

On Fröhlich twisted bundles

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Introduction

A symmetric bundle (E, f) over a noetherian $\mathbf{Z}[\frac{1}{2}]$ -scheme Y is a vector bundle E over Y equipped with a symmetric isomorphism f between E and its Y -dual E^\vee . A symmetric bundle can also be viewed as a quadratic form on E and we write (E, q) if we take this point of view, or, if the form is clear from the context, we might even just write E . It is well known how to describe the set of all *twists* of (E, f) , that is the set of symmetric bundles which become isomorphic to (E, f) after an étale base extension. If $\mathbf{O}(E)$ denotes the orthogonal group (scheme) attached to (E, f) this set is $H^1(Y, \mathbf{O}(E))$ (see [Mi], chapter 3 for a precise definition of this set). For α in $H^1(Y, \mathbf{O}(E))$, let E_α be the twist of E corresponding to α . Every symmetric bundle of rank n is a twist of the standard symmetric bundle $\mathbf{1}_n = (O_Y^n, x_1^2 + \dots + x_n^2)$. Let $\mathbf{O}(n)$ denote the automorphism group of this symmetric bundle. We write α_E for the class of E in $H^1(Y, \mathbf{O}(n))$ ($n = \text{rank}(E)$).

Following Delzant [Dz] and Jardine [J1], for any symmetric bundle E over Y one can define a cohomological invariant, which generalizes the classical invariants of quadratic forms and which is known as the *total Hasse-Witt class*. This is a class $w_t(E)$ in the (graded) étale cohomology group $H^*(Y, \mathbf{Z}/2\mathbf{Z})$:

$$w_t(E) = 1 + w_1(E)t + w_2(E)t^2 + \dots$$

A brief review of the definitions of the Hasse-Witt invariants can be found in [CNET1], section 1.e. The terms w_1 and w_2 in degrees one and two generalize the discriminant and the Hasse-Witt invariant respectively and have the following elementary description. We define δ^1 as the map induced by the determinant map

$$\delta^1 = \delta_E^1 : H^1(Y, \mathbf{O}(E)) \rightarrow H^1(Y, \mathbf{Z}/2\mathbf{Z})$$

and δ^2 as the boundary map

$$\delta^2 = \delta_E^2 : H^1(Y, \mathbf{O}(E)) \rightarrow H^2(Y, \mathbf{Z}/2\mathbf{Z})$$

associated to the exact sequence of étale sheaves of groups

$$1 \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow \widetilde{\mathbf{O}}(E) \rightarrow \mathbf{O}(E) \rightarrow 1, \quad (0.1)$$

where $\widetilde{\mathbf{O}}(E)$ is a certain covering constructed with the help of the Clifford algebra of E (see 1.b for the precise definition). Then $w_1(E) = \delta_n^1(\alpha_E)$ and $w_2(E) = \delta_n^2(\alpha_E)$, where, $\delta_n^i = \delta_{\mathbf{1}_n}^i$.

Let us define a class $\Delta_t(\alpha)$ by the equality

$$w_t(E_\alpha) = w_t(E)\Delta_t(\alpha).$$

The first result of this paper provides a description of $\Delta_t(\alpha)$ up to terms of degree 3. More precisely we prove that the invariants $\delta^1(\alpha)$ and $\delta^2(\alpha)$ permit to compute the classes w_1 and w_2 of the form obtained from E by twisting with α . Our result generalises in this geometric context a result of Serre [Se3] obtained in the case of field extensions.

Theorem 0.2 *Let (E, q) be a symmetric bundle over a scheme Y and let α be an element of the cohomology set $H^1(Y, \mathbf{O}(E))$. Then*

$$i) \quad w_1(E_\alpha) = w_1(E) + \delta^1(\alpha).$$

$$ii) \quad w_2(E_\alpha) = w_2(E) + w_1(E)\delta^1(\alpha) + \delta^2(\alpha).$$

Our proof of this result consists of a cocycle computation in the spirit of the work of Serre and Fröhlich. It is based on an explicit formula for the cup-product of two 1-cocycles and the study of the behaviour of the above exact sequence (0.1) under change of forms up to isomorphism.

Our next aim will be to apply this result to the study of twists of symmetric bundles which are obtained from certain types of coverings of Y . Such twists have been first considered by Fröhlich in the case of fields extensions (see [F] and [Es-K-V]). Let $\pi : X \rightarrow Y$ be a finite Galois cover with group G , i.e π is a G -torsor. Consider an orthogonal representation of G given by a symmetric bundle (E, q) over Y together with a group homomorphism $\rho : G \rightarrow \Gamma(Y, \mathbf{O}(E))$. The G -torsor X defines an element $c(X)$ in $H^1(Y, G)$ and so by push-forward along ρ it defines an element $\rho(X) = \rho_*(c(X))$ in

$H^1(Y, \mathbf{O}(E))$. In terms of cohomology $E_{\rho(X)}$ is the twist of the original bundle E by the class $\rho_*(c(X))$. It is however important to give a simple explicit description of the form $E_{\rho(X)}$. In that purpose we need some more definitions. The group G acts diagonally on the O_Y -bundle $E \otimes \pi_*(O_X)$, namely through ρ on the first component and through the given action on the second. We denote by $E_{\rho, X}$ the submodule of the fixed points under G . Let μ be the product form $\mu(x, y) = xy$ on $\pi_*(O_X)$ and let $q_{\rho, X}$ be the restriction of the form $q \otimes \mu$ to $E_{\rho, X}$. The twist of (E, q) by ρ is defined as the bilinear form $(E_{\rho, X}, q_{\rho, X})$ on Y . We shall prove in section 2:

Proposition 0.3 *i) The twist $(E_{\rho, X}, q_{\rho, X})$ of (E, q) by ρ is a symmetric bundle over Y .*

ii)

$$E_{\rho(X)} = E_{\rho, X} .$$

In comparing the invariants of E and $E_{\rho, X}$, there appear not only the Stiefel-Whitney classes $w_i(\rho)$ of ρ but also a new kind of invariant of an orthogonal representation called the spinor classes. Building on the work by Fröhlich [F], Kahn [K] and Snaith [Sn], Jardine showed that, for Y the spectrum of a field K of characteristic different from 2, there is a class

$$sp_t(\rho) = 1 + sp_1(\rho)t + sp_2(\rho)t^2 + \dots$$

called the *total spinor class*, which satisfies

$$w_t(E_{\rho, X})sp_t(\rho) = w_t(E)w_t(\rho)$$

in $H^*(K, \mathbf{Z}/2\mathbf{Z})$ and whose odd components are all trivial (see [J2]). In fact little else is known about the spinor class except in degree 2.

