

On the Approximations of Multiple target filtering equations

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~ Three joint articles with F. Caron, A. Doucet, M. Pace, B.N. Vo

• **Particle approximations of a class of branching distribution flows arising in multi-target tracking.** Joint work with Fr. Caron, A. Doucet, and M. Pace HAL-INRIA RR-7233 (2010).

Under minor revision

• **On the Conditional Distributions of Spatial Point Processes.** Joint work with Fr. Caron, A. Doucet, and M. Pace. [HAL-INRIA RR-7232 (2010)].

To appear in the Advances in Applied Probability (2011).

• **On the Stability & the Approximation of Branching Distribution Flows, with Applications to Nonlinear Multiple Target Filtering.** Joint work with Fr. Caron, M. Pace, and B.N. Vo. HAL-INRIA RR-7376 (2010).

To appear in Stochastic Analysis and Applications (2011).

- 1 Basic notation
- 2 Spatial Branching models
- 3 Without spontaneous births
- 4 Multiple target branching signals
- 5 Particle approximations
- 6 Multiple target filtering models
- 7 Approximation models

Some notation : E measurable space

$(\mathcal{M}(E), \mathcal{P}(E), \mathcal{B}(E)) =$ (measures, probabilities, bounded functions) on E .

- $\mu(f) = \int \mu(dx) f(x)$ and $\bar{\mu}(dx) := \mu(dx)/\mu(1) \in \mathcal{P}(E)$
- $Q(x, dy)$ **integral operator over E** (composition $(Q_1 Q_2)$)

$$Q(f)(x) = \int Q(x, dy) f(y)$$
$$[\mu Q](dy) = \int \mu(dx) Q(x, dy) \quad (\implies [\mu Q](f) = \gamma[Q(f)])$$

- **Boltzmann-Gibbs transformation** : $G : E \rightarrow [0, \infty[$ with $\mu(G) > 0$

$$\Psi_G(\mu)(dx) = \frac{1}{\mu(G)} G(x) \mu(dx) \quad \exists \text{ Markov transport } \Psi_G(\mu) = \mu S_\mu$$

Example ($G \leq 1$) \rightsquigarrow accept/reject/interacting jump transition

$$S_\mu(x, dy) = G(x)\delta_x(dy) + (1 - G(x)) \Psi_G(\mu)(dy)$$

- 1 Basic notation
- 2 **Spatial Branching models**
 - A branching-exploration model
 - First moment recursion
- 3 Without spontaneous births
- 4 Multiple target branching signals
- 5 Particle approximations
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Spatial Branching models (time index $n \in \mathbb{N}$, state spaces E_n)

- **3 simple ingredients :**

Potential $G_n(x) \geq 1$, measure $\mu_n(dx) \geq 0$ and $M_n(x_{n-1}, dx_n)$ Markov.

- **Branching rule (spawning) :**

$x \rightsquigarrow g_n(x)$ offsprings, with $\mathbb{E}(g_n(x)) = G_n(x)$

▷ *survival rates $e_n(x)$ + cemetery states* : $G_n \rightsquigarrow e_n(x)G_n(x)$

- **Spontaneous births:** Spatial Poisson with intensity $\mu_n(dx)$
- **Free motion between branching times :** M_n -evolutions
- \rightsquigarrow Random occupation measure (after the n -th evolution step)

$$\mathcal{X}_n = \sum_{i=1}^{N_n} \delta_{X_n^i}$$

Spatial Branching models (time index $n \in \mathbb{N}$, state spaces E_n)

- First moment recursion = **branching intensity distribution**

$$\begin{aligned}\gamma_{n+1}(f) &:= \mathbb{E}(\mathcal{X}_{n+1}(f)) \\ &= \gamma_n(Q_{n+1}(f)) + \mu_{n+1}(f) \text{ with } Q_{n+1}(x, dy) = G_n(x)M_{n+1}(x, dy)\end{aligned}$$

