Wavelet techniques for *p*-exponent multifractal analysis

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based on joint works with

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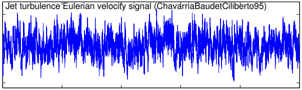
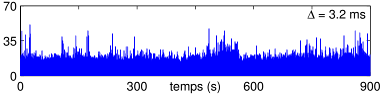
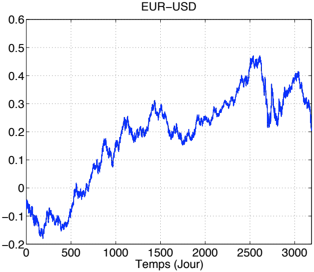
Type of data concerned with Multifractal Analysis



Fully developed turbulence



Internet trafic



Euro vs Dollar (2001-2009)

Everywhere irregular data

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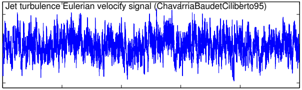

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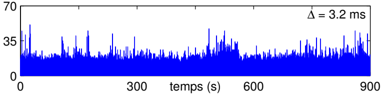

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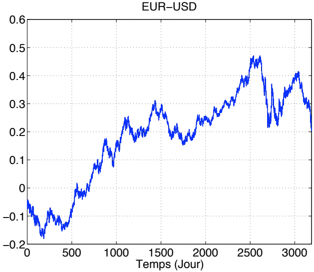

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Everywhere irregular data

Data that share the same statistical properties should be classified as identical

Pointwise exponent

One associates to such data a pointwise regularity exponent $h(x)$ which describes how the regularity fluctuates from point to point

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Examples :

Let μ be a Probability measure on $\mathbb{R}^d$ and $x_0 \in \mathbb{R}^d$
$\mu \in M^\alpha(x_0)$ if there exist $C > 0$ such that

$$|\mu(B(x_0, r))| \leq C\, r^\alpha$$

The local dimension of μ at x_0 is $h_\mu(x_0) = \sup\{\alpha : \quad \mu \in M^\alpha(x_0)\}$

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Let f be a locally bounded function $\mathbb{R}^d \to \mathbb{R}$ and $x_0 \in \mathbb{R}^d$
$f \in C^\alpha(x_0)$ if there exist $C > 0$ and a polynomial P such that

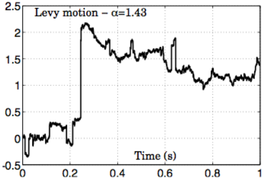
$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$$

The Hölder exponent of f at x_0 is $h_f(x_0) = \sup\{\alpha : \quad f \in C^\alpha(x_0)\}$

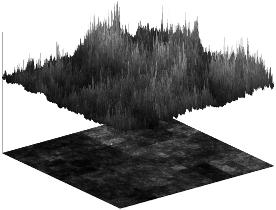
Difficulty to use directly the pointwise regularity exponent for classification

For classical models, such exponents are extremely erratic :

The Hölder exponent
of “most” Lévy processes



The Local dimension
of multiplicative cascades



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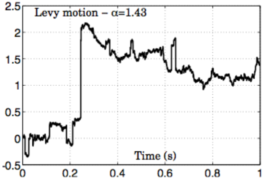

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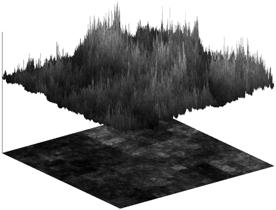

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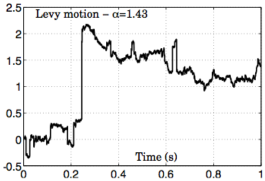

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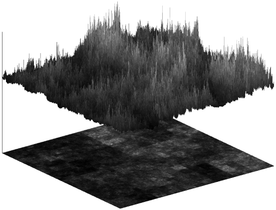

The function h is everywhere discontinuous

Goal : Recover some information on $h(x)$ from (time or space) averaged quantities that are numerically computable on a sample path of the process, or on real-life data

A general framework : Admissible exponents

Dyadic cubes : $\lambda = \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_d}{2^j}, \frac{k_d+1}{2^j}\right)$

$\lambda_j(x_0)$ denotes the dyadic cube of scale j that contains x_0

Dyadic cubes at scale j : $\Lambda_j = \{\lambda : \quad |\lambda| = 2^{-j}\}$

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Definition : A positive sequence (d_λ) is a hierarchical sequence if

$$\exists \alpha \in \mathbb{R} \text{ such that if } \lambda' \subset \lambda \text{ then } 2^{-\alpha j'} d_{\lambda'} \leq 2^{-\alpha j} d_\lambda$$

The exponent h defined by : $\quad h(x_0) = \liminf_{j \to +\infty} \left(\frac{\log\left(d_{\lambda_j(x_0)}\right)}{\log(2^{-j})} \right)$

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Proposition : An admissible exponent $h(x)$ is the liminf of a family of continuous functions

Multifractal Analysis

Notation : 3λ denote the cube of same center as λ and three times wider (it is the union of λ and its $3^d - 1$ immediate neighbours)

Examples :

If μ is a probability measure, then h_μ is admissible : Take $d_\lambda = \mu(3\lambda)$

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Multifractal spectrum :

The isohölder sets are the sets

$$E_H = \{x_0 : \quad h(x_0) = H\}$$

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The multifractal associated with the exponent h is

$$D(H) = \dim\,(E_H)$$

where dim stands for the Hausdorff dimension ($\dim\,(\emptyset) = -\infty$)

