

Concentration \leq for Mean Field Particle Models

P. Del Moral, E. Rio

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- *Concentration Inequalities for Mean Field Particle Models* [HAL-INRIA RR-6901 (09)]. To appear in the Annals of Applied Probability (2010).
- *A Backward Particle Interpretation of Feynman-Kac Formulae*.
 - + A. Doucet and S.S. Singh. RR-7019(09). M2AN, vol 44, no. 5 (2010)
- *An introduction to probabilistic methods with applications*
 - + N. Hadjiconstantinou. M2AN, vol 44, no. 5 (2010)

- 1 Introduction, motivations
- 2 Mean field particle models
- 3 Convergence analysis

- 1 Introduction, motivations
 - Some notation
 - Running ex. : Feynman-Kac models
 - 4 Key observations
- 2 Mean field particle models
- 3 Convergence analysis

Some notation

E measurable state space, $\mathcal{M}(E)$ & $\mathcal{P}(E)$ measures & probabilities on E
 $\mathcal{B}(E)$ bounded meas. functions

- $(\mu, f) \in \mathcal{M}(E) \times \mathcal{B}(E) \mapsto \mu(f) = \int \mu(dx) f(x)$

- $M(x, dy)$ **integral operator over E**

$$M(f)(x) = \int M(x, dy) f(y)$$

$$[\mu M](dy) = \int \mu(dx) M(x, dy) \quad (\implies [\mu M](f) = \mu[M(f)])$$

- **Composition:** $(M_1 M_2)(x, dz) = \int M_1(x, dy) M_2(y, dz)$

- **Boltzmann-Gibbs transformation :** $G : E \rightarrow [0, \infty[$ with $\mu(G) > 0$

$$\Psi_G(\mu)(dx) = \frac{1}{\mu(G)} G(x) \mu(dx)$$

Running example : Feynman-Kac integration models

- Markov chain X_n on some state spaces E_n , n =time index .
- Potential functions $G_n : x_n \in E_n \rightarrow G_n(x_n) \in [0, 1]$

Feynman-Kac path measures:

$$d\mathbb{Q}_n := \frac{1}{Z_n} \left\{ \prod_{0 \leq p < n} G_p(X_p) \right\} d\mathbb{P}_n \quad \text{with} \quad \mathbb{P}_n := \text{Law}(X_0, \dots, X_n)$$

The n -time marginals: $\forall f_n \in \mathcal{B}(E_n)$

$$\eta_n(f_n) := \frac{\gamma_n(f_n)}{\gamma_n(1)} \quad \text{with} \quad \gamma_n(f_n) := \mathbb{E} \left(f_n(X_n) \prod_{0 \leq p < n} G_p(X_p) \right)$$

Updated measures $\longrightarrow \hat{\mathbb{Q}}_n, \hat{\eta}_n,$ and $\hat{\gamma}_n$ w.r.t. the product $\prod_{0 \leq p \leq n}$ up to n .

Ex 1: Filtering, Hidden Markov Chains, Bayesian Inference

Signal-observation type model

$$\mathbb{P}((X_n, Y_n) \in d(x, y) | (X_{n-1}, Y_{n-1})) := M_n(X_{n-1}, dx) g_n(x, y) \lambda_n(dy)$$

- Given the observation sequence $Y = y$ with $G_n(x_n) = g_n(x_n, y_n)$

$$\eta_n = \text{Law}(X_n | \forall 0 \leq p < n \ Y_p = y_p) \quad \text{and} \quad \mathcal{Z}_{n+1} \propto p_n(y_0, \dots, y_n)$$

- In path space settings

$$\mathbb{Q}_n = \text{Law}((X_0, \dots, X_n) | \forall 0 \leq p < n \ Y_p = y_p)$$

Continuous time filtering models

$(X, Y) = (\text{Signal-Observation}) \quad X_n = X'_{[t_n, t_{n+1}]}$ and $dY_t = H(X'_t) dt + \sigma dV_t$

$$G_n(X_n) = e^{\int_{t_n}^{t_{n+1}} [H^*(X'_n(s)) dY_s - \frac{1}{2} H^*(X'_n(s)) H(X'_n(s)) ds]}$$

\Downarrow

$$\mathbb{Q}_n = \text{Law}((X'_{[0, t_{n+1}]}) \mid Y_s, s \leq t_n)$$

The t_n -marginals $\eta_{t_n} \rightsquigarrow$ **Kushner-Stratonovich SPDE** (X' inf. generator L')

$$\eta_{t_n}(f) = \nu_0(f) + \int_0^{t_n} \eta_s(L'(f)) ds + \int_0^{t_n} [\eta_s(H^* f) - \eta_s(H^*) \eta_s(f)] (dY_s - \eta_s(H) ds)$$

Note: Euler time discretization \rightsquigarrow full discrete time model

Ex 2 : Absorption model $G : x \in E^c := E \cup \{c\} \mapsto G(x) \in [0, 1]$

$$\begin{cases} X_n^c \in E_n^c \xrightarrow{\text{absorption} \sim (1-G)} \widehat{X}_n^c \xrightarrow{\text{free-motion} \sim M} X_{n+1}^c \\ T = \inf \{n : \widehat{X}_n^c = c\} \Rightarrow \mathbb{Q}_n = \text{Law}((X_0^c, \dots, X_n^c) | T \geq n) \end{cases}$$

