ESTIMATES FOR THE CUT-OFF RESOLVENT OF THE LAPLACIAN FOR TRAPPING OBSTACLES

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1. Introduction

Let $K \subset \mathbb{R}^n,\ n \geq 2$ be a bounded domain with C^∞ boundary ∂K and connected complement $\Omega = \mathbb{R}^n \setminus \overline{K}$. The set K is called an *obstacle* in $\mathbb{R}^n$. We consider the Dirichlet problem for the wave equation

$$\begin{cases} (\partial_t^2 - \Delta_x)u = 0 \ \text{ in } \mathbb{R} \times \Omega, \\ u = 0 \ \text{ on } \mathbb{R} \times \partial K, \\ u(0,x) = f_0(x),\ \partial_t u(0,x) = f_1(x). \end{cases} \tag{1}$$

Let $K \subset B_a = \{x \in \mathbb{R}^n :\ |x| \leq a\}$ and for $m \geq 0$ set

$$p_m(t) = \sup\Big[\frac{\|\nabla_x u\|_{L^2(B_a\cap\Omega)} + \|\partial_t u\|_{L^2(B_a\cap\Omega)}}{\|\nabla_x f_0\|_{H^m(B_a\cap\Omega)} + \|f_1\|_{H^m(B_1\cap\Omega)}};$$

$$(0,0) \neq (f_0, f_1) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega),\ \operatorname{supp} f_i \subset B_a,\ i = 0,1\Big].$$

For $\operatorname{Im}\lambda > 0$ consider the cut-off resolvent $R_\chi(\lambda) = \chi(-\Delta_D - \lambda^2)^{-1}\chi :\ L^2(\Omega) \longrightarrow L^2(\Omega)$, where $\chi \in C_0^\infty(\mathbb{R}^n),\ \chi = 1$ on B_a and Δ_D is the Dirichlet Laplacian in Ω. The behavior of $R_\chi(\lambda)$ on the real axis is closely related to the decay of the local energy $p_m(t)$ as $t \longrightarrow +\infty$. The following result of Vodev generalizes the classical one of Morawetz [13].

Theorem 1. ([23]) *The following conditions are equivalents:*
(a) $\lim_{t\to+\infty} p_0(t) = 0$,
(b) *There exist* $C_0 > 0,\ C_1 > 0$ *so that*

$$\|\lambda R_\chi(\lambda)\| \leq C_1, \lambda \in \mathbb{R},\ |\lambda| \geq C_0,$$

(c) *There exist constants* $C,\ \gamma > 0$ *such that for* $t \geq 1$ *we have*

$$p_0(t) \leq \begin{cases} Ce^{-\gamma t},\ n \text{ odd}, \\ Ct^{-n}, n \text{ even}. \end{cases}$$

It is known that (a) holds if the obstacle K is non-trapping, which means that the singularities of the solution of the Dirichlet problem with initial data with compact support leave any compact $\omega \subset \Omega$ for $t \geq t(\omega)$. For trapping obstacles without any condition on the geometry of the obstacle N. Burq proved the following

Theorem 2. ([5]) *There exist constants* $C > 0$ *and* $C_0 > 0$ *so that*

$$\|R_\chi(\lambda)\| \leq Ce^{C|\lambda|},\ \lambda \in \mathbb{R},\ |\lambda| \geq C_0$$

and for every integer $m > 1$ we have

$$p_m(t) \leq \frac{C_m}{(\log t)^m},\ t > 1. \tag{2}$$

It is well known that the cut-off resolvent $R_\chi(\lambda)$ has a meromorphic continuation in $\mathbb{C}$ for n odd and in $\mathbb{C}' = \{z \in \mathbb{C} : z \neq -i\mu,\ \mu \in \mathbb{R}^+\}$ for n even. There are many examples when we have a domain

$$D_\delta = \{z \in \mathbb{C} :\ -\delta \leq \operatorname{Im} z \leq 0\},\ \delta > 0$$

without poles (resonances) of $R_\chi(\lambda)$ (see for example [9]). We will show that in this case we have a polynomial bound of the cut-off resolvent $R_\chi(\lambda)$ on $\mathbb{R}$ and a better local energy decay than (2).

A general result says that if the generalized Hamiltonian flow introduced in [12] is *continuous* and if we have *at least* one trapping ray γ, then the condition (b) in Theorem 1 fails and we have

$$\sup_{\lambda \in \mathbb{R}} \|\lambda R_\lambda(\lambda)\|_{L^2(\Omega) \to L^2(\Omega)} = +\infty. \tag{3}$$

This condition (3) is too week and we have no information on the geometry of K outside a small neighborhood of the ray γ. Nevertheless, this condition implies some interesting spectral properties of the Lax-Phillips semigroup $Z(t)$ and we discuss in Sections 3 and 4 this question.