Multifractal formalism

The scaling function associated with a hierarchic sequence (d_λ) is defined by

$$\forall q \in \mathbb{R}, \qquad 2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^q \sim 2^{-\eta(q) j}$$

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Since η is a concave function, there is no loss of information in rather considering the Legendre Spectrum

$$L(H) = \inf_{q \in \mathbb{R}} (d + Hq - \eta(q))$$

Theorem : Let (d_λ) be an admissible sequence

$$\forall H \in \mathbb{R}, \qquad D(H) \leq \inf_{q \in \mathbb{R}} (d + Hq - \eta(q))$$

Alternative admissible sequences for the Hölder exponent

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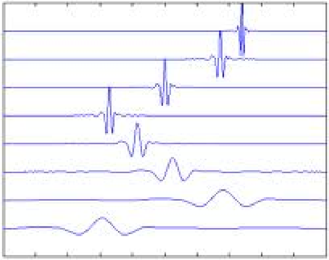
A wavelet basis on $\mathbb{R}$ is generated by a smooth, well localized, oscillating function ψ such that the

$\psi(2^j x - k), \quad j, k \in \mathbb{Z}$

form an orthogonal basis of $L^2(\mathbb{R})$

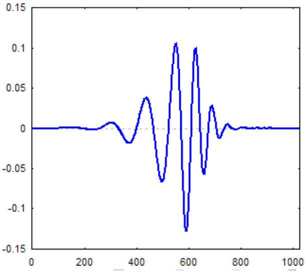


$\forall f \in L^2(\mathbb{R})$,

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}\, \psi(2^j x - k)$$

where

$$c_{j,k} = 2^j \int f(x)\, \psi(2^j x - k)\, dx$$



Daubechies Wavelet

Credit to : http ://www.kfs.oeaw.ac.at/content/blogcategory/0/502/lang,8859-1/

Notations for wavelets on $\mathbb{R}$

Dyadic intervals

$$\lambda = \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)$$

Wavelets

$$\psi_\lambda(x) = \psi(2^j x - k)$$

Wavelet coefficients

$$c_\lambda = 2^j \int_{\mathbb{R}} f(x)\psi(2^j x - k)dx$$

Dyadic intervals at scale j

$$\Lambda_j = \{\lambda : \quad |\lambda| = 2^{-j}\}$$

Wavelet expansion of f

$$f(x) = \sum_j \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda(x)$$

Wavelets in 2 variables

In 2D, the wavelets used are tensor products :

$$\psi^1(x,y) = \psi(x)\varphi(y)$$

$$\psi^2(x,y) = \varphi(x)\psi(y)$$

$$\psi^3(x,y) = \psi(x)\psi(y)$$

Notations

Dyadic squares : $\lambda = \left[\frac{k}{2^j}, \frac{(k+1)}{2^j}\right] \times \left[\frac{l}{2^j}, \frac{(l+1)}{2^j}\right[$

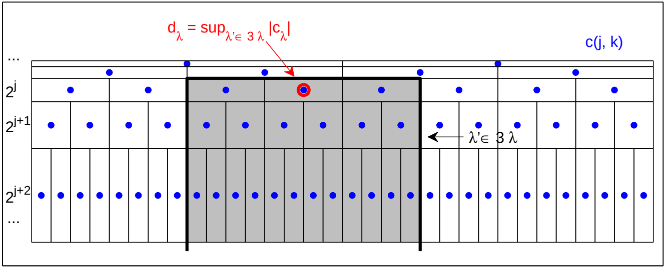
Wavelet coefficients

$$c_\lambda = 2^{2j} \int\int f(x,y)\, \psi^i\left(2^j x - k, 2^j y - l\right)\, dx\, dy$$

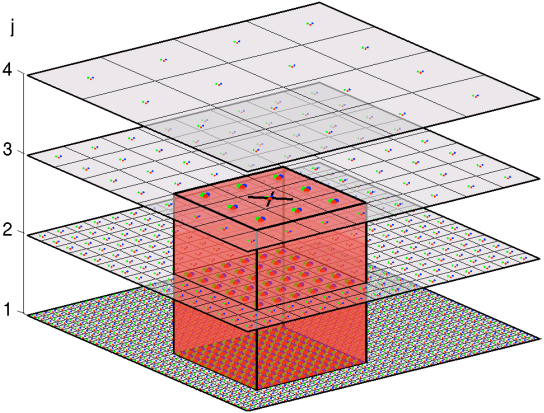
Wavelet leaders

Let f be a locally bounded function ; the wavelet leaders of f are

$$d_\lambda = \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|$$



Computation of 2D wavelet leaders



Proposition : Let f be a uniform Hölder function ($f \in C^{\varepsilon}(\mathbb{R}^d)$ for an $\varepsilon > 0$). If one uses the wavelet leaders d_λ for hierarchical sequence, then the associated pointwise exponent is the Hölder exponent

Uniform Hölder regularity

How can one check that the data correspond to a locally bounded function ?

