sub-lagrangian of (E, q) , then V^\perp/V is a sub-bundle of E/V and the form on V^\perp/V obtained by reducing q modulo V is non-degenerate. A sub-bundle V of (E, q) is called a *lagrangian* if it has the property that $V = V^\perp$. The symmetric bundle (E, q) is called *metabolic* if it contains a lagrangian. If V is a lagrangian in (E, q) , then $\text{rank}(E) = 2 \cdot \text{rank}(V)$ and V is, in a sense, a maximal totally isotropic sub-bundle. One can observe that V is a lagrangian in (E, q) if and only if one has a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & V & \rightarrow & E & \rightarrow & V^\vee & \rightarrow & 0 \\ & & id \downarrow & & \downarrow \varphi_q & & \downarrow id & & \\ 0 & \rightarrow & V \cong V^{\vee\vee} & \rightarrow & E^\vee & \rightarrow & V^\vee & \rightarrow & 0 \end{array} .$$

That is a metabolic form is given by a symmetric/self-dual short exact sequence (see [Kne], Chapt. 3).

1.c The main lemma.

Here we state a result of [EKV], Proposition 5.5, which is one of the main tools in calculating Hasse-Witt invariants. It provides a precise relation between the Hasse-Witt invariants of a metabolic form and the Chern classes of a lagrangian. We will use this lemma and its corollary in Section 3.

Lemma 1.1 *Let (E, q) be a metabolic form with lagrangian V . Then*

$$w_t(E) = d_t(V) .$$

In particular

$$1 + w_1(E)t + w_2(E)t^2 = 1 + n(-1)t + \left(c_1(V) + \binom{n}{2}(-1) \cup (-1) \right) t^2 .$$

Let us assume Y to be irreducible with generic point η . In our framework, we will construct metabolic bundles from symmetric bundles over Y which are isometric when restricted to the generic fiber of Y . Namely if (E, q_E) and (F, q_F) are defined over Y and agree on the generic fiber

$$E|_\eta = F|_\eta ,$$

then, under suitable assumptions, $(E \perp F, q_E \perp -q_F)$ is metabolic. A natural approach is to consider the sub-sheaf \mathcal{G} of $E|_\eta = F|_\eta$ defined as

$$\mathcal{G} := \langle e - f | e \in E, f \in F \rangle ,$$

and the exact sequence

$$0 \rightarrow E \cap F \rightarrow E \perp F \rightarrow \mathcal{G} \rightarrow 0 ,$$

where the maps are obtained by restricting to $E \perp F$ the maps defined at the level of the generic fibers given by the diagonal map and the map sending (x, y) to $(x - y)/2$. Then, if \mathcal{G} is locally free, it follows that $E \cap F$ is a sub-bundle of $E \perp F$, which is a lagrangian and

$$E \perp F / E \cap F \simeq (E \cap F)^\vee \simeq \mathcal{G} .$$

Corollary 1.2 *Let (E, q_E) and (F, q_F) be non-degenerate forms which are isometric when restricted to η .*

a) $w_1(E) = w_1(F)$ and $c_1(E) = c_1(F)$.

b) Consider the exact sequence

$$0 \rightarrow E \cap F \rightarrow E \perp F \rightarrow \mathcal{G} \rightarrow 0 .$$

Assume that \mathcal{G} (and $E \cap F$) is locally free, and that $(E \perp F, q_E \perp - q_F)$ is metabolic with lagrangian $E \cap F = \mathcal{G}^\vee$. Then

$$w_t(E, q_E) \cdot w_t(F, -q_F) = d_t(\mathcal{G}) .$$

c) $w_2(E) - w_2(F) = c_1(\mathcal{G}) - c_1(E)$. In particular this sum belongs to the image of the Picard group modulo 2 in $H^2(Y_{et}, \mathbf{Z}/2\mathbf{Z})$.

Remark

The first Chern class $c_1(L)$ of a vector bundle L over Y is obtained as the image of the class of $\text{Pic}(Y)$ defined by the line bundle $\det(L)$ under the boundary map associated to the Kummer exact sequence deduced from the multiplication by 2 on \mathbf{G}_m .

2 The twist of a bundle à la Fröhlich

The main aim of this section is to prove Theorem 0.4 (i), namely, that the twist of a symmetric bundle by an orthogonal representation of the “Galois group” of a tame covering is again a symmetric bundle. Moreover we provide interesting examples of such twists. It should be observed that in this section we are able to relax some of the hypotheses imposed in subsequent sections.

2.a Tame coverings with odd ramification

In what follows all the schemes will be defined over $\text{Spec}(\mathbf{Z}[\frac{1}{2}])$. We let X be a connected, projective, regular scheme which is either defined over the spectrum $\text{Spec}(\mathbf{F}_p)$ of the prime field of characteristic $p \neq 2$ or is flat over $\text{Spec}(\mathbf{Z}[\frac{1}{2}])$. We assume that X is equipped with a tame action by a finite group G , in the sense of Grothendieck-Murre. In particular the quotient

$$\pi : X \rightarrow Y = X/G$$

exists and is a torsor for G over Y outside a divisor b with normal crossings (see [Gr-M], [CEPT1], [CE] Appendix and [CEPT2] 1.2 and Appendix). We assume that π is a flat morphism of schemes and therefore that Y is regular. The different $\mathcal{D}_{X/Y}$ is defined as the annihilator of the sheaf $\Omega_{X/Y}^1$ of relative differentials of degree 1 (see [Mi] Rem. I.3.7). The reduced closed subscheme of X defined by the support of the different is the ramification locus of π , which we denote by $B = B(X/Y)$. Then $b = b(X/Y)$ is the reduced subscheme of Y defined by the image of B under π . There are decompositions

$$b(X/Y) = \coprod_{1 \leq h \leq m} b_h \quad \text{and} \quad B(X/Y) = \coprod_{h,k} B_{h,k} ,$$

where the b_h are the irreducible components of b and where for any fixed integer h between 1 and m , the $B_{h,k}$ are the irreducible components of B such that $\pi(B_{h,k}) = b_h$. The tameness assumption on the ramification implies that the branch locus $b(X/Y)$ on Y is a divisor with normal crossings. For each irreducible component b_h of b , we denote by ξ_h the generic point of b_h and by $\xi'_{h,k}$ a generic point of the component $B_{h,k}$ of the ramification locus on X . The ramification index (resp. residue class extension degree) of $\xi'_{h,k}$ over ξ_h which is independent of k will be denoted by e_h (resp. f_h). We assume that the inertia group of each closed point of X has odd order, thus every point has odd inertia and in particular the integers e_h are odd. We introduce the divisor of X

$$\omega_{X/Y} = \sum_{h,k} (e_h - 1) B_{h,k}$$

and we define the square root of the codifferent as a vector bundle on X by setting

$$\mathcal{D}_{X/Y}^{-1/2} = \mathcal{O}_X(\omega_{X/Y}/2) .$$

Therefore we obtain a symmetric bundle on Y as defined in 1.a by considering $(\pi_*(\mathcal{D}_{X/Y}^{-1/2}), Tr_{X/Y})$, where $Tr_{X/Y}$ denotes the trace form.

2.b Algebraic preliminaries

The proof of Theorem 0.4(i) will be a consequence of a number of elementary algebraic results. For these auxiliary steps, as in the rest of this paper, we adopt the following conventions: R is an integral domain and M is a left $R[G]$ -module which we assume to be locally free and finitely generated as an R -module. We write $\sigma = \sum_{g \in G} g$.

Lemma 2.1 *If M is a projective $R[G]$ -module, then $M^G = \sigma M$. Furthermore:*

- i) M^G is a locally free R -module.*
- ii) The map $m \mapsto \sigma m$ induces an isomorphism of R -modules from M_G onto M^G .*

PROOF. All the above statements are clear when $M = R[G]$; they therefore hold for any finitely generated free $R[G]$ -module and are then easily seen to hold for a direct summand of a finitely generated free $R[G]$ -module.