Our next main result extends the work of Serre [Se1] and Fröhlich [F] to bundles over schemes and give a proof which is different from that of Kahn in [K] Cor. 6.1.

Theorem 0.4 *Let (E, q) be a symmetric bundle over a scheme Y , let X be a G -torsor over Y and let $\rho : G \rightarrow \Gamma(Y, \mathbf{O}(E))$ be an orthogonal representation of G . Let $(E_{\rho, X}, q_{\rho, X})$ be the twist of (E, q) by ρ . Then, we have the equalities:*

i) $w_1(E_{\rho, X}) = w_1(E) + w_1(\rho)$.

$$ii) w_2(E_{\rho,X}) = w_2(E) + w_1(E)w_1(\rho) + w_2(\rho) + sp_2(\rho) .$$

This theorem will be easily deduced from Theorem 0.2 and Proposition 0.3. This provides a new proof of Theorem 2.3 in [Es-K-V] and Theorem 0.2 in [CNET1] in the étale case. Note that our definition (Def. 2.d) of $sp_2(\rho)$ is different from that in [K], but it is known that the two definitions coincide when Y is the spectrum of a field, by a remark in [S1], p.127.

We should indicate that the twisting results for forms over fields play an important role in the work of Saito on the sign of the functional equation of the L-function of an orthogonal motive (see [S1]). Saito proves a p -adic version of Fröhlich's result: namely, he deals with Galois representations that do not necessarily have finite image. He uses this to prove a result analogous to the Fröhlich-Queyruet Theorem, which states that the global root number of real orthogonal characters equals one ([FQ]). Saito's approach follows that of Deligne, who interpreted this reciprocity result in terms of the Stiefel-Whitney classes of the (local) characters [De].

In [Se2] Serre considered coverings of Riemann surfaces with odd ramification and showed how to obtain analogous formulæ, which involve expressions defined in terms of ramification data. Serre's work has been extended in [Es-K-V], [K] and more recently in [CNET1], [CNET2]. Let

$$\pi : X \rightarrow Y = X/G$$

be a covering which is tamely ramified along a divisor b with normal crossings and whose ramification indices are all odd. In [CNET1] we studied the symmetric bundle $(\pi_*(\mathcal{D}_{X/Y}^{-1/2}), Tr_{X/Y})$, where $\mathcal{D}_{X/Y}^{-1/2}$ is the locally free sheaf over X whose square is the inverse of the different of X/Y and $Tr_{X/Y}$ is the trace form. In this generality we obtained the same formula as Serre, *with no extra terms*. Still the ramification invariant was not so well understood. In a forthcoming paper we will use the inverse different bundle to define the twists of a symmetric bundle by orthogonal representations coming from tame coverings. The case of genuinely tamely ramified actions is geometrically more involved and leads us to introduce a new invariant of ramification which in a sense provides a decomposition in term of representations of the inertia groups of the invariant introduced by Serre for curves.

1 Invariants of a twisted form

1.a Symmetric bundles on schemes

For completeness, we recall the basic definitions concerning forms over schemes.

We let Y be a scheme and we assume that 2 is invertible over Y , then the theory of symmetric bilinear forms over Y is equivalent to that of quadratic forms over Y . A *vector bundle* E on Y is a locally free O_Y -module of finite rank. The dual of a vector bundle E is the vector bundle E^\vee such that, for any open subscheme Z of Y

$$E^\vee(Z) = \text{Hom}_{O_Z}(E|_Z, O_Z) .$$

There is a natural evaluation pairing $\langle \cdot, \cdot \rangle$ between E and E^\vee and one can identify E with the double dual $E^{\vee\vee}$ by

$$\kappa : E \cong E^{\vee\vee} ,$$

where $\langle \alpha, \kappa(u) \rangle = \langle u, \alpha \rangle$. A *symmetric bilinear form* on Y is a vector bundle E on Y equipped with a map of sheaves

$$q : E \times_Y E \rightarrow O_Y ,$$

which on sections over an open subscheme restricts to a symmetric bilinear form. This defines an *adjoint* map

$$\varphi = \varphi_q : E \rightarrow E^\vee ,$$

which because of the symmetry assumption equals its transpose:

$$\varphi = \varphi^t : E \xrightarrow{\kappa} E^{\vee\vee} \xrightarrow{\varphi^\vee} E^\vee .$$

We shall say that (E, q) is *non-degenerate* (or unimodular) if the adjoint φ is an isomorphism. From now on we will call a symmetric bundle any vector bundle endowed with a non-degenerate quadratic form.

1.b Clifford algebras

The properties of the Clifford algebra and the Clifford group associated with a symmetric bundle will play an important role in the proof of Thm. 0.2.

Therefore we start by briefly recalling some of the basic material that we shall need in this section. Our references will be [Es-K-V], section 1.9, [Knu], Chapter 4, in the case of forms over a ring, and [F], Appendix 1, for a brief review in the case of forms over a field. In fact one has to observe that most of the definitions about Clifford algebras associated to a quadratic form over a field, or more generally over a commutative ring, can be generalised in our geometric context. Moreover, by reducing to affine neighbourhoods, we will essentially work with non-degenerate forms over noetherian, integral domains.

