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Cours de M2

Analyse Harmonique, théorie des opérateurs et contrôle

M2 Course

Harmonic analysis, operator theory and control

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Contents

Notation

1. Warning

These notes are intended for the course given in the second year of the masters program "Analysis, PDEs, Probability" at Université de Bordeaux. They are intended for students in that programonly and the pre-requisites are basic Lebesgue integration, L^p spaces, Fourier analysis and minimal distribution theory (that I am trying to avoid for this course). The first 3 chapters (Fourier transform, L^p spaces and convolution) cover in a large part material from the first year of the masters program and are only included here for the convenience of the reader. Note also that taking this into account, the order in which this course has been given may differ from the order of the notes.

Further, these notes do **not pretend to be original in any way**. The path taken is rather classical by now and follows in part lectures I followed as a student. Also, numerous colleagues have made their lecture notes available online. While preparing this course I have often consulted online courses and some material I have read may inconciously have made its way into these notes. I am unable to give a full list of course notes I have consulted, but the following are those that I have used the most:

- Giovanni Leoni, Lecture on harmonic analysis at Carnegie Mellon

  http://giovannileoni.weebly.com/teaching.html
- Ioannis Parissis, Lecture on harmonic analysis

  https://sites.google.com/site/ioannisparissis/teaching?authuser=0
- Terrence Tao, Lecture notes for MATH 247A : Fourier analysis at UCLA

  https://www.math.ucla.edu/~tao/247a.1.06f/

Finally, this is only a short introduction to a vast subject. The following books have been a good source for this course and also provide a good starting point to go deeper into the subject. They have all been used at some stage during the preparation of these lecture notes.

- Javier Duoandikoetxea, Fourier Analysis. Translated and revised from the 1995 Spanish original by David Cruz-Uribe. Graduate Studies in Mathematics, 29. American Mathematical Society, Providence, RI, 2001.
- Loukas Grafakos, Classical Fourier analysis. Graduate Texts in Mathematics, 249. Springer, New York, NY, 2008.
- Loukas Grafakos, Modern Fourier analysis. Graduate Texts in Mathematics, 250. Springer, New York, NY, 2008.
- Yitzhak Katznelson, An introduction to harmonic analysis. Dover Publications, Inc., New York, NY, 1976.
- Elliott H. Lieb and Michael Loss, Analysis. Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.
- Camil Muscalu and Wilhelm Schlag, Classical and multilinear harmonic analysis. Vol. I–II. Cambridge Studies in Advanced Mathematics, 137–8. Cambridge University Press, Cambridge, 2013.
- Elias M. Stein and Guido Weiss, Introduction to Fourier analysis on Euclidean spaces. Princeton University Press, Princeton, NJ, 1971.
- Elias M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton University Press, Princeton, NJ, 1993.
- Thomas H. Wolff, Lectures on harmonic analysis. American Mathematical Society, Providence, RI, 2003.

2. Main notations

2.1. Special functions. The Γ function is

$$\Gamma(x)=\int_0^{+\infty}t^xe^{-t}\,\frac{\mathrm{d}t}{t}.$$

The β function is

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

The Bessel function is

$$\begin{aligned}J_\nu(t) &= \left(\frac{t}{2}\right)^\nu\frac{1}{\Gamma\left(\nu+\frac12\right)\Gamma\left(\frac12\right)}\int_{-1}^1 e^{ist}(1-s^2)^\nu\,\frac{\mathrm{d}t}{\sqrt{1-s^2}}\\ &= \sum_{n=0}^{+\infty}\frac{1}{\Gamma(\nu+n+1)}\frac{(-1)^n}{n!}\left(\frac{t}{2}\right)^{\nu+2n}.\end{aligned}$$

The Newton potential is given by

$$\Gamma(t)=\begin{cases}\dfrac{1}{2\pi}\log|t| & \text{when } d=2\\ \dfrac{1}{d(2-d)\omega_d}|t|^{2-d} & \text{when } d\geq 3\end{cases}$$

and the fundamental solution of the laplacian in $\mathbb{R}^d$ is also denoted by $\Gamma(x,y)=\Gamma(x-y)$:

$$\Gamma(x,y)=\begin{cases}\dfrac{1}{2\pi}\log|x-y| & \text{when } d=2\\ \dfrac{1}{d(2-d)\omega_d}|x-y|^{2-d} & \text{when } d\geq 3\end{cases}.$$

Multiindex notation. For $\alpha=(\alpha_1,\ldots,\alpha_d)\in\mathbb{N}^d$, $x=(x_1,\ldots,x_d)\in\mathbb{R}^d$ and $f:\mathbb{R}^d\to\mathbb{C}$ sufficiently smooth,

– $|\alpha|=\alpha_1+\cdots+\alpha_d$, the lenght of α;