We now consider a symmetric bundle (M, t) over R ; that is to say M is a finitely generated locally free R -module, equipped with a non-degenerate symmetric bilinear form t . We denote by φ_t the adjoint map $\varphi_t : M \rightarrow M^\vee$. We suppose further that M is now a projective $R[G]$ -module and that the pairing t is G -invariant. Under these assumptions we then can use Lemma 2.1 to define the symmetric bilinear form t^G on M^G by

$$t^G(x, y) = t(m, y) ,$$

where m is an arbitrary element of M with the property that $\sigma m = x$. Let I be the R -submodule of M generated by the set $\{(1 - g)m, m \in M, g \in G\}$. Since m is defined up to an element of I (Lemma 2.1 (ii)) and since t is G -invariant, one verifies immediately that t^G is well defined. Moreover one observes that for any x and y in M^G one has

$$| G | t^G(x, y) = t(x, y).$$

Proposition 2.2 *If M is a projective $R[G]$ -module, then (M^G, t^G) is a symmetric R -bundle.*

PROOF. From Lemma 2.1 we know that M^G is a locally free R -module. It remains to prove that φ_{t^G} is an R -module isomorphism from M^G onto $\text{Hom}_R(M^G, R)$. The proof of this result is given in [CNET3], Proposition 2.2.

We conclude this preparatory section by recalling the following well-known result (see [McL] Corollary 3.3, p. 145 and p. 196):

Lemma 2.3 *Let M and N be left $R[G]$ -modules. Assume that M and N are both projective R -modules and that either M or N is projective as an $R[G]$ -module. Then $M \otimes_R N$, endowed with the diagonal action of G , is a projective $R[G]$ -module.*

2.c Proof of Theorem 0.4 (i)

Let ρ be an orthogonal representation of G taking values in the orthogonal group $\mathbf{O}(E)$ of the symmetric bundle E over Y . Under the assumptions that the action of G is tame and that the ramification indices are odd, we may consider the O_Y -module

$$E_{\rho,X} := (E \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G$$

endowed with the form $q_{\rho,X} = (q \otimes \text{Tr}_{X/Y})^G$. Our aim is to show that $(E_{\rho,X}, q_{\rho,X})$ is a symmetric bundle over O_Y . Let U be any open subscheme of Y , then $R = O_Y(U)$ is an integral domain. Since the ramification of the cover is tame and π is flat we know that $\pi_*(\mathcal{D}_{X/Y}^{-1/2})(U)$ is a projective $O_Y(U)[G]$ -module, (see for instance Section 4). It follows from Lemma 2.1 and Lemma 2.3 that $E_{\rho,X}(U)$ is projective and therefore locally free over $O_Y(U)$. We have then proved that $E_{\rho,X}$ is a vector bundle over O_Y . We may now consider the $O_Y(U)$ -symmetric bundle (M, t) with

$M = (E \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))(U)$ and $t = (q \otimes \text{Tr}_{X/Y})$. We deduce from Proposition 2.2 that t^G defines a non-degenerate symmetric form on $E_{\rho,X}(U)$ and this completes the proof of the first part of the theorem. The second part will be proved in Sect. 3, Lemma 3.1.

2.d Examples of twists

We conclude this section by giving examples of twists of symmetric bundles.

Example 2.4. Let μ denote the product form $\mu(x, y) = xy$ on $\pi_*(\mathcal{D}_{X/Y}^{-1/2})$. This is a form in a generalised sense, in that it takes its values in the ideal $\pi_*(\mathcal{D}_{X/Y}^{-1})$ instead of the ring O_Y . We want to show that $q_{\rho,X}$ is nothing other than the restriction to $E_{\rho,X} = (E \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G$ of the form $t_X := q \otimes \mu$ on $(E \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))$.

Proposition 2.5

$$t_X(x, y) = q_{\rho,X}(x, y) .$$

PROOF. In order to prove the equality it suffices to prove that t_X and $q_{\rho,X}$ coincide on $E_{\rho,X}(U)$ for any affine open U in Y . Moreover any element of

$E_{\rho,X}(U)$ can be obtained as a sum of elements of the form $\sigma(m \otimes n)$. So let $x = \sigma(m \otimes n)$ and $y = \sigma(m' \otimes n')$. We have the equalities

$$\begin{aligned} t_X(x, y) &= q \otimes \mu(\sum_{g \in G} (\rho(g)m \otimes gn), \sum_{h \in G} (\rho(h)m' \otimes hn')) \\ &= \sum_{g, h \in G} q(\rho(g)m, \rho(h)m')(gn)(hn') . \end{aligned}$$

Using the invariance of q by G this can be written

$$t_X(x, y) = \sum_{g \in G} q(m, \rho(g)m') Tr_{X/Y}(n(gn')) .$$

We now observe that the right-hand side of this equality can be expressed as

$$(q \otimes Tr_{X/Y})(m \otimes n, \sigma(m' \otimes n')) .$$

Finally it follows from the very definition of $q_{\rho,X}$ that this last quantity is equal to $q_{\rho,X}(x, y)$. Hence we have proved that $t_X(x, y) = q_{\rho,X}(x, y)$.

Example 2.6. Subquotients. Here we show that the square root of the inverse different appears as the twist of the standard form on a permutation module by the natural orthogonal representation attached to a Galois covering. This is to be compared with the interpretation of the trace form of an étale algebra as a twist of the standard form, which lies at the foundation of the original results by Serre and Fröhlich. We keep the hypotheses and the notation of the general set-up as introduced in 2.a of this Section . We fix a subgroup H of G and we let

$$\lambda : X \rightarrow V := X/H$$

denote the quotient map and

$$\gamma : V \rightarrow Y$$

be the induced map, so that $\pi = \gamma \circ \lambda$. We note that V being the quotient of a normal scheme by a finite group is normal but not necessarily regular. Let a run over a left transversal of H in G and let E be the free bundle $O_Y[G/H]$ which has for basis the left cosets aH of H in G . We consider the symmetric O_Y -bundle (E, q) where q is the quadratic form on E which has $\{aH\}$ as an orthonormal basis. From now on we denote by \bar{a} the coset aH .

The left action of G by permuting the cosets $\{\bar{a}\}$ extends to an orthogonal representation

$$\rho : G \rightarrow \Gamma(Y, \mathbf{O}(q)) .$$

Our goal is to describe the twist $(E, q)_{\rho, X}$ of (E, q) by the permutation representation ρ .

Proposition 2.7 *i) The twist $(E, q)_{\rho, X}$ is the symmetric bundle*

$$(\pi_*(\mathcal{D}_{X/Y}^{-1/2})^H, Tr_{V/Y}) \text{ on } Y .$$

ii) Suppose that λ is flat, then $(\pi_(\mathcal{D}_{X/Y}^{-1/2})^H, Tr_{V/Y})$ is the symmetric bundle $(\gamma_*(\mathcal{D}_{V/Y}^{-1/2}), Tr_{V/Y})$.*

PROOF. (i) We can reduce to an affine situation. For any point y of Y we denote by U a sufficiently small open affine neighbourhood of y . One defines a morphism of $\mathcal{O}_Y(U)$ -modules

$$\theta : \mathcal{D}_{X/Y}^{-1/2}(\pi^{-1}(U))^H \rightarrow (\mathcal{O}_Y(U)[G/H] \otimes_{\mathcal{O}_Y(U)} \mathcal{D}_{X/Y}^{-1/2}(\pi^{-1}(U)))^G$$

$$x \rightarrow \sum_{a \in A} a \otimes ax$$

where A is a left transversal of H in G . This is an isomorphism which inverse is given by taking the identity component. Since we have the equalities:

$$\sum_{a, b \in A} q(a, b) \mu(ax, by) = \sum_{a \in A} \mu(ax, ay) = Tr_{V/Y}(xy) ,$$

this isomorphism is as required an isometry.

(ii) In order to complete the proof of the proposition it suffices to show that, if λ is flat, then:

$$(\mathcal{D}_{X/Y}^{-1/2}(\pi^{-1}(U)))^H = \mathcal{D}_{V/Y}^{-1/2}(\gamma^{-1}(U)) .$$

From the transitivity formula of differentials, this last equality reduces us to showing that

$$(\mathcal{D}_{X/V}^{-1/2}(\pi^{-1}(U)))^H = \mathcal{O}_V(\gamma^{-1}(U)) .$$

Since the cover $X \rightarrow V$ is tame and λ is flat, it follows that $\mathcal{D}_{X/V}^{-1/2}(\pi^{-1}(U))$ is a projective $O_V(\gamma^{-1}(U))[H]$ -module. Once again from Lemma 2.1 we deduce that

$$(\mathcal{D}_{X/V}^{-1/2}(\pi^{-1}(U)))^H = \text{Tr}_{X/V}(\mathcal{D}_{X/V}^{-1/2}(\pi^{-1}(U))) .$$

From an easy generalisation of [Se4], III, Proposition 7, we obtain that

$$\text{Tr}_{X/V}(\mathcal{D}_{X/V}^{-1/2}(\pi^{-1}(U))) = O_V(\gamma^{-1}(U)) .$$

The required equality now follows.