To any symmetric bundle (V, q) of constant rank n , one associates a sheaf of algebras $\mathcal{C}(q)$ over O_Y , of constant rank 2^n . As in the classical case one has the notion of odd and even elements of $\mathcal{C}(q)$ and hence a *mod 2* grading. The Clifford group $\mathcal{C}^*(q)$ is the subgroup of homogeneous, invertible elements x in $\mathcal{C}(q)$ such that xvx^{-1} belongs to V for any v of V . The *mod(2)* grading induces a splitting

$$\mathcal{C}^*(q) = \mathcal{C}_+^*(q) \cup \mathcal{C}_-^*(q) .$$

Let σ be the anti-automorphism on $\mathcal{C}(q)$ induced by the identity on V so that $\sigma(v_1 \dots v_m) = v_m \dots v_1$. One verifies that the map N defined on $\mathcal{C}^*(q)$ by $N(x) = \sigma(x)x$ induces an homomorphism,

$$N : \mathcal{C}^*(q) \rightarrow \mathbf{G}_m .$$

We define the algebraic group scheme $\widetilde{\mathbf{O}}(q)$ as the kernel of this homomorphism. As before this group scheme splits as $\widetilde{\mathbf{O}}_+(q) \cup \widetilde{\mathbf{O}}_-(q)$. Let x be in $\widetilde{\mathbf{O}}_\epsilon(q)$ with $\epsilon = \pm 1$, then we define $r_q(x)$ as the element of $\mathbf{O}(q)$

$$\begin{aligned} r_q(x) : V &\rightarrow V \\ v &\mapsto \epsilon x v x^{-1} . \end{aligned}$$

This defines a group homomorphism $r_q : \widetilde{\mathbf{O}}(q) \rightarrow \mathbf{O}(q)$. One can show that for each $x \in \widetilde{\mathbf{O}}_\epsilon(q)$ the element $r_q(x)$ belongs to $\mathbf{O}_+(q) = \mathbf{SO}(q)$ or $\mathbf{O}_-(q) = \mathbf{O}(q) \setminus \mathbf{SO}(q)$ depending on whether $\epsilon = 1$ or -1 . Where there is no risk of ambiguity, we will write r for r_q .

We then have constructed an exact sequence of étale sheaves of groups:

$$1 \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow \widetilde{\mathbf{O}}(q) \rightarrow \mathbf{O}(q) \rightarrow 1 .$$

We recall that we have previously introduced

$$\delta^1 : H^1(Y, \mathbf{O}(q)) \rightarrow H^1(Y, \mathbf{Z}/2\mathbf{Z})$$

as the map induced by the determinant and

$$\delta^2 : H^1(Y, \mathbf{O}(q)) \rightarrow H^2(Y, \mathbf{Z}/2\mathbf{Z})$$

as the boundary map associated to the above exact sequence. We next consider an affine situation, namely $Y = \text{Spec}(R)$ where R is an integral domain. For a symmetric bundle (V, q) over O_Y , by abuse of notation, we shall write V , $O(q)$, $\tilde{O}(q)$, for the corresponding module or groups of the global sections of these objects. For any invertible element a of $\mathcal{C}(q)$ we will denote by ι_a the inner automorphism of $\mathcal{C}(q)$, $x \mapsto axa^{-1}$.

Proposition 1.1 *Let (E, q) and (F, f) be symmetric bundles over O_Y and let $\theta : (E, q) \rightarrow (F, f)$ be an isometry. Then*

- i) θ extends to an isomorphism $\tilde{\theta}$ from $\tilde{\mathbf{O}}(q)$ onto $\tilde{\mathbf{O}}(f)$ which induces the isomorphism $u \rightarrow \theta u \theta^{-1}$ from $\text{Im}r_q$ onto $\text{Im}r_f$.*
- ii) Suppose that $(E, q) = (F, f)$ and that θ belongs to $\text{Im}r_q$. Let $t(\theta)$ denote a lift of θ in $\tilde{\mathbf{O}}(q)$. Then $\tilde{\theta} = \iota_{t(\theta)}$, (resp $\epsilon \iota_{t(\theta)}$ on $\tilde{\mathbf{O}}_\epsilon(q)$), if $\theta \in \mathbf{O}_+(q)$, (resp $\mathbf{O}_-(q)$).*

PROOF. The universal property of the Clifford algebra implies that θ induces a graded isomorphism $\tilde{\theta} : \mathcal{C}(q) \rightarrow \mathcal{C}(f)$, which by restriction induces the required isomorphism. Moreover since it coincides with the identity on $\mathbf{Z}/2\mathbf{Z}$, it follows that $\tilde{\theta}$ induces a group isomorphism $s_\theta : \text{Im}r_q \rightarrow \text{Im}r_f$. Let $u \in \text{Im}r_q$, with $u \in \mathbf{O}_\epsilon(q)$ and let $a \in \tilde{\mathbf{O}}_\epsilon(q)$ denote a lift of u . From the very definition of r we deduce that for any $y \in F$

$$s_\theta(u)(y) = \epsilon \tilde{\theta}(a) y \tilde{\theta}(a^{-1}) .$$

Since $\tilde{\theta}$ is an isomorphism of O_Y -algebras, the right hand side of this equality can be written

$$\epsilon \tilde{\theta}(a \tilde{\theta}^{-1}(y) a^{-1}) = \theta(\epsilon a \theta^{-1}(y) a^{-1}) = (\theta u \theta^{-1})(y) .$$

Hence we have proved that s_θ and $u \rightarrow \theta u \theta^{-1}$ coincide on $\text{Im}r_q$.

Under the hypothesis of *ii)* we obtain two automorphisms of $\mathcal{C}(q)$, namely $\tilde{\theta}$ and $\iota_{t(\theta)}$. Moreover, since $t(\theta) \in \tilde{\mathbf{O}}_\epsilon(q)$, it follows the definition of r_q that $\theta(x) = \epsilon t(\theta) x t(\theta)^{-1}$ for any x in E . If $\theta \in \mathbf{O}_+(q)$, then $\epsilon = 1$ and $\tilde{\theta}$ and $\iota_{t(\theta)}$ which coincides on E coincides on $\mathcal{C}(q)$. If $\theta \in \mathbf{O}_-(q)$, then $\epsilon = -1$ therefore $\tilde{\theta}$ and $\iota_{t(\theta)}$ will coincide on $\mathcal{C}_+(q)$ and will differ by a minus sign on $\mathcal{C}_-(q)$ and the result follows.

