RESONANCES FOR MAGNETIC STARK HAMILTONIANS IN TWO DIMENSIONAL CASE

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Abstract. We study the resonances of the two-dimensional Schrödinger operator $P_1(B;\beta) = (D_x - By)^2 + D_y^2 + \beta x + V(x,y)$, $B > 0, \beta > 0$, with constant magnetic and electric fields. We define the resonances of $P_1(B;\beta)$ and the spectral shift function $\xi(\lambda)$ related to $P_1(B;\beta)$ and $P_0(B;\beta) = P_1(B;\beta) - V(x,y)$ without any restriction on B and β. For strong magnetic fields ($B \to \infty$) we obtain a representation of the derivative of $\xi(\lambda)$, a trace formula for $\mathrm{tr}(f(P_1(B;\beta)) - f(P_0(B;\beta)))$ and an upper bound for the number of the resonances lying in $\{z \in \mathbb{C} : |\Re z - (2n-1)B| \leq \alpha B, \ \mathrm{Im}\, z \geq \mu \mathrm{Im}\, \theta\}$, $0 < \alpha < 1$, $0 < \mu < 1$, $\mathrm{Im}\, \theta < 0$. Moreover, for $B \to \infty$ we examine the free resonances domains and show that the resonances are included in the neighborhoods $\{z \in \mathbb{C} : |\Re z - (2n-1)B| \leq C_0\}$, where $(2n-1)B$ are the Landau levels and $C_0 > 0$ is a constant independent on B and $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

1. Introduction

The two-dimensional Schrödinger operator with homogeneous magnetic and electric fields can be written in the form

$$P_1(B;\beta) = (D_x - By)^2 + D_y^2 + \beta x + V(x,y), \quad D_\nu = -i\frac{\partial}{\partial \nu},$$

where B and β are proportional to the strength of the homogeneous magnetic and electric fields. In this paper we study the spectral shift function of the pair $(P_1(B;\beta), P_0(B;\beta))$, where

$$P_0(B;\beta) = (D_x - By)^2 + D_y^2 + \beta x$$

and $V \in C^\infty(\mathbb{R}^2;\mathbb{R})$. We assume that there exists $\epsilon > 0$ so that

$$|\partial_{x,y}^\alpha V(x,y)| \leq C_\alpha \langle x \rangle^{-2-\epsilon} \langle y \rangle^{-1-\epsilon}, \ \forall \alpha, \tag{1.1}$$

where $\langle X \rangle = (1 + |X|^2)^{1/2}$.

The essential spectrum of $P_1(B;0)$ and $P_0(B;0)$ are the same and it is well known that the spectrum of the operator $P_0(B;0)$ is given by

$$\cup_{n=1}^\infty \{(2n-1)B\}.$$

The numbers $\lambda_n = (2n-1)B$, $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, called Landau levels, are eigenvalues of infinite multiplicity (see [2]). Outside the Landau levels we have discrete eigenvalues caused by the potential V. The presence of electric field creates resonances which will be characterized as the eigenvalues of a distorted operator.

The spectral properties of the 2D Schrödinger operator $P_1(B;0)$ have been intensively studied in the last ten years. In the case of perturbations the Landau levels λ_n become accumulation points of the eigenvalues of $P_1(B;0)$ and the asymptotics of the function counting the number of the eigenvalues lying in a neighborhood of λ_n have been examined by many authors in different aspects. For recent results the reader may consult [25], [17], [14], [18], [26], [23] and the references given there. We would like to mention that it seems difficult to obtain a trace formula involving some summation over the eigenvalues close to a Landau level (see [22] for a result in this direction).

For the 2D Schrödinger operator with crossed magnetic and electric fields ($\beta \neq 0$) the situation completely changes and $\sigma_{\rm ess}(P_0(B;\beta)) = \sigma_{\rm ess}(P_1(B;\beta)) = \mathbb{R}$. For decreasing potentials the operator $P_1(B;\beta)$ can have embedded eigenvalues $\lambda \in \mathbb{R}$, but this question seems not sufficiently investigated. From physical point of view, it is expected that $V(x,y)$ creates resonances $z \in \mathbb{C}$, $\operatorname{Im} z \leq 0$, and it natural to define and to study the spectral shift function (SSF) $\xi(\lambda)$ related to $P_1(B;\beta)$ and $P_0(B;\beta)$. There are only few works treating magnetic Stark resonances. The case $B \to \infty$ was studied in [32], while the case $\beta \to 0$ has been examined in [12], [13]} (see also [19], [20], [31], [33]). In these works the authors study mainly the resonances close to the eigenvalues of the non-perturbed operator $P_0(B;\beta)$. Moreover, in [32], the complex scaling and the definition of the resonances for $B \to \infty$ lead to some difficulties when we try to show that there are no resonances z with $\operatorname{Im} z > 0$ and this was an open problem in [32]. We can define SSF following the general setup [34], but to our best knowledge the SSF for magnetic Stark Hamiltonians has not been investigated, as well as there are no trace formulae involving the resonances lying in a compact domain in $\mathbb{C}$.

In this work we are strongly inspired by the recent progress in the analysis of the resonances, SSF and trace formulae for Schrödinger operators (see [28], [29], [30], [24], [4], [6], [8], [10], [11]). In this direction the role played by the SSF is very important and it was shown in [4] how many applications as Weyl asymptotics of SSF, trace formulae and Breit-Wigner approximations, can be deduced from a representation of the derivative of SSF as a sum of harmonic measures related to resonances z with $\operatorname{Im} z < 0$, Dirac measures associated to embedded eigenvalues $z \in \mathbb{R}$ and, a harmonic function. In [11] we have followed this strategy for Stark Hamiltonians without magnetic fields ($B = 0$). In this paper we study the connexions between the resonances and the SSF for magnetic Stark Hamiltonians and our main goal is to show that the derivative $\xi'(\lambda)$ has the same representation as that mentioned above. Assuming $\beta = 0$, in the 3D case a representation of SSF has been obtained in [5].

In Section 2 we define the SSF for $P_1(B;\beta)$ and $P_0(B;\beta)$ without any restriction on B and β. Next without a restriction on the generality, we assume throughout the paper that $\beta = 1$ and we will use the **notations**

$$P_1(B) = P_1(B;1),\ P_0(B) = P_0(B;1).$$

To define the resonances, we will suppose that V admits a holomorphic extension in the x-variable into the domain

$$\Gamma_{\delta_0} = \{z \in \mathbb{C} :\ 0 \leq |\operatorname{Im} z| \leq \delta_0\}$$

for some $\delta_0 > 0$. We assume also that for some $\epsilon > 0$ we have the estimates

$$|\partial^\alpha V(x,y)| \leq C_\alpha \langle |\Re x| \rangle^{-2-\epsilon} \langle y \rangle^{-1-\epsilon},\ \text{for}\ x \in \Gamma_{\delta_0},\ y \in \mathbb{R},\ \forall \alpha. \tag{1.2}$$

In Section 3, by using a complex scaling in x-direction, $(x,y) \longrightarrow (x+\theta,y)$, we introduce the dilated operators

$$P_j(B,\theta) = \mathcal{U}_\theta^{-1} P_j(B) \mathcal{U}_\theta,\ j = 0,1,$$

where for $\theta \in \mathbb{R}$ we consider the unitary operator

$$\mathcal{U}_\theta : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2),\quad f \to f(x+\theta,y)$$

and for $\theta \in D(0,\theta_0) \subset \mathbb{C}$ we extend $P_j(B,\theta)$ (see Lemma 1 in Section 3). The notations $P_j(B,\theta)$ makes no **confusion** with the notations $P_j(B;\beta)$ given above since $\beta = 1$. The resonances are defined as the eigenvalues of the dilated operator $P_1(B,\theta)$ (see Section 3 for more details) and in this direction we follow the previous works on Stark Hamiltonians [1], [16] (for a more complete list of references see [11]).

Denote by Res $P_1(B)$ the set of resonances of $P_1(B)$. In Sections 4-6 we study the strong magnetic fields characterized by $B \to \infty$. Denote by $\xi(\lambda, B)$ the spectral shift function related to operators $P_1(B)$, $P_0(B)$. Let $0 < \alpha < 2$, $0 < \alpha_1$, $0 < \mu < 1$ be fixed and let

$$\Omega_n = \{z \in \mathbb{C} : |\Re z - (2n+1)B| \le \alpha B,\ \alpha_1 B \ge \operatorname{Im} z \ge \mu \operatorname{Im}\theta\}.$$

Let $\Omega \subset \Omega_n$ and let W be an open relatively compact subset of Ω. Suppose that $J = \Omega \cap \mathbb{R}$, $I = W \cap \mathbb{R}$ are intervals. Our main result is the following

Theorem 1. *Assume that V satisfies the assumption* (1.2). *Then for B large enough and $\lambda \in I$ we have the representation*

$$\xi'(\lambda, B) = \frac{1}{\pi}\operatorname{Im} r(\lambda, B) + \sum_{\substack{\omega \in \operatorname{Res}(P_1(B)) \cap \Omega, \\ \operatorname{Im}\omega < 0}} \frac{-\operatorname{Im}\omega}{\pi|\lambda - \omega|^2} + \sum_{\omega \in \sigma_{pp}(P_1(B)) \cap J} \delta(\lambda - \omega), \tag{1.3}$$

where $r(z, B)$ is a function holomorphic in Ω and

$$|r(z, B)| \le C(W)B,\ z \in W. \tag{1.4}$$

We like to stress that this representation of the derivative of $\xi(B, \lambda)$ is the same as that established for operators with perturbations which decay to 0 as $|x| \to \infty$ ([4], [24]) and for Stark Hamiltonians without magnetic field [11]. As an application we obtain a local trace formula completely similar to those in [30], [24], [11] This formula follows immediately from Theorem 1 (see [24]).

Theorem 2. *Assume that V satisfies the assumption* (1.2). *Let*

$$\Omega \subset \{z \in \mathbb{C} :\ \operatorname{Im} z \ge \mu \operatorname{Im}\theta\},\ 0 < \mu < 1$$

be an open, simply connected, relatively compact such that $I = \Omega \cap \mathbb{R}$ is an interval. Suppose that f is holomorphic on a neighborhood of Ω and that $\psi \in C_0^\infty(\mathbb{R})$ satisfies

$$\psi(\lambda) = \begin{cases} 0, & \operatorname{dist}(I, \lambda) > 2\eta, \\ 1, & \operatorname{dist}(I, \lambda) < \eta, \end{cases}$$

where $\eta > 0$ is sufficiently small. Then for B large enough we have

$$\operatorname{tr}\Big[(\psi f)(P_j(B))\Big]_{j=0}^{1} = \sum_{z \in \operatorname{Res} P_1(B) \cap \Omega} f(z) + E_{\Omega, f, \psi}, \tag{1.5}$$

where $[a_j]_{j=0}^1 = a_1 - a_0$ and

$$|E_{\Omega,f,\psi}| \le M(\psi, \Omega) \sup\{|f(z)|\ :\ 0 \le \operatorname{dist}(\Omega, z) \le 2\eta\,,\ \operatorname{Im} z \le 0\} B\,.$$

Our dilatation is simpler than that exploited by Wang [32] and this enables us to prove that there are no resonances z with $\operatorname{Im} z > 0$. We have not raised the question if our definition of the resonances and that in [32] are equivalent for $B \to \infty$. Nevertheless, we think that our approach is more natural, since the resonances, introduced in Section 3, lie in the "non-physical" plane

$\{z \in \mathbb{C} : \operatorname{Im} z \leq 0\}$. The definition of the SSF is independent on the resonances, and this confirms our choice of complex dilatation.