Example 2.8. The twist of a metabolic bundle is metabolic. Let (E, q) be the underlying symmetric Y -bundle of an orthogonal representation $\rho : G \rightarrow \Gamma(Y, \mathbf{O}(q))$, where X is a scheme endowed with a tame action by the finite group G and $Y = X/G$. We shall say that ρ is a metabolic representation when (E, q) is metabolic with a G -invariant lagrangian V . Our goal is to prove that if (E, q) and ρ are metabolic then $(E, q)_{\rho, X}$ is also metabolic.

Let V be a lagrangian of (E, q) . We then have an exact sequence of G -modules

$$0 \rightarrow V \rightarrow E \rightarrow V^\vee \rightarrow 0 .$$

Since the cover $\pi : X \rightarrow Y$ is tame and π is flat, $\pi_*(\mathcal{D}_{X/Y}^{-1/2})$ is a locally projective $O_Y[G]$ -module. Therefore it follows from Lemma 2.3 that the modules $V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2})$, $E \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2})$ and $V^\vee \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2})$ are all locally projective $O_Y[G]$ -modules. So, tensoring and taking G -fixed points, affords a new exact sequence of locally free $O_Y[G]$ -modules

$$0 \rightarrow (V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G \rightarrow E_{\rho, X} \rightarrow (V^\vee \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G \rightarrow 0 ,$$

(see the proof of Proposition 3.2 for details). It is clear from the definition of $q_{\rho, X}$ that its restriction to $(V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G$ is the null form since V is a lagrangian for q . In order to prove that $E_{\rho, X}$ is metabolic with $(V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G$ as a lagrangian it suffices to show that the rank of $E_{\rho, X}$ is twice the rank of $(V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G$. This will follow at once from the O_Y -module isomorphism

$$((V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G)^\vee \cong (V^\vee \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G .$$

This last isomorphism can be proved as follows. Since $\pi_*(\mathcal{D}_{X/Y}^{-1/2})$ is unimodular it is self-dual and therefore

$$(V^\vee \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2})) \cong (V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^\vee .$$

Hence we deduce that $(V^\vee \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G$ is isomorphic to

$$((V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^\vee)^G = \text{Hom}_{O_Y}((V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))_G, O_Y) .$$

Since we know that $(V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))$ is a locally projective $O_Y[G]$ -module, it follows from Lemma 2.1 (ii) that $(V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))_G$ is isomorphic to $(V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G$. We then conclude that

$$\text{Hom}_{O_Y}((V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))_G, O_Y) \cong ((V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G)^\vee ,$$

as required.

3 Proofs of the main Theorems.

In this section we return to the general tame situation as described in Sect. 2.a. We let X be a connected, regular scheme which is projective over $\text{Spec}(\mathbf{Z}[1/2])$ and endowed with a tame action by a finite group G and let Y denote X/G . We assume that $\pi : X \rightarrow Y$ is flat and thus that Y is regular. We consider a symmetric Y -bundle (E, q) , an orthogonal representation $\rho : G \rightarrow \Gamma(Y, \mathbf{O}(q))$ and we let $(E, q)_{\rho, X}$ be the twist of (E, q) by ρ as defined previously. Our aim is to compare the Hasse-Witt invariants of (E, q) and $(E, q)_{\rho, X}$. In order to obtain an explicit comparison formula, we shall therefore associate to our given tame cover an étale cover to which we can directly apply our previous results. As explained in the introduction, the difference between the formula for the tame cover we started with, and that for the étale cover that we construct, will be reflected in the appearance of a new class (which depends on the decomposition of the representation ρ when restricted to the inertia groups of the generic points of the irreducible components of the branch locus of the covering X/Y). Two technical notions will play an important role here: that of a metabolic bundle and that of normalisation along a divisor, which we now describe.

3.a Normalisation along a divisor

The process of normalisation along a branch divisor was introduced and studied in [CNET1]. Let G_2 be a 2-Sylow subgroup of G . We write $Z = X/G_2$ and we let T be the normalisation of the fiber product $T' = Z \times_Y X$. So we have the diagram

$$\begin{array}{ccccc} T & \longrightarrow & T' = Z \times_Y X & \xrightarrow{\phi'} & X \\ & & \downarrow & & \downarrow \pi \\ & \searrow \pi_Z & & & \\ & & Z & \xrightarrow{\phi} & Y \end{array}$$

In [CNET1] Theorem 2.2, we proved that T is regular and that the map π_Z is étale. In *loc. cit.*, Sect. 3.3 we showed how to decompose the normalisation map $T \rightarrow T'$ into a sequence

$$T = T^{(m)} \rightarrow T^{(m-1)} \rightarrow \dots \rightarrow T^{(0)} = T' ,$$

of flat Z -covers, with each $\pi^{(h)} : T^{(h)} \rightarrow Z$ having the property that $\mathcal{D}_{T^{(h)}/Z}^{-1/2}$ is a well-defined $T^{(h)}$ -vector bundle. We set $\Lambda^{(h)} = \pi_*^{(h)}(\mathcal{D}_{T^{(h)}/Z}^{-1/2})$ and define $\mathcal{I}^{(h)} = \Lambda^{(h)} \cap \Lambda^{(h+1)}$ and $\mathcal{G}^{(h)} = (\Lambda^{(h)} + \Lambda^{(h+1)})/\mathcal{I}^{(h)}$. Then for $0 \leq h \leq m-1$ we have short exact sequences of locally free O_Z -modules

$$0 \rightarrow \mathcal{I}^{(h)} \rightarrow \Lambda^{(h)} \oplus \Lambda^{(h+1)} \rightarrow \mathcal{G}^{(h)} \rightarrow 0 .$$

The $\Lambda^{(h)}$ all coincide on the generic fiber; in [CNET1] we showed that $\mathcal{I}^{(h)}$ and $\mathcal{G}^{(h)}$ are locally free O_Z -modules, and we then deduced that for $1 \leq h \leq m-1$, $(\Lambda^{(h)}, (-1)^h Tr_{T^{(h)}/Z}) \perp (\Lambda^{(h+1)}, (-1)^{(h+1)} Tr_{T^{(h+1)}/Z})$ is a metabolic bundle which satisfies the hypotheses of Corollary 1.2.

3.b Proof of Theorems 0.4 (ii), 0.5 (i) and 0.6

Our strategy consists of considering $w_k(\phi^*(E)_{\phi^*(\rho), T'}) - w_k(\phi^*(E))$ as a sum of two terms, namely:

$$(w_k(\phi^*(E)_{\phi^*(\rho), T'}) - w_k(\phi^*(E)_{\phi^*(\rho), T})) + (w_k(\phi^*(E)_{\phi^*(\rho), T}) - w_k(\phi^*(E))) .$$

The cover T/Z is étale and so T is a G -torsor over Z . Hence the second term is known by Theorem 0.2 for $k = 1, 2$. Therefore the main part of this section will be devoted to computing the first term.

Our first aim is to prove that our twisting process behaves well under flat base change (see Theorem 0.4 (ii)).

Lemma 3.1 *For any orthogonal representation $\rho : G \rightarrow \Gamma(Y, \mathbf{O}(E))$*

$$\phi^*(E)_{\phi^*(\rho), T'} = \phi^*(E_{\rho, X}) .$$

PROOF. It suffices to prove that the sections of these vector bundles coincide over the basis for the topology of Z given by the $\phi^{-1}(V)$ where V runs over the affine open subsets of Y . Let $U = \phi^{-1}(V)$ be one such neighbourhood. Write $R = O_Y(V)$, $S = O_Z(U)$ and $M = E(V) \otimes_R \pi_* (\mathcal{D}_{X/Y}^{-1/2})(V)$. From the very definitions we obtain that $\phi^*(E_{\rho, X})(U) = M^G \otimes_R S$ and that $\phi^*(E)_{\phi^*(\rho), T'}(U) = (M \otimes_R S)^G$, using the naturality of differentials with respect to base change. Since S is flat over R we know that $M^G \otimes_R S = (M \otimes_R S)^G$ and the result follows.

