Resonances and semiclassical trace formulae

Vesselin Petkov

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Operators with point spectrum

Consider the operator $A(h) = -h^2\Delta + V(x), V \in C^\infty(\mathbb{R}^n;\mathbb{R})$, $0 < h \leq 1$. Assume that

$$|\partial^\alpha V| \leq C_\alpha(1+|x|^2)^q,\ q \geq 0,\ \forall\alpha,\ x \in \mathbb{R}^n$$

and set $p(x,\xi) = |\xi|^2 + V(x)$. Let $I = (a,b) \subset \mathbb{R}$ be an open bounded interval and let

$$\underline{\lim}_{|(x,\xi)|\to\infty} p(x,\xi) > b.$$

Then the operator $A(h)$ has only a discret spectrum on I and $\forall f \in C_0^\infty(I)$ we have

$$\boxed{\operatorname{tr} f(A(h)) = \sum_{\lambda_j(h)\in\sigma_{pp}(A(h))} f(\lambda_j) \leq C(f)(h^{-n})}$$

Moreover, we have Weyl asymptotics

$$N(\lambda;h) = \#\{\lambda_j(h) : a \leq \lambda_j(h) \leq \lambda < b\}$$

$$= c_0(\lambda)h^{-n} + \mathcal{O}(h^{-n+1}),$$

$$\boxed{N(\lambda+\epsilon;h) - N(\lambda-\epsilon;h) = \sum_{\lambda_j\in(a,b)} \delta_{\lambda_j}([\lambda+\epsilon,\lambda-\epsilon])}$$

Spectral shift function

Set $A_0(h) = -h^2\Delta$ and assume that

$$|\partial_\alpha V(x)| \leq C_\alpha(1 + |x|^2)^{-m/2},\ m > n,\ \forall \alpha.$$

Then $\sigma_{\text{ess}}(A(h)) = [0, +\infty[$. Let $I = (a, b) \subset \mathbb{R}^+$. We may introduce the spectral shift function $\xi(\lambda; h) \in \mathcal{D}'(\mathbb{R})$ as a distribution

$$\langle \xi', f\rangle = \text{tr}\Big(f(A(h)) - f(A_0(h))\Big),\ f \in C_0^\infty(\mathbb{R}).$$

Problems :
(1) Prove that $\xi'(\lambda, h)$ is a sum of *continuous measures* related to resonances $z_j \in \mathbb{C}_- +$ sum of *delta measures* related to embedded eigenvalues $\mu_j \in \mathbb{R}^+ +$ a regular measure.
(2) Find a (Breit-Wigner) representation of $\xi(\lambda + \delta; h) - \xi(\lambda - \delta; h)$ for $0 < \delta \leq h/C$.
(3) Establish a trace formula relating the trace $\text{tr}\Big((\psi f)(A(h)) - (\psi f)(A_0(h))\Big)$ to the sum $\sum_j f(z_j)$ over the resonances $z_j \in \Omega$ for f holomorphic in Ω and $\psi \in C_0^\infty(\mathbb{R}^+)$.
(4) Find a Weyl asymptotics of $\xi(\lambda, h)$.
(5) Examine the existence of resonances and the lower bounds on the number of resonances.

Compact perturbations

Let $A(h)$ be a compact perturbation of $A_0(h)$ (potentials, obstacles etc.) Consider the scattering operator $S(\lambda,h) = I + K(\lambda,h)$, $\lambda \in \mathbb{R}^+$, where K is a trace class operator. Let $\Omega = [a,b] + i[-c,c]$, $0 < a < b$, $c > 0$,

$$\Omega_\epsilon = \{z \in \mathbb{C} : \ \mathrm{dist}\,(z,\Omega) \leq \epsilon\}.$$

We can write

$$\boxed{\det S(\lambda,h) = e^{g(\lambda,h)} \frac{\prod_{z_j \in \Omega_\epsilon}(\lambda - \overline{z}_j)}{\prod_{z_j \in \Omega_\epsilon}(\lambda - z_j)}}$$

with z_j resonances in $\{z \in \mathbb{C} : \mathrm{Im}\, z < 0\}$ and $g(\lambda,h)$ holomorphic in $\Omega_{\epsilon/2}$.