We establish the following properties of the resonances.

Proposition 1. *Let $0 < \mu < 1$, $n \in \mathbb{N}$ be fixed. Then there exists a constant $C_0 > 0$, independent on B, and B_n such that for $B \geq B_n$, the operator $P_1(B)$ has no resonances z lying in the domain*

$$\{z \in \mathbb{C} : C_0 \leq |\Re z - (2n-1)B| \leq B, \operatorname{Im} z \geq \mu \operatorname{Im} \theta\}.$$

Moreover, we show that there are no resonances z with $\Re z < \alpha B$, $0 < \alpha < 1$ and we establish an upper bound

$$\#\{z \in \operatorname{Res} P_1(B) : |\Re z - \lambda_n| \leq C_0, \operatorname{Im} z \geq \mu \operatorname{Im} \theta\} \leq C_1 B$$

with $C_1 > 0$ independent on B and λ_n. In particular, in every compact subset of $\mathbb{C}$ we have only a finite number of resonances with finite multiplicities.

Remark. From physical point of view, we see that the presence of a constant electric field generated by the potential βx leads to the absence of embedded eigenvalues and resonances with infinite multiplicity. On the other hand, the Landau levels λ_n are the only points that may play the role of attractors of resonances creating the gaps and free resonances regions. For fixed B it is proved in [12] that there are no resonances z of $P_1(B)$ with $|\Re z| \geq R_0 > 0$. In this direction we obtain a stronger result saying that we have no resonances with negative real part.

The main difficulty in the proof of Theorem 1 is the construction of an operator $L(B, \theta)$ and a trace class operator K with $\|K\|_{\text{tr}} = \mathcal{O}(B)$ so that

$$P_1(B, \theta) - z = L(B, \theta) - z + K,$$

where $(L(B, \theta) - z)^{-1} = \mathcal{O}(1)$ for z in a complex neighborhood Ω_n of λ_n. For this purpose we must study for $z \in \Omega_n$ the invertibility of the non-selfadjoint operator $\Big((I - \Pi)P_1(B, \theta) - z\Big)(I - \Pi)$, where Π is the spectral projector on the eigenspace of $(D_x - y)^2 + D_y^2$ related to λ_n. The existence of double characteristics of the operator $(D_x - By)^2 + D_y^2$ which is not globally elliptic, combined with the Stark effects caused by x, lead to several difficulties. The proof of Theorem 1, given in Section 5, works without a reduction to an effective Hamiltonian. Following the same strategy, we will study elsewhere the general case without the assumption $B \gg 1$. On the other hand, in Section 6 we construct an effective Hamiltonian $E_{1,-+}(z)$, related to $P_1(B, \theta)$, and the existence of the resonances is reduced to the invertibility of $E_{1,-+}(z)$ in Ω_n. This leads to Proposition 1 given above. It is possible, applying the argument of Wang [32], to obtain a more precise information of the existence of resonances close to some energy level E associated to the maximum or the minimum of V.

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2. Spectral shift function

Throughout this work we will use the notations of [9] for symbols and pseudodifferential operators. In particular, if $m : \mathbb{R}^d \to [0, \infty[$ is an order function (see Definition 7.5 in [9]), we say that

$a(X,\Xi) \in S^0(\mathbb{R}^d;m)$ if $a(X,\Xi) \in C^\infty(\mathbb{R}^d)$ is such that for every $\alpha \in \mathbb{N}^d$, there exists $C_\alpha > 0$ such that

$$|\partial^\alpha a(X,\Xi)| \le C_\alpha m(X,\Xi)\,.$$

In the special case when $m=1$, we will write $S^0(\mathbb{R}^d)$ instead of $S^0(\mathbb{R}^d;1)$. We will use the standard Weyl quantization of symbols. More precisely, if $P(y,\eta)$ is a symbol in $S^0(\mathbb{R}^4;m)$, then $P^w(y,D_y)$ is the operator defined by

$$P^w(y,D_y)u(y) = (2\pi)^{-2}\iint e^{i(y-y')\cdot\eta}P\Big(\frac{y+y'}{2},\eta\Big)u(y')dy'd\eta, \text{ for } u \in S(\mathbb{R}^2).$$

Sometimes we will quantize a function $P(x,y,\xi,\eta)$ only with respect to the variable (y,η). In this case we will denote by $P^w(x,y,\xi,D_y)$ the operator obtained as above, considering (x,ξ) as a parameter. Finally, when $P(y,\eta)$ is a function on $T^*(\mathbb{R}^2)$ (possibly operator-valued), we denote by $P^w(y,hD_y)$ the semiclassical quantization obtained as above by quantizing $P(y,h\eta)$.

In this section we assume that $V(x,y)$ satisfies only the assumption (1.1). The operators $P_1(B)$, $P_0(B)$ are essentially self-adjoint with domain $C_0^\infty(\mathbb{R}^2)$. In this section we define the spectral shift function related to $P_1(B)$ and $P_0(B)$.

Introduce the unitary operator $U : L^2(\mathbb{R}^2) \to L^2(R^2)$ by

$$(Uu)(x,y) = \frac{B^{\frac{3}{4}}}{2\pi}\iint_{\mathbb{R}^2} e^{i\varphi_B(x,y,x',y')}\,u(x',y')\,dx'dy'$$

where

$$\varphi_B(x,y,x',y') = Bxy - \sqrt{B}xy' - Bx'y + \sqrt{B}x'y' - \frac{1}{2\sqrt{B^3}}y'.$$

A simple calculus shows that

$$\tilde{P}_0(B) = U^{-1}P_0(B)U = B(D_y^2+y^2) + x - \frac{1}{4B^2}\,,$$

$$\tilde{P}_1(B) = U^{-1}P_1(B)U = \tilde{P}_0(B) + V^\omega\Big(x - B^{-1/2}D_y - \frac{1}{2B^2}, B^{-1/2}y + B^{-1}D_x\Big)\,.$$

The fact that U is unitary can be easily obtained by a direct calculation, but a deeper reason for this is the following observation. Since U is a *metaplectic operator* (i.e. operator associated with a linear canonical transformation), it follows from a classical result of the theory of Fourier integral operators that U is unitary (see [10], Theorem A.2, Chapter 7). The reader could consult [3], [15], [32], [7]), for more details concerning the construction of U. We have the following

Proposition 2. *Assume that V satisfies the estimate* (1.1). *Then*

(*i*) *The operator $(P_1(B) \pm i)^{-1} - (P_0(B) \pm i)^{-1}$ is a trace class one.*

(*ii*) *For* $\operatorname{Im} z \neq 0$ *we have*

$$\|(i - P_1(B))^{-1}(z-P_1(B))^{-1} - (i-P_0(B))^{-1}(z-P_0(B))^{-1}\|_{\rm tr} = \mathcal{O}(|\operatorname{Im} z|^{-2}). \qquad (2.1)$$

Proof. Since U is unitary, it is sufficient to show that the operator

$$(\tilde{P}_1(B)+i)^{-1} - (\tilde{P}_0(B)+i)^{-1}$$

is trace class. In the following we will write $\tilde{P}_j, j = 0,1$, instead of $\tilde{P}_j(B)$. By applying the resolvent equality, we get

$$(\tilde{P}_1+i)^{-1} - (\tilde{P}_0+i)^{-1} = -(\tilde{P}_1+i)^{-1}V^\omega(\tilde{P}_0+i)^{-1}$$

$$= -(\tilde{P}_0 + i)^{-1} V^\omega (\tilde{P}_0 + i)^{-1} + (\tilde{P}_1 + i)^{-1} V^\omega (\tilde{P}_0 + i)^{-1} V^\omega (\tilde{P}_0 + i)^{-1} .$$

The operator $(\tilde{P}_1 + i)^{-1} V^\omega$ is bounded and the proof is reduced to show that

$$(\tilde{P}_0 + i)^{-1} V^\omega (\tilde{P}_0 + i)^{-1}$$

is trace class. Next we assume that $B \geq \beta_0 > 0$. For simplicity suppose that $\beta_0 = 1$. Let $\chi(t) \in C_0^\infty(\mathbb{R}; [0, 1])$ be a cut-off function such that $\chi(t) = 1$ for $|t| \leq 1$ and $\chi(t) = 0$ for $|t| \geq 2$. Fix a number k, $\max\{1, \frac{2}{1+\epsilon}\} < k < 2$, and introduce the symbol

$$q(x, y, \eta) = \chi\Big(\frac{\langle y, \eta\rangle^k}{|\eta^2 + y^2 + B^{-1}(x + i)|}\Big),$$

where $\langle y, \eta\rangle = (1 + y^2 + \eta^2)^{1/2}$. It clear that $q(x, y, \eta) \in S^0(\mathbb{R}^4_{(x,\xi,y,\eta)})$ and we set $A = q^\omega(x, y, D_y)$. We decompose

$$\begin{aligned}
&(\tilde{P}_0 + i)^{-1} V^\omega (\tilde{P}_0 + i)^{-1} \\
&= (\tilde{P}_0 + i)^{-1} A V^\omega A (\tilde{P}_0 + i)^{-1} + (\tilde{P}_0 + i)^{-1} (I - A) V^\omega A (\tilde{P}_0 + i)^{-1} \\
&\quad + (\tilde{P}_0 + i)^{-1} (I - A) V^\omega (I - A)(\tilde{P}_0 + i)^{-1} \\
&\quad + (\tilde{P}_0 + i)^{-1} A V^\omega (I - A)(\tilde{P}_0 + i)^{-1} = L_1 + L_2 + L_3 + L_4 .
\end{aligned}$$

To treat L_1, notice that on the support of $q(x, y, \eta)$ we have

$$(B(\eta^2 + y^2) + x + i)^{-1} \in S^0(\mathbb{R}^4; \langle y, \eta\rangle^{-k}) .$$