Hölder spaces : Let $\alpha \in (0,1)$; $f \in C^\alpha(\mathbb{R}^d)$ if $f \in L^\infty$ and

$$\exists C, \ \ \forall x, y : \qquad |f(x) - f(y)| \leq C \cdot |x-y|^\alpha$$

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Numerical computation from the wavelet coefficients

Let $\omega_j = \sup_{\lambda \in \Lambda_j} |c_\lambda|$ then $H_f^{min} = \liminf_{j \to +\infty} \frac{\log(\omega_j)}{\log(2^{-j})}$

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$$H_f^{min} > 0 \implies f \text{ is continuous}$$

$$H_f^{min} < 0 \implies f \text{ is not locally bounded}$$

Is $H_{min} > 0$ fulfilled in applications ?

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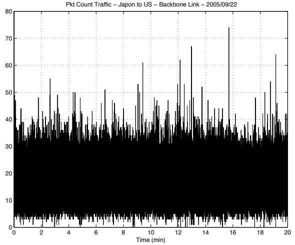
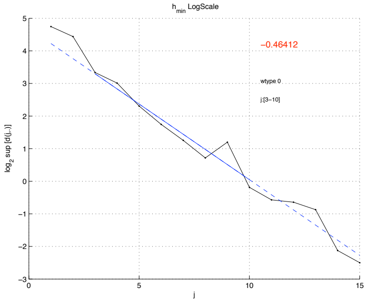
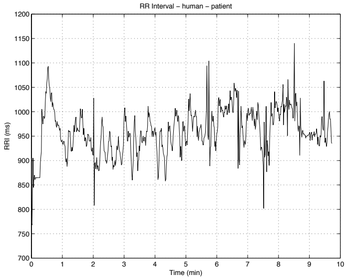
Internet Traffic

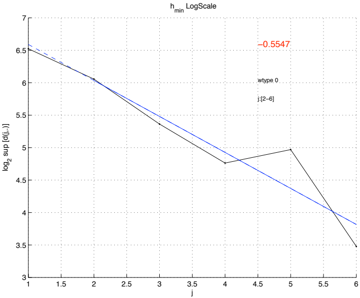


$$H_{min} = -0.46$$

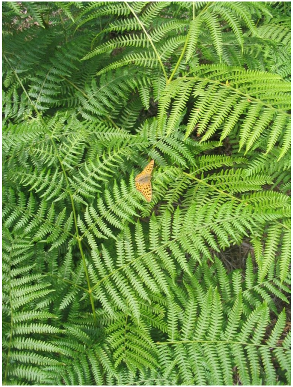


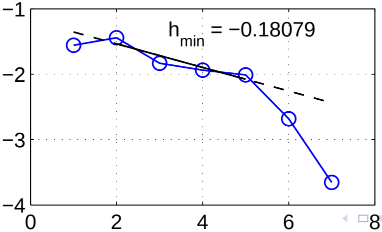
Heartbeat Intervals



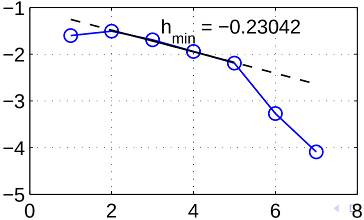


$H_{min} = -0.55$





h min = −0.18079
−1
−2
−3
−4
0
2
4
6
8



h_min = −0.23042
−1
−2
−3
−4
−5
0
2
4
6
8

Pointwise regularity with negative exponents ?

Pointwise Hölder regularity : $f \in C^\alpha(x_0)$ if

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$$

If $\alpha < 0$, this definition implies that, outside of x_0, f is locally bounded

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Clue : The definition of pointwise Hölder regularity can be rewritten

$$f \in C^\alpha(x_0) \iff \sup_{B(x_0,r)} |f(x) - P(x - x_0)| \le Cr^\alpha$$

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Definition (Calderón and Zygmund) : Let $f \in L^p(\mathbb{R}^d)$; $f \in T^p_\alpha(x_0)$ if there exists a polynomial P such that for r small enough,

$$\left(\frac{1}{r^d} \int_{B(x_0,r)} |f(x) - P(x - x_0)|^p dx \right)^{1/p} \leq Cr^\alpha$$

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The p-exponent of f at x_0 is $h_p(x_0) = \sup\{\alpha : f \in T^p_\alpha(x_0)\}$

The p-spectrum of f is $d^p(H) = dim \ (\{x_0 : \quad h_p(x_0) = H\})$

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Remarks :

- The case $p = +\infty$ corresponds to pointwise Hölder regularity
- The normalization is chosen so that a cusp $|x - x_0|^\alpha$ has the same p-exponent for all p : $h_p(x_0) = \alpha$ (as long as $\alpha \geq -d/p$)

How can one check that the data belong to L^p ?

The wavelet scaling function is informally defined by

$$\forall p > 0 \qquad 2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \sim 2^{-\zeta_f(p) j}$$

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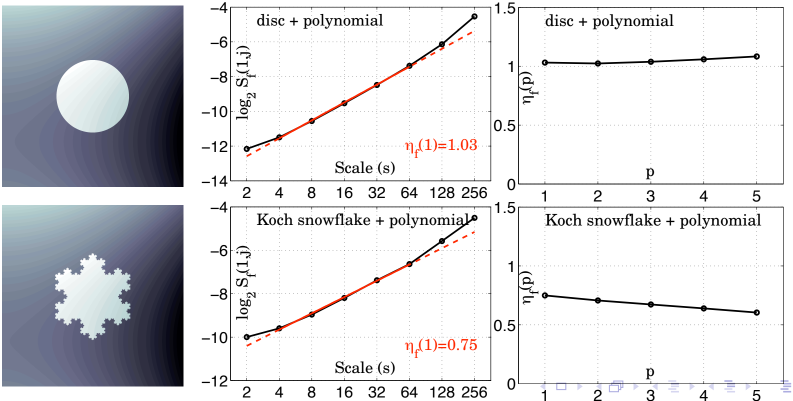
- If $\zeta_f(p) > 0$, then $f \in L^p$
- If $\zeta_f(p) < 0$, then $f \notin L^p$