2. Estimates of $R_\chi(\lambda)$ and local energy decay

Under the hypothesis that there exists a resonances free region we obtain the following

Theorem 3. *Assume that the cut-off resolvent $R_\chi(\lambda)$ has no poles in the domain* $\operatorname{Im} \lambda \geq -\delta$, $\delta > 0$. *Then*

$$\|R_\chi(\lambda)\|_{L^2 \to L^2} \leq C|\lambda|^{n-2},\ \lambda \in \mathbb{R}, |\lambda| \geq C_0. \tag{4}$$

The proof is based on a semiclassical approach. Setting $\lambda = \frac{\sqrt{z}}{h}$, $0 < h \leq 1$, we have

$$(-\Delta_D - \lambda^2)^{-1} = h^2(-h^2\Delta_D - z)^{-1}$$

and we study the operator $\chi(P(h) - z)^{-1}\chi$ with $P(h) = -h^2\Delta_D,\ h > 0$, in the domain

$$\mathcal{D}_{a,c} = \{z \in \mathbb{C} :\ a^{-1} < |\operatorname{Re} z| < a,\ -ch < \operatorname{Im} z < c,\ a > 2,\ c > 0\}.$$

We will work in the “black box” setup ([19], [20]). For this purpose define $\mathcal{H}_R = L^2(\Omega \cap B_R)$ and set

$$\mathcal{L} = \mathcal{H}_R \oplus L^2(\mathbb{R}^n \setminus B_R).$$

We consider $P(h)$ as an operator $P(h) : \mathcal{L} \longrightarrow \mathcal{L}$ with domain $\mathcal{D}(P) \subset \mathcal{L}$ and the hypothesis in [19], [20] for a “black box” framework are satisfied. In particular, setting

$$\mathcal{H}^\sharp = \mathcal{H}_R \oplus L^2(\mathbb{T}^n_R \setminus B_1),\ \mathbb{T}^n_R = \mathbb{R}^n/(R\mathbb{Z}^n),$$

we introduce $P^\sharp(h)$ by replacing $-h^2\Delta_D$ by $-h^2\Delta_{\mathbb{T}^n_R}$. The operator $P^\sharp(h)$ has a discrete spectrum and we denote by $N(P^\sharp(h), \lambda)$ the number of eigenvalues of $P^\sharp(h)$ in $[-\lambda, \lambda]$. Then we have

$$N(P^\sharp(h), \lambda) = \mathcal{O}\Big(\Big(\frac{\lambda}{h^2}\Big)^{n/2}\Big),\ \text{for } \lambda \geq 1.$$

We examine the resolvent of the complex dilated operator $P_\theta(h)$ (see [19]) and we take $\theta = \widetilde{c}h$, $\widetilde{c} \gg c$ so that in the domain

$$\Omega_{\omega,c} = \{z \in \mathbb{C} :\ |z - \omega| \leq \theta,\ -ch \leq \operatorname{Im} z \leq c\} \subset \mathcal{D}_{a,c},$$

for $a^{-1} < |\operatorname{Re}\omega| < a$, there are no eigenvalues of P_θ. Note that the eigenvalues of P_θ coincide with their multiplicities with the resonances of P ([19], [20]). By using the construction of a suitable finite rank perturbation (see [1]) and a Grushin type operator we show that

$$\|(P_\theta - z)^{-1}\| \leq C_1 e^{C_2 h^{-n+1}},\ z \in \Omega_{\omega, c/2}.$$

This estimate is uniform with respect to the choice of ω with $2a^{-1} < |\operatorname{Re}\omega| < a/2$. Thus we obtain

$$\|(P_\theta - z)^{-1}\| \leq C_3 e^{C_3 h^{-n+1}},\ z \in \mathcal{D}_{a/2, c/2}.$$

The complex scaling is chosen so that for $\operatorname{supp}\chi \subset B_{R+1}$ we have

$$\chi(P-z)^{-1}\chi = \chi(P_\theta - z)^{-1}\chi,$$

hence

$$\|\chi(-h^2\Delta_D - z)^{-1}\chi\|_{L^2(\Omega)\longrightarrow L^2(\Omega)} \leq C_4 e^{C_4 h^{-n+1}},\ z \in \mathcal{D}_{a/2,c/2}.$$

Taking into account $\lambda = \frac{\sqrt{z}}{h}$, for $z \in \mathcal{D}_{a/2,c/2}$ we get

$$\|R_\chi(\lambda)\| \leq C_5 e^{C_5|\lambda|^{n-1}},\ \operatorname{Re}\lambda \geq b,\ \operatorname{Im}\lambda \geq -b,\ b > 0. \tag{5}$$

In the same way we treat the domain $\operatorname{Re}\lambda \leq -b,\ \operatorname{Im}\lambda \geq -b$ and we obtain (5) for $|\operatorname{Re}\lambda| \geq b > 0$. To establish the estimate (4) on $\mathbb{R}$, we apply Pragmen-Lindelöf theorem to prove the following

Proposition 1. *Let $f(z)$ be a holomorphic function in*

$$U_\alpha = \{z \in \mathbb{C}:\ \operatorname{Im} z \geq -\alpha\},\ \alpha > 0,$$

such that

$$|f(z)| \leq A_0 e^{A|z|^m},\ z \in U_\alpha,\ m \geq 1,$$

$$|f(z)| \leq \frac{A_1}{|z|\operatorname{Im} z},\ \operatorname{Im} z > 0.$$

Then we have $|f(z)| \leq A_2(1+|z|)^{m-1},\ z \in \mathbb{R}$.