In fact, on the support of q we obtain

$$\langle y, \eta\rangle^k \leq 2B^{-1} |B(\eta^2 + y^2) + x + i)| \leq 2|B(\eta^2 + y^2) + x + i|$$

and it is easy to estimate the derivatives of $(B(\eta^2 + y^2) + x + i)^{-1}$. According to the calculus of pseudodifferential operators, L_1 becomes a pseudodifferentail operator with symbol in

$$S^0(\mathbb{R}^4; \langle y, \eta\rangle^{-k} \langle x - B^{-1/2}\eta\rangle^{-2-\epsilon} \langle B^{-1/2} y + B^{-1}\xi\rangle^{-1-\epsilon}),$$

and the trace of L_1 can be estimated (see for instance, Theorem 9.4 in [9]) by

$$\begin{aligned}
\|L_1\|_{\rm tr} &\leq C_0 \iiiint \langle y, \eta\rangle^{-2k} \langle x - B^{-1/2}\eta\rangle^{-2-\epsilon} \langle B^{-1/2} y + B^{-1}\xi\rangle^{-1-\epsilon} dx d\xi dy d\eta \\
&\leq C_0' B \iint \langle y, \eta\rangle^{-2k} dy d\eta \leq C_0'' B
\end{aligned}$$

with constants C_0', C_0'', independent on B. To deal with L_j, $j = 2, 3, 4$, we will show that $(I - A)V^\omega$ and $V^\omega(I - A)$ are trace class operators. For our analysis in Sections 4-6 we examine the dependence on B of the trace estimates. Notice that on the support of the symbol of $(I - A)$ we have

$$\langle y, \eta\rangle^k \geq |(\eta^2 + y^2) + B^{-1}(x + i)| .$$

Taking into account the estimate (1.1), we get

$$\begin{aligned}
&\|(I - A)V^\omega\|_{\rm tr} \leq \\
&C_1 \iiiint_{\langle y,\eta\rangle^k \geq |\eta^2 + y^2 + B^{-1}(x+i)|} \langle x - B^{-1/2}\eta\rangle^{-2-\epsilon} \langle B^{-1/2} y + B^{-1}\xi\rangle^{-1-\epsilon} dx d\xi dy d\eta \\
&\leq C_2 B \iiint_{\langle y,\eta\rangle^k \geq |\eta^2 + y^2 + B^{-1}(x+i)|} \langle x - B^{-1/2}\eta\rangle^{-2-\epsilon} dx dy d\eta \\
&\leq C_2 B^2 \iiint_{\langle y,\eta\rangle^k \geq |\eta^2 + y^2 + B^{-3/2}\eta + u + B^{-1} i|} \langle Bu\rangle^{-2-\epsilon} du dy d\eta
\end{aligned}$$

$$\le C_2' B^2 \iiint_{\substack{\langle y,\eta\rangle^k \ge |\eta^2+y^2+B^{-3/2}\eta+u|,\\ |u|\le \frac{1}{2}\langle y,\eta\rangle^k}} \langle Bu\rangle^{-2-\epsilon} du dy d\eta$$

$$+ C_2' B^2 \iiint_{\substack{\langle y,\eta\rangle^k \ge |\eta^2+y^2+B^{-3/2}\eta+u|,\\ |u|\ge \frac{1}{2}\langle y,\eta\rangle^k}} \langle Bu\rangle^{-2-\epsilon} du dy d\eta$$

$$\le C_2' B^2 \Big(\iiint_{|u|\le C_3, |y|\le C_3, |\eta|\le C_3} \langle Bu\rangle^{-2-\epsilon} du dy d\eta + \iiint_{|u|\ge \frac{1}{2}\langle y,\eta\rangle^k} \langle Bu\rangle^{-2-\epsilon} du dy d\eta \Big)$$

$$\le C_4 B + C_5 B^2 \int \langle Bu\rangle^{-2-\epsilon} \Big(\int_0^{(2|u|)^{\frac{1}{k}}} r dr \Big) du \le C_4 B + C_6 B^2 \int \langle Bu\rangle^{-2-\epsilon+2/k} du \le \tilde{C} B\,,$$

since $-2-\epsilon+2/k < -1$. The analysis of $V^\omega(I-A)$ is completely similar. Finally, we obtain the estimate

$$\|(\tilde{P}_0+i)^{-1} V^\omega (\tilde{P}_0+i)^{-1}\|_{\rm tr} \le A_0 B,\ B\ge 1$$

with a constant $A_0>0$ independent on B.

To establish (2.1), we write the left-hand side in the form

$$\Big((i-P_1(B))^{-1} - (i-P_0(B))^{-1}\Big)(z-P_1(B))^{-1}$$

$$+(i-P_0(B))^{-1}\Big((z-P_1(B))^{-1} - (z-P_0(B))^{-1}\Big)$$

$$= \Big((i-P_1(B))^{-1} - (i-P_0(B))^{-1}\Big)(z-P_1(B))^{-1}$$

$$-(z-P_1(B))^{-1}(i-P_0(B))^{-1} V (z-P_1(B))^{-1}\,.$$

The first term at the right-hand side of the last equality is trace class. To estimate the second one, we replace $(z-P_1(B))^{-1}$ by

$$(i-P_1(B))^{-1} - (z-i)(i-P_1(B))^{-1}(z-P_1(B))^{-1}$$

and, as above, we write $(i-P_0(B))^{-1}V(z-P_1(B))^{-1}$ as a product of $(i-P_0(B))^{-1}V(i-P_0(B))^{-1}$ and a bounded operator. Combining this with the estimate $\|(z-P_j(B))^{-1}\| = \mathcal{O}(|{\rm Im}\, z|^{-1})$, we complete the proof of (2.1). □

The property (i) of Proposition 2 enables us to define the spectral shift function $\xi(B,\lambda)\in\mathcal{D}'(\mathbb{R})$ related to operators $P_1(B)$ and $P_0(B)$ following the general theory (see for instance, p.297-303, [34]) by the equality

$$\langle \xi', f\rangle = {\rm tr}\Big(f(P_1(B)) - f(P_0(B))\Big),\ f\in C_0^\infty(\mathbb{R})\,.$$

For our analysis it is important to have a representation of $\xi(B,\lambda)$ involving the resolvents of $P_j(B)$. Let $\tilde{f}(z)\in C_0^\infty(\mathbb{C})$ be an almost analytic extension of f. Set $g(z)=f(z)(z-i)$. By the Helffer-Sjöstrand formula we have

$$g(P_j(B)) = -\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)(z-i)(z-P_j(B))^{-1} L(dz),\ j=0,1\,,$$

and we obtain

$${\rm tr}\Big(f(P_1(B)) - f(P_0(B))\Big) = -\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)(z-i){\rm tr}\Big[(P_j(B)-i)^{-1}(z-P_j(B))^{-1}\Big]_{j=0}^1 L(dz)\,,$$

where $L(dz)$ is the Lebesgue measure on $\mathbb{C}$ and $[a_j]_{j=0}^1 = a_1 - a_0$. Since $\bar{\partial}_z f(z) = \mathcal{O}(|{\rm Im} z|^N),\ \forall N\in\mathbb{N}$, the trace is well defined.

3. Resonances for magnetic Stark hamiltonians

In this and in the following sections we assume that $V(x,y)$ satisfies the estimates (1.2). Let $D(0,\theta_0)$ be the disk in $\mathbb{C}$ of center 0 and radius $\theta_0 > 0$. For $\theta \in D(0,\theta_0)$, $\theta_0 > 0$ small, we will use the dilation $(x,y) \longrightarrow (x+\theta, y)$. For $\theta \in \mathbb{R}$, consider the unitary operator

$$\mathcal{U}_\theta : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \quad f \to f(x+\theta, y).$$

Let U be the unitary operator introduced in Section 2. Setting $\tilde{\mathcal{U}}_\theta = U\mathcal{U}_\theta$, we introduce the operators

$$\mathcal{U}_\theta^{-1} P_j(B)\mathcal{U}_\theta := P_j(B,\theta),\ j = 0,1, \tag{3.1}$$

$$\tilde{\mathcal{U}}_\theta^{-1} P_0(B)\tilde{\mathcal{U}}_\theta := \tilde{P}_0(B,\theta) = B(D_y^2 + y^2) + x + \theta - \frac{1}{4B^2}, \tag{3.2}$$

$$\tilde{\mathcal{U}}_\theta^{-1} P_1(B)\tilde{\mathcal{U}}_\theta := \tilde{P}_1(B,\theta) \tag{3.3}$$

$$= \tilde{P}_0(B,\theta) + V^w(x + \theta - B^{-\frac{1}{2}}D_y - \frac{1}{2B^2}, B^{-\frac{1}{2}}y + B^{-1}D_x),$$

Recall that throughout the paper we will use the notations $P_j(B,\theta)$, $\tilde{P}_j(B,\theta)$, $j = 0,1$, for the operators defined above and this makes no confusion with the notation $P_j(B;\beta)$ given in the Introduction.

Lemma 1. *There exists $\theta_0 > 0$ such that the self-adjoint operators $P_1(B,\theta)$, $\tilde{P}_1(B,\theta)$, defined for $\theta \in]-\theta_0, \theta_0[$, extend to an analytic type-A family of operators on $D(0,\theta_0)$ with the same domain $\mathcal{D}$ as that of $P_0(B,0), \tilde{P}_0(B,0)$. Moreover,*

$$\sigma_{\text{ess}}(\tilde{P}_1(B,\theta)) = \sigma_{\text{ess}}(\tilde{P}_0(B,\theta)) = \sigma(\tilde{P}_0(B,\theta)) = \{\lambda + \theta;\ \ \lambda \in \mathbb{R}\}.$$

Proof. Clearly, the domain $\mathcal{D}$ of $\tilde{P}_0(B,\theta)$ is independent of θ and $\theta \to \tilde{P}_0(B,\theta)u$ is analytic for all $u \in \mathcal{D}$. On the other hand, the analytic assumption on V implies that there exists $\theta_0 > 0$ small enough such that

$$D(0,\theta_0) \ni \theta \to V^w\Big(x + \theta - B^{-\frac{1}{2}}D_y - \frac{1}{2B^2}, B^{-\frac{1}{2}}y + B^{-1}D_x\Big)u$$

is analytic for any $u \in L^2(\mathbb{R}^2)$. Following [21], this gives the first statement of the lemma. For the second one, notice that

$$\sigma_{\text{ess}}(\tilde{P}_0(B,\theta)) = \sigma(\tilde{P}_0(B,\theta)) = \{\lambda + \theta :\ \lambda \in \mathbb{R}\}.$$

Using (1.2) and Lemma 3 of [27], p. 111, we deduce that $\sigma_{\text{ess}}(\tilde{P}_1(B,\theta)) = \sigma_{\text{ess}}(\tilde{P}_0(B,\theta))$ and this completes the proof. □

Below we take $\theta \in D(0,\theta_0)$, $\operatorname{Im}\theta \le 0$, and consider the domain

$$\Omega_\theta = \{z \in \mathbb{C} :\ \operatorname{Im} z > \operatorname{Im}\theta\}.$$

It is easy to see that there exist $\theta_0 > 0$ small enough such that for $\theta \in D(0,\theta_0)$ with $\operatorname{Im}\theta \le 0$ we have

$$\|(z - \tilde{P}_0(B,\theta))^{-1}\| \le \frac{1}{|\operatorname{Im}\theta - \operatorname{Im} z|} \tag{3.4}$$

for $z \in \Omega_\theta$.