Wavelet scaling functions of synthetic images

Wavelet scaling function $\zeta_f(p)$:

$$2^{-2j} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \sim 2^{-\zeta_f(p)\, j}$$

Disk : $\zeta_f(p) = 1$ Van Koch snowflake : $\zeta_f(p) = 2 - \frac{\log 4}{\log 3} \sim 0.74$



Properties of p-exponents

Gives a mathematical framework to the notion of negative regularity exponents

The p-exponent satisfies : $h_p(x_0) \geq -\frac{d}{p}$

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p-exponents may differ :

Theorem : Let f be an L^1 function, and $x_0 \in \mathbb{R}^d$. Let

$$p_0 = \sup\{p : f \in L^p_{loc}(\mathbb{R}^d) \text{ in a neighborhood of } x_0\}$$

The function $p \to h_p(x_0)$ is defined on $[1, p_0)$ and possesses the following properties :

1. It takes values in $[-d/p, \infty]$
2. It is a decreasing function of p.
3. The function $r \to h_{1/r}(x_0)$ is concave.

Furthermore, Conditions 1 to 3 are optimal.

When do p-exponents coincide ?

Notation : $h_{p,\gamma}(x_0)$ denotes the p-exponent of the fractional integral of f of order s at x_0

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Theorem : This notion is independent of p and γ

Types of pointwise singularities (Yves Meyer)

Typical pointwise singularities :

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After one integration :

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More generally, after a fractional integration of order s,

- If f has a cusp at x_0, then $h_{\mathcal{I}^s f}(x_0) = h_f(x_0) + s$
- If f has an oscillating singularity at x_0, then $h_{\mathcal{I}^s f}(x_0) = h_f(x_0) + (1 + \beta)s$

Further classification

Two types of oscillating singularities :

"Full singularities" : $|x - x_0|^H \sin\left(\frac{1}{|x - x_0|^\beta}\right)$ (p-exponents coincide)

"Skinny singularities " $|x - x_0|^H 1_{E_\gamma}$ where

$$E_\gamma = \bigcup \left[\frac{1}{n}, \frac{1}{n} + \frac{1}{n^\gamma}\right] \quad \text{for } \gamma > 1$$

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oscillating singularities : $F(x) = |x - x_0|^H g\left(\frac{1}{|x - x_0|^\beta}\right) + r(x)$

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For "full singularities" g is "large at infinity"

For "skinny singularities " g is "small at infinity"

Admissible sequences for the p-exponent

Definition : Let $f : \mathbb{R}^d \to \mathbb{R}$ be locally in L^p ; the p-leaders of f are

$$d_\lambda^p = \left(\sum_{\lambda' \subset 3\lambda} |c_{\lambda'}|^p 2^{d(j-j')} \right)^{1/p}$$

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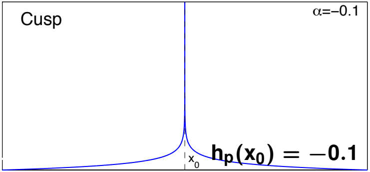
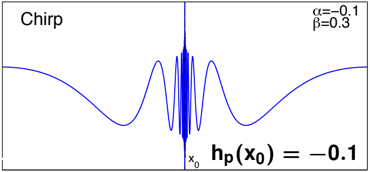
Theorem : (C. Melot)
If $\eta_f(p) > 0$, then h_p is the admissible exponent associated with the sequence d_λ^p

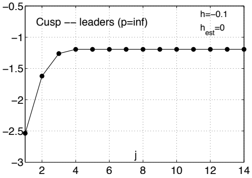
$$h_p(x_0) = \liminf_{j \to +\infty} \left(\frac{\log\left(d^p_{\lambda_j(x_0)}\right)}{\log(2^{-j})} \right)$$

p-leaders and negative regularity

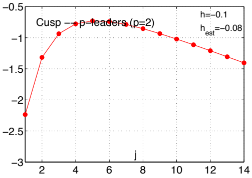




leaders : $\hat{h}(x_0) = \mathbf{0.00}$



p-leaders : $\hat{h}(x_0) = \mathbf{-0.08}$



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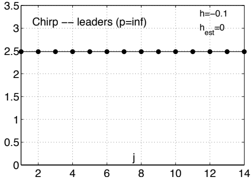
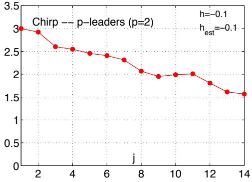


p-leaders : $\hat{h}(x_0) = \mathbf{-0.10}$



p-Multifractal Formalism

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Stability properties :