Spectral radius-Lyapunov exponents : $Q(x, dy) = G(x)M(x, dy)$ sub-Markov

$$\gamma_n(1) = \prod_{0 \leq p < n} \eta_p(G) = \mathbb{P}(T \geq n) \simeq e^{-\lambda n}$$

with $e^{-\lambda} \stackrel{M \text{ reg.}}{=} \mathbf{Q}$ -top eigenvalue & eigenfunction $Q(h) = e^{-\lambda h}$

$$\lambda = \lim_{n \rightarrow \infty} -\frac{1}{n} \log \|Q^n\| = \lim_{n \rightarrow \infty} -\frac{1}{n} \sum_{0 \leq p \leq n} \log \eta_p(G) = -\log \eta_\infty(G)$$

Limiting FK measures $\eta_\infty = \text{Yaglom limits} = \text{quasi-stationary measures} :$

$$M \mu - \text{reversible} \Rightarrow \eta_n(f) = \mathbb{E}(f(X_n^c) | T \geq n) \simeq \eta_\infty(f) := \frac{\mu(h M(f))}{\mu(h)}$$

\sim Contraction of nonlinear FK sg \rightsquigarrow Slides Prob & Analysis, Nice 2008

Continuous time models

$\frac{d}{dt} \gamma_t(f) := \gamma_t(L^V(f))$ with the Schrodinger operator $L^V = L - V$

$$L(f) = \frac{1}{2} \sum_{i,j=1}^d a^{i,j} \partial_{x_i} \partial_{x_j}(f) + \sum_{i=1}^d b^i \partial_{x_i}(f)$$

Feynman-Kac representation $\gamma_t = \gamma_0 P_t^V$ with

$$\begin{aligned} P_t^V(f)(x) &= \mathbb{E}_x \left(f(X'_t) \exp \left\{ - \int_0^t V(X'_s) ds \right\} \right) \\ &\stackrel{t=t_n}{=} \mathbb{E}_x \left(f_n(X'_{[t_{n-1}, t_n]}) \prod_{0 \leq p < n} G_p(X_p) \right) \end{aligned}$$

with $X_n = X'_{[t_n, t_{n+1}]}$ and $G_n(X_n) = e^{-\int_{t_n}^{t_{n+1}} V(X'_s) ds}$ (alternative = Euler schemes).

Sensitivity analysis

$$\theta \mapsto M_n(x_{n-1}, dx_n) = p_n^\theta(x_{n-1}, x_n) \lambda_n^X(dx_n) \quad \text{and/or} \quad G_n(x_n) = G_n^\theta(x_n)$$

↓

θ -Feynman-Kac models : $(G_n, \mathbb{Q}_n, \gamma_n, \eta_n, \lambda, \dots) \rightsquigarrow (G_n^\theta, \mathbb{Q}_n^\theta, \gamma_n^\theta, \eta_n^\theta, \lambda^\theta, \dots)$

Eigenvalues derivatives :

$$-\frac{\partial \lambda^\theta}{\partial \theta} \simeq \frac{1}{n+1} \frac{\partial}{\partial \theta} \log \hat{\gamma}_n^\theta(1) = \hat{\mathbb{Q}}_n^\theta(F_n^\theta)$$

with the additive functional

$$F_n^\theta(x_0, \dots, x_n) := \frac{1}{n+1} \sum_{0 \leq k \leq n} \left(\frac{\partial \log G_k^\theta(x_k)}{\partial \theta} + \frac{\partial \log p_k^\theta(x_{k-1}, x_k)}{\partial \theta} \right)$$

⊕ High order derivatives using the covariance formulae

$$\frac{\partial}{\partial \theta} \hat{\mathbb{Q}}_n^\theta(\varphi_n) = (n+1) \hat{\mathbb{Q}}_n^\theta \left(F_n^\theta [\varphi_n - \hat{\mathbb{Q}}_n^\theta(\varphi_n)] \right)$$

4 Key observations

$$\eta_n(f_n) := \frac{\gamma_n(f_n)}{\gamma_n(\mathbf{1})} \quad \text{with} \quad \gamma_n(f_n) := \mathbb{E} \left(f_n(X_n) \prod_{0 \leq p < n} G_p(X_p) \right)$$

- 1st important observation:

$$[X_n := (X'_0, \dots, X'_n) \ \& \ G_n(X_n) := G'_n(X'_n)] \implies \eta_n = \mathbb{Q}'_n$$

- 2nd important observation:

$$\mathcal{Z}_n = \gamma_n(\mathbf{1}) = \mathbb{E} \left(\prod_{0 \leq p < n} G_p(X_p) \right) = \prod_{0 \leq p < n} \eta_p(G_p)$$

Proof:

$$\gamma_n(\mathbf{1}) = \gamma_{n-1}(G_{n-1}) = \eta_{n-1}(G_{n-1}) \gamma_{n-1}(\mathbf{1})$$