Now, repeating the argument in [11], we prove the following

Lemma 2. *Let $\operatorname{Im} z_0 > \operatorname{Im}\theta$. Then the operator $P_1(B,\theta) - z_0$ is a Fredholm one with index 0.*

Proof. Let $\Phi_\theta(x,y) = (x+\theta, y)$. For $z \in \Omega$ we have

$$P_1(B,\theta) - z = \Big(I + \Big(V \circ \Phi_\theta\Big)(1-\psi_z(u))\Big[P_0(B,\theta) - z$$
$$+ \Big(V \circ \Phi_\theta\Big)\psi_z(u)\Big]^{-1}\Big)\Big[P_0(B,\theta) - z + \Big(V \circ \Phi_\theta\Big)\psi_z(u)\Big],$$

where $u = (x,y)$, $\psi_z(u) \in C^\infty(\mathbb{R}^2; [0,1])$ is a function such that $\psi_z(u) = 0$ for $|u| \le C_1$, $\psi_z(u) = 1$ for $|u| \ge C_1 + 1$. Choosing $C_1 > 0$ (depending on z) large enough, we may assume that $|\Big(V \circ \Phi_\theta\Big)|\psi_z(u)$ is small, so the operator

$$A_\theta(z) = P_0(B,\theta) - z + \Big(V \circ \Phi_\theta\Big)\psi_z(x)$$

is invertible for $z \in \Omega_\theta$. On the other hand,

$$K_\theta(z) = \Big(V \circ \Phi_\theta\Big)(1-\psi_z(u))A_\theta(z)^{-1}$$

is compact. Then

$$\dim \mathrm{Ker}\,(\tilde{P}_1(B,\theta) - z_0) = \dim \mathrm{Ker}\,(I + K_\theta(z_0)),$$

provided

$$\mathrm{Im}\, z_0 > \mathrm{Im}\, \theta.$$

A simple argument shows that Image $(P_1(B,\theta) - z_0)$ is closed and

$$\mathrm{codim}\,(P_1(B,\theta) - z_0) = \dim \mathrm{Ker}\ \ (I + K_\theta^*(z_0)).$$

Thus $P_1(B,\theta) - z$ is a Fredholm operator with index 0 and the proof is complete. □

Definition 1. *Let* $\mathrm{Im}\,\theta < 0$. *We say that* $z \in \Omega_\theta$ *is a resonance of* $P_1(B)$ *if*

$$\dim \mathrm{Ker}\ \ (P_1(B,\theta) - z) > 0.$$

As in [11], we show that $P_1(B)$ has no resonances z with $\mathrm{Im}\, z > 0$, as well as, that the resonances in $\{z \in \mathbb{C} :\ \mathrm{Im} z > \mathrm{Im}\,\theta_2 > \mathrm{Im}\,\theta_1\}$ are independent of the choice of θ satisfying the condition $0 > \mathrm{Im}\,\theta_2 \ge \mathrm{Im}\,\theta \ge \mathrm{Im}\,\theta_1$.

Following [27] and repeating the argument in [11], we can establish a link between the eigenvalues of the complex scaling operator $P_1(B,\theta)$ and the poles of the suitably regularized resolvent. For this purpose, notice that

$$L^2(\mathbb{R}^2) \ni f \longrightarrow f(x+\theta, y) \in L^2(\mathbb{R}^2),\ \theta \in \mathbb{R}$$

form an unitary group. Then there exists a dense set $\mathcal{A} \subset L^2(\mathbb{R}^2)$ of analytic vectors so that

$$\sum_{n=0}^{\infty} \frac{\theta^n}{n!}\Big\|\frac{\partial^n f}{\partial x^n}\Big\|,\ f \in \mathcal{A}$$

is convergent for $\theta \in D(0,\theta_0)$. This implies that for θ_0 small and for $f \in \mathcal{A}$ the functions $\mathcal{U}_\theta f = f(x+\theta, y)$ admit a holomorphic extention in $D(0,\theta_0)$. The same is true for $\mathcal{U}_\theta^{-1} f$. Now suppose that $\lambda \in \Omega_n$ is an eigenvalue for $P_1(B,\theta)$. Then we can find $\varphi \ne 0$ and $\psi \ne 0$ so that $(\psi, (P_1(B,\theta) - z)^{-1}\varphi)$ has a pole at $z = \lambda$. By approximation, we construct functions $\psi_n \in \mathcal{A}$, $\varphi_n \in \mathcal{A}$ so that $\psi_n \longrightarrow \psi$, $\varphi_n \longrightarrow \varphi$. For n large enough $(\psi_n, (P_1(B,\theta) - z)^{-1}\varphi_n)$ will have a pole at $z = \lambda$. We fix a such n and setting $F = \mathcal{U}_\theta^{-1}\psi_n$, $G = \mathcal{U}_\theta^{-1}\varphi_n$, we deduce that $(F, (P_1(B) - z)^{-1}G)$ has a pole at $z = \lambda$.

We define the multiplicity of a resonance z_0 by

$$m(z_0) = \text{rank}\, \frac{1}{2\pi i} \int_{\gamma_\nu(z_0)} (z - P_1(B,\theta))^{-1} dz,$$

where $\gamma_\epsilon(z_0) = \{z = z_0 + \nu e^{i\varphi},\ 0 \leq \varphi \leq 2\pi\}$ and $\nu > 0$ is small enough. In the following we fix $\theta \in D(0, \theta_0)$ with $\text{Im}\,\theta < 0$ and we denote the set of resonances of $P_1(B)$ by $\text{Res}\, P_1(B)$.

Remark. Clearly, the operators

$$P_1(B,\theta) = \mathcal{U}_\theta P_1(B) \mathcal{U}_\theta^{-1} = (D_x - By)^2 + D_x^2 + x + \theta + V(x+\theta, y),$$

and $\tilde{P}_1(B,\theta)$ have the same eigenvalues in Ω_θ with the same multiplicity. In a such way, we can work directly with $\tilde{P}_1(B,\theta)$.

Proposition 3. *Let V satisfy* (1.2) *and let the condition*

$$1 + \partial_x V(x,y) > 0$$

be fulfilled. Then, there exists θ_0 , $\text{Im}\,\theta_0 < 0$, such that $P_1(B)$ has no resonances in Ω_{θ_0}.

Proof. First, since $\partial_x V(x,y)$ tends to 0 when $|(x,y)|$ tends to infinity, it follows from our assumptions that

$$1 + \partial_x V(x,y) \geq \eta > 0,$$

uniformly on $(x,y) \in \mathbb{R}^2$. For u in the domain of $P_0(B)$ we have

$$-\text{Im}\,((P_1(B,\theta) - z)u, u) = (\text{Im}\, z - \text{Im}\,\theta)\|u\|^2 - \text{Im}\,(V(\cdot + \theta, \cdot)u, u).$$

Applying Taylor's formula for the function $\theta \mapsto V(x+\theta, y)$, we obtain

$$\text{Im}\, V(x+\theta, y) = \text{Im}\,\theta\, \partial_x V(x + \Re\theta, y) + \mathcal{O}(|\text{Im}\,\theta|^2).$$

Thus

$$-\text{Im}\,((P_1(B,\theta) - z)u, u) = \text{Im}\, z\|u\|^2 - \text{Im}\,\theta((1 + \partial_x V(\cdot + \Re\theta, \cdot))u, u) + \mathcal{O}(|\text{Im}\,\theta|^2)\|u\|^2.$$

Next, we choose $\text{Im}\,\theta < 0$ small enough, and using the above inequality, we get the proposition. □

4. Estimates of the resolvent for strong magnetic fields

In this section we will examine the case of strong magnetic field characterized by $B \to \infty$. For simplicity we assume $\theta \in i\mathbb{R}$. Let φ_n be the n-th real normalized Hermite function given by

$$(D_y^2 + y^2)\varphi_n = (2n-1)\varphi_n,\ \ \|\varphi_n\| = 1,\ n \in \mathbb{N}^*.$$

To examine the resolvent $(P_1(B,\theta) - z)^{-1}$, we will study the resolvent of the operator

$$\tilde{P}_1(B,\theta) = B(D_y^2 + y^2) + x + \theta - \frac{1}{4B^2} + V^\omega.$$

Recall that V^ω is a bounded pseudodifferential operator in $L^2(\mathbb{R}^2)$ with Weyl symbol

$$V\Big(x - B^{-1/2}\eta - \frac{1}{2B^2}, B^{-1/2}y + B^{-1}\xi\Big).$$

We fix an integer $n \geq 1$ and let Π be the spectral projection of the operator $D_y^2 + y^2$ associated to the interval $[2n-2, 2n]$. Introduce the operator

$$Q(B,\theta) = (I - \Pi)\Big[B(D_y^2 + y^2) + x + \theta - \frac{1}{4B^2} + V^\omega\Big](I - \Pi).$$

The main result in this section is the following

Proposition 4. *Let $0 < \alpha < 2$, $0 < \alpha_1$, $0 < \mu < 1$ be fixed and let*

$$\Omega_n = \{z \in \mathbb{C} : |\Re z - (2n+1)B| \leq \alpha B,\ \alpha_1 B \geq \operatorname{Im} z \geq \mu \operatorname{Im}\theta\}.$$

Then for $B \gg 1$ sufficiently large and $z \in \Omega_n$ the operator $(Q(B,\theta) - z)^{-1}(I - \Pi)$ is well defined and there exists a constant $\gamma > 0$, independent on B, such that

$$\|(Q(B,\theta) - z)(I - \Pi)u\| \geq \gamma |\operatorname{Im}\theta| \|(I - \Pi)u\|,\ u \in \mathcal{D} \tag{4.1}$$

uniformly with respect to $z \in \Omega_n$.

Consider a partition of unity $G_1^2(x) + G_2^2(x) = 1$ with $G_i \in C^\infty(\mathbb{R}; [0,1]),\ i = 1,2,$

$$\operatorname{supp} G_1 \subset \{x \in \mathbb{R} :\ |x| \leq 2\}$$

and $G_1(x) = 1$ for $|x| \leq 1$. Choose $1/2 < \delta < 1$ and introduce the operators

$$A_1 = G_1^\omega\Big(\frac{x - B^{-1/2}D_y}{B^\delta}\Big),\ A_2 = G_2^\omega\Big(\frac{x - B^{-1/2}D_y}{B^\delta}\Big)$$

with Weyl symbols $G_i\Big(\frac{x - B^{-1/2}\eta}{B^\delta}\Big)$, $i = 1,2$. By a partial Fourier transform with respect to y, we can view A_i as an multiplication operator. Then, it is easy to see that $A_i^* = A_i$, $i = 1,2$, and

$$A_i^2 = Op^\omega G_i^2\Big(\frac{x - B^{-1/2}D_y}{B^\delta}\Big),\ A_1^2 + A_2^2 = 1\,.$$

Here $Op^\omega g(x, D_y)$ denotes the Weyl pseudodifferential operator with symbol $g(x,\eta)$.

Lemma 3. *Let $G \in C_0^\infty(\mathbb{R})$. Then*

$$\Big[Op^\omega G\Big(\frac{x - B^{-1/2}D_y}{B^\delta}\Big), \Pi\Big] = \mathcal{O}\Big(B^{-1/2-\delta}\Big)$$

in the space of bounded operators $\mathcal{L}(L^2(\mathbb{R}^2))$.