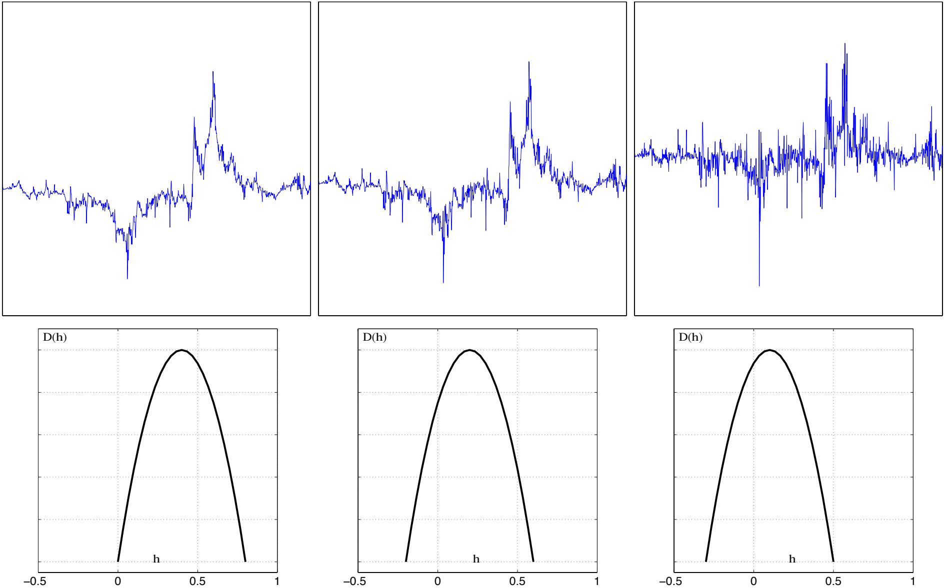
- Invariant with respect to deformations
- independent of the wavelet basis

The p-Legendre Spectrum is

$$L_p(H) = \inf_{q \in \mathbb{R}} \left(d + Hq - \eta_p(q)\right)$$

MRW $(H = 0.72, \lambda = \sqrt{0.08})$

$p_0 = +\infty \; h_X^{(min)} = 0$ | $p_0 = 10 \; h_X^{(min)} = -0.2$ | $p_0 = 2.5 \; h_X^{(min)} = -0.3$



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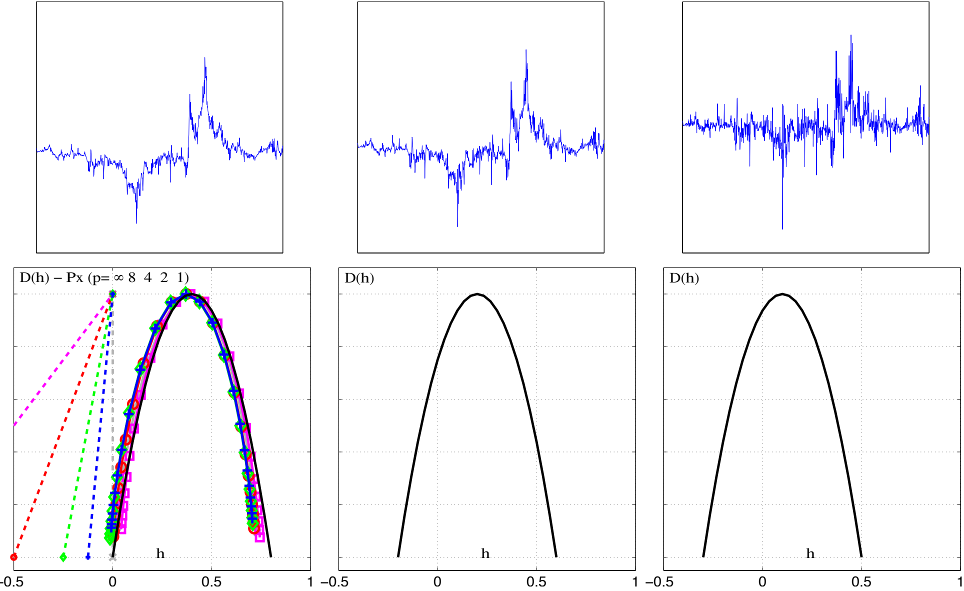

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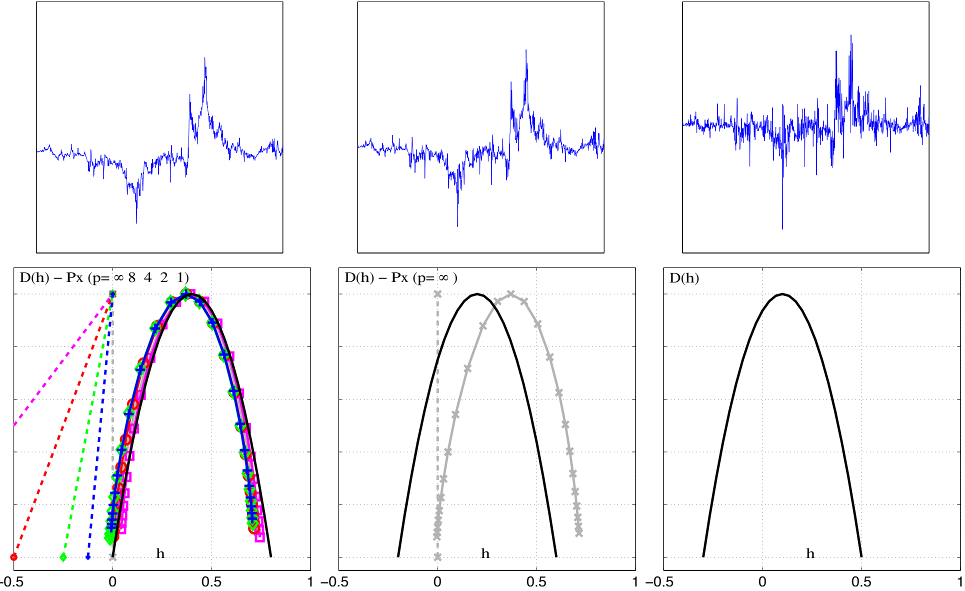