Sketched proof for $\mu_n = 0$:

$$\begin{aligned}\mathcal{X}_{n+1} &= \sum_{i=1}^{N_{n+1}} \delta_{X_{n+1}^i} = \sum_{i=1}^{N_n} \sum_{j=1}^{g_n^i(X_n^i)} \delta_{X_{n+1}^{i,j}} \\ &\Downarrow \\ \mathbb{E}(\mathcal{X}_{n+1}(f) \mid \mathcal{X}_n, g_n(X_n)) &= \sum_{i=1}^{N_{n+1}} g_n^i(X_n^i) M_{n+1}(f)(X_n^i) \\ &\Downarrow \\ \mathbb{E}(\mathcal{X}_{n+1}(f) \mid \mathcal{X}_n) &= \sum_{i=1}^{N_{n+1}} G_n(X_n^i) M_{n+1}(f)(X_n^i) = \mathcal{X}_n(G_n M_{n+1}(f))\end{aligned}$$

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$\mu_n = 0 \Rightarrow$ Conventional Feynman-Kac models

- First moment recursion

$$\gamma_{n+1}(f) = \gamma_n(Q_{n+1}(f)) = \gamma_{n-1}(Q_n(Q_{n+1}(f))) = \dots$$

- Feynman-Kac representation ($\supset \uparrow$ Application domains)

$$\gamma_{n+1}(f) = \gamma_0(1) \mathbb{E}_{\eta_0} \left(f(X_{n+1}) \prod_{0 \leq p \leq n} G_p(X_p) \right)$$

- Particle approximations = Genetic type algorithms = Particle filters = ...

$$Q_{n+1}(x, dy) = \underbrace{G_n(x)}_{\text{Selection potential}} \times \underbrace{M_{n+1}(x, y)}_{\text{Mutation transition}}$$

Summary

- 1 Basic notation
- 2 Spatial Branching models
- 3 Without spontaneous births
- 4 Multiple target branching signals**
 - Some problems & 3 scenarios
 - Nonlinear equations
 - Nonlinear filtering type model
- 5 Particle approximations
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More general spatial Branching models

$$\gamma_n = \gamma_{n-1} Q_n + \mu_n \quad \text{and} \quad \eta_n := \gamma_n / \gamma_n(\mathbf{1}) \quad (\& \text{ hypothesis: } \gamma_0 = \mu_0)$$



$$(\gamma_n(\mathbf{1}), \eta_n) = \Gamma_n(\gamma_{n-1}(\mathbf{1}), \eta_{n-1}) := \Gamma_{p,n}(\gamma_p(\mathbf{1}), \eta_p) \quad \text{Nonlinear semigroup}$$

Some problems

- **Problem 1:** Mass process $\gamma_n(\mathbf{1})$ "unstable" $\gamma_n(\mathbf{1}) \uparrow \infty$ or $\gamma_n(\mathbf{1}) \downarrow 0$ as $n \uparrow \infty$
- **Problem 2:** $\mathcal{X}_n = \sum_{i=1}^{N_n} \delta_{X_n^i}$ generally **NOT POISSON** random field.
- **Problem 3:** \exists non generate numerical sampling method?
- **Problem 4:** \exists non generate approximation of γ_n ?

Some answers:

\supset Particle approximations of a class of branching distribution flows arising in multi-target tracking. Joint work with Fr. Caron, A. Doucet, and M. Pace HAL-INRIA RR-7233 (March 2010).

Three scenarios $\gamma_n = \gamma_{n-1}Q + \mu$ with $Q(x, \cdot) = G(x)M(x, \cdot)$

$$g_- := \inf G \leq G = Q(1) \leq \sup G := g_+$$

- $G = 1$ & $\eta_\infty := \eta_\infty M$

$$\gamma_n(1) = \gamma_0(1) + n\mu(1) \quad \text{and} \quad \|\eta_n - \eta_\infty\|_{\text{tv}} = O(1/n)$$

- $g_+ < 1$ & $\eta_\infty := \gamma_\infty / \gamma_\infty(1)$ with $\gamma_\infty := \sum_{n \geq 0} \mu Q^n$ (when $\gamma_0 = \mu$)

$$|\gamma_n(f) - \gamma_\infty(f)| \vee |\eta_n(f) - \eta_\infty(f)| \leq c g_+^n \|f\|$$

- $g_- > 1$ & $\eta_\infty(f) := \eta_\infty Q(f) / \eta_\infty Q(1)$

$$\eta_\infty =$$

[quasi-invariant measures, Yaglom measures, ground states, Feynman-Kac semigroup fixed points, infinite population stationary measure, etc.]