- 3rd important observation: (H) $M_n(x, dx') = H_n(x, x') \lambda_n(dx')$

$$\mathbb{Q}_n(d(x_0, \dots, x_n)) = \eta_n(dx_n) M_{n, \eta_{n-1}}(x_n, dx_{n-1}) \dots M_{1, \eta_0}(x_1, dx_0)$$

with the backward transitions :

$$M_{n, \eta}(x, dx') \propto \eta(dx') G_{n-1}(x') H_n(x', x)$$

Absorption models : \rightsquigarrow h -process $X_n^{(h)} \sim M^{(h)}(x, dy) \propto M(x, dy)h(y)$

$$\mathbb{P}\left(\left(X_0^{(h)}, \dots, X_n^{(h)}\right) \in d(x_0, \dots, x_n)\right) \frac{h^{-1}(x_n)}{\mathbb{E}(h^{-1}(X_n^{(h)}))} = \mathbb{Q}_n(d(x_0, \dots, x_n))$$

Additive functionals

$$\mathbb{E}_{\mathbb{Q}_n} \left(\frac{1}{n} \sum_{0 \leq p < n} f(X_p) \right) \xrightarrow{n \uparrow \infty} \eta_\infty^{(h)}(f) = \eta_\infty^{(h)} M^{(h)}(f) = \Psi_h(\eta_\infty)(f)$$

4th Key observation

- A two step correction prediction model

$$\eta_n \xrightarrow{\text{Updating-correction}} \hat{\eta}_n = \Psi_{G_n}(\eta_n) \xrightarrow{\text{Prediction/Markov transport}} \eta_{n+1} = \hat{\eta}_n M_{n+1}$$

- Selection nonlinear transport formulae

$$\Psi_{G_n}(\eta_n) = \eta_n S_{n,\eta_n}$$

with, for any $\epsilon_n \in [0, 1]$

$$S_{n,\eta_n}(x, \cdot) := \epsilon_n G_n(x) \delta_x + (1 - \epsilon_n G_n(x)) \Psi_{G_n}(\eta_n)$$

⇓

$$\eta_{n+1} = \eta_n (S_{n,\eta_n} M_{n+1}) := \eta_n K_{n+1,\eta_n}$$

Note : Continuous time models = nonlinear Moran type interacting jump proc.

1 Introduction, motivations

2 Mean field particle models

- Nonlinear McKean distribution flows
- Mean field particle interpretations
- The 4 types of particle approximation measures
- Some key advantages

3 Convergence analysis

Nonlinear Markov chains $\eta_n = \text{Law}(\bar{X}_n) = \text{Perfect sampling algorithm}$

- **Nonlinear transport formulae :**

$$\eta_{n+1} = \eta_n K_{n+1, \eta_n}$$

- **Local transitions :**

$$\mathbb{P}(\bar{X}_n \in dx_n \mid \bar{X}_{n-1}) = K_{n, \eta_{n-1}}(\bar{X}_{n-1}, dx_n) \quad \text{avec} \quad \eta_{n-1} = \text{Law}(\bar{X}_{n-1})$$

- **McKean measures (canonical process) :**

$$\mathbb{P}_n(d(x_0, \dots, x_n)) = \eta_0(dx_0) K_{1, \eta_0}(x_0, dx_1) \dots K_{n, \eta_{n-1}}(x_{n-1}, dx_n)$$

Other examples : Gaussian-McKean Vlasov type transitions

$$\bar{X}_{n+1} := d_n(\bar{X}_n, \eta_n) + \mathcal{N}(0, Q_n)$$

Sampling pb \Rightarrow Mean field particle interpretations

- Markov Chain $\xi_n = (\xi_n^1, \dots, \xi_n^N) \in E_n^N$ s.t.

$$\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq_{N \uparrow \infty} \eta_n$$

- Approximated local transitions ($\forall 1 \leq i \leq N$)

$$\xi_{n-1}^i \rightsquigarrow \xi_n^i \sim K_{n, \eta_{n-1}^N}(\xi_{n-1}^i, dx_n)$$

Feynman-Kac models \Leftrightarrow Genetic type stochastic algo.

$$\begin{bmatrix} \xi_n^1 \\ \vdots \\ \xi_n^i \\ \vdots \\ \xi_n^N \end{bmatrix} \xrightarrow{S_{n,\eta_n^N}} \begin{bmatrix} \widehat{\xi}_n^1 \\ \vdots \\ \widehat{\xi}_n^i \\ \vdots \\ \widehat{\xi}_n^N \end{bmatrix} \xrightarrow{M_{n+1}} \begin{bmatrix} \xi_{n+1}^1 \\ \vdots \\ \xi_{n+1}^i \\ \vdots \\ \xi_{n+1}^N \end{bmatrix}$$

Accept/Reject-Selection : [Geometric clocks] [Confinement ex. : $G_n = 1_A$]