In our case, the resolvent $R(z) = (-\Delta_D - z^2)^{-1}$ of the positive operator $-\Delta_D$ satisfies the estimate

$$\|R(z)\|_{L^2\to L^2} \leq \frac{C}{|z|\operatorname{Im} z},\ \operatorname{Im} z > 0$$

and using Proposition 1 we complete the proof of Theorem 3. We refer to [4] for more details.

Remark 1. Notice that if for some $M \geq 0$ we have the estimate

$$\|R_\chi(\lambda)\|_{L^2\to L^2} \leq C_1(1+|\lambda|)^M,\ \operatorname{Im}\lambda \geq -\delta, |\operatorname{Re}\lambda| \geq C_0,$$

then a result of N. Burq [8] says that

$$\|R_\chi(\lambda)\|_{L^2\to L^2} \leq C_2 \frac{\log(2+|\lambda|^2)}{|\lambda|}, \lambda \in \mathbb{R}, |\lambda| \geq C_0.$$

In particular, a such estimate holds for two strictly convex disjoint obstacles and under some conditions for several strictly convex disjoint obstacles (see [9]). It is interesting to examine the question if it is possible under the hypothesis of Theorem 3 to obtain an estimate of $R_\chi(\lambda)$ on $\mathbb{R}$ independent on the dimension n. For the semiclassical Schrödinger operators $-h^2\Delta + V(x)$ in the case of dimension 1 a polynomial bound $\mathcal{O}(h^{-M})$ of the cut-off resolvent in

$$W = \{z \in \mathbb{C} : 0 < a_0 \leq \operatorname{Re} z \leq a_1,\ \operatorname{Im} z \geq -a_2 h, a_i > 0, i = 0,1,2\}$$

has been obtained in [2], provided that we have no resonances in W.

Theorem 3 combined with a result of G.Popov and G. Vodev [17] leads to the following

Theorem 4. *Under the hypothesis of Theorem 3 for every $m > 0$ and $t > 1$ we have for n odd the estimate*
$$p_m(t) \leq C(t^{-1}\log t)^{m/(n-1)},$$
while for n even and $t > 1$ we have
$$p_m(t) \leq \begin{cases} C(t^{-1}\log t)^{m/(n-1)}, \text{ for } 0 < m \leq n(n-1), \\ Ct^{-n} \text{ for } m > n(n-1). \end{cases}$$

3. Spectrum of the semigroup $Z(t)$

Throughout this and the following sections we assume $n \geq 3$, n odd, and we examine the spectrum of the Lax-Phillips semigroup $Z^b(t) = P_+^b U(t) P_-^b$, $t \geq 0$, where $U(t) = e^{itG}$ is the unitary group related to the Dirichlet problem for the wave equation in Ω and $P_\pm^a$ are the orthogonal projections on the orthogonal complements of the spaces
$$D_\pm^b = \{f \in \mathcal{H} : U_0(t)f = 0,\ |x| < \pm t + b, \pm t > 0\}.$$
Here $b > a$ and $U_0(t)$ is the unitary group related to the Cauchy problem for the wave equation in $\mathbb{R}_t \times \mathbb{R}^n$. We choose $\chi \in C_0^\infty(\mathbb{R}^n)$ so that $\chi = 1$ for $|x| \leq a$, $\chi = 0$ for $|x| \geq b$. We fix $b > a$ with this property and note that $P_\pm^b \chi = \chi = \chi P_\pm^b$. For simplicity of the notations we will write $Z(t)$ instead of $Z^b(t)$ and $P_\pm$ instead of $P_\pm^b$. Let B be the generator of $Z(t)$. Therefore,
$$\sigma(B) \subset \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$$
and the eigenvalues z_j of iB coincide with their multiplicities with the poles of $R_\chi(\lambda)$ (see [14]).

The condition (3) implies
$$\sup_{\lambda \in \mathbb{R}} \|(B + i\lambda)^{-1}\|_{\mathcal{H} \to \mathcal{H}} = +\infty. \tag{6}$$
In fact, for $\operatorname{Re}\lambda > 0$ we have
$$\chi(iG - \lambda)^{-1}\chi = -\int_0^\infty e^{-\lambda t}\chi e^{itG}\chi dt = \chi(B - \lambda)^{-1}\chi.$$
By analytic continuation for $\operatorname{Re}\lambda \geq 0$ we obtain
$$\chi(iG + i\lambda)^{-1}\chi = \chi(B + i\lambda)^{-1}\chi,\ \forall \lambda \in \mathbb{R}$$
and we may exploit the representation
$$(G - \lambda)^{-1} = \begin{pmatrix} \lambda R(\lambda) & -iR(\lambda) \\ -i\Delta_D R(\lambda) & \lambda R(\lambda) \end{pmatrix}.$$

The condition (3) is typical for trapping obstacles. To make a precise definition we must consider the generalized bicharacteristics of the wave operator $\Box = \partial_t^2 - \Delta_x$ determined as the trajectories of the generalized Hamiltonian flow $\mathcal{F}_t$ in Ω generated by the symbol $\sum_{i=1}^n \xi_i^2 - \tau^2$ of $\Box$ (see [12] for a precise definition). In general, $\mathcal{F}_t$ is not smooth and in some cases there may exist two different integral curves issued from the same point in the phase space. To avoid this situation we introduce the following generic condition

($\mathcal{G}$) If for $(x, \xi) \in T^*(\partial K)$ the normal curvature of ∂K vanishes of infinite order in direction ξ, then ∂K is convex at x in direction ξ.