To compare the Hasse-Witt invariants of $\phi^*(E)_{\phi^*(\rho), T'}$ and $\phi^*(E)_{\phi^*(\rho), T}$ we now use the decomposition of the normalisation map

$$T = T^{(m)} \rightarrow T^{(m-1)} \rightarrow \dots \rightarrow T^{(0)} = T' ,$$

recalled in the previous sub-section, in order to construct a new family of metabolic bundles which allows us to use the Main Lemma. For any $0 \leq h \leq m$, the group G acts on $T^{(h)}$ and ρ induces an orthogonal representation $\rho : G \rightarrow \Gamma(Z, \mathbf{O}(\phi^*(q)))$. Therefore we can consider the twist of $(\phi^*(E), \phi^*(q))$ by this representation. For simplicity, when there is no ambiguity on the choice of the representation, we will denote by $(\phi^*(E), \phi^*(q))_{T^{(h)}}$ the symmetric bundle $(\phi^*(E), \phi^*(q))_{\phi^*(\rho), T^{(h)}}$. The principal advantage in considering the decomposition of normalisation into m steps is that we will be able to compare the Hasse-Witt invariants of two consecutive terms $\phi^*(E)_{T^{(h)}}$ and $\phi^*(E)_{T^{(h+1)}}$.

For $0 \leq h \leq m - 1$ we have the short exact sequence of locally free O_Z -modules

$$0 \rightarrow \mathcal{I}^{(h)} \rightarrow \Lambda^{(h)} \oplus \Lambda^{(h+1)} \rightarrow \mathcal{G}^{(h)} \rightarrow 0 .$$

Since $\phi^*(E)$ is a locally free O_Z -module, we obtain a further exact sequence

$$0 \rightarrow \phi^*(E) \otimes_{O_Z} \mathcal{I}^{(h)} \rightarrow \phi^*(E) \otimes_{O_Z} \Lambda^{(h)} \oplus \phi^*(E) \otimes_{O_Z} \Lambda^{(h+1)} \rightarrow \phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h)} \rightarrow 0 .$$

From the definitions of $\mathcal{I}^{(h)}$ and $\mathcal{G}^{(h)}$ it is clear that these modules are G -modules and that the morphisms in this sequence respect the action of G

when G acts diagonally on the tensor products. Next we consider the sequence obtained by taking G -fixed points.

Proposition 3.2 *For $0 \leq h \leq m - 1$*

$$0 \rightarrow (\phi^*(E) \otimes_{O_Z} \mathcal{I}^{(h)})^G \rightarrow \phi^*(E)_{T^{(h)}} \oplus \phi^*(E)_{T^{(h+1)}} \rightarrow (\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h)})^G \rightarrow 0$$

is an exact sequence of locally free O_Z -modules.

PROOF. In Lemma 4.2 we shall show that $\mathcal{I}^{(h)}, \Lambda^{(h)}$ and $\mathcal{G}^{(h)}$ are locally projective $O_Z[G]$ -modules for any $h, 0 \leq h \leq m - 1$. Therefore, for any (closed) point z of Z we may choose a sufficiently small affine neighbourhood U of z such that $\mathcal{I}^{(h)}(U), \mathcal{G}^{(h)}(U), \Lambda^{(h)}(U)$ and $\Lambda^{(h+1)}(U)$ are projective $O_Z(U)[G]$ -modules. From Lemma 2.3 it follows that the modules obtained by tensoring these modules with $\phi^*(E)(U)$ over $O_Z(U)$ are all projective $O_Z(U)[G]$ -modules. Hence, by Lemma 2.1, we know that their G -fixed points are locally free $O_Z(U)$ -modules. We now consider the exact sequence of left $O_Z(U)[G]$ -modules. The result then follows by considering the exact sequence

$$0 \rightarrow \mathcal{I}^{(h)}(U) \rightarrow \Lambda^{(h)}(U) \oplus \Lambda^{(h+1)}(U) \rightarrow \mathcal{G}^{(h)}(U) \rightarrow 0,$$

tensoring with the $O_Z(U)$ -locally free module $\phi^*(E)(U)$ and taking G -fixed points.

Remark. We observe that in the case where E is a projective $O_Y[G]$ -module it follows that $\phi^*(E)$ is a projective $O_Z[G]$ -module and thus the proof of the proposition is complete without having to check the local projectivity of the modules $\mathcal{I}^{(h)}, \Lambda^{(h)}$ and $\mathcal{G}^{(h)}$.

From Proposition 3.2 we deduce that for $0 \leq h \leq m - 1$ the symmetric bundle obtained as the orthogonal sum of $(\phi^*(E), (-1)^h \phi^*(q))_{T^{(h)}}$ and $(\phi^*(E), (-1)^{h+1} \phi^*(q))_{T^{(h+1)}}$ is metabolic with lagrangian $(\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h)})^G$. Hence, using the Main Lemma, we deduce that the product

$$w_t((\phi^*(E), (-1)^h \phi^*(q))_{T^{(h)}}) w_t((\phi^*(E), (-1)^{h+1} \phi^*(q))_{T^{(h+1)}})$$

is equal to $d_t((\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h)})^G)$. By taking the product of these equalities we then deduce that $w_t((\phi^*(E)_{T^{(0)}}, \phi^*(q)_{T^{(0)}}))$ equals to

$$w_t((\phi^*(E), (-1)^m \phi^*(q))_{T^{(m)}}) \prod_{0 \leq h \leq m-1} d_t((\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h)})^G)^{(-1)^h}.$$

In order to deduce Theorem 0.6 from this equality we simply have to remember that $T^{(m)} = T$ by definition and that $(\phi^*(E), \phi^*(q))_{T^{(0)}} = \phi^*(E, q)_{\rho, X}$ comes from Lemma 3.1.

We now establish the first part of Theorem 0.5 together with a preliminary version of the second part. Then in the next section we shall conclude the proof of the second part.

As a consequence of the fact that for $0 \leq h \leq m$ the restrictions of the forms $(\phi^*(E), \phi^*(q))_{T^{(h)}}$ to the generic fiber of Z coincide, we deduce that $w_1(\phi^*(E)_{T^{(h)}})$ and $c_1(\phi^*(E)_{T^{(h)}})$ do not depend on the choice of h . Moreover it also follows from Corollary 1.2 (c) that

$$w_2(\phi^*(E)_{T^{(h)}}) - w_2(\phi^*(E)_{T^{(h+1)}}) = c_1((\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h)})^G) - c_1(\phi^*(E)_{T^{(h+1)}})$$

for $0 \leq h \leq m-1$. Therefore, by adding these equalities, we obtain that

$$w_2(\phi^*(E_{\rho, X})) - w_2(\phi^*(E)_T) = \delta(\phi^*(E), Z)$$

with

$$\delta(\phi^*(E), Z) = \sum_{1 \leq h \leq m} \left[c_1((\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h-1)})^G) - c_1(\phi^*(E)_{T^{(h)}}) \right].$$

Using Theorem 0.2 to evaluate $w_k(\phi^*(E)_T) - w_k(\phi^*(E))$, we finally obtain that

$$w_1(\phi^*(E_{\rho, X})) = w_1(\phi^*(E)_T) = w_1(\phi^*(E)) + w_1(\phi^*(\rho))$$

and

$$\begin{aligned} w_2(\phi^*(E_{\rho, X})) &= w_2(\phi^*(E)) + w_1(\phi^*(E))w_1(\phi^*(\rho)) + w_2(\phi^*(\rho)) + \\ &+ sp_2(\phi^*(\rho)) + \delta(\phi^*(E), Z). \end{aligned}$$

Remark. Since 2 is invertible in Z , the squaring map on \mathbf{G}_m affords an exact Kummer sequence of étale sheaves of groups

$$0 \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow \mathbf{G}_m \rightarrow \mathbf{G}_m \rightarrow 0$$

and therefore we obtain an exact sequence of groups

$$0 \rightarrow \text{Pic}(Z)/2 \rightarrow H^2(Z_{et}, \mathbf{Z}/2\mathbf{Z}) \rightarrow H^2(Z_{et}, \mathbf{G}_m)_2 \rightarrow 0.$$

The group $H^2(Z_{et}, \mathbf{G}_m)$ is known as the cohomological Brauer group of Z and is denoted by $Br'(Z)$, (see for instance [Mi], Chapter 4). It now follows from the definition of the first Chern class that $\delta(\phi^*(E), Z)$ belongs to $\text{Pic}(Z)/2$ (see below). Therefore by projecting our formula into $Br'(Z)$ we obtain a formula of Fröhlich-type in this group (see [F], Theorems 2 and 3). It should be observed that in our case— Z regular and integral—one believes that $Br'(Z)$ and the Brauer group of Z coincide. Furthermore, since Z is integral and regular, we have an exact sequence, [Es-K-V] 5.4

$$0 \rightarrow \text{Pic}(Z)/2 \rightarrow H^2(Z_{et}, \mathbf{Z}/2\mathbf{Z}) \rightarrow H^2(K, \mathbf{Z}/2\mathbf{Z}) \rightarrow 0 ,$$

where K denotes the function field of Z . Therefore, when we restrict our formula to the generic fiber of Z , the term $\delta(\phi^*(E), Z)$ disappears and so we obtain Fröhlich's original formula in the Galois cohomology group $H^2(K, \mathbf{Z}/2\mathbf{Z})$.