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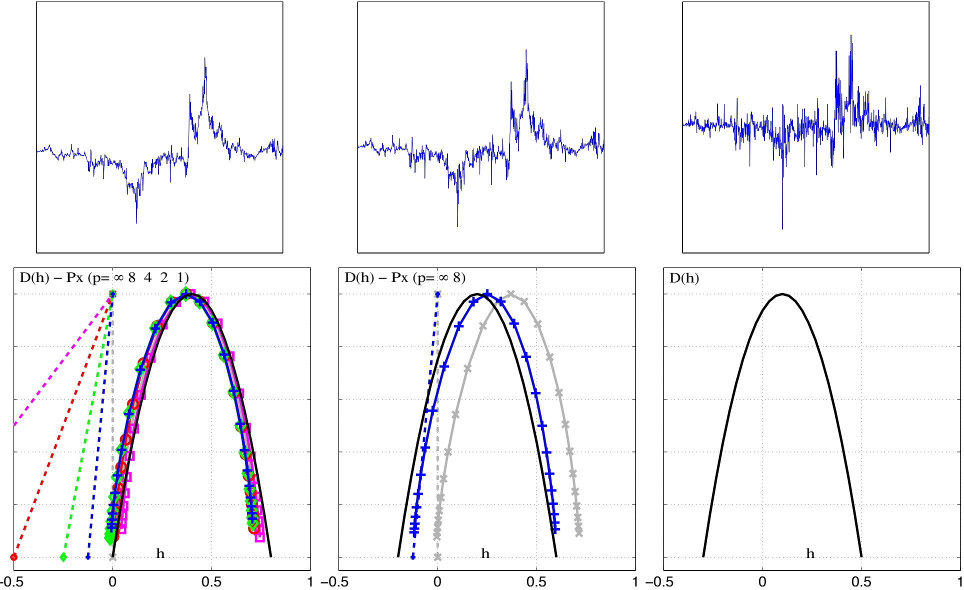

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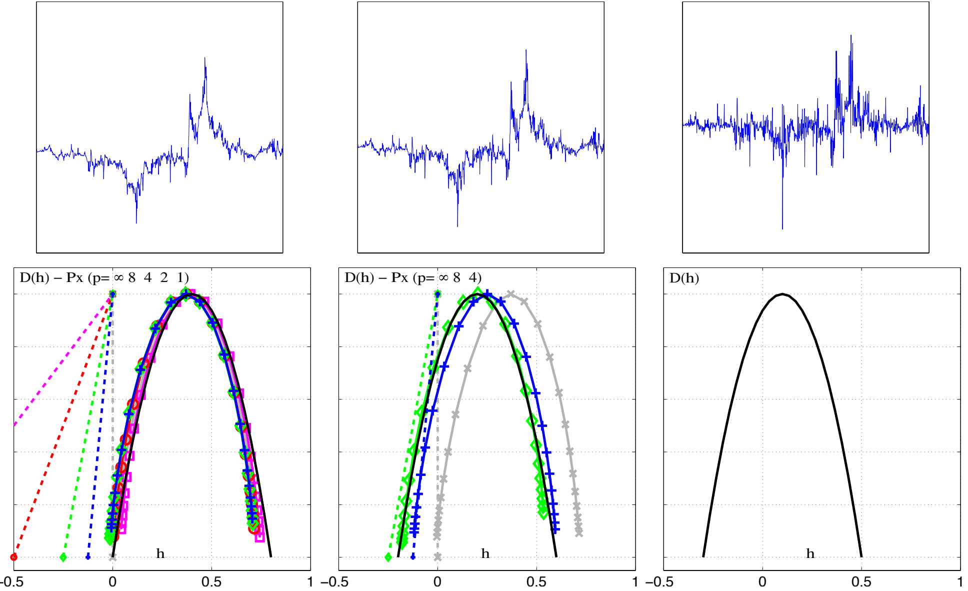

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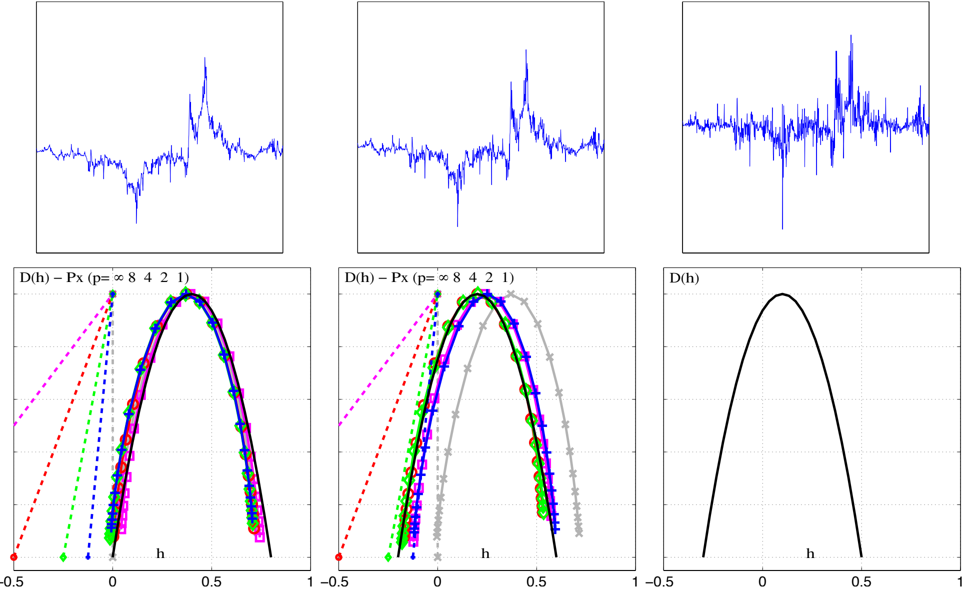