1.c Proof of Theorem 0.2

Let (E, q) be a symmetric bundle and α a cohomology class of $H^1(Y, \mathbf{O}(q))$. Since the set $H^1(Y, \mathbf{O}(q))$ classifies the isometry classes of twisted forms of (E, q) , we may consider any twisted form (E_α, q_α) whose class corresponds to α . Finally we denote by (T_n, q_n) the standard form $(O_Y^n, x_1^2 + x_2^2 + \dots + x_n^2)$ and as usual we write $\mathbf{O}(n)$ for $\mathbf{O}(q_n)$. Since both (E, q) and (E_α, q_α) are twisted forms of (T_n, q_n) , there exists an affine covering $\mathcal{U} = (\mathcal{U}_i \rightarrow Y)$ for the étale topology and isomorphisms

$$\varphi_i : (E_\alpha, q_\alpha) \times U_i \rightarrow (E, q) \times U_i$$

$$\psi_i : (E, q) \times U_i \rightarrow (T_n, q_n) \times U_i .$$

Therefore, following [Mi], section 4, we deduce that $(\alpha_{ij}) = \varphi_i \varphi_j^{-1}$ and $(\gamma_{ij}) = \psi_i \psi_j^{-1}$ are 1-cocycles representing (E_α, q_α) in $H^1(\mathcal{U}/Y, \mathbf{O}(q))$ and (E, q) in $H^1(\mathcal{U}/Y, \mathbf{O}(n))$ respectively. By considering $(\delta_{ij}) = (\psi_i \varphi_i)(\psi_j \varphi_j)^{-1}$ we obtain a 1-cocycle representative of (E_α, q_α) in $H^1(\mathcal{U}/Y, \mathbf{O}(n))$. From the previous equalities, we observe that we can write

$$\delta_{ij} = \psi_i \varphi_i \varphi_j^{-1} \psi_j^{-1} = \psi_i \alpha_{ij} \psi_j^{-1} = \psi_i \psi_j^{-1} \psi_j \alpha_{ij} \psi_j^{-1} = \gamma_{ij} (\psi_j \alpha_{ij} \psi_j^{-1}) .$$

In order to obtain a representative of $w_1(E_\alpha)$, it suffices to take the image by the determinant map of the cocycle (δ_{ij}) . From the previous equalities we deduce that $\det(\delta_{ij}) = \det(\gamma_{ij}) \det(\alpha_{ij})$ which immediately implies that

$$w_1(E_\alpha) = w_1(E) + \delta^1(\alpha) ,$$

as required.

We now want to compare $w_2(E_\alpha)$ and $w_2(E)$. After refining (U_i) we may assume, [Mi], III.2.19, that each α_{ij} , (resp. γ_{ij}), is the image of an element of $\widetilde{\mathbf{O}}(q)(U_{ij})$, (resp. $\widetilde{\mathbf{O}}(n)(U_{ij})$) that we denote by α'_{ij} , (resp. γ'_{ij}). By abuse of notation for any $l \in \{i, j, k\}$ we will still denote by ψ_l the restriction of ψ_l , considered previously, to U_{ijk} . We now deduce from Proposition 3.2. *i*) that each δ_{ij} is the image of the element $\delta'_{ij} = \gamma'_{ij} \tilde{\psi}_j(\alpha'_{ij})$. Therefore $w_2(E_\alpha)$ is the class of the 2-cocycle (b_{ijk}) where

$$b_{ijk} = \delta'_{jk} \delta'_{ik}{}^{-1} \delta'_{ij} \in \Gamma(U_{ijk}, \mathbf{Z}/2\mathbf{Z}) .$$

Using repeatedly the previous equalities we obtain that

$$b_{ijk} = \gamma'_{jk} \tilde{\psi}_k(\alpha'_{jk}) \tilde{\psi}_k(\alpha'_{ik}{}^{-1}) \gamma'_{ik}{}^{-1} \gamma'_{ij} \tilde{\psi}_j(\alpha'_{ij}) ,$$

that we write

$$b_{ijk} = \gamma'_{jk} \tilde{\psi}_k(\alpha'_{jk}) \tilde{\psi}_k(\alpha'_{ik}{}^{-1}) \gamma'_{jk}{}^{-1} (\gamma'_{jk} \gamma'_{ik}{}^{-1} \gamma'_{ij}) \tilde{\psi}_j(\alpha'_{ij}) .$$

We now observe on the one hand that $(\gamma'_{jk} \gamma'_{ik}{}^{-1} \gamma'_{ij}) = \pm 1$ and thus commutes with every factor of the product, while on the other hand, since $(\alpha'_{jk} \alpha'_{ik}{}^{-1} \alpha'_{ij}) = \pm 1$, then $\tilde{\psi}_k(\alpha'_{jk}) \tilde{\psi}_k(\alpha'_{ik}{}^{-1}) = (\alpha'_{jk} \alpha'_{ik}{}^{-1} \alpha'_{ij}) \tilde{\psi}_k(\alpha'_{ij}{}^{-1})$. Piecing these observations together we obtain the equality:

$$b_{ijk} = (\gamma'_{jk} \gamma'_{ik}{}^{-1} \gamma'_{ij}) (\alpha'_{jk} \alpha'_{ik}{}^{-1} \alpha'_{ij}) \gamma'_{jk} \tilde{\psi}_k(\alpha'_{ij}{}^{-1}) \gamma'_{jk}{}^{-1} \tilde{\psi}_j(\alpha'_{ij}) .$$

We consider this equality as a product of three factors namely

$$b_{ijk} = (\gamma'_{jk} \gamma'_{ik}{}^{-1} \gamma'_{ij}) (\alpha'_{jk} \alpha'_{ik}{}^{-1} \alpha'_{ij}) \epsilon_{ijk}$$

where $\epsilon_{ijk} = \gamma'_{jk} \tilde{\psi}_k(\alpha'_{ij}{}^{-1}) \gamma'_{jk}{}^{-1} \tilde{\psi}_j(\alpha'_{ij})$. The first two factors are 2-cocycles which respectively represent $w_2(E)$ and $\delta^2(\alpha)$.