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \gamma_n(1) = \log \eta_\infty(G) \quad \text{and} \quad \|\eta_n - \eta_\infty\|_{\text{tv}} \leq c e^{-\lambda n}$$

Nonlinear equations

$$\eta_{n+1}(f) \propto \gamma_n(1) \eta_n(Q_{n+1}(f)) + \mu_{n+1}(1) \bar{\mu}_{n+1}(f)$$

↓

Nonlinear & interacting mass + proba measures equations

$$\begin{cases} \gamma_{n+1}(1) &= \gamma_n(1) \eta_n(G_n) + \mu_{n+1}(1) \\ \eta_{n+1} &= \Psi_{G_n}(\eta_n) M_{n+1,(\gamma_n(1), \eta_n)} \end{cases}$$

with the Markov transitions:

$$M_{n+1,(m,\eta)}(x, dy) := \alpha_n(m, \eta) M_{n+1}(x, dy) + (1 - \alpha_n(m, \eta)) \bar{\mu}_{n+1}(dy)$$

with the collection of $[0, 1]$ -parameters

$$\alpha_n(m, \eta) = \frac{m \eta(G_n)}{m \eta(G_n) + \mu_{n+1}(1)}$$

A nonlinear filtering/Feynman-Kac type model

$$\eta_n \xrightarrow{\text{updating}} \hat{\eta}_n := \Psi_{G_n}(\eta_n) = \eta_n S_{n,\eta_n} \xrightarrow{\text{prediction}} \eta_{n+1} := \hat{\eta}_n M_{n+1,(\gamma_n(1),\eta_n)}$$

↓

A couple of equations:

- the total mass evolution

$$\gamma_{n+1}(1) = \gamma_n(1) \eta_n(G_n) + \mu_{n+1}(1)$$

- The "nonlinear filtering/Feynman-Kac type" conservative equations

$$\eta_{n+1} = \eta_n S_{n,\eta_n} M_{n+1,(\gamma_n(1),\eta_n)} := \eta_n \underbrace{K_{n+1,(\gamma_n(1),\eta_n)}}_{\text{Markov transition}}$$

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 - Mean field particle systems
 - Theoretical convergence results
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Mean field interacting particle models

$$\eta_n^N = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^{(N,i)}} \simeq_{N \uparrow \infty} \eta_n \quad \text{and} \quad \gamma_n^N(\mathbf{1}) \simeq_{N \uparrow \infty} \gamma_n(\mathbf{1})$$

Genetic type population evolution :

- the total mass evolution [**"deterministic"**]

$$\gamma_{n+1}^N(\mathbf{1}) := \gamma_n^N(\mathbf{1}) \eta_n^N(G_n) + \mu_{n+1}(\mathbf{1})$$

- Mean field particle model

$$\forall 1 \leq i \leq N \quad \xi_{n+1}^{(N,i)} = \text{random var. with law } K_{n+1,(\gamma_n^N(\mathbf{1}), \eta_n^N)}(\xi_n^{(N,i)}, dx_{n+1})$$

↓

with sampling error fluctuations:

$$\eta_{n+1}^N := \eta_n^N K_{n+1,(\gamma_n^N(\mathbf{1}), \eta_n^N)} + \frac{1}{\sqrt{N}} W_{n+1}^N$$

Theoretical convergence results

- **Independent local sampling error fluctuations**

$$(W_n^N)_{n \geq 0} \simeq_{N \uparrow \infty} \text{iid centered Gaussian fields } (W_n)_{n \geq 0}$$

- **Functional CLT(s) (with $[\gamma_n^N := \gamma_n^N(1) \times \eta_n^N]$)**

$$V_n^{\gamma, N} := \sqrt{N} (\gamma_n^N - \gamma_n) \quad \& \quad V_n^{\eta, N} := \sqrt{N} (\eta_n^N - \eta_n) \quad \rightarrow_N \quad V_n^\gamma \quad \& \quad V_n^\eta$$

- **Uniform convergence results (under some mixing conditions on M_n)**

$$\sup_{n \geq 0} \mathbb{E} \left(\left| [\eta_n^N - \eta_n](f) \right|^p \right) \leq c(p)/N^{p/2} \quad (\oplus \text{ uniform expo. concentration})$$

- **Unbiased particle total mass with variance**

$$N \mathbb{E} \left(\left[\frac{\gamma_n^N(1)}{\gamma_n(1)} - 1 \right]^2 \right) \leq c n \left(1 + \frac{c}{N} \right)^{n-1}$$

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 - Conditioning principles
 - PHD filtering equation
 - Stability properties
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Conditioning principles for marked point processes