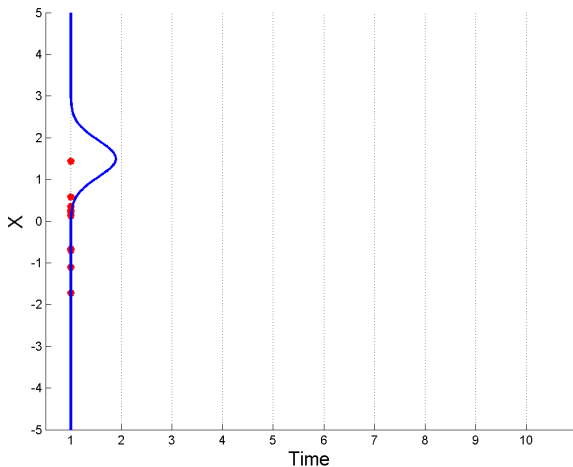
$$S_{n,\eta_n^N}(\xi_n^i, \cdot) := \epsilon_n G_n(\xi_n^i) \delta_{\xi_n^i} + (1 - \epsilon_n G_n(\xi_n^i)) \sum_{j=1}^N \frac{G_n(\xi_n^j)}{\sum_{k=1}^N G_n(\xi_n^k)} \delta_{\xi_n^j}$$

\oplus Unbias particle normalizing Cts

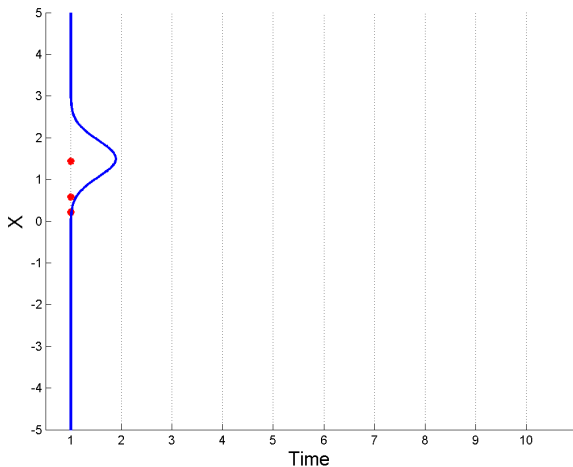
$$\mathcal{Z}_n^N := \prod_{0 \leq p < n} \eta_p^N(G_p) \simeq \mathcal{Z}_n$$

\supset Particle filters, Diffusion Monte Carlo (DMC), Quantum Monte Carlo (QMC), Sequential Monte Carlo methods (SMC), ...

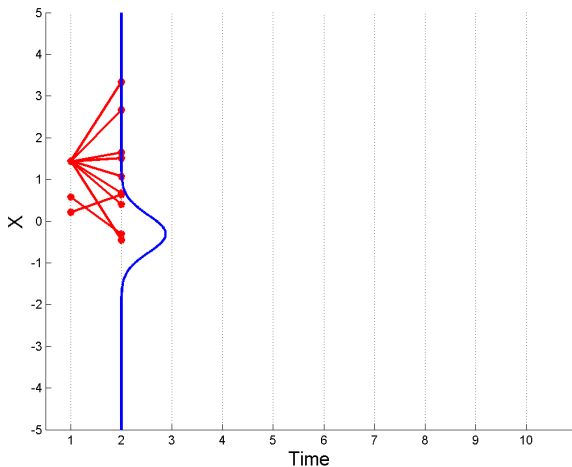
Genealogical tree evolution in dimension 1



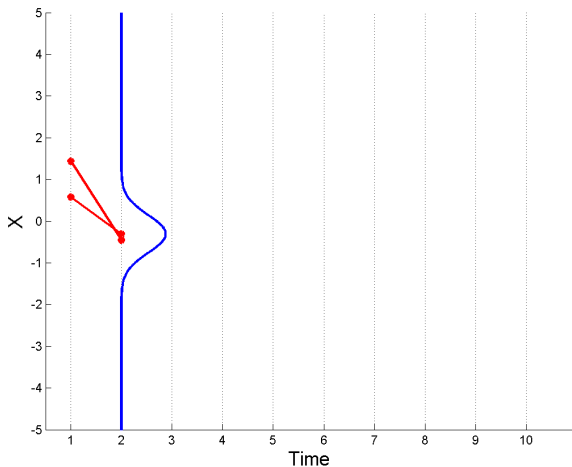
Genealogical tree evolution in dimension 1



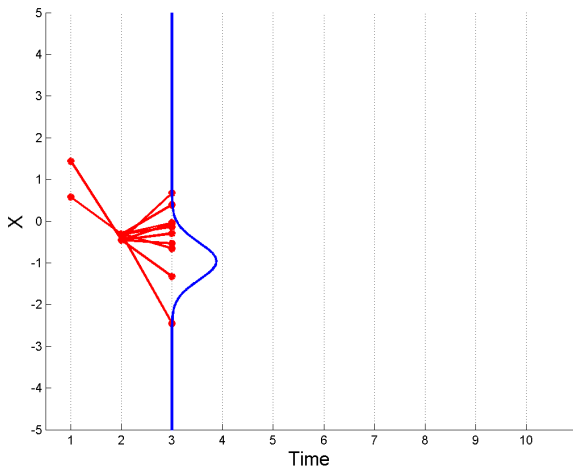
Genealogical tree evolution in dimension 1