Assuming $(\mathcal{G})$, given $\sigma = (x, \xi) \in T^*(\Omega) \setminus \{0\} = \dot{T}^*(\Omega)$, there exists a unique generalized bicharacteristic $(x(t), \xi(t)) \in \dot{T}^*(\Omega)$ such that $x(0) = x$, $\xi(0) = \xi$ and we define $\mathcal{F}_t(x, \xi) = (x(t), \xi(t))$ for all $t \in \mathbb{R}$ (see [12]). We obtain a flow $\mathcal{F}_t : \dot{T}^*(\Omega) \longrightarrow \dot{T}^*(\Omega)$ which is called the *generalized geodesic flow* on $\dot{T}^*(\Omega)$. The flow $\mathcal{F}_t$ is discontinuous at points of transversal reflection at $\dot{T}^*_{\partial K}(\Omega)$ and to make it continuous, consider the *quotient space* $\dot{T}^*(\Omega)/\sim$ of $\dot{T}^*(\Omega)$ with respect to the following equivalence relation: $\rho \sim \sigma$ if and only if $\rho = \sigma$ or $\rho, \sigma \in T^*_{\partial K}(\Omega)$ and either $\lim_{t \nearrow 0} \mathcal{F}_t(\rho) = \sigma$ or $\lim_{t \searrow 0} \mathcal{F}_t(\rho) = \sigma$. Let Σ_b be the image of $S^*(\Omega)$ in $\dot{T}^*(\Omega)/\sim$. The set Σ_b is called the *compressed characteristic set*. Melrose and Sjöstrand ([12]) proved that the natural projection of $\mathcal{F}_t$ on $\dot{T}^*(\Omega)/\sim$ is continuous. Thus if $(\mathcal{G})$ holds, the *compressed Hamiltonian flow* is continuous.

Proposition 2. *If the generalized compressed Hamiltonian flow is continuous and if we have at least one (generalized) trapping ray the condition* (3) *is fulfilled.*

Proof. Our hypothesis imply the existence of a sequence of ordinary reflecting rays γ_n with sojourn times $T_{\gamma_n} \to \infty$ (see for instance [12], [15]) and we may apply the result of Ralston [18] which says that the condition (a) of Theorem 1 is not fulfilled.

In the following we suppose the condition (3) fulfilled. Assume that there are only finite number of resonances in the domain

$$\{z \in \mathbb{C} : \operatorname{Im} z \geq -\delta\},\ \delta > 0.$$

Choose $0 \leq \alpha \leq \delta$ so that we have no resonances on the line $\{z \in \mathbb{C} : \operatorname{Im} z = -\alpha\}$. Then the resolvent $(B + \alpha + i\lambda)^{-1}$ exists for every $\lambda \in \mathbb{R}$ and it is easy to see that

$$\sup_{\lambda \in \mathbb{R}} \|(B + \alpha + i\lambda)^{-1}\|_{\mathcal{H} \to \mathcal{H}} = +\infty. \tag{7}$$

Indeed, if the resolvent $(B + \alpha + i\lambda)^{-1}$ is uniformly bounded with respect to $\lambda \in \mathbb{R}$, the cut-off resolvent $\|\lambda R_\chi(-i\alpha + \lambda)\|_{L^2 \to L^2}$ will be also bounded uniformly with respect to $\lambda \in \mathbb{R}$. Consider the domain

$$\{z \in \mathbb{C} : -\alpha \leq \operatorname{Im} z \leq c_0,\ |\operatorname{Re} z| \geq c_1,\ c_i > 0,\ i = 0, 1\}$$

with sufficiently large c_1. For each z in this domain we have the estimate (see [22])

$$\|z R_\chi(z)\|_{L^2(\Omega) \to L^2(\Omega)} \leq C e^{C|z|^{n+1}}$$

and an application of the Pragmen-Lindelöf theorem leads to a contradiction with (3). Next, assume that

$$e^{-\alpha - i\beta} \notin \sigma(e^B),\ \forall \beta \in \mathbb{R}.$$

Then $\|(e^{-\alpha - i\beta} - e^B)^{-1}\| \leq C_\alpha$, $\forall \beta \in \mathbb{R}$ and we deduce

$$(B + \alpha + i\beta)^{-1} = -\int_0^1 e^{t(B + \alpha + i\beta)} dt (I - e^{B + \alpha + i\beta})^{-1}.$$

Consequently, the resolvent $(B + \alpha + i\beta)^{-1}$ is uniformly bounded with respect to $\beta \in \mathbb{R}$ and we obtain a contradiction with (7).