4 Ramification divisor of a representation

The aim of this section is to complete the proof of Theorem 0.5 and to show how the ramification divisor attached to an orthogonal representation, that we mentioned in the introduction, naturally appears in this proof. Here we shall often need to refer to [CNET1], Sect. 4.

4.a Proof of Theorem 0.5 (ii)

We first observe that, from the definition of the first Chern class, it follows that for any $1 \leq h \leq m$, the element of $H^2(Z_{et}, \mathbf{Z}/2\mathbf{Z})$

$$c_1((\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h-1)})^G) - c_1(\phi^*(E)_{T^{(h)}})$$

is the image of the element $[\det(\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h-1)})^G] - [\det(\phi^*(E)_{T^{(h)}})]$ of $\text{Pic}(Z)$, where we denote by $[D]$ the class in $\text{Pic}(Z)$ of the divisor D of Z . Let $\alpha^{(h)} : \phi^*(E)_{T^{(h)}} \rightarrow (\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h-1)})^G$ be the inclusion map (see Proposition 3.2). We have the short exact sequence of O_Z -modules

$$0 \rightarrow \det(\phi^*(E)_{T^{(h)}}) \xrightarrow{\det(\alpha^{(h)})} \det((\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h-1)})^G) \rightarrow \text{coker}(\det(\alpha^{(h)})) \rightarrow 0 .$$

Therefore there exists a divisor $\Delta^{(h)}(E)$ such that

$$[\Delta^{(h)}(E)] = [\det(\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h-1)})^G] - [\det(\phi^*(E)_{T^{(h)}})] .$$

One of the main ingredients of the proof of the theorem is to provide an explicit description of each divisor $\Delta^{(h)}(E)$ and thus of the sum $\Delta(E) = \sum_{h=1}^m \Delta^{(h)}(E)$ at least when restricted to each U_i of some étale covering $(U_i \rightarrow Z)$ of Z . Since $\Delta(E)$ has image $\delta(\phi^*(E), Z)$ in $H^2(Z_{\text{ét}}, \mathbf{Z}/2\mathbf{Z})$, the theorem will follow from the congruence in $\text{Div}(Z)$

$$\Delta(E) \equiv \phi^*(R(\rho, X)) \pmod{2} .$$

This congruence will be deduced from the congruence mod 2 of the restrictions of both terms to each U_i of the covering. The final part of this section is devoted to the construction for any (closed) point of Z of an étale neighbourhood on which we can evaluate and compare as divisors the restrictions of $\Delta(E)$ and $\phi^*(R(\rho, X))$.

Before proceeding to these constructions and computations, we start by fixing some notation and by recalling the results of [CNET1] that we shall require. Let y be a point of Y of arbitrary dimension and $\sigma : S \rightarrow Y$ be an étale neighbourhood of y . For any scheme $v : V \rightarrow Y$ we write $V_S = V \times_Y S$, we define $O_V(S)$ as the ring of its global sections and denote by $v_S : V_S \rightarrow S$ and $\sigma_V : V_S \rightarrow V$ the projection maps. For a V -vector bundle \mathcal{F} we obtain the vector bundle $\mathcal{F}_S = \sigma_V^*(\mathcal{F})$ over V_S . We denote by $\mathcal{F}(S)$ the module of its global sections. Let now z be point of Z and $y = \phi(z)$. The étale neighbourhoods of z that we shall consider will always be of the form $\sigma_Z : Z_S \rightarrow Z$ where $\sigma : S \rightarrow Y$ is a well-chosen étale affine neighbourhood of y . As mentioned previously, the objects that we consider all have good functorial properties under base change. To be more precise: suppose that $\sigma_Z : Z_S \rightarrow Z$ is an étale neighbourhood of the type introduced above. For any $0 \leq h \leq m$ we can consider on the one hand $(T^{(h)})_S$ as defined previously and on the other hand $T_S^{(h)}$ as the normalisation of T_S' along the divisor $\sigma^*(b_1) \cup \dots \cup \sigma^*(b_h)$. From the fact that they coincide ([CNET1], Proposition 3.6), we deduce as in Theorem 0.4 that

$$(\phi^*(E)_{T^{(h)}})_S = ((\phi^*(E))_S)_{T_S^{(h)}} ,$$

and that we have an exact sequence of Z_S -vector bundles

$$0 \rightarrow (F \otimes_{O_{Z_S}} \mathcal{I}_S^{(h)})^G \rightarrow F_{T_S^{(h)}} \oplus F_{T_S^{(h+1)}} \rightarrow (F \otimes_{O_{Z_S}} \mathcal{G}_S^{(h)})^G \rightarrow 0 ,$$

where for simplicity we denote by F the O_{Z_S} -vector bundle $\phi^*(E)_S$. Therefore, using the fact that base-change commutes with taking determinants,

for any $1 \leq h \leq m$, the restriction $\sigma_Z^*(\Delta^{(h)}(E))$ of $\Delta^{(h)}(E)$ to the étale neighbourhood Z_S of z will be obtained via the exact sequence

$$0 \rightarrow \det(F_{T_S^{(h)}}) \rightarrow \det(F \otimes_{O_{Z_S}} \mathcal{G}_S^{(h-1)})^G \rightarrow \text{coker}(\det(\alpha_S^{(h)})) \rightarrow 0 ,$$

with $\alpha_S^{(h)} : F_{T_S^{(h)}} \rightarrow (F \otimes_{O_{Z_S}} \mathcal{G}_S^{(h-1)})^G$ again being the inclusion map.

We now want to make precise the choice of the neighbourhood S which allows us to obtain explicit descriptions of the global sections of $F_{T_S^{(h)}}$ and $(F \otimes_{O_{Z(S)}} \mathcal{G}_S^{(h-1)})^G$ which lead themselves to determinantal computations.

We fix z a point of Z , we choose a point x of X whose image in Z is z and we write y for $\phi(z)$. For any $1 \leq h \leq m$, the inertia group of a generic point of $B_{h,k}$ (see 2.(a)) only depends on h up to conjugacy, thus we may denote by I_h this group and by e_h the order of I_h . The cover X/Y being tame, I_h is cyclic and therefore will often be identified with $\mathbf{Z}/e_h\mathbf{Z}$. Let $I(x)$ be the inertia group of the point x . Then we know (see for instance Sect. 2 of [CNET1]), that

$$I(x) \cong \prod_{\ell \in J(x)} I_\ell \cong \prod_{\ell \in J(x)} \mathbf{Z}/e_\ell\mathbf{Z} ,$$

where

$$J(x) = \{\ell \mid 1 \leq \ell \leq m, \exists k : x \in B_{\ell,k}\} .$$

After reordering if necessary, we shall take $J(x) = \{1, 2, \dots, n\}$. For any such ℓ , we denote by χ_ℓ the character giving the action of I_ℓ on the cotangent space at the generic point of $B_{\ell,k}$. It follows from [CNET1], Lemma 2.3, that there exists an integral affine étale neighbourhood of y ,

$$S = \text{Spec}(\mathcal{A}_y) ,$$

where \mathcal{A}_y is an algebra containing a sequence a_1, a_2, \dots, a_n of regular parameters and enough roots of unity of order coprime with the residue characteristic of y and an isomorphism of schemes with G -action

$$X_S \cong \text{Spec}(O_X(S))$$

where

$$O_X(S) := \text{Map}_{I(x)}(G, \mathcal{B}_x)$$

and

$$\mathcal{B}_x := \mathcal{A}_y[t_1, \dots, t_n] = \mathcal{A}_y[T_1, \dots, T_n]/(T_1^{e_1} - a_1, \dots, T_n^{e_n} - a_n) .$$

Moreover, for $1 \leq \ell \leq n$, the action of I_ℓ on the image of T_ℓ in \mathcal{B}_x that we denote by t_ℓ , is given by the character χ_ℓ . We may now obtain a description of Z_S and a description of T'_S considered as a scheme with G -action, from the above description of X_S . More precisely, if s denotes the cardinality of $G_2 \backslash G/I(x)$, we obtain that

$$Z_S = \text{Spec}(O_Z(S)) \quad \text{with} \quad O_Z(S) = \prod_{1 \leq j \leq s} \mathcal{B}_{j,x} , \quad (4.1)$$

which is a product of s copies of $\mathcal{B}_{j,x} = \mathcal{B}_x$ and

$$T'_S = \text{Spec}(O_{T'}(S)) \quad \text{with} \quad O_{T'}(S) = \text{Map}_{I(x)}(G, \prod_{1 \leq j \leq s} \mathcal{B}_{j,x} \otimes_{\mathcal{A}_y} \mathcal{B}_x) ,$$

and $I(x)$ acting on the second factor of the tensor products. From now on the schemes S and Z_S given above will be the neighbourhoods of y and z that we shall consider.