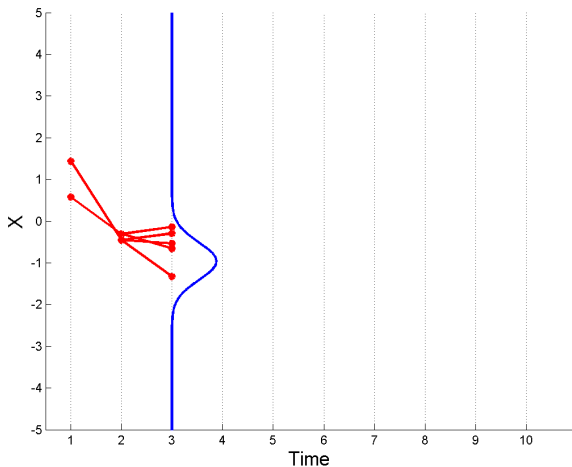
Genealogical tree evolution in dimension 1



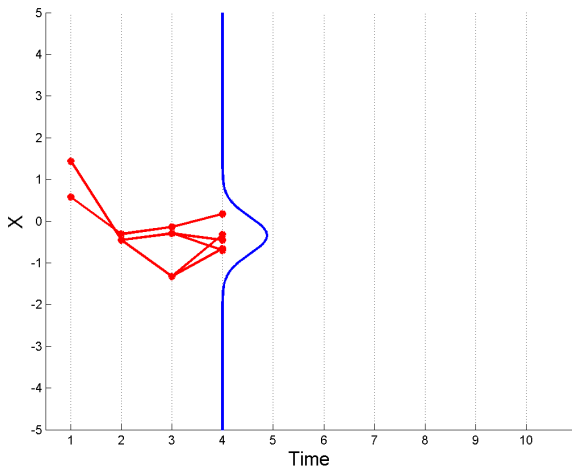
Genealogical tree evolution in dimension 1



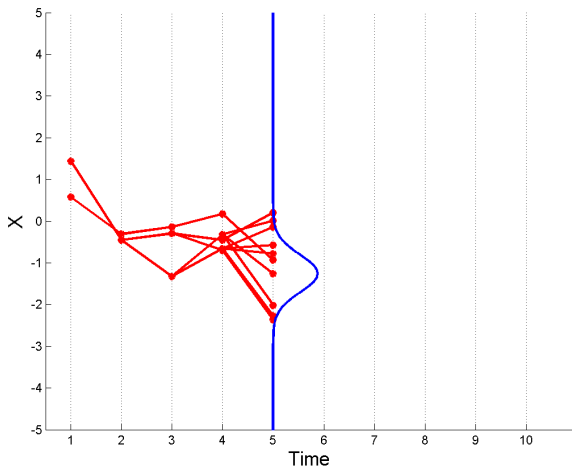
Genealogical tree evolution in dimension 1



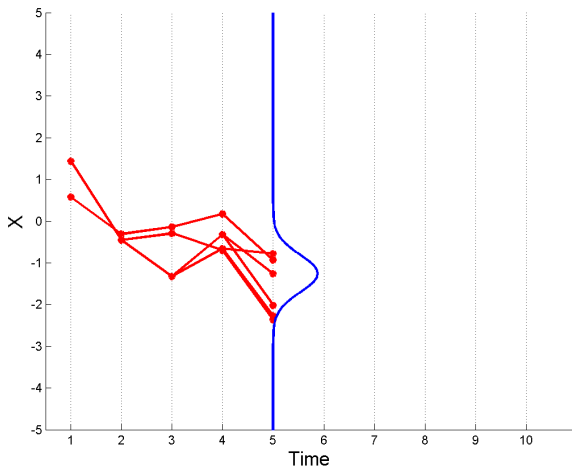
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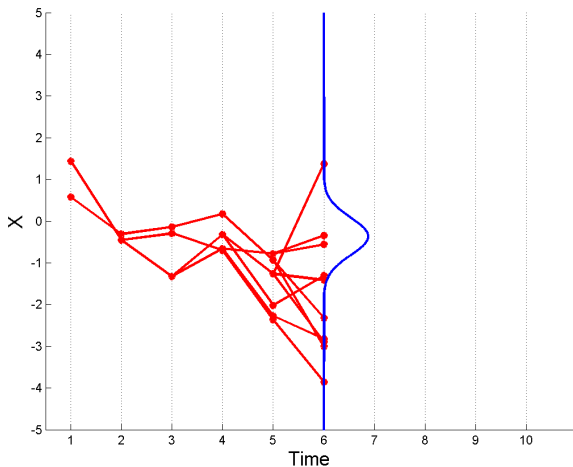
Genealogical tree evolution in dimension 1