Theorem 1 *(P.- Zworski, 2001). The derivative of the scattering phase*

$$s(\lambda,h) = \frac{1}{2\pi i} \log\det S(\lambda,h)$$

admits for $\lambda \in \Omega \cap \mathbb{R}$ *the representation*

$$\boxed{s'(\lambda,h) = \frac{1}{2\pi} \mathrm{Im}\, g'(\lambda,h) - \frac{1}{\pi} \sum_{z_j \in \Omega_\epsilon} \frac{\mathrm{Im}\, z_j}{|\lambda - z_j|^2}}$$

and $|g(\lambda,h)| \leq C(\Omega) h^{-n}$, $\lambda \in \Omega$.

Long range perturbations

Consider self-adjoint operators $L_j = L_j(h), j = 1, 2$ and assume that

$$L_j u = \sum_{|\nu| \le 2} a_{j,\nu}(x)(hD_x)^\nu u, \; u \in C_0^\infty(\mathbb{R}^n)$$

There exists $C > 0$ such that

$$l_{j,0}(x, \xi) = \sum_{|\nu|=2} a_{j,\nu}(x)\xi^\nu \ge C|\xi|^2, \qquad (1)$$

$$\sum_{|\nu| \le 2} a_{j,\nu}(x)\xi^\nu \longrightarrow |\xi|^2, \; |x| \longrightarrow \infty \qquad (2)$$

Suppose that for $m > n$, $|\nu| \le 2$ we have

$$\Big| a_{1,\nu}(x) - a_{2,\nu}(x) \Big| \le C(1 + |x|^2)^{-m/2} \qquad (3)$$

The *spectral shift function* $\xi(\lambda, h)$ is defined for $f(\lambda) \in C_0^\infty(\mathbb{R})$ by

$$< \xi'(\lambda, h), f(\lambda) >= \text{tr}\Big(f(L_2) - f(L_1) \Big).$$

There exist $\theta_0 \in]0, \frac{\pi}{2}[$, $\epsilon > 0$ and $R_1 > R_0$ so that the coefficients $a_{j,\nu}(x)$ of L_j can be extended holomorphically in x to

$$\Gamma = \{r\omega;\ \omega \in \mathbb{C}^n,\ \text{dist}\,(\omega, S^{n-1}) < \epsilon,$$

$$r \in \mathbb{C}, r \in e^{i[0,\theta_0]}]R_1, +\infty[\}$$

and (2), (3) extend to Γ. Next we define the resonances $w \in \overline{\mathbb{C}}_-$ by the complex scaling method as the eigenvalues of the complex scaling operators $L_{j,\theta}$, $j = 1, 2$. We consider a map $\kappa(\theta) : \mathbb{R}^n \ni t\omega \to f_\theta(t)\omega \in \mathbb{C}^n$, $t = |x|$,

$$f_\omega(t) = t,\ 0 \le t \le R_1,\ 0 \le \arg f_\theta(t) \le \theta,$$

$$\arg f_\theta(t) \le \arg \partial_t f_\theta \le \arg f_\theta(t) + \epsilon,$$

$$f_\theta(t) = e^{i\theta} t,\ t \ge T_0,\ \partial_t f_\theta \neq 0.$$

We change the variables and for $\text{Im}\,\theta > 0$, the operators $L_{j,\theta}$ become non-selfadjoint operators and

$$\dim \text{Ker}\,(L_{j,\theta} - z) < \infty$$

for $-2\theta < \text{Im}\, z \le 0$. Denote by $\text{Res}\, L_j(h)$, $j = 1, 2$, the set of resonances of $L_j(h)$.