Proof. Choose a function $f \in C_0^\infty(]2n-2, 2n[)$ such that $f = 1$ near $2n-1$. Obviously, $f(D_y^2 + y^2) = \Pi$ and the pseudodifferential calculus yields $yf'(D_y^2 + y^2) = \mathcal{O}(1)$ in $\mathcal{L}(L^2(\mathbb{R}^2))$. Thus

$$\Big[Op^\omega G\Big(\frac{x - B^{-1/2}D_y}{B^\delta}\Big), \Pi\Big] = \Big[Op^\omega G\Big(\frac{x - B^{-1/2}D_y}{B^\delta}\Big), f(D_y^2 + y^2)\Big]$$

$$= B^{-1/2-\delta}\mathcal{O}\Big(Op^\omega(G')\Big(\frac{x - B^{-1/2}D_y}{B^\delta}\Big) yf'(D_y^2 + y^2)\Big) = \mathcal{O}(B^{-1/2-\delta})\,.$$

□

To estimate the norms of the commutators, we need the following

Lemma 4. *There exist constants $C_0 > 0$, $C_1 > 0$, independent on B, and $B_0 \gg 1$ so that for $B \geq B_0$ we have*

$$\|y \operatorname{Op}^\omega\Big(G_i'\Big(\frac{x - B^{-1/2}D_y}{B^\delta}\Big)\Big)(I - \Pi)u\|$$

$$\leq C_0\|(Q(B,\theta) - z)u\| + C_1\|(I - \Pi)u\|,\ i = 1,2,\ \forall z \in \Omega_n,\ u \in \mathcal{D}. \tag{4.2}$$

Proof. Introduce the symbol

$$g_i(x,y,\eta) = \frac{yG_i'\Big(B^{-\delta}(x - B^{-1/2}\eta)\Big)}{\eta^2 + y^2 + B^{-1}x + i}\,.$$

We will show that this symbol is in the class $S^0(\mathbb{R}^4)$. In fact, the derivative $\partial_x^l\partial_y^p\partial_\eta^k g_i(x,y,\eta)$ can be written as a sum of terms

$$B^{-(q-1)\delta}\frac{y^{p'+1}\eta^{k'}}{(\eta^2+y^2+B^{-1}x+i)^{k+p+l'}}G_i^{(q)}\Big(B^{-\delta}(x-B^{-1/2}\eta)\Big), q \geq 1$$

with $p' \leq p,\ k' \leq k,\ l' \geq l+1$. Setting $u = B^{-\delta}(x - B^{-1/2}\eta)$, we need to estimate

$$T_{l,p,k}(B,y,\eta,u) = \frac{y^{p+1}\eta^k}{\Big(\eta^2+y^2+B^{-1+\delta}u + B^{-3/2}\eta + i\Big)^{k+p+l}}G_i^{(q)}(u)$$

uniformly with respect to B, y, η, u. For $B^{-1+\delta}|u| \leq \frac{1}{2}(\eta^2+y^2)$, we have

$$|\eta^2+y^2+B^{-1+\delta}u+i| \geq c_0(\eta^2+y^2+1),\ c_0 > 0$$

and we get $T_{l,p,k} = \mathcal{O}(1)$ with respect to B, y, η, u. On the other hand, the support of $G_i^{(q)}(u)$ is bounded and $B^{-1+\delta}|u| \geq \frac{1}{2}(\eta^2+y^2)$ leads to $(\eta^2+y^2) \leq c_1B^{-1+\delta} \leq c_2$. Thus we obtain again $T_{l,p,k} = \mathcal{O}(1)$. Now consider the operator $Op^\omega g_i(x,y,D_y)$ with Weyl symbol $g_i(x,y,\eta)$. We have

$$Op^\omega g_i(x,y,D_y)(D_y^2+y^2+B^{-1}x+i) = yOp^\omega\Big(G_i'\Big(\frac{x-B^{-1/2}D_y}{B^\delta}\Big)\Big) + R_i(x,y,D_y)\,.$$

Using the explicit formulae of R_i given by the calculus of pseudodifferential operators, and repeating the above arguments, we see that the symbol of R_i is in the class $S^0(\mathbb{R}^4)$. It follows from the Calderon-Vaillancourt's theorem (see for instance, Theorem 7.11 of [9]) that $Op^\omega g_i(x,y,D_y)$ and R_i are bounded. Thus,

$$\|y\ Op^\omega\Big(G_i'\Big(\frac{x-B^{-1/2}D_y}{B^\delta}\Big)\Big)(I-\Pi)u\|$$
$$\leq \|Op^\omega g_i(x,y,D_y)(D_y^2+y^2+B^{-1}x+i)(I-\Pi)u\| + C_2\|(I-\Pi)u\|$$
$$\leq C_3\|(D_y^2+y^2+B^{-1}x+i)(I-\Pi)u\| + C_2\|(I-\Pi)u\|$$
$$\leq C_3\|(D_y^2+y^2+B^{-1}(x-z))(I-\Pi)u\| + C_2'\|(I-\Pi)u\|$$
$$\leq C_3\|\Big(B(D_y^2+y^2)+x+\theta-\frac{1}{4B^2}+V^\omega-z\Big)(I-\Pi)u\| + C_4\|(I-\Pi)u\|$$
$$\leq C_3\|(Q(B,\theta)-z)u\| + C_1\|(I-\Pi)u\|,\ \forall z \in \Omega_n,\ u \in \mathcal{D}\,.$$

Here we have used the fact that $B^{-1}z$ is bounded for $z \in \Omega_n$ as well as the estimate $[\Pi, V^\omega] = \mathcal{O}(B^{-1/2})$. □.

To estimate the action of $Q(B,\theta)$ on $A_i(I-\Pi)u,\ i = 1,2$, we need the following

Lemma 5. *There exists $a_1 > 0$, independent on B, and $B_0 \gg 1$ so that for $B \geq B_0$ and $z \in \Omega_n$ we have*

$$\Big\| \Big[(I-\Pi)\Big(B(D_y^2+y^2)+x+\theta+V^\omega\Big)(I-\Pi)-z\Big]A_1(I-\Pi)u\Big\| \geq a_1 B\|A_1(I-\Pi)u\| - \mathcal{O}(B^{1/2-\delta})\|(I-\Pi)u\|,\ u \in \mathcal{D}. \tag{4.3}$$

Lemma 6. *There exists $a_2 > 0$, independent on B, and $B_0 \gg 1$ so that for $B \geq B_0$ and $z \in \Omega_n$ we have*

$$\Big\| \Big[(I-\Pi)\Big(B(D_y^2+y^2)+x+\theta+V^\omega\Big)(I-\Pi)-z\Big]A_2(I-\Pi)u\Big\| \geq a_2|\mathrm{Im}\,\theta|\|A_2(I-\Pi)u\| - \mathcal{O}(B^{-1/2-\delta})\|(I-\Pi)u\|,\ u \in \mathcal{D}. \tag{4.4}$$

Assuming the above estimates established, we will complete the proof of Proposition 4.

Proof of Proposition 4. We have

$$\begin{aligned} F(u) &= \|(Q(B,\theta)-z)(I-\Pi)u\|^2 \\ &= \Big((A_1^2+A_2^2)(Q(B,\theta)-z)(I-\Pi)u, (Q(B,\theta)-z)(I-\Pi)u\Big) \\ &= \sum_{i=1,2} \|A_i(Q(B,\theta)-z)(I-\Pi)u\|^2\,. \end{aligned}$$

Thus

$$F(u) \geq \frac{1}{2}\sum_{i=1,2}\|(Q(B,\theta)-z)A_i(I-\Pi)u\|^2 - 2\sum_{i=1,2}\|[A_i,Q(B,\theta)](I-\Pi)u\|^2\,.$$

The operators A_i, Π commute with x and

$$\begin{aligned} [A_i,Q(B,\theta)] &= [A_i,(I-\Pi)B(D_y^2+y^2)(I-\Pi)] + [A_i,(I-\Pi)V^\omega(I-\Pi)] \\ &= [A_i,(I-\Pi)B(D_y^2+y^2)(I-\Pi)] + \mathcal{O}(B^{-1/2-\delta})\,, \end{aligned}$$

since by the pseudodifferential calculus we get

$$[A_i,\Pi] = \mathcal{O}(B^{-1/2-\delta}),\ [A_i,V^\omega] = \mathcal{O}(B^{-1/2-\delta}).$$

Next

$$[A_i,(I-\Pi)B(D_y^2+y^2)(I-\Pi)] = [A_i,B(D_y^2+y^2)](I-\Pi) - B(D_y^2+y^2)[A_i,\Pi]\,.$$

Then we have

$$B(D_y^2+y^2)[A_i,\Pi] = B^{1/2-\delta}(D_y^2+y^2)L_i f'(D_y^2+y^2),\ i=1,2$$

with operators L_i having symbols uniformly bounded with respect to B. The symbol of the operator on the right hand side of the above equality is bounded with its derivatives and we deduce

$$B(D_y^2+y^2)[A_i,\Pi] = \mathcal{O}(B^{1/2-\delta})$$

in the space of bounded operators $\mathcal{L}(L^2(\mathbb{R}^2))$. To treat the commutator with $B(D_y^2+y^2)$, we apply Lemma 4. It is clear that

$$[A_i,B(D_y^2+y^2)](I-\Pi) = \mathcal{O}(1)B^{1/2-\delta}y\,\mathrm{Op}^\omega\Big(G_i'(B^{-\delta}(x-B^{-1/2}D_y)\Big)(I-\Pi) + \mathcal{O}(B^{1/2-\delta})\,,$$

so for B large enough, according to (4.2), we obtain

$$\|(I-\Pi)[A_i,B(D_y^2+y^2)](I-\Pi)u\|$$

$$\leq C_5 B^{1/2-\delta}\|(Q(B,\theta)-z)(I-\Pi)u\| + C_6 B^{1/2-\delta}\|(I-\Pi)u\|,\ z\in\Omega_n,\ u\in\mathcal{D}\,,$$

with constants $C_5>0$, $C_6>0$ independent on z and B. Finally, taking into account the estimates (4.3), (4.4), we deduce

$$2(1-C_5' B^{1-2\delta})F(u) \geq \Big[a_1^2 B^2\|A_1(I-\Pi)u\|^2 + a_2^2|\mathrm{Im}\,\theta|^2\|A_2(I-\Pi)u\|^2\Big]$$
$$-C_7 B^{1-2\delta}\|(I-\Pi)u\|^2$$
$$\geq \Big[\Big(\min\{a_1 B, a_2|\mathrm{Im}\,\theta|\}\Big)^2 - C_7 B^{1-2\delta}\Big]\|(I-\Pi)u\|^2\,.$$

For B sufficiently large this implies the estimate (4.1). □

Proof of Lemma 5. First notice that the operator $D_y^2+y^2+1$ is elliptic, so

$$\|D_y u\| \leq \|(D_y^2+y^2+1)u\| + C_8\|u\| \leq \|(I-\Pi)(D_y^2+y^2)u\| + (C_8+2n+2)\|u\|$$
$$\leq B^{-1}\|(I-\Pi)\Big[B(D_y^2+y^2)-z\Big]u\| + C_8'\|u\|, \tag{4.5}$$

where we have used that $B^{-1}z$ is bounded for $z\in\Omega_n$. Second, applying the estimate (4.5) for the term

$$(I-\Pi)B^{-1/2}D_y A_1(I-\Pi)u,$$

we obtain

$$\|(I-\Pi)\Big[B(D_y^2+y^2)+x+\theta-\frac{1}{4B^2}+V^\omega-z\Big]A_1(I-\Pi)u\|$$
$$=\|(I-\Pi)\Big[B(D_y^2+y^2)+B^\delta\Big(\frac{x-B^{-1/2}D_y}{B^\delta}\Big)+B^{-1/2}D_y-\frac{1}{4B^2}+V^\omega-z\Big]A_1(I-\Pi)u\|$$
$$\geq (1-B^{-3/2})\|(I-\Pi)\Big[B(D_y^2+y^2)-z\Big]A_1(I-\Pi)u\|$$
$$-B^\delta\Big\|\Big(\frac{x-B^{-1/2}D_y}{B^\delta}\Big)A_1(I-\Pi)u\Big\| - C_9\|A_1(I-\Pi)u\|\,.$$