- **Poisson point process** \mathcal{X} with intensity $\gamma(dx_1) Q(x_1, dx_2)$ on $E = (E_1 \times E_2)$

$$\mathcal{X} := m_N(X_1, X_2) = \sum_{1 \leq i \leq N} \delta_{(X_1^i, X_2^i)} \quad \text{and} \quad \mathcal{X}_j := m_N(X_j) = \sum_{1 \leq i \leq N} \delta_{X_j^i}$$

- **2 Bayes' rules:**

Normalization $p(x_2|x_1) \oplus$ Reversal Markov operator $p(x_1|x_2)$

$$\bar{Q}(x_1, dx_2) = \frac{Q(x_1, dx_2)}{Q(x_1, E_2)} \quad \text{and} \quad \gamma(dx_1) Q(x_1, dx_2) = (\gamma Q)(dx_2) Q_\gamma(x_2, dx_1)$$

- \Rightarrow **2 conditional distributions formulae:**

$$\mathbb{E}(F_1(\mathcal{X}_1) \mid \mathcal{X}_2) = \int F_1(m_N(x_1)) \prod_{1 \leq i \leq N} Q_\gamma(X_2^i, dx_1^i)$$

$$\mathbb{E}(F_2(\mathcal{X}_2) \mid \mathcal{X}_1) = \int F_2(m_N(x_2)) \prod_{1 \leq i \leq N} \bar{Q}(X_1^i, dx_2^i)$$

Conditioning principles for marked point processes

- $(\mathcal{X}_1, \mathcal{X}_2) = (\mathcal{X}, \mathcal{Y})$, $F(\mathcal{X}) = \mathcal{X}(f)$ and $P(y, dx)$ Markov transition

$$\begin{aligned}\widehat{\gamma}(f) &:= \mathbb{E}(F(\mathcal{X}) \mid \mathcal{Y}) \\ &= \int F(m_N(x)) \prod_{1 \leq i \leq N} P(Y^i, dx^i) = \mathcal{Y}(P(f))\end{aligned}$$

- **Example with \mathcal{X} Poisson Signal $\gamma(dx) \rightsquigarrow \mathcal{Y}$ Poisson Observation**

$$(X^i = x) \rightsquigarrow (Y_i = y) \sim \alpha(x) g(x, y) \lambda(dy) + (1 - \alpha(x)) \delta_c(dy)$$

$$\oplus \text{ Clutter } \mathcal{Y}' \text{ Poisson with intensity } \nu(dy) = h(y) \lambda(dy)$$

- **Only observable $\mathcal{Y}^0 = \mathcal{Y} \times 1_{\neq c}$**

$$\widehat{\gamma}(f) := \mathbb{E}(\mathcal{X}(f) \mid \mathcal{Y}^0) = \gamma((1-\alpha)f) + \int \mathcal{Y}^0(dy) (1 - \beta_\gamma(y)) \Psi_{\alpha g(y, \cdot)}(\eta)(f)$$

with "the conditional clutter probability density"

$$\beta_\gamma(y) = h(y) / [h(y) + \gamma(\alpha g(y, \cdot))]$$

Example full detection and no clutter $\alpha = 1$ & $h = 0$

$$\hat{\gamma}(f) := \int \mathcal{Y}^{\circ}(dy) \Psi_{g(y, \cdot)}(\eta)(f)$$

↓

Conditional mean number of targets and "their conditional distributions"

$$\hat{\gamma}(1) = \mathcal{Y}^{\circ}(1) \quad \text{and} \quad \hat{\eta}(f) := \frac{\hat{\gamma}(f)}{\hat{\gamma}(1)} = \int \frac{\mathcal{Y}^{\circ}(dy)}{\mathcal{Y}^{\circ}(1)} \underbrace{\Psi_{g(y, \cdot)}(\eta)(f)}_{\text{Bayes' formula}}$$

PHD filtering equation

[with the prediction stage as the signal branching model (Q_n, μ_n)]

Hypothesis:

\mathcal{X}_{n+1} **Poisson with intensity** $\gamma_{n+1} = \hat{\gamma}_n Q_n + \mu_n \rightsquigarrow$ **with obs.** \mathcal{Y}_{n+1}^0 **as before**