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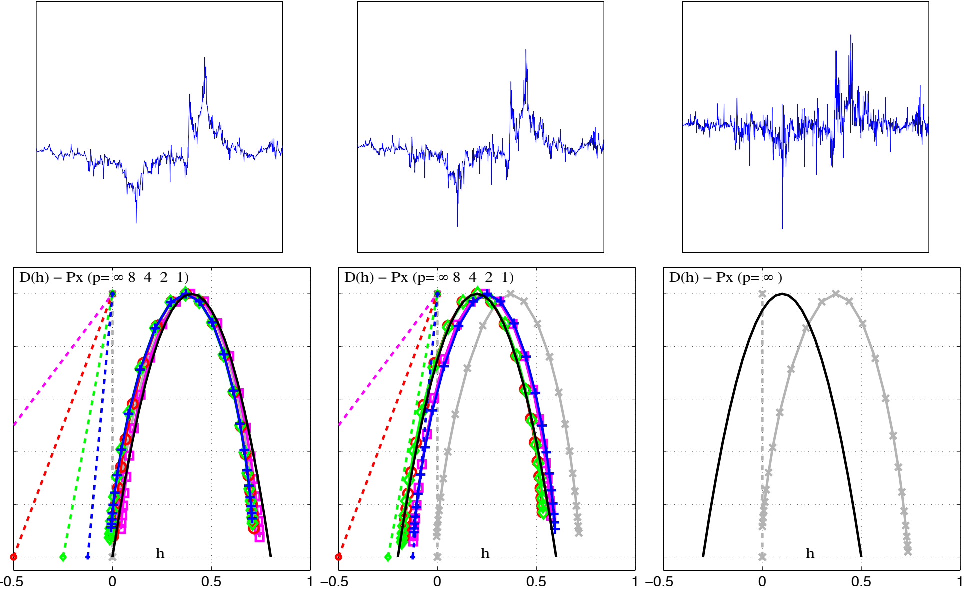

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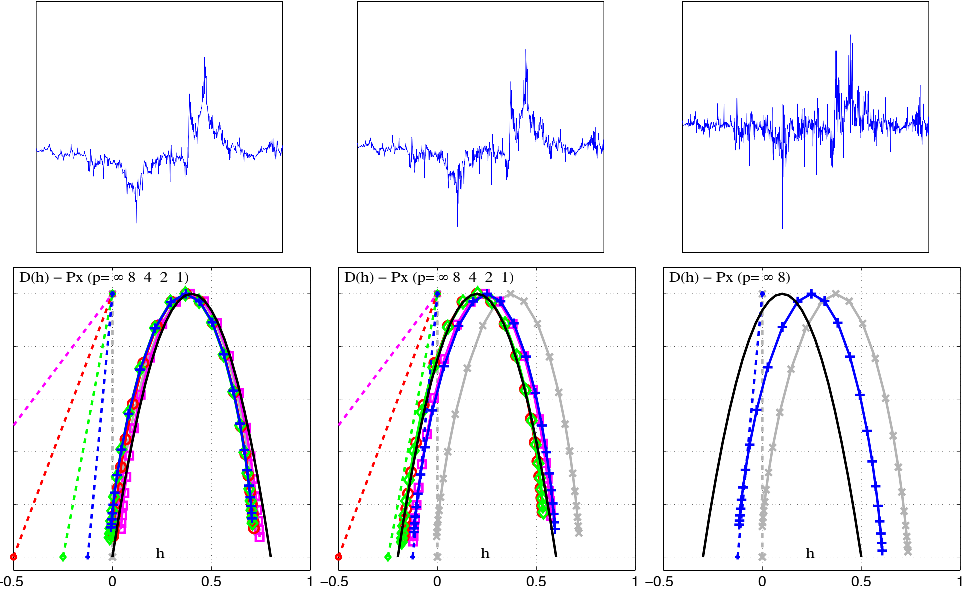