Our assumptions on Ω_n and the spectral theorem for $D_y^2+y^2$ imply

$$\|(I-\Pi)\Big[B(D_y^2+y^2)-z\Big]A_1(I-\Pi)u\| \geq a_1 B\|(I-\Pi)A_1(I-\Pi)\|$$
$$\geq a_1 B\|A_1(I-\Pi)u\| - \mathcal{O}(B^{1/2-\delta})\|(I-\Pi)u\|,\ a_1>0\,.$$

Next choose a function $\tilde{G}_1\in C_0^\infty(\mathbb{R};[0,1])$ with support close to that of G_1 and such that $\tilde{G}_1=1$ on supp G_1. Then

$$B^\delta\Big(\frac{x-B^{-1/2}D_y}{B^\delta}\Big)A_1(I-\Pi)u$$
$$=B^\delta\Big(\frac{x-B^{-1/2}D_y}{B^\delta}\Big)Op^\omega\tilde{G}_1\Big(\frac{x-B^{-1/2}D_y}{B^\delta}\Big)A_1(I-\Pi)u$$

and, since $u\tilde{G}_1(u)$ is bounded, we can estimate this term by $C_{10}B^\delta\|A_1(I-\Pi)u\|$. Taking B large enough, we complete the proof. □

Proof of Lemma 6. We have

$$\Big|\Big(\Big[(I-\Pi)\Big(B(D_y^2+y^2)+x+\theta-\frac{1}{4B^2}+V^\omega\Big)(I-\Pi)-z\Big]A_2(I-\Pi)u, A_2(I-\Pi)u\Big)\Big|$$

$$\geq -\mathrm{Im}\Big(\Big[(I-\Pi)\Big(B(D_y^2+y^2)+x-\frac{1}{4B^2}+\theta+V^\omega\Big)(I-\Pi)-z\Big]A_2(I-\Pi)u, A_2(I-\Pi)u\Big)$$

$$= (\mathrm{Im}\, z-\mathrm{Im}\,\theta)\|A_2(I-\Pi)u\|^2-\mathrm{Im}\Big((I-\Pi)V^\omega(I-\Pi)A_2(I-\Pi)u, A_2(I-\Pi)u\Big)$$

$$\geq \Big((1-\mu)|\mathrm{Im}\,\theta|-C_{11}B^{-1/2-\delta}\Big)\|A_2(I-\Pi)u\|^2$$

$$-\mathrm{Im}\Big((I-\Pi)V^\omega A_2(I-\Pi)u, A_2(I-\Pi)u\Big)-C_{11}B^{-1/2-\delta}\|(I-\Pi)u\|^2 .$$

Now we choose a function $\tilde{G}_2 \in C_b^\infty(\mathbb{R};[0,1])$ with support close to that of G_2 and such that $\tilde{G}_2=1$ on supp G_2, and replace $V^\omega A_2$ by $V^\omega Op^\omega \tilde{G}_2(B^{-\delta}(x-B^{-1/2}D_y))A_2$. On the support of $\tilde{G}_2(u)$ we have $|u|\geq 1$. Thus to treat the term

$$\mathrm{Im}\Big((I-\Pi)V^\omega Op^\omega\tilde{G}_2(B^{-\delta}(x-B^{-1/2}D_y))A_2(I-\Pi)u, A_2(I-\Pi)u\Big),$$

we take B large enough in order to arrange the estimate

$$\|V^\omega(x-B^{-1/2}D_y-\frac{1}{2B^2}, B^{-1/2}y+B^{-1}D_x)Op^\omega\tilde{G}_2(B^{-\delta}(x-B^{-1/2}D_y))\|\leq\frac{(1-\mu)}{2}|\mathrm{Im}\,\theta| .$$

The decay properties of the potential V and its derivatives and the pseudodifferential calculus make this possible, since on the support of the symbol $\tilde{G}_2(B^{-\delta}(x-B^{-1/2}\eta))$ we have

$$|x-B^{-1/2}\eta|\geq B^\delta .$$

Combining this with the above estimates, we complete the proof. □

Remark. By the same argument, we obtain the estimate

$$\|(Q^*(B,\theta)-\bar{z})(I-\Pi)u\|\geq\gamma|\mathrm{Im}\,\theta|\|(I-\Pi)u\|,\ z\in\Omega_n,\ u\in\mathcal{D} .$$

Thus the operator $Q(B,\theta)-z$ is bijective on Image $(I-\Pi)$ and we denote its inverse by

$$\hat{R}(z)=\Big((I-\Pi)\tilde{P}_1(B,\theta)(I-\Pi)-z\Big)^{-1}(I-\Pi). \tag{4.6}$$

A modification of the proof of Proposition 4 yields the following

Proposition 5. *Let $0<\alpha<1$, $\alpha_1>0$, $0<\mu<1$, $n\geq 0$ be fixed. Then for B large enough the operator $P_1(B)$ has no resonances z lying in the domains*

$$\Re z\leq\alpha B,\ \alpha_1 B\geq\ \mathrm{Im}\, z\geq\mu\,\mathrm{Im}\,\theta, \tag{4.7}$$

$$((2n+1)+\alpha)B\leq|\Re z|\leq((2n+3)-\alpha)B,\ \alpha_1 B\geq\mathrm{Im}\, z\geq\mu\,\mathrm{Im}\,\theta. \tag{4.8}$$

Proof. To treat the domain (4.8), it is sufficient to repeat the proof of Proposition 4 without the projector Π. For example, in the proof of Lemma 5 we estimate the term

$$\|\Big[B(D_y^2+y^2)-z\Big]A_1u\|\geq\alpha B\|A_1u\|$$

and we follow the same argument. To deal with z lying in (4.7), first assume that

$$\nu B\leq\Re z\leq\alpha B,\ \nu<0.$$

Then we can repeat the argument of the proof of Proposition 4 with Π replaced by 0, since $B^{-1}z$ is bounded. Now suppose that $\Re z\leq\nu B<0$. There are two points, where we have used the fact that $B^{-1}z$ is bounded. The first one is the estimate (4.3) in the proof of Lemma 5. For $\Re z < 0$, the operator $D_y^2 + y^2 - B^{-1}\Re z + 1$ is elliptic and since $B^{-1}\mathrm{Im}\, z$ is bounded, we get

$$\begin{aligned}\|D_y u\| &\leq \|(D_y^2 + y^2 - B^{-1}\Re z)u\| + C_9\|u\| \\ &\leq B^{-1}\|(B(D_y^2 + y^2) - z)u)\| + C_9\|u\|\,.\end{aligned}$$

Next we estimate the term

$$B^{-1/2} D_y A_1 u,$$

exploiting the fact that D_y and A_1 commutes and apply the above estimate. The second point is related to Lemma 4. For $\Re z < 0$ consider the symbol

$$g_i(x, y, \eta) = \frac{y G_i'\Big(B^{-\delta}(x - B^{-1/2}\eta)\Big)}{\eta^2 + y^2 + B^{-1}(x - \Re z) + i}\,.$$

It is easy to show that this symbol is in $S^0(\mathbb{R}^4)$. Then the proof of Lemma 4 goes without any change and we obtain (4.2). Finally, we get the estimate

$$\|\tilde{P}_1(B, \theta) - z)u\| \geq \gamma|\mathrm{Im}\,\theta|\|u\|,\ u \in \mathcal{D}\,,$$

and we conclude that $P_1(B)$ has no resonances z with $\Re z < 0$. □

5. Representation of the derivative of the spectral shift function for strong magnetic fields

Our purpose in this section is to prove Theorem 1 given in the Introduction. We use the notations of the previous sections and we work in the domain Ω_n. Consider the operators

$$L_1(B, \theta) = (I - \Pi)\Big(B(D_y^2 + y^2) + x + \theta - \frac{1}{4B^2} + V^\omega\Big)(I - \Pi)\,,$$

$$L_2(B, \theta) = \Pi\Big(B(D_y^2 + y^2) + x + \theta - \frac{1}{4B^2}\Big)\Pi\,,$$

$$W^\omega = (I - \Pi)V^\omega\Pi + \Pi V^\omega(I - \Pi) + \Pi V^\omega \Pi\,.$$

It is clear that

$$L_1(B, \theta) + L_2(B, \theta) - z + W^\omega = \tilde{P}_1(B, \theta) - z\,.$$

The operator $\tilde{L}(B, \theta) - z := L_1(B, \theta) + L_2(B, \theta) - z$ is invertible for $z \in \Omega_n$. In fact, we have

$$\|(\tilde{L}(B, \theta) - z)u\|^2 = \|(L_1(B, \theta) - z)(I - \Pi)u\|^2 + \|(L_2(B, \theta) - z)\Pi u\|^2\,.$$

For the first term at the right hand side we apply Proposition 4, while for the second one we estimate the imaginary part of $(L_2(B, \theta) - z)\Pi u, \Pi u)$. Thus for $z \in \Omega_n$ we obtain

$$\|(L(B, \theta) - z)u\|^2 \geq \gamma_1\|(I - \Pi)u\|^2 + \gamma_2\|\Pi u\|^2 \geq \gamma_3\|u\|^2,\ \gamma_j > 0,\ j = 1, 2, 3\,.$$

Since $[\Pi, V^\omega] = \mathcal{O}(B^{-1/2})$, for B large enough, the operator

$$L(B, \theta) - z := \tilde{L}(B, \theta) + (I - \Pi)V^\omega\Pi + \Pi V^\omega(I - \Pi) - z$$

is invertible for $z \in \Omega_n$. On the other hand, $K = \Pi V^\omega \Pi$ is a pseudodifferential operator in $L^2(\mathbb{R}^2)$ with principal symbol

$$f(\eta^2 + y^2)V\Big(x - B^{-1/2}\eta - \frac{1}{2B^2}, B^{-1/2}y + B^{-1}\xi\Big)f(\eta^2 + y^2),$$

and we conclude that K is a trace class one. Moreover, one obtains immediately the estimate

$$\|K\|_{\mathrm{tr}} \le CB \tag{5.1}$$

with a constant $C > 0$, independent on B. Thus we have the following

Theorem 3. *Let B be sufficiently large. Then for $z \in \Omega_n$ we have*

$$z - \tilde{P}_1(B,\theta) = z - L(B,\theta) - K \tag{5.2}$$

and the operator $z - L(B,\theta)$ is invertible for $z \in \Omega_n$.

By using the above theorem, it is easy to establish directly the existence of a meromorphic continuation of the resolvent $(\tilde{P}_1(B,\theta) - z)^{-1}$ for $z \in \Omega_n$. In fact, we write

$$z - \tilde{P}_1(B,\theta) = [I - K(z - L(B,\theta))^{-1}](z - L(B,\theta)),$$

and we conclude that the operator $[I - K(z - L(B,\theta))^{-1}]$ has a meromorphic continuation for $z \in \Omega_n$. In the next section we will construct an effective operator $E_{1,-+}(z)$ so that the eigenvalues of $\tilde{P}_1(B,\theta)$, and hence those of $P_1(B,\theta)$, coincide with the zero eigenvalues of $E_{1,-+}(z)$. This will be more convenient for the analysis of the free resonances regions.

