## SPEECH: SIEGEL ZERO

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#### Abstract

We talk about the effect of the positions of the zeros of Dirichlet L-function in the distribution of prime numbers in arithmetic progressions. We point to effect of possible existence of Siegel zero on above distribution, distribution of twin primes, entropy of black holes, and finally in the size of least prime in arithmetic progressions.


Out line:

1. Introducing the problem of existence of infinite primes in Arithmetic Progressions (APs),
2. Dirichlet characters and Dirichlet L-functions,
3. Brief review of Dirichlet's proof, and pointing importance of $L(1, \chi) \neq 0$ in proof,
4. About the zeros of Dirichlet L-functions: Zero Free Regions (ZFR), Siegel zero and Siegel's theorem,
5. Connection between zeros of L-functions and distribution of primes in APs: Explicit formula,
6. Prime Number Theorem (PNT) for APs,
7. More about Siegel zero and the problem of least prime in APs.

Extended abstract:
After introducing the problem of existence of infinite primes in the arithmetic progression $a, a+q, a+2 q, \cdots$ with $(a, q)=1$, we give some fast information about Dirichlet characters and Dirichlet L-functions: mainly we write orthogonality relations for Dirichlet characters and Euler product for L-fucntions, and after taking logarithm we arrive at

$$
\begin{equation*}
\frac{1}{\varphi(q)} \sum_{\chi} \bar{\chi}(a) \log L(s, \chi)=\sum_{\substack{p, m \\ p^{m} \equiv a[q]}} m^{-1} p^{-m s} \tag{1}
\end{equation*}
$$

where the left sum is over all $\varphi(q)$ characters molulo $q$, and in the right sum $m$ rins over all positive integers and $p$ runs over all primes. In (1), the right side is

$$
\sum_{p \equiv a[q]} p^{-s}+O(1) \quad \text { as } s \rightarrow 1
$$

and the left hand side is

$$
\frac{1}{\varphi(q)} \log L\left(s, \chi_{0}\right)+\frac{1}{\varphi(q)} \sum_{\chi \neq \chi_{0}} \bar{\chi}(a) \log L(s, \chi)
$$

[^0]Euler product formula gives

$$
L\left(s, \chi_{0}\right)=\zeta(s) \prod_{p \mid q}\left(1-p^{-s}\right) \rightarrow \infty \quad \text { as } s \rightarrow 1
$$

The important and hard point is that for $\chi \neq \chi_{0}$ we have $L(1, \chi) \neq 0$, and so

$$
\frac{1}{\varphi(q)} \sum_{\chi \neq \chi_{0}} \bar{\chi}(a) \log L(s, \chi)=O(1) \quad \text { as } s \rightarrow 1
$$

So, we obtain $\lim _{s \rightarrow 1} \sum_{p \equiv a[q]} p^{-s}=\infty$, and this means that the series $\sum_{p \equiv a[q]} 1 / p$ diverges, and so, Dirichlet's theorem. This shows the effect of non-vanishing Dirichlet L-functions on the line $\Re(s)=1$, on the distribution of primes in APs. If we can say more about the zeros of Dirichlet L-functions, then we may get more information about primes in APs. In this direction, we have:
Theorem (Zero Free Regions for Dirichlet L-functions) Let $\mathcal{L}_{q}(s)$ be the product of Dirichlet L-functions modulo $q$. Then, there exists a positive absolute constant $c$ such that $\mathcal{L}_{q}(s)$ has at most one zero in the region

$$
\Re(s) \geq 1-\frac{c}{\log \max (q, q|\Im(s)|)}
$$

Note 1 (Kadiri 2005). There are some known numerical values for $c$. The best known is due to H. Kadiri with $c=1 / 6.397$.
Note 2 (Landau 1918). The only possible zero is real and simple (single), and related by only one nonprincipal real character, namely exeptional character, which we denote it by $\chi_{1}$, and traditionaly, we denote its possible mentioned real zero by $\beta$.
Note 3. There exists an effective constant $c_{1}$, such that $\beta<1-c_{1} / q^{1 / 2} \log ^{2} q$, if it exists.
Theorem (Carl Ludwig Siegel: 1896-1981) For every $\varepsilon>0$ there exists positive real $c_{2}(\varepsilon)$, but non-effective, such that $\beta \leq 1-c_{2}(\varepsilon) / q^{\varepsilon}$.
Now, we wish to use abouve information on the zeros to get information about the distribution of primes in APs. We define the following prime counting functions:

$$
\pi(x ; q, a)=\sum_{\substack{p \leq x \\ p \equiv a[q]}} 1
$$

and

$$
\psi(x ; q, a)=\sum_{\substack{n \leq x \\ n \equiv a[q]}} \Lambda(n)
$$

For technical reasons, it is better to work with $\psi(x ; q, a)$ and then translate obtained results for $\pi(x ; q, a)$ using the relation

$$
\pi(x ; q, a)=\int_{2}^{x} \frac{\psi(t ; q, a)}{t \log ^{2} t} d t+\frac{\psi(x ; q, a)}{\log x}+O(\sqrt{x})
$$