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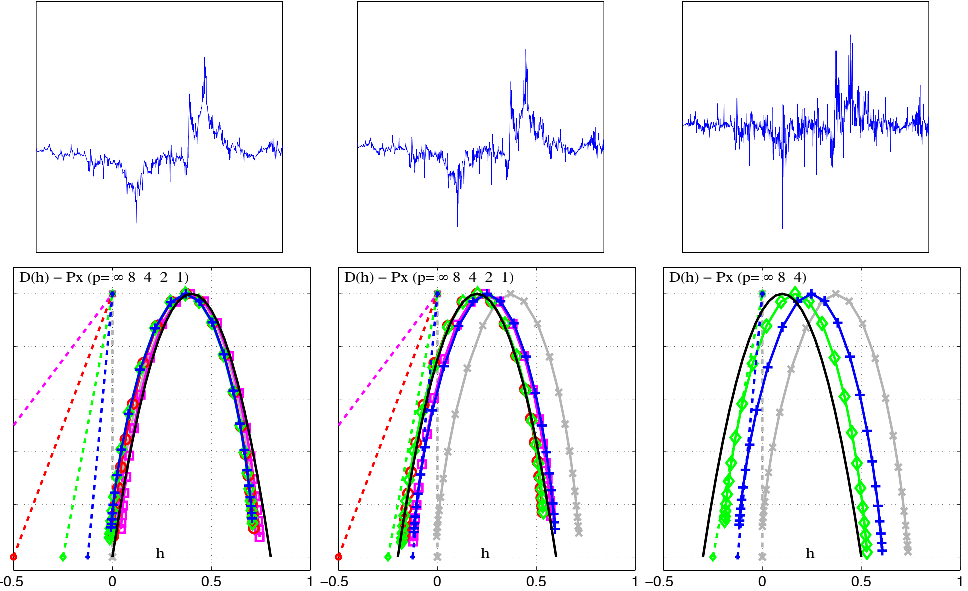

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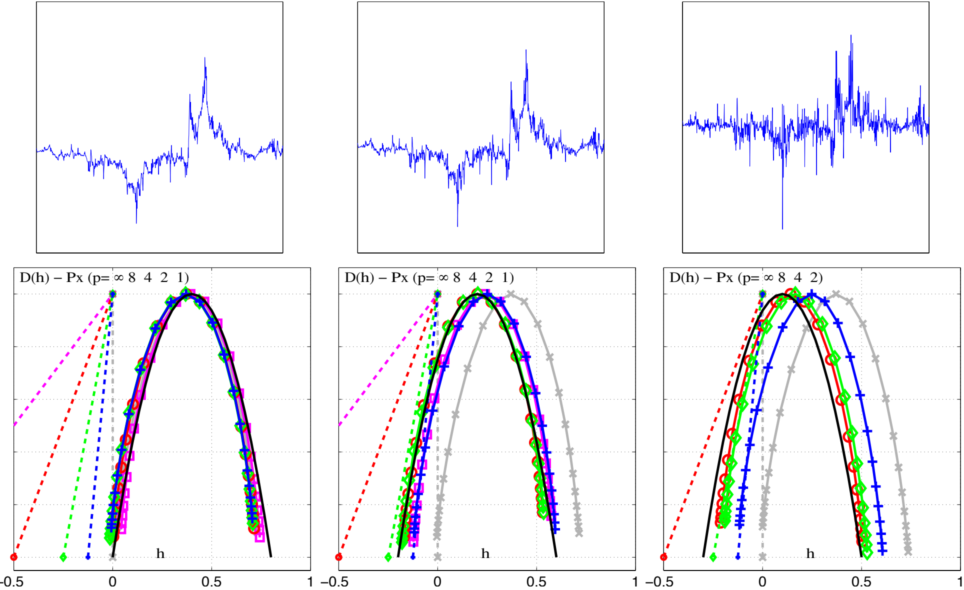

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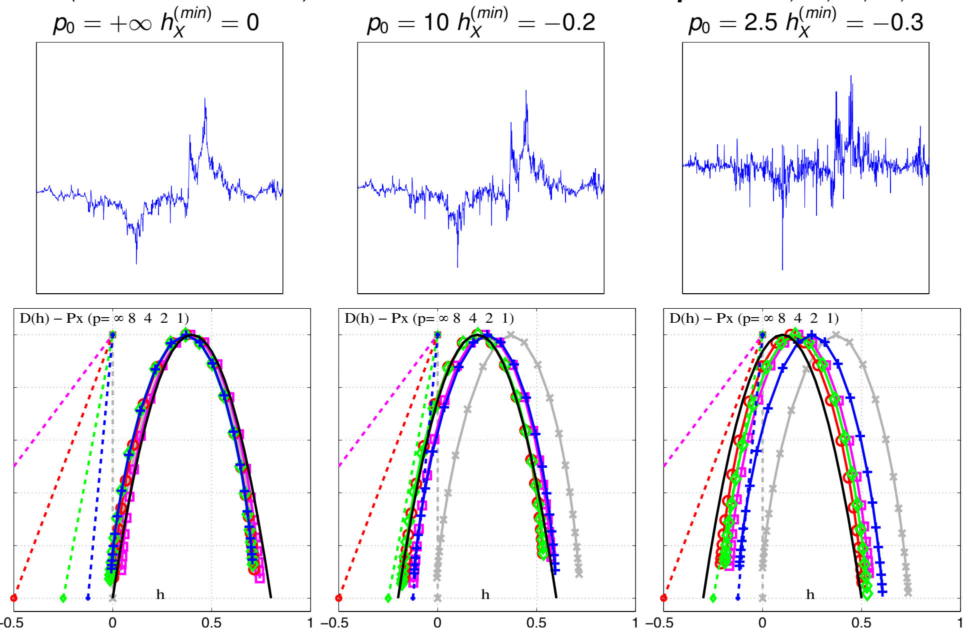

Advantages and drawbacks of multifractal analysis based on the p-exponent

Advantages and drawbacks of multifractal analysis based on the *p*-exponent

- Allows to deal with larger collections of data
- The estimation is not based on a unique extremal value, but on an l^p average $\Rightarrow$ better statistical properties
- Systematic bias in the estimation of *p*-leaders

Thank you for your attention !