Theorem 2 *(Bruneau - P., Dimassi - P., 2003)* *Under the above assumptions let*

$$\Omega \subset\subset e^{]-2\theta,2\theta[}]0,+\infty[,\ 0<\theta\leq\theta_0<\pi/2$$

be an open simply connected set and let $W \subset\subset \Omega$ *be an open simply connected relatively compact set which is symmetric with respect to* $\mathbb{R}$*. Assume that* $J=\Omega\cap\mathbb{R}^+$, $I=W\cap\mathbb{R}^+$ *are intervals. Then for* $\lambda\in I$ *we have*

$$\xi'(\lambda,h)=\frac{1}{\pi}\operatorname{Im} r(\lambda,h)+\Big[\sum_{\substack{w\in\operatorname{Res} L_j\cap\Omega,\\ \operatorname{Im} w\neq 0}}\frac{-\operatorname{Im} w}{\pi|\lambda-w|^2}$$

$$+\sum_{w\in\operatorname{Res} L_j\cap J}\delta(\lambda-w)\Big]_{j=1}^2,$$

where $[a_j]_{j=1}^2=a_2-a_1$, $r(z,h)$ *is a function holomorphic in* Ω *and* $r(z,h)$ *satisfies the estimate*

$$|r(z,h)|\leq C(W)h^{-n},\ \ z\in W.$$

Given $z\in\mathbb{C}$, $\operatorname{Im} z<0$, and a Borel set $J\subset\mathbb{R}=\partial C_-$ we have a harmonic measure

$$\omega(z;J)=\int_J\frac{-\operatorname{Im} z}{\pi|t-z|^2}dt.$$

Applications

- local trace formula of Sjöstrand, 1996

Theorem. Suppose that f is holomorphic on a neighborhood of Ω. Let $I = \Omega \cap \mathbb{R}$ and let $\psi \in C_0^\infty(\mathbb{R})$ satisfies

$$\psi(\lambda) = \begin{cases} 0, & d(I,\lambda) > 2\eta, \\ 1, & d(I,\lambda) < \eta, \end{cases}$$

where $\eta > 0$ is sufficiently small. Then

$$\mathrm{tr}\Big[(\psi f)(L_j(h))\Big]_{j=1}^{2}$$

$$= \Big[\sum_{z \in \, \mathrm{Res}\, P_j(h) \,\cap\, \Omega} f(z)\Big]_{j=1}^{2} + E_{\Omega,f,\psi}(h)$$

with

$$|E_{\Omega,f,\psi}(h)| \leq M(\psi,\Omega)$$

$$\times \sup\,\{|f(z)| \;:\; 0 \leq d(\Omega,z) \leq 2\eta\,,\ \mathrm{Im}\, z \leq 0\} h^{-n}\,.$$

- (W) Weyl type asymptotics for the spectral shift function $\xi(\lambda,h) = c(\lambda)h^{-n} + \mathcal{O}(h^{1-n})$.

• (BW) Breit-Wigner approximation

Let $0 < E_1 < E_2$ and suppose that each $\lambda \in [E_0, E_1]$ is a non-critical energy level for L_j, $j = 1, 2$.
(H1) There exist positive constants B, ϵ_1, C_1, h_1 such that for any $\lambda \in [E_0 - \epsilon_1, E_1 + \epsilon_1]$, $h/B \leq \delta \leq B$ and $h \in]0, h_1]$ we have

$$\xi(\lambda + \delta, h) - \xi(\lambda + \delta, h) \leq C_1 \delta h^{-n}$$

Theorem 3 *(Bruneau - P.) Assume* $(H1)$ *and suppose that* $L_j(h)$, $j = 1, 2$, *have no embedded eigenvalues in* $[E_1, E_2]$. *Then with* $B_1 > 0$ *we have*

$$\xi(\lambda + \delta, h) - \xi(\lambda - \delta, h)$$

$$= \Big[\sum_{\substack{w \in \mathrm{Res}\, L_j(h), \\ \mathrm{Im}\, w \neq 0,\ |w - \lambda| < h/B_1}} \int_{\lambda - \delta}^{\lambda + \delta} \frac{-\mathrm{Im}\, w}{\pi |t - w|^2} dt \Big]_{j=1}^{2} + \mathcal{O}(\delta) h^{-n},$$

for $0 < \delta \leq h/C$.