This shows that there exists $\beta_0 \in \mathbb{R}$ such that

$$e^{-\alpha - i\beta_0} \in \sigma(e^B) \setminus e^{\sigma(B)}.$$

Now we are in position to apply the result of I. Herbst [11] saying that there exists a set $\mathcal{M}_\alpha \subset \mathbb{R}^+$ with Lebesgue measure zero so that for all $t \in]0, \infty[\backslash \mathcal{M}_\alpha$ we have

$$e^{t(-\alpha - i\beta_0)} e^{i\omega} \in \sigma(Z(t)),\ \forall \omega \in \mathbb{R},$$

hence

$$e^{-\alpha t + i\omega} \in \sigma(Z(t)),\ \forall \omega \in \mathbb{R},$$

where $\sigma(Z(t))$ denotes the spectrum of $Z(t)$.

Assume that for $\frac{p_n}{q_n} \in \mathbb{Q}$, $0 < \frac{p_n}{q_n} \leq \delta$, we have no resonances on the line

$$\{z \in \mathbb{C} :\ \operatorname{Im} z = -\frac{p_n}{q_n}\}.$$

The above argument implies the existence of a set $\mathcal{M}_n \subset \mathbb{R}^+$ with Lebesgue measure zero such that for $t \in]0, \infty[\backslash \mathcal{M}_n$ we have

$$e^{-t \frac{p_n}{q_n} + i\omega} \in \sigma(Z(t)).$$

The rationals are dense in $]0, \delta[$ and the spectrum $\sigma(Z(t))$ is closed. Thus for

$$t \in]0, \infty[\backslash \Big(\bigcup_{n \in \mathbb{N}} \mathcal{M}_n \Big)$$

we get the relation

$$\{z = e^{-ty + i\omega} \in \sigma(Z(t)) :\ 0 \leq y \leq \delta,\ \omega \in \mathbb{R}\}.$$

Thus we have proved the following

Theorem 5. *Suppose the condition* (3) *fulfilled. Assume that we have only a finite number of resonances z with* $\operatorname{Im} z \geq -\delta$, $\delta > 0$. *Then there exists a set $\mathcal{R} \subset \mathbb{R}^+$ with Lebesgue measure zero so that for all $t \in]0, \infty[\backslash \mathcal{R}$ we have*

$$\{z \in \mathbb{C} : e^{-t\delta} \leq |z| \leq 1\} \subset \sigma(Z(t)).$$

Moreover, if for all $\delta > 0$ we have only a finite number of resonances z with $\operatorname{Im} z \geq -\delta$, *then there exists $\mathcal{M}$ with Lebesgue measure zero so that for all $t \in]0, \infty[\backslash \mathcal{M}$ we have*

$$\{z \in \mathbb{C} :\ |z| \leq 1\} = \sigma(Z(t)). \tag{8}$$

Remark 2. The argument used above shows that if the condition (3) holds, then for almost all $t \in \mathbb{R}^+$ we have

$$\mathbb{S}^1 \subset \sigma(Z(t)).$$

Moreover, this relation is true without any hypothesis on the distribution of the resonances and on the geometry of K.

Remark 3. The above theorem shows that if (3) holds, we have at least one of the following properties:

(i) For some $\delta > 0$ we have infinite number of resonances in the domain $\{z \in \mathbb{C} : \operatorname{Im} z \geq -\delta\}$.

(ii) For $t \in]0, \infty[\backslash \mathcal{M}$ we have (8).

The condition (i) is known as the modified Lax-Phillips conjecture.

It is interesting to see that in some cases both properties (i) and (ii) are satisfied.

Theorem 6. *Suppose the condition* (3) *fulfilled and let* $|\operatorname{Im}\lambda_j| \geq \epsilon > 0$ *for all resonances* λ_j. *Assume that for every* $\alpha > 0$ *and all* $r \in \mathbb{R}$ *the resonances counted with their multiplicities satisfy the estimate*

$$\sharp\{z \in \operatorname{Res}(-\Delta_D) : r \leq \operatorname{Re} z \leq r+1, \operatorname{Im} z \geq -\alpha\} \leq C_\alpha \tag{9}$$

with $C_\alpha > 0$ *depending only on* α. *Then there exists* $\mathcal{M}$ *with Lebesgue measure zero so that for all* $t \in]0, \infty[\backslash \mathcal{M}$ *we have* (8). *In particular,* (8) *holds if* $K = K_1 \cup K_2$, *where* $K_1 \cap K_2 = \emptyset$ *and* K_i, $i = 1, 2$, *are strictly convex obstacles.*

The proof is based on the construction of a holomorphic function $f_\alpha(z)$ such that $f_\alpha(z)$ has as zeros with their multiplicities the resonances lying in $\{z :\ \operatorname{Im} z \geq -\alpha\}$ and, moreover,

$$|f_\alpha(z)| \leq A_\alpha,\ |\operatorname{Im} z| \leq \alpha,$$
$$|f_\alpha(z)| \geq B_\alpha > 0,\ z \in \mathbb{R}.$$

Let $\{\lambda_j\}_{j\in\mathbb{Z}} \in \mathbb{C}$ counted with their multiplicities be such that $-\alpha \leq \operatorname{Im}\lambda_j \leq \beta$, $\alpha > 0, \beta > 0$, and assume that with some integer $N = N(\alpha, \beta)$, depending on α, β, we have

$$\sharp\{\lambda_j :\ \operatorname{Re}\lambda_j \in [x, x+1],\ -\alpha \leq \operatorname{Im}\lambda_j \leq \beta\} \leq N,\ \forall x \in \mathbb{R}. \tag{10}$$