We now want to simplify the expression for ϵ_{ijk} . This is achieved by considering the various possible signs of $\det(\gamma_{jk})$ and $\det(\alpha_{i,j})$. We first observe that the equality $\gamma_{jk} \psi_k = \psi_j$ implies that $\tilde{\gamma}_{jk} \tilde{\psi}_k = \tilde{\psi}_j$. Moreover, with the notation of Prop. 1.1, we may write $\gamma'_{jk} \tilde{\psi}_k(x) \gamma'_{jk}{}^{-1} = \iota_{\gamma'_{jk}} \tilde{\psi}_k(x)$. We start by supposing that $\det(\gamma_{jk}) = 1$. Then it follows Prop. 1.1 *ii*) that $\iota_{\gamma'_{jk}} = \tilde{\gamma}_{jk}$, hence $\gamma'_{jk} \tilde{\psi}_k(x) \gamma'_{jk}{}^{-1} = \tilde{\gamma}_{jk} \tilde{\psi}_k(x) = \tilde{\psi}_j(x)$ and we conclude that $\epsilon_{ijk} = 1$. We now suppose that $\det(\gamma_{jk}) = -1$. First we assume that $\det(\alpha_{ij}) = 1$. Then, in this case, $\alpha'_{ij} \in \mathcal{C}_+(q)$ and therefore $\tilde{\psi}_k(\alpha'_{ij}{}^{-1}) \in \mathcal{C}_+(q_n)$. Using as before Prop. 1.1 *ii*), we obtain that

$$\gamma'_{jk} \tilde{\psi}_k(\alpha'_{ij}{}^{-1}) \gamma'_{jk}{}^{-1} = \tilde{\gamma}_{jk} \tilde{\psi}_k(\alpha'_{ij}{}^{-1}) = \tilde{\psi}_j(\alpha'_{ij}{}^{-1})$$

and we conclude that $\epsilon_{ijk} = 1$. We now suppose that $\det(\alpha_{ij}) = -1$ then $\tilde{\psi}_k(\alpha'_{ij}{}^{-1}) \in \mathcal{C}_-(q_n)$. Therefore, using once again Prop. 1.1 *ii*), we deduce that

$$\gamma'_{jk} \tilde{\psi}_k(\alpha'_{ij}{}^{-1}) \gamma'_{jk}{}^{-1} = -\tilde{\gamma}_{jk} \tilde{\psi}_k(\alpha'_{ij}{}^{-1}) = -\tilde{\psi}_j(\alpha'_{ij}{}^{-1}) .$$

It then follows from the definition that $\epsilon_{ijk} = -1$. As an immediate consequence of the study of these different cases we obtain the equality

$$\epsilon_{ijk} = (-1)^{\epsilon(\det(\alpha_{ij}))\epsilon(\det(\gamma_{jk}))} .$$

where for x in $\{\pm 1\}$ we define $\epsilon(x) \in \mathbf{Z}/2\mathbf{Z}$ by $x = (-1)^{\epsilon(x)}$. Therefore we conclude that (ϵ_{ijk}) is a 2-cocycle representative of the cup product $w_1(E)\delta^1(\alpha)$. This completes the proof of the theorem.

2 The étale case

Let X be a torsor of the constant group scheme G over Y , so that X is an étale covering of Y . We consider, as earlier, a symmetric bundle (E, q) , an orthogonal representation $\rho : G \rightarrow \Gamma(Y, \mathbf{O}(q))$ and the twist $(E_{\rho, X}, q_{\rho, X})$ of (E, q) by ρ as defined in the introduction.

Our aim in this section is to prove, as announced in Theorem 0.4, comparison formulæ between the first and the second Hasse-Witt invariants of (E, q) and $(E_{\rho, X}, q_{\rho, X})$ which generalise those obtained by Fröhlich, [F], Theorem 2 and Theorem 3, in the case of fields extensions. Before establishing this theorem we first have to prove Proposition 0.3. We observe that Proposition 0.3 (i), (resp. Proposition 0.3 (ii)) is Proposition 2.3, (resp. Proposition 2.4) of this section.

2.a Algebraic preliminaries

For this auxiliary step we adopt the following conventions: R is a commutative integral ring and M is a left $R[G]$ -module which we assume to be locally free and finitely generated as an R -module. We write $\sigma = \sum_{g \in G} g$.

Lemma 2.1 *If M is a projective $R[G]$ -module, then $M^G = \sigma M$. Furthermore:*

- i) M^G is a locally free R -module.*
- ii) The map $m \mapsto \sigma m$ induces an isomorphism of R -module from M_G onto M^G .*

PROOF. All the above statements are clear when $M = R[G]$; they therefore hold for any finitely generated free $R[G]$ -module and are then easily seen to hold for a direct summand of a finitely generated free $R[G]$ -module.

We now consider a symmetric bundle (M, t) over R ; that is to say M is a finitely generated locally free R -module, equipped with a non degenerate symmetric bilinear form t . We denote by φ_t the adjoint map $\varphi_t : M \rightarrow M^\vee$. We suppose further that M is now a projective $R[G]$ -module and that the pairing t is G -invariant. Under these assumptions we then can use Lemma 2.1 to define the symmetric bilinear form t^G on M^G by

$$t^G(x, y) = t(m, y)$$

where m is any arbitrary element of M such that $\sigma m = x$. Let I be the R -submodule of M generated by the set $\{(1-g)m, m \in M, g \in G\}$. Since m is defined up to an element of I , (Lemma 2.1 ii)) and since t is G invariant one verifies immediately that t^G is well defined. Moreover one observes that for any x and y in M^G one has

$$|G| t^G(x, y) = t(x, y).$$

Proposition 2.2 *If M is a projective $R[G]$ -module, then (M^G, t^G) is a symmetric R -bundle.*

PROOF. From Lemma 2.1 we know that M^G is a locally free R -module. It remains to prove that φ_{t^G} is an R -module isomorphism from M^G onto $Hom_R(M^G, R)$.