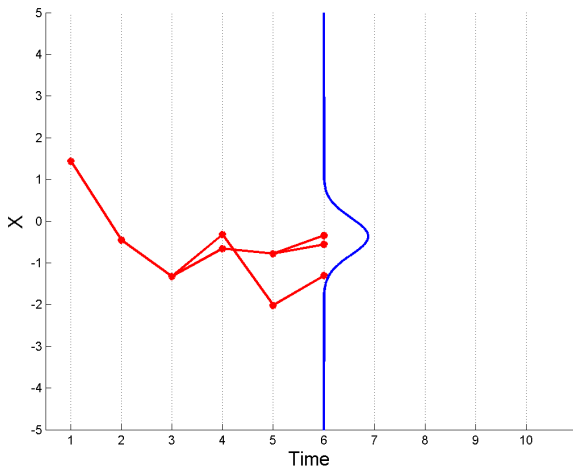
Genealogical tree evolution in dimension 1



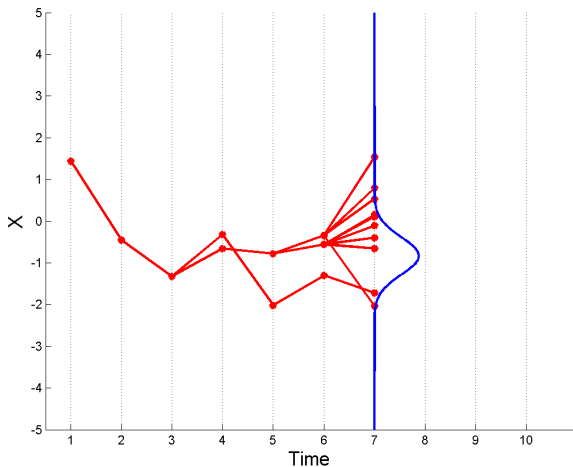
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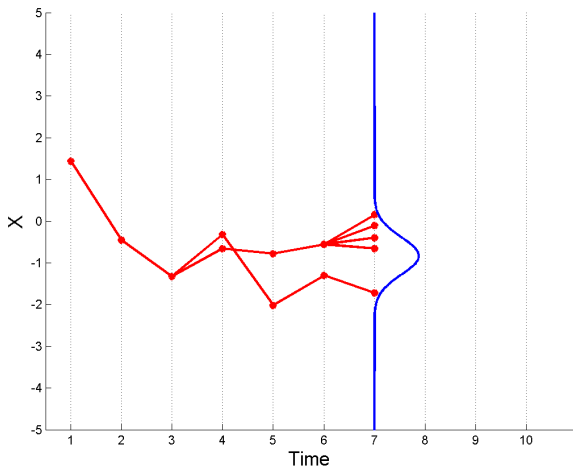
Genealogical tree evolution in dimension 1



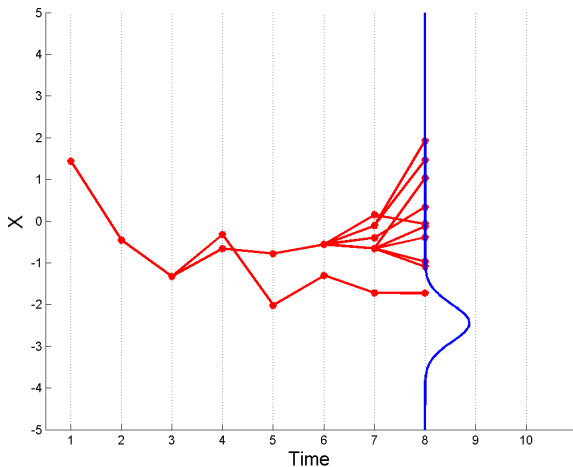
Genealogical tree evolution in dimension 1



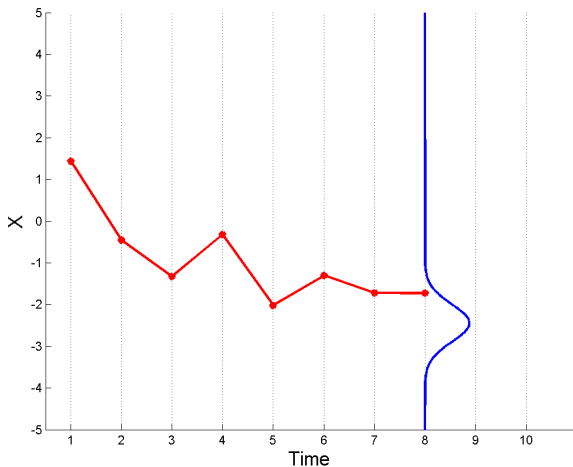
Genealogical tree evolution in dimension 1



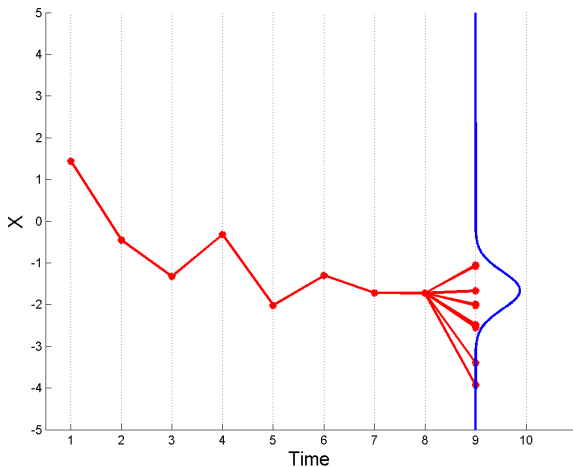
Genealogical tree evolution in dimension 1