- **Estimates and asymptotics of** $M(\lambda, h)$

Let $\omega_{\mathbb{C}_-}(w, J) = \int_J \frac{-\operatorname{Im} w}{\pi |t-w|^2} dt$. Let $\Omega \subset \{\operatorname{Re} z > 0\}$ be a complex relatively compact neighborhood of $[E_0, E_1]$. Consider the function

$$M(\lambda, h) = \sum_{\substack{w \in \operatorname{Res} L,\, w \in \Omega, \\ \operatorname{Im} w \neq 0}} \omega_{\mathbb{C}_-}(w,]-\infty, \lambda])$$

$$+ \#\{\mu \in]-\infty, \lambda] \cap \Omega : \ \mu \in \operatorname{sp}_{pp} L(h)\}$$

Theorem 4 *(Bruneau - P.) Assume that each* $\lambda \in [E_0, E_1]$ *is a non-critical level for* $L(h)$. *Then the condition* $(H1)$ *is equivalent to the estimate*

$$|M(\lambda + \delta, h) - M(\lambda - \delta, h)| \leq C_2 \delta h^{-n}, \quad (4)$$

for $h/B_2 \leq \delta \leq B_2$, $\lambda \in [E_0, E_1]$.

Remark. The estimate (4) implies the result of J.-F. Bony:

$$\#\{w \in \operatorname{Res} L(h) : \ |w - \lambda| \leq \delta\} \leq C \delta h^{-n}.$$

Strategy

I. Representation of the derivative of SSF as a sum of measures

$\Rightarrow$ local trace formula in the spirit of Sjöstrand,

$\Rightarrow$ Weyl asymptotics of SSF,

$\Rightarrow$ weak Weyl asymptotics (H1).

II. Week Weyl asymptotics (H1)

$\Rightarrow$ Estimate for the number of the resonances $\#\{z \in \mathbb{C} : |z - \lambda| \leq Ch\} < C_1 h^{1-n}$,

$\Rightarrow$ Breit-Wigner approximaiton for $\xi(\lambda+\delta, h) - \xi(\lambda - \delta, h)$ with remainder $\mathcal{O}(\delta)h^{-n}$.

III. Local trace formula

$\Rightarrow$ Existence of resonances $\mathcal{O}(h^{-n})$ in every neighborhood W of the energy levels E such that a measure related to $V(x)$ has analytic singularity at E.

$\Rightarrow$ Existence of clusters of resonances related to the positive measure of the set of the periodic trajectories in the phase space and to some quantization conditions involving the Maslov index of the periodic trajectories. (Application of a Gutzwiller type trace fromula without any restriction on the periodic trajectories (Popov - P,)).

Gutzwiller trace formula for isolated periodic trajectories

Let $\lambda \in [E_0 - a, E_0 + a], a > 0$. Assume that for all such λ the Hamiltonian field H_p has an isolated non-degenerate periodic trajectory $\gamma(\lambda)$ with period $T(\lambda)$. Let

$$f(z) = \int e^{-it(z-\lambda)/h} g(t) e^{-(t-T(\lambda))^2 C \ln(1/h)/2} dt,$$

where $g(t) \in C_0^\infty(J)$, $g(t) = 1$ in a neighborhood of $T([E_0 - a, E_0 + a))$. Let $S(\gamma(\lambda)) = \int_{\gamma(\lambda)} \xi dx$ be the action, let $P_{\gamma(\lambda)}$ be the linear Poincaré map and let $\sigma(\gamma(\lambda))$ be the corresponding Maslov index. Finally, let $\chi \in C_0^\infty[E_0 - 3a, E_0 + 3a]$, $\chi = 1$ in a neighborhood of $[E_0 - 2a, E_0 + 2a]$ and $T^*(\gamma(\lambda))$ be the primitive period of $\gamma(\lambda)$.

Theorem 5 *(Robert, J.F.Bony) For* $\lambda \in [E_0 - a, E_0 + a]$ *and* h *small we have*

$$\mathrm{tr}[\chi^2(A_j(h)) f(A_j(h))]_{j=0}^{1} = \mathcal{O}(h \ln(1/h))$$

$$+ e^{iS(\gamma(\lambda))/h} e^{i\sigma(\gamma(\lambda))} T^*(\gamma(\lambda)) |\det(I - P_{\gamma(\lambda)})|^{-1/2}$$

with a remainder uniform with repsect to $\lambda \in [E_0 - a, E_0 + a]$ *and* C *bounded in* $\mathbb{R}$.