We denote by $\mathcal{S}(x)$ the set of sequences $\alpha = (\alpha_\ell)$, $\ell \in J(x)$ where for each $\ell \in J(x)$ we have an integer α_ℓ such that $0 \leq \alpha_\ell < e_\ell$. For any $1 \leq h \leq m$, we consider the partition of $J = J(x)$ into

$$J'_h = J'_h(x) = \{\ell \in J : 1 \leq \ell \leq h\}$$

and

$$J''_h = J''_h(x) = \{\ell \in J : h+1 \leq \ell \leq m\} .$$

So for $h \geq n$, the set J''_h is empty. For $\alpha = (\alpha_\ell)$, we write

$$\partial^{(h)}(\alpha) := \left(\prod_{\ell \in J'_h} t_\ell^{-\alpha_\ell} \prod_{\ell \in J''_h} t_\ell^{-\epsilon_\ell} \right) \otimes \prod_{\ell \in J} t_\ell^{\alpha_\ell} ,$$

where $\epsilon_\ell = 0$ or e_ℓ depending on whether α_ℓ is strictly smaller or strictly larger than $e_\ell/2$. Then we define

$$\mathcal{D}_j^{(h)}(\alpha) = \mathcal{B}_{j,x} \partial^{(h)}(\alpha) \quad \text{and} \quad \mathcal{D}_j^{(h)}(S) = \bigoplus_{\alpha \in \mathcal{S}(x)} \mathcal{D}_j^{(h)}(\alpha) ,$$

$$\mathcal{G}_j^{(h-1)}(\alpha) = \mathcal{D}_j^{(h-1)}(\alpha) + \mathcal{D}_j^{(h)}(\alpha) \quad \text{and} \quad \mathcal{G}_j^{(h-1)}(S) = \bigoplus_{\alpha \in \mathcal{S}(x)} \mathcal{G}_j^{(h-1)}(\alpha),$$

and

$$\mathcal{I}_j^{(h-1)}(\alpha) = \mathcal{D}_j^{(h-1)}(\alpha) \cap \mathcal{D}_j^{(h)}(\alpha) \quad \text{and} \quad \mathcal{I}_j^{(h-1)}(S) = \bigoplus_{\alpha \in \mathcal{S}(x)} \mathcal{I}_j^{(h-1)}(\alpha).$$

Let us denote by χ_x^α the character $\prod_{\ell \in J(x)} \chi_\ell^{\alpha_\ell}$ of $I(x)$; then we shall observe that the previous decompositions correspond to the decomposition according to the characters of $I(x)$. From now on, for any $I(x)$ -module M , we denote by $M(\alpha)$ the submodule of M on which the action of $I(x)$ is given by χ_x^α . It now follows from [CNET1], Sect. 3.e. that we have the isomorphisms of G -modules

$$\begin{aligned} \mathcal{I}^{(h)}(S) &\cong \text{Map}_{I(x)}(G, \prod_{1 \leq j \leq s} \mathcal{I}_j^{(h)}(S)), \\ \Lambda^{(h)}(S) &\cong \text{Map}_{I(x)}(G, \prod_{1 \leq j \leq s} \mathcal{D}_j^{(h)}(S)) \end{aligned}$$

and

$$\mathcal{G}^{(h-1)}(S) \cong \text{Map}_{I(x)}(G, \prod_{1 \leq j \leq s} \mathcal{G}_j^{(h-1)}(S)).$$

We start by deducing from these descriptions, that $\mathcal{I}^{(h)}$, $\Lambda^{(h)}$ and $\mathcal{G}^{(h)}$ are locally projective $O_Z[G]$ -modules (recall that this was used in the proof of Proposition 3.2).

Lemma 4.2 *For any h , $1 \leq h \leq m$, the $O_Z[G]$ -modules $\mathcal{I}^{(h)}$, $\Lambda^{(h)}$ and $\mathcal{G}^{(h)}$ are locally projective.*

PROOF. Let z be a point of Z , $y = \phi(z)$ and let $\sigma : S \rightarrow Y$ be the étale neighbourhood of y introduced above. Since σ is of finite type it is open. Let us denote by U the image $\sigma(S)$. We may assume that U is affine. In fact, since σ is étale and of finite type, it is smooth and quasi-finite, [Mi], I, Remark 3.25. Therefore it follows from Zariski's Main Theorem that it can be decomposed as the product of an open immersion and a finite map. Hence we are reduced to the case where $\sigma : S \rightarrow U$ is finite. It now follows from a theorem of Chevalley [EGA II], Theorem. 6.7-1, that U is affine. Let us denote by V the affine open neighbourhood $\sigma^{-1}(U)$ of z and let us prove

for instance that $\mathcal{I}^{(h)}(V)$ is a projective $O_Z(V)[G]$ -module (the proof of the projectivity of $\Lambda^{(h)}(V)$ and $\mathcal{G}^{(h)}(V)$ will follow exactly the same lines). Let $\sigma_Z : Z_S \rightarrow Z$ be the morphism obtained from σ by base change. It is étale and moreover $\sigma_Z(Z_S) = V$. It follows that σ_Z induces a faithfully flat morphism of affine schemes from Z_S onto V . Therefore it suffices to prove that $\mathcal{I}^{(h)}(S)$ is a projective $O_Z(S)[G]$ -module. From the previous description of $\mathcal{I}^{(h)}(S)$ we observe that this module is induced from the $O_Z(S)[I(x)]$ -module $\prod_{1 \leq j \leq s} \mathcal{I}_j^{(h)}(S)$. From [CNET1], Section 3.e., we know that this last module is finitely generated and free as an $O_Z(S)$ -module. Since the order of $I(x)$ is a unit in $O_Z(S)$, this is enough to conclude that it is projective as an $O_Z(S)[I(x)]$ -module and hence that $\mathcal{I}^{(h)}(S)$ is a projective $O_Z(S)[G]$ -module. This completes the proof of the lemma.

We now return to the proof of Theorem. 0.5. We recall that F has been defined at the beginning of this section as the O_{Z_S} -vector bundle $\phi(E)_S$. Denoting respectively by $F(S)$, $E(S)$ and $F_{T_S^{(h)}}(S)$ the global sections of F , $\phi^*(E)$ and $F_{T_S^{(h)}}$, from the previous equalities we deduce that

$$(F(S) \otimes_{O_Z(S)} \mathcal{G}^{(h-1)}(S))^G \cong \left(\text{Map}_{I(x)}(G, \prod_{1 \leq j \leq s} (E(S) \otimes_{\mathcal{A}_y} \mathcal{G}_j^{(h-1)}(S))) \right)^G$$

and

$$F_{T_S^{(h)}}(S) \cong \left(\text{Map}_{I(x)}(G, \prod_{1 \leq j \leq s} (E(S) \otimes_{\mathcal{A}_y} \mathcal{D}_j^{(h)}(S))) \right)^G .$$

For any $R[G]$ -module M and any subgroup H of G we recall that we denote by $\text{Map}_H(G, M)$ the set of maps $u : G \rightarrow M$ such that for any $x \in G$ and any $h \in H$, $u(hx) = hu(x)$, endowed with its structure of G -module defined by $gu : x \mapsto u(xg)$. It is now easy to check that for any such module M and any such subgroup H of G the map $f \mapsto f(1)$ induces an isomorphism of R -modules from $(\text{Map}_H(G, M))^G$ onto M^H . Therefore we conclude that we have the following isomorphisms

$$(F(S) \otimes_{O_Z(S)} \mathcal{G}^{(h-1)}(S))^G \cong \prod_{1 \leq j \leq s} (E(S) \otimes_{\mathcal{A}_y} \mathcal{G}_j^{(h-1)}(S))^{I(x)}$$

and

$$F_{T_S^{(h)}}(S) \cong \prod_{1 \leq j \leq s} (E(S) \otimes_{\mathcal{A}_y} \mathcal{D}_j^{(h)}(S))^{I(x)} .$$