Given $A > 0$, define the function

$$f(z) = \prod_{|\operatorname{Im}\lambda_j| \leq \alpha} \Big(\frac{z-\lambda_j}{z-\lambda_j - Ai}\Big) \exp\Big(-\frac{Ai}{z-\lambda_j-Ai}\Big).$$

Lemma 1. *For all* $M > 0$ *and* $A > M + \alpha$ *we have* $|f(z)| = \mathcal{O}_{A,M}(1)$ *in the domain*

$$D_M = \{z \in \mathbb{C} :\ |\operatorname{Im} z| \leq M\}.$$

Proof. For $z \in D_M$ we have $\operatorname{Im}(z - \lambda_j - Ai) < M + \alpha - A < 0$ and

$$\Big|\frac{Ai}{z-\lambda_j-Ai}\Big| \leq C_{A,M}. \tag{11}$$

Then

$$\begin{aligned} f(z) =& \prod_j \Big(1 + \frac{Ai}{z-\lambda_j-Ai}\Big)\Big[1 - \frac{Ai}{z-\lambda_j-Ai} + \mathcal{O}\Big(\Big(\frac{Ai}{z-\lambda_j-Ai}\Big)^2\Big)\Big] \\ =& \prod_j \Big(1 + \mathcal{O}\Big(\Big(\frac{Ai}{z-\lambda_j-Ai}\Big)^2\Big)\Big). \end{aligned}$$

We deduce

$$\begin{aligned} |f(z)| \leq& \exp\Big(\sum_j \Big(\frac{C}{|z-\lambda_j-Ai|^2}\Big) \\ =& \exp\Big(\sum_{k=0}^{\infty} \sum_{k \leq |\operatorname{Re}(z-\lambda_j)| < k+1} \frac{C}{|z-\lambda_j-Ai|^2}\Big) \\ \leq& \exp\Big(\sum_{|\operatorname{Re}(z-\lambda_j)|<1} \Big(\frac{C}{|z-\lambda_j-Ai|^2}\Big) + \sum_{k=1}^{\infty} \frac{2NC}{k^2}\Big) = \mathcal{O}(1), \end{aligned}$$

where we have used (11) and we denote by C different positive constants which may change from line to line.

Lemma 2. *For all $M > 0, \eta > 0$ and $A > M + \alpha$, we have $|f(z)|^{-1} = \mathcal{O}_{A,M,\eta}(1)$ in the domain*

$$W_\eta = \{z \in \mathbb{C} : |\operatorname{Im} z| \leq M\} \setminus \bigcup_j B(\lambda_j, \eta).$$

Proof. We have

$$\begin{aligned}
f(z)^{-1} =& \prod_j \Big(\frac{z - \lambda_j - Ai}{z - \lambda_j} \Big) \exp\Big(\frac{Ai}{z - \lambda_j - Ai} \Big) \\
=& \prod_j \Big(1 - \frac{Ai}{z - \lambda_j} \Big) \Big[1 + \frac{Ai}{z - \lambda_j - Ai} + \mathcal{O}\Big(\Big(\frac{Ai}{|z - \lambda_j - Ai|} \Big)^2 \Big) \Big] \\
=& \prod_j \Big(1 + \mathcal{O}\Big(\frac{1}{|z - \lambda_j - Ai||z - \lambda_j|} + \frac{1}{|z - \lambda_j - Ai|^2} \Big) \Big) \\
=& \prod_j \Big(1 + \mathcal{O}\Big(\frac{1}{|z - \lambda_j|^2} + \frac{1}{|z - \lambda_j - Ai|^2} \Big) \Big).
\end{aligned}$$

Consequently,

$$\begin{aligned}
|f(z)|^{-1} \leq& \exp\Big(\sum_j \frac{C}{|z - \lambda_j - Ai|^2} \Big) \exp\Big(\sum_j \frac{C}{|z - \lambda_j|^2} \Big) \\
=& \mathcal{O}(1) \exp\Big(\sum_{|\operatorname{Re}(z - \lambda_j)| \leq 1} \frac{C}{|z - \lambda_j|^2} + \sum_{k=1}^{\infty} \sum_{k \leq |\operatorname{Re}(z - \lambda_j)| < k+1} \frac{C}{|z - \lambda_j|^2} \Big) = \mathcal{O}_{A,M,\eta}(1),
\end{aligned}$$

since $|z - \lambda_j| > \eta$ on W_η. $\square$

To obtain (8), we must show that the function $zR_\chi(z)$ is not bounded on every line $\operatorname{Im} z = -\alpha$ on which we have no resonances and to repeat the argument of the proof of Theorem 5. To do this, assume that the operator-valued function $zR_\chi(z)$ is bounded for $\operatorname{Im} z = -\alpha < 0$. Clearly, this function is also bounded for $\operatorname{Im} z = \beta > 0$. Consider in

$$\mathcal{D}_{\alpha,\beta} = \{z \in \mathbb{C} : -\alpha \leq \operatorname{Im} z \leq \beta\}$$