From the exact sequence

$$0 \rightarrow I \rightarrow M \rightarrow M_G \rightarrow 0 ,$$

we deduce the exact sequence

$$0 \rightarrow Hom_R(M_G, R) \rightarrow Hom_R(M, R) \rightarrow Hom_R(I, R) \rightarrow Ext_R(M_G, R) .$$

By Lemma 2.1 we know that M_G is isomorphic to M^G and is therefore R -projective. Hence we conclude that $Ext_R(M_G, R) = \{0\}$. We consider the following diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & M^G & \rightarrow & M & \rightarrow & N & \rightarrow & 0 \\ & & \varphi' \downarrow & & \downarrow \varphi & & \downarrow \varphi'' & & \\ 0 & \rightarrow & Hom_R(M^G, R) & \rightarrow & Hom_R(M, R) & \rightarrow & Hom_R(I, R) & \rightarrow & 0 \end{array}$$

where $N = M/M^G$, $\varphi = \varphi_t$ and φ' is defined for $x \in M^G$ and $y \in M_G$ by

$$\varphi'(x)(y) = \varphi(x)(m) ,$$

where m denotes any representative of y in M . Once again, since m is defined up to an element of I , φ' is well-defined and moreover the first square and thus the diagram itself are both commutative. Applying the snake lemma to the previous diagram we obtain the exact sequence

$$0 \rightarrow Ker\varphi' \rightarrow Ker\varphi \rightarrow Ker\varphi'' \rightarrow Coker\varphi' \rightarrow Coker\varphi .$$

Since φ is an isomorphism, the previous sequence reduces to

$$0 \rightarrow \text{Ker}\varphi'' \rightarrow \text{Coker}\varphi' \rightarrow 0 .$$

Since M_G is isomorphic to M^G and φ' is injective, $\varphi'(M^G)$ is an R -submodule of $\text{Hom}_R(M_G, R)$ of the same rank. Hence $\text{Coker}\varphi'$ is an R -torsion module. We now show that N and hence $\text{Ker}\varphi''$ is torsion free. Once again we can reduce consideration to the case where M is a free $R[G]$ -module with $\{e_1, e_2, \dots, e_n\}$ as a basis. Let m be an element of M , $m = \sum_{1 \leq i \leq n} a_i e_i$, and $d \in R$, $d \neq 0$, such that $dm \in M^G$. Since $\{\sigma e_1, \sigma e_2, \dots, \sigma e_n\}$ is a basis of M^G as an R -module, there exist $\{b_i, 1 \leq i \leq n\}$ in R such that

$$dm = \sum_{1 \leq i \leq n} da_i e_i = \sum_{1 \leq i \leq n} b_i \sigma e_i .$$

It follows that $da_i = b_i \sigma$, $1 \leq i \leq n$ and thus $b_i = dc_i$ with $c_i \in R$, $1 \leq i \leq n$. Since M is torsion free we conclude that $\sum_{1 \leq i \leq n} a_i e_i = \sum_{1 \leq i \leq n} c_i \sigma e_i$ and hence that m belongs to M^G . Since $\text{Ker}\varphi''$ is torsion free and $\text{Coker}\varphi'$ is torsion, it follows that $\text{Ker}\varphi'' = \{0\} = \text{Coker}\varphi'$. We then have therefore shown that φ' is an isomorphism.

Let θ be the isomorphism

$$\theta : \text{Hom}_R(M_G, R) \rightarrow \text{Hom}_R(M^G, R)$$

induced by the isomorphism from M^G onto M_G described in Lemma 2.1. We want to describe $\theta \circ \varphi'$. Let $x = \sigma m$ and $y = \sigma n$ be elements of M^G , then we have the equalities

$$\theta \circ \varphi'(x)(y) = \varphi'(x)(n) = t(x, n) = t(m, y) = t^G(x, y) .$$

We conclude that $\varphi_{t^G} = \theta \circ \varphi'$ and therefore that φ_{t^G} is an isomorphism as required.

2.b Proof of Proposition 0.3 (i)

The proof of Proposition 0.3. (i) is an easy consequence of Proposition 2.2.

Proposition 2.3 *The bilinear form $(E_{\rho, X}, q_{\rho, X})$ over Y is a symmetric bundle.*

PROOF. Let U be any open subscheme of Y , then $R = O_Y(U)$ is an integral domain. Since π is a G torsor we know that $\pi_*(O_X)(U)$ is a projective $O_Y(U)[G]$ -module. This implies that $\pi_*\pi^*(E)(U)$ is a projective $O_Y(U)[G]$ -module and, using Lemma 2.1, that $E_{\rho,X}(U)$ is projective and therefore locally free over $O_Y(U)$. We then have proved that $E_{\rho,X}$ is a vector bundle over O_Y . We now endow the vector bundle $E_{\rho,X}$ with the form

$$t_X = (q \otimes Tr_{X/Y})^G ,$$

where $Tr_{X/Y}$ is the trace form. It follows from Proposition 2.2 that t_X defines a non-degenerate symmetric form on $E_{\rho,X}(U)$. We then have proved that $(E_{\rho,X}, t_X)$ is a symmetric bundle over Y . In order to conclude the proof of the proposition it suffices to prove that the forms t_X and $q_{\rho,X}$ coincide. Let U denote any affine open subscheme of Y . Any element of $E_{\rho,X}(U)$ can be obtained as a sum of elements $\sigma(m \otimes n)$. So let $x = \sigma(m \otimes n)$ and $y = \sigma(m' \otimes n')$. We have the equality

$$\begin{aligned} q_{\rho,X}(x, y) &= q_{\rho,X}(\sum_{g \in G} (\rho(g)m \otimes gn), \sum_{h \in G} \rho(h)m' \otimes hn') \\ &= \sum_{g, h \in G} q(\rho(g)m, \rho(h)m')(gn)(hn') . \end{aligned}$$

Using the invariance of q by G this can be written

$$q_{\rho,X}(x, y) = \sum_{g \in G} q(m, \rho(g)m' Tr_{X/Y}(n(gn'))) .$$

We now observe that the right hand side of this equality can be expressed as

$$(q \otimes Tr_{X/Y})(m \otimes n, \sigma(m' \otimes n')) .$$

Finally it follows from the very definition of t_X that this last quantity is equal to $t_X(x, y)$. Hence we have proved that $q_{\rho,X}(x, y) = t_X(x, y)$.