We define the following auxiliary function for every character $\chi$ modulo $q$ :

$$
\psi(x, \chi)=\sum_{n \leq x} \chi(n) \Lambda(n)
$$

Then orthogonality relation for characters gives the following linear combination

$$
\begin{equation*}
\psi(x ; q, a)=\frac{1}{\varphi(q)} \sum_{\chi} \bar{\chi}(a) \psi(x, \chi) \tag{2}
\end{equation*}
$$

Now, we need a connection between above defined prime counting functions and the zeros of related Lfunctions.
Theorem (Explicit formula) For $\chi \neq \chi_{0}(\bmod q)$, and $2 \leq T \leq x$, we have

$$
\psi(x, \chi)=-\frac{x^{\beta}}{\beta}-\sum_{0<|\gamma|<T} \frac{x^{\rho}}{\rho}+R(x, T)
$$

where $R(x, T) \ll x T^{-1} \log ^{q} x+x^{1 / 4} \log x$, and the term containing $\beta$ can be omitted if the there is no exceptional zero.
The main term of $\psi(x ; q, a)$ in (2) comes from $\chi_{0}$. We can melt the possible term containing $\beta$ in two ways effective and non-effective, using $\beta<1-c_{1} / q^{1 / 2} \log ^{2} q$ and Siegel's theorem, respectively. In each case, we should load some conditions on the modulo $q$ : this is important point in the theory.
The Siegel zeros $\beta$ has also other effects, which we mention them bellow:

1. Black holes and Siegel zero [Stephen Miller and Gregory Moore, 2000: MR1803719] The WH and WSV conjectures (about the entropy of black holes) together imply that there are no Landau-Siegel zeros.
2. Siegel zeros and twin primes [D. R. Heath-Brown, 1983: MR703977] The existence of Siegel zeros would imply the existence of infinitely many prime twins.
3. Siegel zeros and the least prime in an arithmetic progression [D. R. Heath-Brown, 1990: MR1081103] As we saw, the arithmetic progression $a, a+q, a+2 q, \cdots$ contains infinite many primes, provided $\operatorname{gcd}(a, q)=1$. Naturally we ask how big is the first such prime, $P(a, q)$ say? In 1944 , Yu. V. Linnik showed that $P(a, q) \ll$ $q^{L}$, for some absolute constant $L$. Estimating Linnik constant $L$ is a challenging problem and bellow is a briefly stated of known result:

| $L$ | Date | Author | $L$ | Date | Author |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 10000 | 1957 | Pan | 36 | 1977 | Graham |
| 5448 | 1958 | Pan | 20 | 1981 | Graham |
| 777 | 1965 | Chen | 17 | 1979 | Chen |
| 630 | 1971 | Jutila | 16 | 1986 | Wang |
| 550 | 1970 | Jutila | 13.5 | 1989 | Chen \& Liu |
| 168 | 1977 | Chen | 8 | 1991 | Wang |
| 80 | 1977 | Jutila | 5.5 | 1992 | Heath-Brown |

For any $\delta>0$ there exists an effectively computable constant $c(\delta)>0$ such that if $\beta>1 c(\delta) / \log q$ then $P(a, q) \leq q^{2+\delta} L(1, \chi)^{2}$ if $\chi(a)=1$ and $P(a, q) \leq q^{2+\delta}$ if $\chi(a)=1$. This result plays an important auxiliary role in the authors important work on Linniks theorem holds unconditionally with the constant $L=5.5$, which is the best known value for $L$ until now.

## References

[1] H. Davenport, Multiplicative Number Theory (Third Edition), Springer, 2000.
[2] A. Karatsuba, Basic Analytic Number Theory, Springer, 1993.
[3] G. Tenenbaum, Introdunction to analytic and probabilistic number theory, Cambridge, 2004.


[^0]:    Date: Tuesday, 9 Dec 2008