the holomorphic function $g(z) = zR_\chi(z) f_\alpha(z)$. We know (see [22]) that for

$$z \in \mathcal{D}_{\alpha,\beta} \setminus \bigcup_j B(\lambda_j; \eta)$$

we have the estimate

$$\|zR_\chi(z)\| \leq C_\eta e^{C_\eta |z|^{n+1}},\ \eta > 0.$$

By the maximum principle we deduce

$$\|g(z)\| \leq Ce^{C|z|^{n+1}},\ \forall z \in \mathcal{D}_{\alpha,\beta}.$$

An application of the Pragmen-Lindelöf theorem in $\mathcal{D}_{\alpha,\beta}$ yields $\|g(z)\| \leq B_{\alpha,\beta}$ and for $z \in \mathbb{R}$ we get

$$\|zR_\chi(z)\| \leq B'_{\alpha,\beta}$$

which is a contradiction with (3). In the case of two strictly convex disjoint obstacles the hypothesis (10) follows form the results of C. Gérard [10]. In particular, for every fixed $\alpha > 0$ the resonances in the domain $\operatorname{Im} \geq -\alpha$ have multiplicities bounded by an integer $m_\alpha \in \mathbb{N}$ depending only on α.

On the other hand, it is interesting to mention that the existence of a holomorphic function $F(z)$ with the properties given in Lemmas 1 and 2 implies a restriction on the distribution of the zeros of $F(z)$ and hence on that of the resonances. More precisely, if we have

$$|F(z)| \leq C_{\alpha,\beta},\ -\alpha \leq \operatorname{Im} z \leq \beta,$$

$$|F(z)| \geq c_0 > 0,\ \forall z \in \mathbb{R},$$

we obtain by the Jensen formula for $r_0 \in \mathbb{R}$, $0 < R \leq \min\{\alpha, \beta\}$ and $0 < \delta < 1$ the estimate

$$\sharp\{z \in \mathbb{C} :\ F(z) = 0,\ |z - r_0| \leq \delta R\} \leq \frac{1}{\log \frac{1}{\delta}} \log \frac{C_{\alpha,\beta}}{c_0}$$

and this condition is uniform with respect to r_0 and R. Consequently, we get a restriction equivalent to (10).

In order to cover some cases when we have a different distribution of resonances and (10) fails it seems convenient to search a function $G(z)$ holomorphic and bounded in the domain

$$\mathcal{J} = \{z \in \mathbb{C} :\ -\alpha \leq \operatorname{Im} z \leq \frac{C}{|z|}\}$$

such that

$$|G(z)| \geq c_0 > 0,\ \ \forall z \in \mathbb{R},$$

provided that the resonances $z \in \operatorname{Res}(-\Delta_D)$ with $\operatorname{Im} z \geq -\alpha$ are between the zeros of $G(z)$. In fact, if a such function exists, we may consider $g(z) = G(z) z R_\chi(z)$ in $\mathcal{J}$ and apply the above argument since $zR_\chi(z)$ is bounded on the line $\{z :\ \operatorname{Im} z = C/|z|,\ |\operatorname{Re} z| > C_0 > 0,\ C > 0\}$. The function

$$h(z) = e^{-iAz^2} - 1/2,\ A > 0$$

is an example of functions having the first two properties. It is easy to see that $h(z)$ has a sequence of zeros converging to the real axis. Moreover, the zeros of $h(z)$ have not the property (10).

4. Singularities of the cut-off resolvent $\chi(U(t) - z)^{-1}\chi$

In the analysis of the resonances for time-periodic perturbations of the wave equations [3], [16] the analytic properties of the cut-off resolvent $\chi(U(T) - z)^{-1}\chi$ of the *monodromy* operator $U(T) = U(T, 0)$ play an essential role. Here $T > 0$ is the period of the perturbation and $U(t, s)$ denotes the propagator of the corresponding problem. For stationary obstacles K we can consider $U(t)$ instead of $U(T)$ since the obstacle is periodic with any $t > 0$. Consider the cut-off resolvent

$$\chi(U(t) - z)^{-1}\chi,$$

where $\chi \in C_0^\infty(\Omega)$ is equal to 1 on K. If K is non-trapping, the operator $Z(t)$ is compact for $t \geq t_0 > 0$ (see [14]) and this implies that $\chi(U(t) - z)^{-1}\chi$ for $t \geq t_0$ has a meromorphic continuation from

$$\{z \in \mathbb{C} : |z| > \beta > 1\}\ \text{to}\ \{z \in \mathbb{C} : |z| \leq \beta\}.$$

For trapping obstacles satisfying the condition (3) the situation is dramatically different. Let $\Psi \in C_0^\infty(|x| \leq b + 1)$, $\Psi = 1$ for $|x| \leq b$, where $b > a$ is large and fixed.

Theorem 7. *Assume the condition* (3) *fulfilled. Then for almost all* $t \in]0, \infty[$ *and all* $z_0 \in \mathbb{S}^1$ *we have*

$$\lim_{z \to z_0,\, |z|>1} \|\Psi(U(t) - z)^{-1}\Psi\|_{\mathcal{H} \to \mathcal{H}} = +\infty,$$

where $\mathcal{H} = H_D(\Omega) \oplus L^2(\Omega)$ *is the energy space for* (1).