\Rightarrow **PHD filtering equations:**

$$\gamma_{n+1} := \hat{\gamma}_n Q_n + \mu_n$$

$$\hat{\gamma}_n(f) := \gamma_n((1 - \alpha_n)f) + \int \mathcal{Y}_n^0(dy) (1 - \beta_{\gamma_n}(y)) \Psi_{\alpha_n \mathcal{G}_n(y, \cdot)}(\gamma_n)(f)$$

\subset **A class of positive measure valued equations** \supset **Bernoulli filters, etc.**

$$\gamma_{n+1}(dx') = \gamma_n Q_{n+1, \gamma_n}(dx') = \int \gamma_n(dx) Q_{n, \gamma_n}(x, dx')$$

with

$$Q_{n+1, \gamma_n}(x, dx') = Q_{n+1, \gamma_n}(1)(x) \times \frac{Q_{n+1, \gamma_n}(x, dx')}{Q_{n+1, \gamma_n}(1)(x)} = \underbrace{G_{n, \gamma_n}(x)}_{\text{potential function}} \times \underbrace{K_{n+1, \gamma_n}(x, dx')}_{\text{Markov transition}}$$

Stability properties of meas. valued equations

$$\eta_n = \gamma_n / \gamma_n(1) \rightsquigarrow \text{Nonlinear semigroup} \quad (\gamma_n(1), \eta_n) = \Gamma_{p,n}(\gamma_p(1), \eta_p)$$

- **Hypothesis on** $(m, \eta) \mapsto G_{n,m\eta}$ **s.t.** $m_n = \gamma_n(1) \in I_n = \text{compact } \subset]0, \infty[$
- **Stability Theorem for** $\Gamma_{p,n} = (\Gamma_{p,n}^1, \Gamma_{p,n}^2)$

- ① $\forall \eta_n$, the sg of total masses $\gamma_n(1)$ is expo. stable.
- ② $\forall m_n \in I_n$, the the sg of proba η_n is expo. stable.
- ③ $m \in I_n \mapsto \Gamma_{n+1}^1(m, \eta_n)$ and $\eta \mapsto \Gamma_{n+1}^2(m_n, \eta)$ are Lipschitz continuous

$$(3) \geq (1) \ \& \ (2) \Rightarrow \forall i = 1, 2 \quad \|\Gamma_{p,n}^i(m', \eta') - \Gamma_{p,n}^i(m, \eta)\| \leq c e^{-\lambda(n-p)}$$

- **Applications:**

3 natural conditions for the exponential stability of the PHD filter

- ① **small clutter intensities**
- ② **high detection probability**
- ③ **high spontaneous birth rates**

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 - Mean field particle approximation

Nonlinear equations

$$\gamma_{n+1} = \gamma_n Q_{n+1, \gamma_n} \rightsquigarrow \eta_n := \gamma_n / \gamma_n(1) \quad \text{and} \quad G_{n, \gamma_n} = Q_{n+1, \gamma_n}(1)$$
$$\Downarrow$$

- **The total mass evolution**

$$\gamma_{n+1}(1) = \gamma_n(1) \eta_n(G_{n, \gamma_n(1)\eta_n})$$

- **The "nonlinear filtering" conservative equations**

$$\eta_{n+1}(f) = \frac{\eta_n Q_{n, \gamma_n(1)\eta_n}(f)}{\eta_n Q_{n, \gamma_n(1)\eta_n}(1)} := \eta_n K_{n, \gamma_n(1)\eta_n}(f)$$

Mean field particle models

$$\eta_n^N = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^{(N,i)}} \simeq_{N \uparrow \infty} \eta_n \quad \text{and} \quad \gamma_n^N(\mathbf{1}) \simeq_{N \uparrow \infty} \gamma_n(\mathbf{1})$$

with

$$\begin{aligned} \gamma_{n+1}^N(\mathbf{1}) &= \gamma_n^N(\mathbf{1}) \times \eta_n^N(G_{n, \gamma_n^N(\mathbf{1}), \eta_n^N}) \\ \xi_{n+1}^{(N,i)} &= \text{random var. with law } K_{n+1, (\gamma_n^N(\mathbf{1}), \eta_n^N)}(\xi_n^{(N,i)}, dx) \end{aligned}$$

⇓

Same theorems as before with uniform convergence estimates

- $\supset \forall$ scheme with local error controls
- \supset Interacting Kalman type filters \rightsquigarrow particle associations (\simeq GM-PHD)
- \supset Particle association \oplus local branching models