Denoting by $E_j(S)$ the tensor product $E(S) \otimes_{\mathcal{A}_y} \mathcal{B}_{j,x}$, we finally obtain that

$$(F(S) \otimes_{O_Z(S)} \mathcal{G}^{(h-1)}(S))^G \cong \prod_{1 \leq j \leq s} (E_j(S) \otimes_{\mathcal{B}_{j,x}} \mathcal{G}_j^{(h-1)}(S))^{I(x)}$$

and

$$F_{T_S^{(h)}}(S) \cong \prod_{1 \leq j \leq s} (E_j(S) \otimes_{\mathcal{B}_{j,x}} \mathcal{D}_j^{(h)}(S))^{I(x)} .$$

We now want to consider the structure of E as a module when restricted to $I(x)$. To that end we consider E_y as an $O_{Y,y}[I(x)]$ -module. Since the order of $I(x)$ and the residue characteristic of y are coprime, E_y can be decomposed according to the characters of $I(x)$ as a direct sum

$$E_y = \sum_{\alpha \in \mathcal{S}(x)} E_y(\alpha) .$$

For any α , let us denote by $l_y(\alpha)$ the $O_{Y,y}$ -rank of $E_y(\alpha)$. Since E_y is an orthogonal representation of $I(x)$ we observe that for any $\alpha \in \mathcal{S}(x)$ it follows that $l_y(\alpha) = l_y(e - \alpha)$ where $e = (e_\ell)$, $\ell \in J(x)$. Therefore for any $1 \leq j \leq s$, we obtain a decomposition for $E_j(S)$, namely

$$E_j(S) = \sum_{\alpha \in \mathcal{S}(x)} E_j(S)(\alpha) ,$$

where $E_j(S)(\alpha)$ is a free $\mathcal{B}_{j,x}$ -module of rank $l_y(\alpha)$. Hence we deduce that

$$(E_j(S) \otimes_{\mathcal{B}_{j,x}} \mathcal{G}_j^{(h-1)}(S))^{I(x)} \cong \bigoplus_{\alpha \in \mathcal{S}(x)} (E_j(S)(\alpha) \otimes_{\mathcal{B}_{j,x}} \mathcal{G}_j^{(h-1)}(S)(e - \alpha))$$

and

$$(E_j(S) \otimes_{\mathcal{B}_{j,x}} \mathcal{D}_j^{(h)}(S))^{I(x)} \cong \bigoplus_{\alpha \in \mathcal{S}(x)} (E_j(S)(\alpha) \otimes_{\mathcal{B}_{j,x}} \mathcal{D}_j^{(h)}(S)(e - \alpha)) .$$

We now return to the computations of determinants. For $h \notin J(x)$ and for any j and α we know from [CNET1], Proposition 3.14, that $\mathcal{D}_j^{(h)}(S)(\alpha) = \mathcal{D}_j^{(h-1)}(S)(\alpha)$ and so is equal to $\mathcal{G}_j^{(h-1)}(S)(\alpha)$. Therefore

$$\det(F_{T_S^{(h)}}) = \det(F \otimes_{O_Z(S)} \mathcal{G}_S^{(h-1)})^G .$$

Assuming now that $h \in J(x)$ we introduce the partition of $\mathcal{S}(x)$ into $\mathcal{S}_h(x)$ and $\mathcal{S}'_h(x)$ where $\mathcal{S}_h(x)$ (resp. $\mathcal{S}'_h(x)$) denotes the set of sequences α such that

$e_h/2 < \alpha_h$ (resp. $\alpha_h < e_h/2$). We recall from [CNET1], Proposition 3.14 that $\mathcal{D}_j^{(h)}(S)(\alpha) = (t_h^{(e_h - \alpha_h)} \otimes 1) \mathcal{G}_j^{(h-1)}(S)(\alpha)$ for $\alpha \in \mathcal{S}_h(x)$ and is equal to $\mathcal{G}_j^{(h-1)}(S)(\alpha)$ otherwise. Hence we deduce from these equalities that, if $\alpha \in \mathcal{S}_h(x)$, then $(e - \alpha) \in \mathcal{S}'_h(x)$ and thus we have

$$\det(E_j(\alpha) \otimes_{\mathcal{B}_{j,x}} \mathcal{G}_j^{(h-1)}(S)(e - \alpha)) = \det(E_j(\alpha) \otimes_{\mathcal{B}_{j,x}} \mathcal{D}_j^{(h)}(S)(e - \alpha)) .$$

If now $\alpha \in \mathcal{S}'_h(x)$, then $(e - \alpha) \in \mathcal{S}_h(x)$ and therefore

$$\det(E_j(\alpha) \otimes_{\mathcal{B}_{j,x}} \mathcal{D}_j^{(h)}(S)(e - \alpha)) = \det(E_j(\alpha) \otimes_{\mathcal{B}_{j,x}} t_h^{\alpha_h} \mathcal{G}_j^{(h-1)}(S)(e - \alpha)) .$$

Using the fact that the $E_j(S)(\alpha)$ are free $\mathcal{B}_{j,x}$ -modules of rank $l_y(\alpha)$ and that $\mathcal{D}_j^{(h)}(S)(\alpha)$ and $\mathcal{G}_j^{(h-1)}(S)(\alpha)$ are free $\mathcal{B}_{j,x}$ rank one modules ([CNET1], Proposition 3.14), we deduce from above that in this last case

$$\det(E_j(\alpha) \otimes_{\mathcal{B}_{j,x}} \mathcal{D}_j^{(h)}(S)(e - \alpha)) = \det(E_j(\alpha) \otimes_{\mathcal{B}_{j,x}} \mathcal{G}_j^{(h-1)}(S)(e - \alpha)) t_h^{\alpha_h l_y(\alpha)} .$$

Thus for any $1 \leq j \leq s$,

$$\det((E_j(S) \otimes_{\mathcal{B}_{j,x}} \mathcal{D}_j^{(h)}(S))^{I(x)}) = \det((E_j(S) \otimes_{\mathcal{B}_{j,x}} \mathcal{G}_j^{(h-1)}(S))^{I(x)}) t_h^{\sum_{\alpha \in \mathcal{S}'_h(x)} \alpha_h l_y(\alpha)} ,$$

and therefore from the previous equalities we may finally conclude that

$$\det(F_{T_S^{(h)}}) = \det((F \otimes_{O_{Z_S}} \mathcal{G}_S^{(h-1)})^G) \prod_{1 \leq j \leq s} t_h^{\sum_{\alpha \in \mathcal{S}'_h(x)} \alpha_h l_y(\alpha)} .$$

It follows from this equality that, for $h \in J(x)$, the restriction of $\Delta^{(h)}(E)$ to Z_S is defined as a Cartier divisor by

$$\Gamma^{(h)}(E) = \prod_{1 \leq j \leq s} t_h^{-\gamma^{(h)}(E)} ,$$

where we write $\gamma^{(h)}(E) = \sum_{\alpha \in \mathcal{S}'_h(x)} \alpha_h l_y(\alpha)$.

We now want to give an interpretation of $\gamma^{(h)}(E)$. We start by observing that, from its very definition,

$$\gamma^{(h)}(E) = \sum_{0 \leq k \leq e_h/2} k \sum_{\substack{\alpha \in \mathcal{S}_h(x) \\ \alpha_h = k}} l_y(\alpha) .$$

For any $1 \leq h \leq m$, let us denote by $d_k^{(h)}(E)$ the rank over O_{Y,ξ_h} of the χ_h^k component of E_{ξ_h} , when considered as an I_h -module. When $h \in J(x)$, since O_{Y,ξ_h} contains $O_{Y,y}$, we can write E_{ξ_h} as the tensor product $E_y \otimes_{O_{Y,y}} O_{Y,\xi_h}$. We then deduce from this equality the following decomposition of E_{ξ_h} into a direct sum of $I(x)$ -modules:

$$E_{\xi_h} = \bigoplus_{\alpha \in \mathcal{S}(x)} ((E_y)(\alpha) \otimes_{O_{Y,y}} O_{Y,\xi_h}) .$$

This therefore implies that the χ_h^k -component of E_{ξ_h} , when considered as an I_h -module, is the direct sum of the $E_y(\alpha) \otimes_{O_{Y,y}} O_{Y,\xi_h}$ when α runs through the elements of $\mathcal{S}_h(x)$ such that $\alpha_h = k$. It then follows from this decomposition that, for $0 \leq k < e_h/2$,

$$d_k^{(h)}(E) = \sum_{\substack{\alpha \in \mathcal{S}_h(x) \\ \alpha_h = k}} l_y(\alpha) .$$