The proof is based on a representation of $\sum_{j=0}^{\infty} z^{-j-1} Z(jt)$, $|z| > 1$ as a sum of terms involving the cut-off resolvent $\Psi \sum_{j=0}^{\infty} z^{-j-1} U(jt)\Psi$.

Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a function such that $\psi(x) = 1$ for $|x| \leq a+1$, $\psi(x) = 0$ for $|x| \geq a+2$. Introduce the operator

$$L_\psi(g, h) = \Big(0, \langle \nabla_x \psi, \nabla_x g\rangle + (\Delta\psi) g\Big).$$

In particular, we define $L_\psi(U(t)f)$ and $L_\psi(U_0(t)f)$ and will write simply $L_\psi U(t)$ and $L_\psi U_0(t)$. It is easy to see that we have

$$\begin{aligned}
(1-\psi)U(t) &= U_0(t)(1-\psi) + \int_0^t U_0(t) L_\psi U(t-s) ds,\\
U(t)(1-\psi) &= (1-\psi)U_0(t) + \int_0^t U(t-s) L_\psi U_0(s) ds.
\end{aligned}$$

Applying these equalities, we get

$$\begin{aligned}
U(t) =& U(t)\psi + (1-\psi)U_0(t) + \int_0^t \psi U(t-s) L_\psi U_0(s) ds\\
&+ \int_0^t U_0(t-s)(1-\psi) L_\psi U_0(s) ds + \int_0^t \int_0^{t-s} U_0(\tau) L_\psi U(t-s-\tau) L_\psi U_0(s) ds d\tau\\
=& \psi U(t)\psi + U_0(t)\psi(1-\psi) + (1-\psi)U_0(t) + \int_0^t \psi U(t-s) L_\psi U_0(s) ds\\
&+ \int_0^t U_0(s) L_\psi U(t-s)\psi ds + \int_0^t U_0(t-s)(1-\psi) L_\psi U_0(s) ds\\
&+ \int_0^t \int_0^{t-s} U_0(\tau) L_\psi U(t-s-\tau) L_\psi U_0(s) ds d\tau.
\end{aligned}$$

Now let $z \in \mathbb{C}$ be such that $|z| > 1$. Let $g \in C_0^\infty(B_{a+2})$ be a cut-off function equal to 1 on B_{a+1}. We choose the projectors $P_\pm^b = P_\pm$ so that

$$P_\pm \psi = \psi = \psi P_\pm,\ P_\pm g = g = g P_\pm.$$

Next we fix $b > 0$ and the projectors $P_\pm$ with these properties and note that $gL_\psi = L_\psi = L_\psi g$. Let $T_0 > 0$ be chosen so that $P_+ U_0(t) P_- = 0$ for $t \geq T_0$. Given a $t > 0$, we have

$$\begin{aligned}(Z(t)-z)^{-1} = & -\sum_{j=0}^{\infty} z^{-j-1} P_+ U(jt) P_- \\ = & P_+\psi(U(t)-z)^{-1}\psi P_- - \sum_{jt\leq T_0} z^{-j-1} P_+ U_0(jt)\psi(1-\psi)P_- \\ & - \sum_{jt\leq T_0} z^{-j-1} P_+(1-\psi)U_0(jt)P_- \\ & + \int_0^{T_0} P_+ U_0(s) L_\psi (U(t)-z)^{-1}\Phi U(-s)\psi P_- ds \\ & + \int_0^{T_0} P_+\psi(U(t)-z)^{-1}\Phi U(-s) L_\psi U_0(s) P_- ds \\ & - \sum_{jt\leq T_1}\int_0^{\min(jt,T_0)} z^{-j-1} P_+ U_0(jt-s)(1-\psi)L_\psi U_0(s)P_- ds \\ & + \int_0^{T_0}\int_0^{T_0} P_+ U_0(\tau) L_\psi (U(t)-z)^{-1}\Phi U(-s-\tau) L_\psi U_0(s) P_- ds d\tau + G(z)\end{aligned}$$

with an operator $G(z)$ holomorphic for $z \neq 0$. Here Φ is a cut-off function with compact support determined by the finite speed of propagation so that

$$(1-\Phi)U_0(t)g = 0 \text{ and } (1-\Phi)U(t)g = 0 \text{ for } |t| \leq 2T_0.$$

The terms in the above presentation of $(Z(t)-z)^{-1}$ given by finite sums are holomorphic operators with respect to $z \neq 0$. Consequently, if

$$\lim_{z\to z_0, |z|>1} \|\Psi(U(t)-z)^{-1}\Psi\|_{\mathcal{H}\to\mathcal{H}} < \infty$$

exists for $\Psi \in C_0^\infty(x \in \mathbb{R}^n : |x| \leq c+1)$ and equal to 1 for $|x| \leq c$ for some suitably large and fixed $c > 0$, we conclude that $(Z(t)-z)^{-1}$ is not singular at $z_0 \in \mathbb{S}^1$. As we mentioned in Remark 2, this gives a contradiction with the condition (3) which implies that $\mathbb{S}^1 \subset \sigma(Z(t))$ for almost all $t \in R^+$.

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