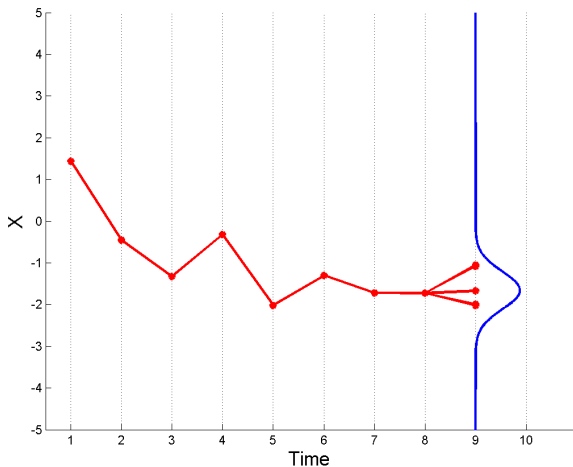
Genealogical tree evolution in dimension 1



Genealogical tree evolution in dimension 1

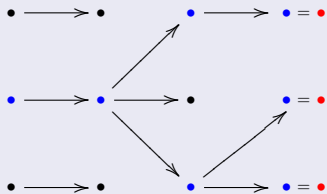


Genealogical tree evolution in dimension 1



Interaction/branch. process \hookrightarrow 4 types of occupation measures

($N = 3$)



- **Current population** $\hookrightarrow \frac{1}{N} \sum_{i=1}^N \delta_{\xi_n^i} \leftarrow i\text{-th individual at time } n \simeq \eta_n$
- **Genealogical tree** $\hookrightarrow \frac{1}{N} \sum_{i=1}^N \delta_{(\xi_{0,n}^i, \xi_{1,n}^i, \dots, \xi_{n,n}^i)} \leftarrow i\text{-th ancestral line} \simeq \mathbb{Q}_n$
- **Complete genealogical tree** $\hookrightarrow \frac{1}{N} \sum_{i=1}^N \delta_{(\xi_0^i, \xi_1^i, \dots, \xi_n^i)} \simeq$ McKean meas.
- **Forward particle approximation** \sim complete genealogical tree :

$$\mathbb{Q}_n^N(d(x_0, \dots, x_n)) = \eta_n^N(dx_n) M_{n, \eta_{n-1}^N}(x_n, dx_{n-1}) \dots M_{1, \eta_0^N}(x_1, dx_0)$$

Some key advantages

- Mean field models = **stochastic linearization/perturbation technique** :

$$\eta_n^N = \eta_{n-1}^N K_{n, \eta_{n-1}^N} + \frac{1}{\sqrt{N}} W_n^N$$

with [theorem] $(W_n^N)_n \simeq (W_n)_n$ Indep. Centered Gaussian Fields.

- $\eta_n = \eta_{n-1} K_{n, \eta_{n-1}}$ stable \Rightarrow No propagation of local sampling errors

\Rightarrow **Uniform control w.r.t. the time horizon**

- "No burning, no need to study the stability of MCMC type models".
- PDE viewpoint : Stochastic adaptive grid approximation
- Nonlinear system \rightsquigarrow "positive-benefic interactions.
- Simple and natural sampling algorithm.

1 Introduction, motivations

2 Mean field particle models

3 **Convergence analysis**

- Feynman-Kac type models
- Concentration inequalities

Feynman-Kac models = rather complete analysis [LDP,CLT, Propagation of chaos] (FK Springer 2004 + refs)

- Example: Empirical processes \mathbb{L}_p -mean error estimates

$$\sup_{n \geq 0} \sup_{N \geq 1} \sqrt{N} \mathbb{E} \left(\sup_{f \in \mathcal{F}_n} |\eta_n^N(f) - \eta_n(f)|^p \right) < \infty$$

- (New) Propagations of chaos (+Patras & Rubenthaler (AAP 10))

$$\begin{aligned} \mathbb{P}_{n,q}^N &:= \text{Law}(\xi_n^1, \dots, \xi_n^q) \\ &\simeq \eta_n^{\otimes q} + \frac{1}{N} \partial^1 \mathbb{P}_{n,q} + \dots + \frac{1}{N^k} \partial^k \mathbb{P}_{n,q} + \frac{1}{N^{k+1}} \partial^{k+1} \mathbb{P}_{n,q} \end{aligned}$$

with $\sup_{N \geq 1} \|\partial^{k+1} \mathbb{P}_{n,q}^N\|_{\text{tv}} < \infty$ & $\sup_{n \geq 0} \|\partial^1 \mathbb{P}_{n,q}\|_{\text{tv}} \leq c q^2$.