2.c Proof of Proposition 0.3 (ii)

On the one hand, following Milne [Mi], III, Proposition 4.6, we note that the isomorphism class of X , considered as a sheaf torsor for G , defines an element $c(X)$ in the cohomology set $H^1(Y, G)$. On the other hand we know that $(E_{\rho,X}, q_{\rho,X})$ is a twisted form of (E, q) , see for instance [CNET1], section 1.d for precise definitions; therefore its isometry class defines an element denoted

by $[E_{\rho,X}, q_{\rho,X}]$ in $H^1(Y, \mathbf{O}(q))$. Finally the morphism ρ induces a natural map $\rho_* : H^1(Y, G) \rightarrow H^1(Y, \mathbf{O}(q))$. We establish the following connection between these objects:

Proposition 2.4 *In $H^1(Y, \mathbf{O}(q))$ one has the equality*

$$\rho_*(c(X)) = [E_{\rho,X}, q_{\rho,X}] .$$

PROOF. For any sheaf of groups in the étale topology, \mathcal{F} on Y , the set $H^1(Y, \mathcal{F})$ is defined to be $\varinjlim H^1(\mathcal{U}, \mathcal{F})$ where the limit is taken over all étale coverings \mathcal{U} of Y . The strategy of the proof is to show that for any “fine enough” étale covering $\mathcal{U} = (U_i \rightarrow Y)_{i \in I}$ which trivializes X as a torsor for G , we obtain an isometry of symmetric bundles from $(E_{\rho,X} \times_Y U_i, q_{\rho,X})$ onto $(E \times_Y U_i, q)$. For such coverings \mathcal{U} we will first obtain a 1-cocycle representing $c(X)$ with values in G , then from this we obtain a 1-cocycle representing $[E_{\rho,X}, q_{\rho,X}]$ which takes values in $\mathbf{O}(q)$.

Let $\mathcal{U} = (U_i)_{i \in I}$ be a sufficiently fine, affine, étale cover which trivialises X as a G -torsor. We denote respectively by $O_Y(U_i)$ and $O_X(U_i)$ the global sections of U_i and $X \times_Y U_i$. Now $X \times_Y U_i \rightarrow U_i$ is finite and U_i affine, thus $X \times_Y U_i$ is affine. Therefore for any i the isomorphism of U_i schemes $G_{(U_i)} \simeq X \times_Y U_i$ is induced from an $O_Y(U_i) - G$ isomorphism of algebras

$$\Phi_i : O_X(U_i) \rightarrow \text{Map}(G, O_Y(U_i)) .$$

Furthermore we note that $g_{ij} = \Phi_j \Phi_i^{-1}$ is the 1-cocycle representing $c(X)$, ([Mi], III, section 4).

We again denote by $Tr_{X/Y}$ the form induced by the trace on $O_X(U_i)$. For any elements x and y of $O_X(U_i)$ then $Tr_{X/Y}(xy)$ belongs to $O_Y(U_i)$. Therefore, since Φ_i is an isomorphism of $O_Y(U_i)$ -algebras, we will have

$$Tr_{X/Y}(xy) = \Phi_i(Tr_{X/Y}(xy)) = \Phi_i(Tr_{X/Y}(xy))(1) .$$

Using now that Φ_i is G -equivariant we then deduce from the previous equalities that

$$Tr_{X/Y}(xy) = \sum_{g \in G} \Phi_i(x)(g) \Phi_i(y)(g) .$$

Denoting by μ_G the standard G -invariant form on $\text{Map}(G, O_Y(U_i))$ given by

$$\mu_G(f, f') = \sum_{g \in G} f(g) f'(g) ,$$

we have then proved that Φ_i is an equivariant isometry between the $O_Y(U_i)$ - G symmetric bundles $(O_X(U_i), Tr_{X/Y})$ and $(Map(G, O_Y(U_i)), \mu_G)$.

Let $(E(U_i), q)$ be the $O_Y(U_i)$ symmetric bundle defined by considering the global sections of the inverse image of (E, q) by the morphism $U_i \rightarrow Y$. After tensoring over $O_Y(U_i)$ and taking fixed points by G we deduce from Φ_i an isometry that we again denote by Φ_i

$$\Phi_i : (E_{\rho, X}(U_i), q_{\rho, X}) \rightarrow (E(U_i) \otimes_{O_Y(U_i)} Map(G, O_Y(U_i)))^G, t^G ,$$

where t is the form $q \otimes \mu_G$ and t^G is obtained from t by following the recipe described in section 2. We now let ν_i be the morphism of $O_Y(U_i)$ -modules induced by

$$\nu_i : E(U_i) \otimes_{O_Y(U_i)} Map(G, O_Y(U_i)) \rightarrow Map(G, E(U_i))$$

$$\nu_i : e \otimes f \rightarrow (g \rightarrow f(g)(\rho(g)e)) .$$

It is easy to check that ν_i is an isomorphism, (use for instance that $Map(G, O_Y(U_i))$ is a free $O_Y(U_i)[G]$ rank 1 module with basis l where l is defined by $l(g) = 1$ if $g = 1$ and 0 otherwise). The group G acts diagonally on the left hand side while on the right hand side it acts by

$$uf : g \rightarrow f(gu) .$$

It follows from the definitions of these actions that ν_i is a G -isomorphism and thus induces an isomorphism, again denoted ν_i

$$\nu_i : (E(U_i) \otimes_{O_Y(U_i)} Map(G, O_Y(U_i)))^G \rightarrow Map(G, E(U_i))^G .$$

We now observe that the map $f \rightarrow f(1)$ is clearly an isomorphism from $Map(G, E(U_i))^G$ onto $E(U_i)$. Hence, finally composing this map with ν_i we have defined an isomorphism of $O_Y(U_i)$ -modules

$$\gamma_i : (E(U_i) \otimes_{O_Y(U_i)} Map(G, O_Y(U_i)))^G \rightarrow E(U_i) .$$

We now want to describe in some details the map γ_i . We note that, since the set $\{gl, g \in G\}$ is a free basis of the $O_Y(U_i)$ -module $Map(G, O_Y(U_i))$, thus every element of $(E(U_i) \otimes_{O_Y(U_i)} Map(G, O_Y(U_i)))^G$ can be written as a sum $\sum_{g \in G} a_g \otimes gl$ and therefore that every element of the subgroup of the

fixed points by G as a sum $\sum_{g \in G} \rho(g)e \otimes gl$. Let $x = \sum_{g \in G} \rho(g)e \otimes gl$ be an element of $(E(U_i) \otimes_{O_Y(U_i)} \text{Map}(G, O_Y(U_i)))^G$ we then have:

$$\gamma_i(x) = \nu_i(x)(1) = \sum_{g \in G} \nu_i(\rho(g)e \otimes gl)(1) = \sum_{g \in G} (gl)(1)\rho(g)e = e .$$