Introduce the functions

$$\sigma_\pm(z) = (z^2+1)\mathrm{tr}\Big[(P_j(B) - i)^{-1}(P_j(B)+i)^{-1}(z - P_j(B))^{-1}\Big]_0^1, \ \pm \mathrm{Im}\, z > 0,$$

where $[a_j]_0^1 = a_1 - a_0$. It follows from Proposition 2 that $\sigma_\pm(z)$ are well defined and we have

$$\sigma_-(z) = \overline{\sigma_+(\bar{z})},\ \mathrm{Im}\, z < 0\,.$$

For θ real the operator $(P_j(B) - i)^{-1}(P_j(B)+i)^{-1}(z - P_j(B))^{-1}$ is unitary equivalent to

$$(\tilde{P}_j(B,\theta) - i)^{-1}(\tilde{P}_j(B,\theta)+i)^{-1}(z - \tilde{P}_j(B,\theta))^{-1}.$$

Consequently, the cyclicity of the trace yields

$$\sigma_+(z) = (z^2+1)\mathrm{tr}\Big[(\tilde{P}_j(B,\theta) - i)^{-1}(\tilde{P}_j(B,\theta)+i)^{-1}(z - \tilde{P}_j(B,\theta))^{-1}\Big]_0^1 \tag{5.3}$$

for all $z \in \Omega_+ = \Omega_n \cap \{\mathrm{Im}\, z > 0\}$, $\theta \in D(0,\theta_0) \cap \mathbb{R}$.

Now, fix $\delta > 0$ and let $z \in \Omega_\delta = \Omega_n \cap \{\mathrm{Im}\, z \ge \delta\}$. Since $\tilde{P}_j(B,\theta)$ extends to an analytic type-A family of operators on $D(0,\theta_0)$, for sufficiently small θ_0 and $z \in \Omega_\delta$, the r.h.s of (5.3) extends by analytic continuation in θ to the disk $D(0,\theta_0)$. For $\theta \in D(0,\theta_0)$ with $\mathrm{Im}\,\theta < 0$, both terms of (5.3) are holomorphic on Ω_+, and, consequently, (5.3) remains true for all z in Ω_+.

From now on, the number θ will be fixed in $D(0,\theta_0)$ with $\mathrm{Im}\,\theta < 0$. We drop the subscript B, θ, most of the time and write P_j, L instead of $\tilde{P}_j(B,\theta)$, $L(B,\theta)$. For simplicity of the notations we set $B = h^{-1}$. As we have proved, there exists a trace class operator K, $\|K\|_{\mathrm{tr}} = \mathcal{O}(h^{-1})$, such that $P_1 = L - K$ and

$$(L-z)^{-1} = \mathcal{O}(1): \ \ L^2(\mathbb{R}^2) \to \mathcal{D} \text{ uniformly for } z \in \Omega_n\,.$$

Then $(z - P_1) = (I + \tilde{K}(z))(z - L)$ with $\tilde{K} = K(z-L)^{-1}$ and the resonances $z \in \mathrm{Res}\, P_1(B)$ coincide with their multiplicities with the zeros of the function

$$D(z,h) = \det(I + \tilde{K}(z)).$$

Then, as in Proposition 3 in [11], we obtain the upper bound

$$\#\{z \in \mathrm{Res}\, P_1 : \ z \in \Omega_n\} \le C(\Omega_n)h^{-1}. \tag{5.4}$$

For the function $\sigma_+(z)$ we have the following

Proposition 6. *There exists a function $a_+(z,h)$, holomorphic in Ω_n, such that for $z \in \Omega_n \cap \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ we have*

$$\sigma_+(z) = \operatorname{tr}\Big((P_1 - z)^{-1} K (L-z)^{-1}\Big) + a_+(z,h). \tag{5.5}$$

Moreover,

$$|a_+(z,h)| \leq C_1(\Omega_n) h^{-1},\ z \in \Omega_n. \tag{5.6}$$

Proof. The proof is similar to that of Proposition 3 in [11], so we will omit some details. We write $\sigma_+(z) = I_1(z) + I_2(z)$, where

$$I_1(z) = (z^2+1)\operatorname{tr}\Big((P_1 - i)^{-1}(P_1+i)^{-1}(z-L)^{-1}$$

$$-(P_0-i)^{-1}(P_0+i)^{-1}(z-P_0)^{-1}\Big),$$

$$I_2(z) = (z^2+1)\operatorname{tr}\Big((P_1-i)^{-1}(P_1+i)^{-1}(P_1-z)^{-1}K(z-L)^{-1}\Big).$$

As in [11], Section 3, by using the resolvent equation and the cyclicity of the trace, we show that $I_2(z)$ is equal to $\operatorname{tr}((P_1-z)^{-1}K(L-z)^{-1})$ modulo a function holomorphic in a neighborhood of Ω_n and satisfies (5.6). Next we decompose $I_1(z)$ as follows

$$I_1(z) = (z^2+1)\operatorname{tr}\Big(\Big[(P_j-i)^{-1}(P_j+i)^{-1}\Big]_{j=0}^{1}(z-L)^{-1}\Big)$$

$$+(z^2+1)\operatorname{tr}\Big((P_0-i)^{-1}(P_0+i)(z-P_0)^{-1}K(z-L)^{-1}\Big)$$

$$+(z^2+1)\operatorname{tr}\Big((P_0-i)^{-1}(P_0+i)^{-1}(z-P_0)^{-1}V^{\omega}(z-L)^{-1}\Big),$$

and we conclude that $I_1(z)$ is holomorphic in Ω_n. For the first and the second terms in the above equality we obtain bounds $\mathcal{O}(h^{-1})$. In fact, for the first term we apply the argument of the proof of Proposition 2 with $B = h^{-1}$, while for the second one we use the estimate (5.1) combined with the estimate of the resolvent $(z-L)^{-1}$. Finally, to estimate the trace of the term

$$(P_0-i)^{-1}(P_0+i)^{-1}V^{\omega}\Big(x+\theta-h^{1/2}D_y - \frac{1}{2}h^2, h^{1/2}y + hD_x\Big),$$

we write $V^{\omega} = AV^{\omega}A + (I-A)V^{\omega}A + AV^{\omega}(I-A) + (I-A)V^{\omega}(I-A)$ with the operator A having symbol $q(x,y,\eta)$, introduced in the proof of Proposition 2, and we follow the argument of this proposition. □

According to Lemma 1 in [11], for every $f \in C_0^{\infty}(\mathbb{R})$ we have

$$\langle \xi', f\rangle = \lim_{\epsilon \searrow 0} \frac{i}{2\pi} \int f(\lambda)\Big[\sigma_+(\lambda + i\epsilon) - \sigma_-(\lambda - i\epsilon)\Big] d\lambda\,,$$

where the limit is taken in the sense of distributions. Following without any change the proof in Section 6 of [11], we obtain Theorem 1.

6. Effective Hamiltonian for strong magnetic fields

In this section we use freely the notations of the previous section. In particular, the set Ω_n and the projection Π are associated to a Landau level $(2n-1)B$ with a fixed integer $n \geq 1$. Let us introduce the operators

$$R_+ : \; L^2(\mathbb{R}^2_{x,y}) \ni v \longrightarrow (\langle v(x,\cdot), \psi_n(\cdot)\rangle_{L^2(\mathbb{R}_y)} = \int_{\mathbb{R}_y} v(x,y)\overline{\psi_n(y)}dy \in L^2(\mathbb{R}_x),$$

$$R_- = R_+^* : \; L^2(\mathbb{R}_x) \ni u \longrightarrow u(x)\psi_n(y) \in L^2(\mathbb{R}^2_{x,y}).$$

These operators satisfy

$$R_+R_- = I_{L^2(\mathbb{R}_x)}, \qquad R_-R_+ = \Pi.$$

Consider the following Grushin problem:

$$\mathcal{P}_0(z) = \begin{pmatrix} \tilde{P}_0(B,\theta) - z & R_- \\ R_+ & 0 \end{pmatrix} : \; \mathcal{D} \times L^2(\mathbb{R}_{x,y}) \to L^2(\mathbb{R}^2_{x,y}) \times L^2(\mathbb{R}_x).$$

By a simple computation, we have the following

Lemma 7. *The operator $\mathcal{P}_0(z)$ is uniformly invertible for $z \in \Omega_n$ and $\theta \in D(0,\theta_0)$. Its inverse is holomorphic in (z,θ) and has the form*

$$\mathcal{E}_0(z) = \begin{pmatrix} E_0(z) & E_{0,+} \\ E_{0,-} & E_{0,-+}(z) \end{pmatrix},$$

where

$$E_0(z) = \widehat{R}_0(z), \; E_{0,+} = R_-, \; E_{0,-} = R_+, \; E_{0,-+}(z) = z - (2n-1)B - x - \theta + \frac{1}{4B^2},$$

$$\widehat{R}_0(z) = \Big((I-\Pi)\tilde{P}_0(B,\theta)(I-\Pi) - z\Big)^{-1}(I-\Pi).$$

Now consider the Grushin problem for the perturbed Hamiltonian $\tilde{P}_1(B,\theta) - z$:

$$\mathcal{P}_1(z) = \begin{pmatrix} \tilde{P}_1(B,\theta) - z & R_- \\ R_+ & 0 \end{pmatrix}. \tag{6.1}$$

Proposition 7. *For B large enough the operator $\mathcal{P}_1(z)$ is invertible for $z \in \Omega_n$ and its inverse is given by*

$$\mathcal{E}_1(z) = \begin{pmatrix} E_1(z) & E_{1,+}(z) \\ E_{1,-}(z) & E_{1,-+}(z) \end{pmatrix},$$

where

$$E_1(z) = \hat{R}(z)a(z), \tag{6.2}$$

$$E_{1,-}(z) = R_+a(z), \tag{6.3}$$

$$E_{1,+}(z) = -\hat{R}(z)a(z)[V^\omega,\Pi]R_- + R_-, \tag{6.4}$$

$$E_{1,-+}(z) = -R_+a(z)[V^\omega,\Pi]R_- + \Big[z - (2n-1)B - x - \theta + \frac{1}{4B^2} - R_+V^\omega R_-\Big]. \tag{6.5}$$

Here $a(z) = \Big(I + [\Pi, V^\omega]\hat{R}(z)\Big)^{-1}$ and $\hat{R}(z)$ is given by (4.6).