We have then shown that for $h \in J(x)$

$$\gamma^{(h)}(E) = d^{(h)}(E) = \sum_{k=0}^{e_h/2} k d_k^{(h)}(E) .$$

We now return to the function $\Gamma^{(h)}(E)$, given above, which defines $\Delta^{(h)}(E)$. Writing $N = \prod_{h \in J(x)} e_h$ and, for $h \in J(x)$, $N_h = N/e_h$ and using that $t_h^{e_h} = a_h$ for any such h , we obtain that

$$\Gamma^{(h)}(E)^N = a_h^{-N_h d^{(h)}(E)} .$$

Since by hypothesis the ramification indices are odd we conclude that the restriction of $\Delta(E)$ to Z_S , namely $\sigma_Z^*(\Delta(E))$, is defined as a Cartier divisor by the function

$$\Gamma(E) = \prod_{h \in J(x)} \Gamma^{(h)}(E) \equiv \prod_{h \in J(x)} a_h^{d^{(h)}(E)} \pmod{2} .$$

Let us now consider the ramification divisor

$$R(\rho, X) = \sum_{1 \leq h \leq m} d^{(h)}(E) b_h$$

defined in the introduction. We write U for the image of σ . We assume that S has been chosen sufficiently small to ensure that the irreducible components

that U intersects are precisely those containing y , namely $\{b_h, h \in J(x)\}$. Since $\sigma : S \rightarrow U$ is étale, it follows that

$$\sigma^*(R(\rho, X)) = \sum_{h \in \mathcal{J}(x)} d^h(E) \sum_{\sigma(\eta) = \xi_h} \{\bar{\eta}\},$$

where the η run through the points of S of codimension 1 over ξ_h . Therefore, since a_h is a local equation of b_h for any such h , we obtain that $\sigma^*(R(\rho, X))$ and therefore $\phi_S^*(\sigma^*(R(\rho, X)))$ is defined by the function $\prod_{h \in \mathcal{J}(x)} a_h^{d^h(E)}$. Using the equality $\phi_S^*(\sigma^*(R(\rho, X))(E)) = \sigma_Z^*(\phi^*(R(\rho, X)))$ and the congruence satisfied by the function $\Gamma(E)$ we conclude that the restrictions of $\phi^*(R(\rho, X))$ and $\Delta(E)$ to Z_S are indeed congruent mod 2 as required.

4.b Example

Our final goal is to compute the divisor $R(\rho, X)$ in a special case considered in Sect. 2.d. We keep the hypotheses and the notations of Example 2.6. So we consider the symmetric bundle (E, q) where $E = O_Y[G/H]$ and q is the symmetric form which has the cosets $\{\bar{a} = aH\}$ as an orthonormal basis; ρ is the tame orthogonal representation of G which permutes the cosets G/H by left multiplication.

By the above work the computation of this divisor $R(\rho, X)$ reduces to evaluating the integers $d_k^{(h)}(E)$ for any $1 \leq h \leq m$ and $1 \leq k \leq e_h/2$. We now fix such an h , we choose once for all a codimension one point ξ_h'' of X above ξ_h and we assume for simplicity that $O_{Y, \xi_h''}$ contains the values of the character χ_h . We let I_h (resp. Δ_h) denote the inertia group (resp. decomposition group) of ξ_h'' . Since there is no risk of ambiguity we make no further mention from now of the dependance upon h of the objects we consider and therefore we will write I for I_h , Δ for Δ_h , ξ for ξ_h , etc.

If $G = \cup_{1 \leq i \leq r} \Delta \gamma_i H$ is a double coset decomposition of G then, by standard theory (see for instance [FT], Chap. 8, Sect. 7) we have an isomorphism of left $O_{Y, \xi}[\Delta]$ -modules

$$E_\xi \cong \oplus_{1 \leq i \leq r} O_{Y, \xi}[\Delta / (\gamma_i H \cap \Delta)],$$

where for $\gamma \in G$ we write ${}^\gamma H = \gamma H \gamma^{-1}$. We observe that the above double cosets parametrise the codimension one points of $V = X/H$ above ξ by the rule $\gamma_i \mapsto \lambda(\xi^{\gamma_i})$ (we recall that G acts on the right on X). Then $\Delta_i = H \cap \gamma_i^{-1} \Delta$ (resp. $I_i = H \cap \gamma_i^{-1} I$) is the decomposition group (resp. the inertia

group) of ξ''^{γ_i} over V . We recall that we write e (resp. f) for the ramification index (resp. the residue class extension degree) of ξ'' over Y . Let us write e'_i (resp. f'_i) for the ramification index (resp. the residue class extension degree) of ξ''^{γ_i} over V , thus the codimension one point on V corresponding to γ_i , namely $\lambda(\xi^{\gamma_i})$, has ramification $e_i = ee'_i{}^{-1}$ and residue class extension degree $f_i = ff'_i{}^{-1}$. It follows from the above isomorphism that in order to decompose E_ξ as a direct sum of $O_{Y,\xi}[I]$ -modules we have to decompose each $O_{Y,\xi}[\Delta/(\gamma_i H \cap \Delta)]$. With this in mind we consider the double coset decomposition of $I \backslash \Delta / (\gamma_i H \cap \Delta)$. Using the fact that I is a normal subgroup of Δ we observe that each component of the direct sum decomposition is isomorphic to $O_{Y,\xi}[I/(\gamma_i H \cap I)]$ and that the number of components, namely the number of the sets of double cosets, is equal to f_i . Therefore we have proved that there is an isomorphism of $O_{Y,\xi}[I]$ -modules

$$E_\xi \cong \bigoplus_{1 \leq i \leq r} \bigoplus_{1 \leq j_i \leq f_i} O_{Y,\xi}[I/(\gamma_i H \cap I)] .$$

For any $O_{Y,\xi}[I]$ -module M and for any integer $1 \leq k \leq e_h/2$, let us denote by $M(k)$ the χ^k -component of M . For each k we deduce from above the following decomposition

$$E_\xi(k) \cong \bigoplus_{1 \leq i \leq r} \bigoplus_{1 \leq j_i \leq f_i} O_{Y,\xi}[I/(\gamma_i H \cap I)](k) .$$

Therefore the computation of $d_k(E)$ reduces to the computation of the rank $r_{\gamma_i}(k)$ over $O_{Y,\xi}$ of each $O_{Y,\xi}[I/(\gamma_i H \cap I)](k)$.

It is now easily checked that $O_{Y,\xi}[I/(\gamma_i H \cap I)](k)$ is different from 0 if and only if χ^k is trivial when restricted to $\gamma_i H \cap I$; namely when k belongs to the set of integers $\{e'_i t, 1 \leq t \leq e_i/2\}$. It follows that for $1 \leq t \leq e_i/2$, $O_{Y,\xi}[I/(\gamma_i H \cap I)](e'_i t)$ is a free, rank 1, $O_{Y,\xi}$ -module. Hence we have proved that $r_{\gamma_i}(k) = r_i(k) = 1$ if e'_i divides k and is 0 otherwise. In summary we have shown that

$$d(E) = \sum_{0 \leq k \leq e/2} k d_k(E) = \sum_{0 \leq k \leq e/2} k \sum_{1 \leq i \leq r} f_i r_i(k) .$$

and thus

$$d(E) = \sum_{1 \leq i \leq r} e'_i f_i \sum_{0 \leq k \leq e_i/2} k = \sum_{1 \leq i \leq r} e'_i f_i \frac{(e_i^2 - 1)}{8} .$$

Since the ramification indices are all odd, we conclude that

$$R(\rho, X) \equiv \sum_h \left(\sum_{\eta_h \rightarrow \xi_h} f(\eta_h) \frac{(e(\eta_h)^2 - 1)}{8} \right) b_h \pmod{2} ,$$

where η_h ranges over the codimension one points on V above ξ_h . The right hand side of this congruence is, of course, precisely the divisor obtained in [CNET1] and the divisor obtained by Serre in [Se2].

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AUTHORS' ADDRESSES:

Ph. C-N. & B. E.: Laboratoire "Théorie des nombres et algorithmique arithmétique" (UMR CNRS 5465), Institut de Mathématiques de Bordeaux (FR CNRS 2254), Université Bordeaux 1, 351, cours de la Libération, F-33405 Talence, France

e-mail : phcassou@math.u-bordeaux1.fr erez@math.u-bordeaux1.fr

M.J. T.: Department of Mathematics, U.M.I.S.T., P.O. Box 88, GB-Manchester M 60 1 QD, England, U.K.

e-mail : martin.taylor@umist.ac.uk