We now consider $x = \sum_{g \in G} \rho(g)e \otimes gl$ and $y = \rho(g)e' \otimes gl$ elements of $(E(U_i) \otimes_{O_Y(U_i)} \text{Map}(G, O_Y(U_i)))^G$. Since $x = \sigma(e \otimes l)$ and $y = \sigma(e' \otimes l)$ it follows from the very definition of t^G , prior to Prop. 2.2, that

$$t^G(x, y) = t(x, e' \otimes l) = \sum_{g \in G} q(\rho(g)e, e')\mu_G(gl, l) .$$

Since $\{gl, g \in G\}$ is an orthonormal basis of $\text{Map}(G, O_Y)$, the right hand side of the last equality is equal to $q(e, e')$. We conclude that γ_i is an isometry and therefore that $\Phi_i^{-1}\gamma_i^{-1}$ is an isometry θ_i

$$\theta_i : (E(U_i), q) \rightarrow (E_{\rho, X}(U_i), q_{\rho, X}) .$$

We now must evaluate $\theta_i^{-1}\theta_j$ in order to obtain a 1-cocycle representing $(E_{\rho, X}, q_{\rho, X})$ as a twisted form of (E, q) . Starting with $e \in E(U_{ij})$ we obtain that

$$(\gamma_i\Phi_i\Phi_j^{-1}\gamma_j^{-1})(e) = \gamma_i\left(\sum_{g \in G} \rho(g)e \otimes (gg_{ij}^{-1})l\right) = \gamma_i\left(\sum_{u \in G} (\rho(ug_{ij})e) \otimes ul\right) = \rho(g_{ij})e$$

This concludes the proof of the proposition.

Remark. The twist of a symmetric bundle (E, q) by an orthogonal representation ρ is always a twisted form of (E, q) and therefore is given by a class in $H^1(Y, \mathbf{O}(q))$. In the étale situation Proposition 2.4 tells us precisely that this class is the image by ρ_* of the class defining X as a torsor for G . Therefore the determination of the Hasse-Witt invariants of the twisted bundle will be obtained as an application of Theorem 0.2.

2.d Proof of Theorem 0.4

We now return to the situation considered in Proposition 2.4 and we generalise Theorems 2 and 3 in [F], from field extensions to étale covers. We start by recalling some notation and definitions. Let $\pi_1(Y)$ be the fundamental group of Y based at some chosen geometric point. We consider

a representation $\rho : \pi_1(Y) \rightarrow \Gamma(Y, \mathbf{O}(q))$, where (E, q) is a symmetric Y -bundle. We assume ρ to have an open kernel N . Then N defines a finite Galois étale cover X/Y with Galois group $G = \pi_1(Y)/N$. The cohomology class $c(X)$ of $H^1(Y, G)$ defined by X , considered as a G -torsor, only depends on ρ and therefore will be denoted by $c(\rho)$. The representation ρ factorises into a homomorphism $\rho : G \rightarrow \Gamma(Y, \mathbf{O}(q))$ which in turn induces $\rho_* : H^1(Y, G) \rightarrow H^1(Y, \mathbf{O}(q))$ and $\rho_{*\bar{K}} : H^1(Y, G) \rightarrow H^1(Y, \mathbf{O}(q)(\bar{K}))$ with \bar{K} denoting the separable closure of the residue field at some point. We also consider the two exact sequences of étale sheaves

$$1 \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow \widetilde{\mathbf{O}}(q) \rightarrow \mathbf{O}(q) \rightarrow 1 ,$$

and

$$1 \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow \widetilde{\mathbf{O}}(q)(\bar{K}) \rightarrow \mathbf{O}(q)(\bar{K}) \rightarrow 1 ,$$

where the second one is the \bar{K} -points of the first one, viewed as a sequence of étale sheaves of groups. These sequences induce the boundary maps

$$\delta^2 : H^1(Y, \mathbf{O}(q)) \rightarrow H^2(Y, \mathbf{Z}/2\mathbf{Z})$$

and

$$\delta_{\bar{K}}^2 : H^1(Y, \mathbf{O}(q)(\bar{K})) \rightarrow H^2(Y, \mathbf{Z}/2\mathbf{Z}) .$$

Definition 2.5. The *first Stiefel-Whitney class* of ρ is defined to be $w_1(\rho) = \delta^1(\rho_*(c(\rho)))$. The *second Stiefel-Whitney class* of ρ is defined by $w_2(\rho) = (\delta_{\bar{K}}^2 \rho_{*\bar{K}})(c(\rho))$. The *spinor class* is defined to be the difference

$$sp_2(\rho) = (\delta^2 \rho_*)(c(\rho)) - (\delta_{\bar{K}}^2 \rho_{*\bar{K}})(c(\rho)) .$$

Remark. The notion of the spinor class was first introduced by Fröhlich for the case of field extensions. Fröhlich's initial definition was generalised by Khan in a geometric context. In fact it is not immediately clear that Kahn's definition coincides with ours. Nevertheless one has to observe that in the case of field extension, when $Y = \text{Spec}(K)$ and K is a field of characteristic different from 2, then T. Saito has proved, [Sa1], Lemma 3, that the spinor class we consider here is indeed equal to Fröhlich's original one.

The proof of Theorem 0.3 is now an easy consequence of Proposition 2.4 and Theorem 0.2. Let us denote by α the cohomology class $\rho_*(c(\rho))$. It follows from Proposition 2.4 that $(E_{\rho, X}, q_{\rho, X})$ can be taken as a representative

of the isometry class of twisted forms of (E, q) whose image is α . Therefore it follows from Theorem 0.2 that $w_1(E_{\rho, X}) = w_1(E) + \delta^1(\alpha)$ and $w_2(E_{\rho, X}) = w_2(E) + w_1(E)\delta^1(\alpha) + \delta^2(\alpha)$. From the previous definitions we deduce that $\delta^1(\alpha) = w_1(\rho)$ and that $\delta^2(\alpha) = w_2(\rho) + sp_2(\rho)$. It now suffices to replace $\delta^1(\alpha)$ and $\delta^2(\alpha)$ by their values in order to obtain Theorem 0.4.

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