Proof. Set

$$\tilde{\mathcal{E}}(z) = \begin{pmatrix} \hat{R}(z) & R_- \\ R_+ & z - \left[(2n-1)B + x + \theta - \frac{1}{4B^2} + R_+ V^\omega R_-\right] \end{pmatrix}.$$

We have

$$V^\omega R_- - R_- R_+ V^\omega R_- = V^\omega R_- R_+ R_- - \Pi V^\omega R_- = [V^\omega, \Pi] R_- = \mathcal{O}(B^{-1/2}).$$

A simple calculus implies

$$\mathcal{P}_1(z) \circ \tilde{\mathcal{E}}(z) = \begin{pmatrix} I + [\Pi, V^\omega]\hat{R}(z) & [V^\omega, \Pi] R_- \\ 0 & I \end{pmatrix}$$

$$= I + \mathcal{O}(B^{-1/2}),$$

since $[V^\omega, \Pi] = \mathcal{O}(B^{-1/2})$. Consequently, for B large enough, the operator $\mathcal{P}_1(z)$ has a right inverse and we get

$$\mathcal{E}_1(z) = \tilde{\mathcal{E}}(z) \circ \begin{pmatrix} (I + [\Pi, V^\omega]\hat{R}(z))^{-1} & -(I + [\Pi, V^\omega]\hat{R}(z))^{-1}[V^\omega, \Pi] R_- \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} \hat{R}(z)a(z) & -\hat{R}(z)a(z)[V^\omega, \pi]\mathbb{R}_- + R_- \\ R_+ a(z) & -R_+ a(z)[V^\omega, \Pi]R_- + z - (2n-1)B - x - \theta + \frac{1}{4B^2} - R_+ V^\omega R_- \end{pmatrix}.$$

This completes the proof. □

To study the properties of the operators $E_{j,-+}(z)$, $j = 0, 1$, we set $B^{-1} = h$. We need the following

Lemma 8. *We have*

$$[V^\omega, \Pi] : L^2(\mathbb{R}^2) \longrightarrow L^2(\mathbb{R}^2) \in \mathrm{Op}^{\,\omega}\Big(S^0\Big(\mathbb{R}^4; \langle x\rangle^{-2-\epsilon}\langle B\xi\rangle^{-1-\epsilon}\langle y\rangle^{-\infty}\langle \eta\rangle^{-\infty}\Big)\Big). \tag{6.6}$$

Proof. Recall that we may write $\Pi = f(D_y^2 + y^2)$ with $f \in C_0^\infty(](2n-2, 2n[)$. According to Theorem 8.7 in [9], we have

$$\Pi \in \mathrm{Op}^\omega\big(S^0(\mathbb{R}^2; \langle y\rangle^{-\infty}\langle\eta\rangle^{-\infty})\big).$$

Since the operator $V^\omega \in \mathrm{Op}^\omega\Big(S^0(\mathbb{R}^4; \langle x\rangle^{-2-\epsilon}\langle B\xi\rangle^{-1-\epsilon})\Big)$, we obtain (6.6) by the calculus of pseudo-differential operators. □

Proposition 8. *The operator $E_{1,-+}(z) - E_{0,-+}(z)$ is a h-pseudodifferential one with Weyl symbol $a(x,\xi;z,h) \in S^0(\mathbb{R}^2; \langle x\rangle^{-2-\epsilon}\langle\xi\rangle^{-1-\epsilon})$ such that*

$$a(x,\xi;z,h) \sim \sum_{j=0}^{\infty} a_j(x,\xi;z)h^j, \tag{6.7}$$

where

$$a_0(x,\xi;z) = -V(x+\theta,\xi),\ a_1(x,\xi;z) = -(2n-1)\Delta_{x,\xi}V(x+\theta,\xi)/4. \tag{6.8}$$

Proof. The proof is similar to that of Proposition 2.5 in [7], so we will omit some details. We have

$$E_{1,-+}(z) - E_{0,-+}(z) = -R_+ a(z)[V^\omega, \Pi]R_- - R_+ V^\omega R_-.$$

The operator $R_+ V^\omega R_- : L^2(\mathbb{R}_x) \longrightarrow L^2(\mathbb{R}_x)$ has the form

$$(R_+ V^\omega R_- u)(x) = \Big\langle V^\omega \Big(x + \theta - \frac{1}{2}h^2 - h^{1/2}D_y, h^{1/2}y + hD_x\Big)(u(x)\psi_n(y)), \psi_n(y)\Big\rangle_{L^2(\mathbb{R}_y)}.$$

This implies that the symbol of $R_+ V^\omega R_-$ is given by

$$J(h^{1/2}) = \langle V^\omega(x + \theta - \frac{1}{2}h^2 - h^{1/2}D_y,\ h^{1/2}y + \xi)\psi_n(y), \psi_n(y)\rangle_{L^2(\mathbb{R}_y)}.$$

The estimate

$$\begin{aligned} &\left|\partial_x^\alpha \partial_\xi^\beta V^\omega(x + \theta - h^{1/2}\eta, h^{1/2}y + \xi)\right| \\ &\le C_{\alpha,\beta}\langle x\rangle^{-2-\epsilon}\langle \xi\rangle^{-1-\epsilon}\langle \eta\rangle^{2+\epsilon}\langle y\rangle^{1+\epsilon} \end{aligned}$$

and the fact that $\psi_n(y) = e^{-\frac{y^2}{2}}P_n(y)$, $P_n(y)$ being a polynomial, imply

$$(x,\xi) \to V^\omega(x + \theta - \frac{1}{2}h^2 - h^{1/2}D_y, h^{1/2}y + \xi) \in S^0\Big(\mathbb{R}^2; \langle x\rangle^{-2-\epsilon}\langle\xi\rangle^{-1-\epsilon}\Big).$$

Applying Taylor's formula, we obtain

$$\begin{aligned} &V^\omega(x + \theta - h^{1/2}D_y - \frac{1}{2}h^2, h^{1/2}y + \xi) = V^\omega(x + \theta, \xi) \\ &- h^{1/2}D_y\partial_x V^\omega(x + \theta, \xi) + h^{1/2}y\partial_\xi V^\omega(x + \theta, \xi) + \dots. \end{aligned}$$

Since $\psi_n(-y) = (-1)^n\psi_n(y)$, we have $\langle D_y^{2k+1}\psi_n, \psi_n\rangle = 0$ for all $k \in \mathbb{N}$. This implies that $J(h^{1/2}) = J(-h^{1/2})$, so $J(h^{1/2})$ has an asymptotic expansion in power of h (see Proposition 4.3 in [7] for more details.) Thus the symbol of $R_+ V^\omega R_-$ satisfies (6.7). To show that

$$R_+ a(z)[V^\omega, \Pi]R_- \in \mathrm{Op}^\omega\Big(S^0(\mathbb{R}^2; \langle x\rangle^{-2-\epsilon}\langle\xi\rangle^{-1-\epsilon})\Big), \tag{6.9}$$

first we prove that $a(z)[V^\omega, \Pi]$ is a h-pseudodifferential operator and next we repeat the above argument combined with Lemma 8. This completes the proof. □

From the construction of the Grushin operators one obtains the following well known formulae (see for instance, [15], [6], [7]):

$$\begin{aligned} (z - \tilde{P}_j(B,\theta))^{-1} &= -E_j(z) + E_{j,+}(z)E_{j,-+}^{-1}(z)E_{j,-}(z), \\ E_{j,-+}(z)^{-1} &= -R_+(\tilde{P}_j(B,\theta) - z)^{-1}R_+^*, j = 0, 1. \end{aligned}$$

Consequently,

$$z \in \sigma(\tilde{P}_j(B,\theta)) \Leftrightarrow 0 \in \sigma(E_{j,-+}(z)),\ j = 0, 1. \tag{6.10}$$

Recall that the operators $\tilde{P}_j(B,\theta)$ are unitarily equivalent to the operators $P_j(B,\theta)$, so the eigenvalues of these operators coincide. According to Proposition 8, the analysis of the invertibility of the operator $E_{1,-+}(z)$ is reduced to that of the operator

$$z - (2n-1)h^{-1} - x - \theta + \frac{1}{4}h^2 + a^\omega(x, hD_x; z),$$

where $a^\omega(x, hD_x; z)$ is a h-pseudodifferential operator with Weyl symbol

$$a(x, \xi; z, h) \sim \sum_{j=0}^{\infty} a_j(x, \xi; z) h^j$$

given by (6.7).

Proof of Proposition 1. Set

$$\begin{aligned} A^w(x, hD_x, z) &:= z - (2n-1)B - x - \theta - \frac{1}{4B^2} - V^w(x+\theta, hD_x) \\ &= E_{0,-+}(z) - V^w(x+\theta, hD_x)\,. \end{aligned}$$

Clearly,

$$E_{0,-+}^{-1}(z) = \Big(z - (2n-1)B - x - \theta - \frac{1}{4B^2}\Big)^{-1} \in Op^\omega(S^0(\mathbb{R}^2)),$$

and

$$\|E_{0,-+}^{-1}(z)\| \le \frac{1}{(1-\mu)|\mathrm{Im}\,\theta|}, \tag{6.11}$$

uniformly with respect to $z \in \Omega_n$.

Let $R > 0$ be a large constant such that

$$\sup_{|x|>R,\, \xi\in\mathbb{R}} |\partial_{x,\xi}^\alpha V(x+\theta, \xi)| < \frac{(1-\mu)|\theta|}{2},\ |\alpha| \le N_0,$$

where N_0 is an integer independent on B and n. In fact, N_0 depends on the choice of a semi-norm in the space of symbols $S^0(\mathbb{R}^2)$ which by the Calderon -Vaillancourt's theorem concerning L^2 continuity of Weyl pseudodifferential operators is equivalent to the norm in the space of bounded operator $\mathcal{L}(L^2(\mathbb{R}^2))$ (see Theorem 7.11 in [9]). For

$$z \in \tilde{\Omega}_n := \{z \in \mathbb{C} :\ |\Re z - (2n-1)| > 2 \sup_{(x,\xi)\in\mathbb{R}^2,\, |\alpha|\le N_0} |\partial_{x,\xi}^\alpha V(x+\theta,\xi)| + |\Re\theta| + R\},$$

we have

$$\sup_{(x,\xi)\in\mathbb{R}^2,\, |\alpha|\le N_0} \Big|\partial_{x,\xi}^\alpha\Big(\Big(z - (2n-1)B - x - \theta - x\Big)^{-1} V(x+\theta,\xi)\Big)\Big| < \frac{1}{2}. \tag{6.12}$$

To see this, it suffices to notice that, for $z \in \tilde{\Omega}_n$ and $|x| < R$, we have

$$|z - (2n-1)B - \theta - x| > 2 \sup_{(x,\xi)\in\mathbb{R}^2,\, |\alpha|\le N_0} |\partial_{x,\xi}^\alpha V(x+\theta,\xi)|.$$

It follows from the Calderon- Vaillancourt theorem that for h small enough

$$\|V^w(x+\theta, hD_x) E_{0,-+}^{-1}(z)\| \le \frac{1}{2}.$$

Combining this with (6.11) and using the equality

$$A^w(x, hD_x, z) = (I - V^w(x+\theta, hD_x) E_{0,-+}^{-1}(z)) E_{0,-+}(z),$$

we deduce that, for h small enough we have

$$\|A^w(x, hD_x, z)^{-1}\| \le \frac{2}{(1-\mu)|\mathrm{Im}\,\theta|}.$$

By Proposition 8, we get $E_{1,-+}(z) = A^w(x, hD_x, z) + \mathcal{O}_n(h)$. Here the estimate of the norm $\mathcal{O}_n(h)$ depends on n, since the lower order symbol of $a^\omega(x, hD_x)$ is given by

$$-(2n-1)\Delta V(x+\theta, y)/4 .$$

So for $0 < h \leq h(n)$ we obtain the invertibility of $E_{1,-+}(z)$, and according to (6.10), this completes the proof. □

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