- (New) Additive functional (+ Doucet & Singh M2AN 10) :

$$N \mathbb{E} \left([(\mathbb{Q}_n^N - \mathbb{Q}_n)(F_n)]^2 \right) \leq c \times (1/n + 1/N)$$

Concentration inequalities

- **Large deviation principles**

[joint works with D. Dawson (Springer 05), A. Guionnet (SPA 98, IHP 01), T. Zajic (Bernoulli 03)]

$$\rightsquigarrow \sup_{n \geq 0} \limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P} \left(\sup_{p \leq n} \frac{1}{2} \sum_{1 \leq k \leq d} |\eta_p^N(A_k) - \eta_p(A_k)| \geq \epsilon \right) \leq -\epsilon^2 / \beta(\Phi)$$

with $\beta(\Phi)$ related to the contraction properties of the sg $\Phi_{p,n}(\eta_p) = \eta_n$.

- **Non asymptotic estimates**

[joint works with M. Ledoux (JTP 00), A. Doucet & S. Singh
(Feynman-Kac path space models & Additive functionals) (M2AN-10)]

$$\rightsquigarrow \sup_{n \geq 0} \frac{1}{N} \log \mathbb{P} \left(|[Q_n^N - Q_n](F_n)| \geq \frac{b}{\sqrt{N}} + \epsilon \right) \leq -\epsilon^2 / (2b^2)$$

Concentration inequalities \sim stability prop. semigroups $\Phi_{p,n}(\eta_\rho) = \eta_n$

[+ E. Rio (HAL-INRIA 09 & AAP 10) \supset FK and McKean-Vlasov type models]

$$(\mathbf{H}\Phi) \forall \Phi_{p,n} \text{ s.t. } \Phi_{p,n}(\eta) - \Phi_{p,n}(\mu) = [\eta - \mu] \underbrace{D_\mu \Phi_{p,n}}_{\text{1st order op.}} + \underbrace{\mathcal{R}_{p,n}(\eta, \mu)}_{\text{2nd order } (\eta - \mu)^{\otimes 2}(\dots)}$$

Example : FK semigroups

$$\Phi_{p,n}(\eta)(f) \propto \eta Q_{p,n}(f) \quad \text{with} \quad Q_{p,n}(f) = \mathbb{E}_{p, X_p} \left(f(X_n) \prod_{p \leq q < n} G_q(X_q) \right)$$

Taylor expansion \Rightarrow ($\mathbf{H}\Phi$)

$$\begin{aligned} [\Phi_{p,n}(\mu + \epsilon\nu) - \Phi_{p,n}(\mu)](f) &\propto (\cancel{\mu} + \epsilon\nu) Q_{p,n}[f - \Phi_{p,n}(\mu)(f)] \\ &\propto \epsilon \times \nu D_\mu \Phi_{p,n}(f) \end{aligned}$$

with the operator $D_\mu \Phi_{p,n} \propto Q_{p,n}[Id - \Phi_{p,n}(\mu)]$

Concentration inequalities \sim stability prop. semigroups $\Phi_{p,n}(\eta_p) = \eta_n$

[Theo:] (**H Φ**) $\Rightarrow \forall x \geq 0$, the probability of the next events is $\geq 1 - e^{-x}$

$$(\eta_n^N - \eta_n)(f) \leq \frac{r_n}{N} (1 + \epsilon_0^{-1}(x)) + \bar{\sigma}_n^2 b_n^* \epsilon_1^{-1} \left(\frac{x}{N \bar{\sigma}_n^2} \right) \quad [\text{Bennett } (r = 0)]$$

$$(\eta_n^N - \eta_n)(f) \leq \frac{r_n}{N} (1 + \epsilon_0^{-1}(x)) + \sqrt{\frac{2x}{N}} \beta_n \quad [\text{Hoeffding } (r = 0)]$$

with

$$\epsilon_0(\lambda) = \frac{1}{2} (\lambda - \log(1 + \lambda)), \quad \epsilon_1(\lambda) = (1 + \lambda) \log(1 + \lambda) - \lambda$$

Cts :

$r_n \sim$ bias second order & $\bar{\sigma}_n, \beta_n, b_n^* \sim$ Dobrushin coef. 1st order operators.

Note : Crude bounds

$$\epsilon_0^{-1}(x) \leq 2x + 2\sqrt{x} \quad \epsilon_1^{-1}(x) \leq x/3 + \sqrt{2x} \quad \rightsquigarrow [\text{Bernstein}]$$

Application : Time homogeneous Feynman-Kac models (M, G)

$$(H) \exists m : M^m(x, \cdot) \geq \epsilon M^m(y, \cdot) \quad \text{and} \quad \delta := \sup \prod_{0 \leq p < m} (G(x_p)/G(y_p)) < \infty$$

[Corollary:]

$\forall x \geq 0$, \forall time horizon n , the probability of the next events is $\geq 1 - e^{-x}$

$$(\eta_n^N - \eta_n)(f) \leq \frac{4a}{N} (1 + \epsilon_0^{-1}(x)) + 8a \epsilon_1^{-1} \left(\frac{x}{4bN} \right)$$

$$(\eta_n^N - \eta_n)(f) \leq \frac{4a}{N} (1 + \epsilon_0^{-1}(x)) + 2\sqrt{\frac{2bx}{N}}$$

with $a \leq m (\delta/\epsilon)^5$ and $b \leq m (\delta/\epsilon)^4$.