Let $[a(h), b(h)] \subset [E_0 - a, E_0 + a]$, $0 < a < 1/2$ and let $W \subset \mathbb{C}$ be defined by

$$\begin{cases} \text{Re}\, z \in [a(h) - C_0 h \ln(1/h), b(h) + C_0 h \ln(1/h)], \\ \text{Im}\, z \geq -\frac{h \ln(1/h)}{T(\text{Re}\, z)}\left(n - 1 + \frac{\ln((b(h)-a(h))}{\ln h}\right). \end{cases}$$

Theorem 6 *(J.F.Bony) Under the above assumptions we have with $\beta > 0$ the following lower bound*

$$\#(\text{Res} A_1(h) \cap W)$$

$$\geq \frac{1}{2\pi h} \int_{a(h)}^{b(h)} T^*(\lambda) |\det(I - P_{\gamma(\lambda)})|^{-1/2} d\lambda$$

$$-\mathcal{O}(h^{\beta-1})(b(h) - a(h)).$$

We have a generalization when we have finite number periodic trajectories $\gamma_j(\lambda)$ on $p(x, \xi) = \lambda$ with $S(\gamma_j(\lambda))$, $\sigma(\gamma_j(\lambda))$, $T^*(\gamma_j(\lambda))$ independent on $j = 1, ..., N$

Let E be a non-critical value of $p(x,\xi)$ and let the energy surface

$$\Sigma = \{(x,\xi) \in T^*(\mathbb{R}^n) : \, p(x,\xi) = E\}$$

be compact smooth hypersurface. Let $g(x, hD_x)$ be a h-pseudodifferentail operator representing $f(A(h))$. Let $\widehat{\rho}(t) \in C_0^\infty(\mathbb{R})$ and let $\widehat{\rho}(t)$ vanish in a neighborhood of 0. Let $0 < \delta \leq 1$.

Theorem 7 *(Popov -P., 1998) For any $|r| < r_0$ and $0 < h \leq h_0$ we have*

$$\text{tr} \int \text{exp}(ith^{-1}(E + rh))\widehat{\rho}(\delta t) g(x, hD_x)$$

$$\times exp\Big(-ith^{-1}A(h)\Big)g(x, hD_x)dt$$

$$= (2\pi h)^{1-n} \sum_{k \in Z \setminus \{0\}} \int_{\Pi} \widehat{\rho}(k\delta T^*(\nu))$$

$$\times \, \text{exp}\Big(ik(h^{-1}S(\nu) + rT^*(\nu) - \sigma(\nu))\Big)d\nu + o_\delta(h^{1-n}),$$

where Π is the set of absolutely periodic trajectories of H_p on Σ.

We have $\mu(\mathcal{P} \setminus \Pi) = 0$, where $\mathcal{P}$ is the set of periodic trajectories of H_p on Σ.

Gutzwiller trace formula without assumptions on periodic trajectories

Let E be a non-critical value of $p(x,\xi)$ and let the energy surface

$$\Sigma = \{(x,\xi) \in T^*(\mathbb{R}^n) : \, p(x,\xi) = E\}$$

be compact smooth hypersurface with Liouville measure $\mu(\Sigma)$.

Theorem 8 *(Popov -P., 1998) Suppose that $E < \lambda < \lambda_0$ and let E be a non-critical value of $p(x,\xi)$. Then for any function $\rho(\tau) \in \mathcal{S}(\mathbb{R})$ with Fourier transform $\widehat{\rho}(t) \in C_0^\infty(\mathbb{R})$ we have*

$$\sum_{\lambda_j(h)\le\lambda} \rho\Big(\frac{E-\lambda_j(h)}{h}\Big) = \widehat{\rho}(0)\frac{\mu(\Sigma)}{(2\pi)^n}h^{1-n} + h(2\pi h)^{-n}$$

$$\times \int_{\Pi} \sum_{k\in\mathbb{Z}\setminus\{0\}} \exp\Big(ik(h^{-1}S(\nu) - \sigma(\nu))\Big)\widehat{\rho}(kT^*(\nu))d\nu$$

$$+o_\rho(h^{1-n}),$$

where Π is the set of absolutely periodic trajectories of H_p on Σ.

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