– $\alpha!=\alpha_1!\cdots\alpha_d!$ and $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}\cdots\begin{pmatrix}\alpha_d\\ \beta_d\end{pmatrix}=\dfrac{\alpha!}{\beta!(\alpha-\beta)!}$

– $x^\alpha=x_1^{\alpha_1}\cdots x_d^{\alpha_d}$ and

$$\partial^\alpha f=\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}.$$

Measures, norms, sets. Throughout this notes, $(\Omega,\mathcal{B},\mu)$ is a σ-finite measure space. Usually Ω is an open domain in $\mathbb{R}^d$ in which case $\mathcal{B}$ is the σ-algebra of Borel sets and μ is the Lebesgue measure $\mathrm{d}x$.

We will denote by $|\cdot|$ different things that depend on the context:

– If $x=(x_1,\ldots,x_d)\in\mathbb{R}^d$ is a vector, $|x|=\sqrt{x_1^2+\cdots+x_d^2}$ is the Euclidean norm of x. The associated scalar product in denoted $\langle x,y\rangle$.

– If F is a finite set, $|F|$ is the nulber of elements of F.

– If $E\subset\mathbb{R}^d$ is a Borel set, $|E|$ is its Lebesgue measure.

– When $z\in\mathbb{C}$, $|z|$ is its modulus.

At occasion, we may prefer to use an other norm on $\mathbb{R}^d$, most often $\|x\|_\infty=\max_{i=1,\ldots,d}|x_i|$. The open ball centered at c and of radius r associated to $|\cdot|$ (or any other norm) are denoted $B(c,r)=\{x\in\mathbb{R}^d\,:\,|x-c|<r\}$. Wehn we use the $\|\cdot\|_\infty$ norm, we will rather write this ball $Q(c,r)=\{x\in\mathbb{R}^d\,:\,\|x-c\|_\infty<r\}$ and call it the cube Q centered at c of length $\ell(Q)=2r$. For a ball $B=B(c,r)$ or a cube $Q=Q(c,t)$ we will write $3B=B(c,3r)$ and $3Q=Q(c,3r)$ (and more generally aB, aQ).

We write $\mathbb{S}^{d-1}=\{x\in\mathbb{R}^d\,:\,|x|=1\}$ is the unit sphere of $\mathbb{R}^d$ and σ_{d-1} is the surface measure on $\mathbb{S}^{d-1}$.

$$\mathbb{R}_+^{d+1}=\{(x_1,\ldots,x_d,t)\int\mathbb{R}^d\times\mathbb{R}_+^*\}.$$

A dyadic interval is an interval of the form $I_{j,k} = [2^{-k}j, 2^{-k}(j+1)[$ and a dyadic cube is a set of the form $Q_{j,k} = \prod_{\ell=1}^d [2^{-k}j_\ell, 2^{-k}(j_\ell+1)[$. Every dyadic cube can be divided into 2^d disjoint dyadic cubes

$$Q_{j+\varepsilon,k+1} = \prod_{\ell=1}^d [2^{-k-1}(2j_\ell+\varepsilon_\ell), 2^{-k-1}(2j_\ell+\varepsilon_\ell+1)[$$

where $(\varepsilon_\ell)_{\ell=1,\ldots,f} \in \{0,1\}^d$ called the daughters of $Q_{j,k}$. Note that $j_\ell = \left[\frac{2j_\ell+\varepsilon_\ell}{2}\right]$. It follows that, in the opposite direction, to each dyadic cube $Q_{j,k}$ corresponds a unique dyadic cube $Q_{\tilde{j},k-1}$ such that $Q_{j,k}$ is a daughter of $Q_{\tilde{j},k-1}$ and $\tilde{j}_\ell = \left[\frac{j_\ell}{2}\right]$.

Function spaces. Various function spaces will be used throughout. All functions considered here (unless specified otherwise) are *complex valued*.

When $\Omega \subset \mathbb{R}^d$ is an open (or closed) set (with distance induced by the euclidean norm)

– $\mathcal{C}(\Omega)$ is the set of continuous functions on $\mathbb{R}^d$, $\mathcal{C}_0(\Omega)$ is the subset of $\mathcal{C}(\Omega)$ of functions with compact support.

– $\mathcal{C}_0(\mathbb{R}^d)$ is the set of continuos functions that go to 0 at infinity and $\mathcal{C}_b(\mathbb{R}^d)$ is the set of bounded continuous functions.

– For k an integer, $\mathcal{C}^k(\Omega)$ —resp. $\mathcal{C}^k_c(\mathbb{R}^d)$— is the subset of $\mathcal{C}(\Omega)$ —resp. $\mathcal{C}_c(\mathbb{R}^d)$— of functions that are $\mathcal{C}^k$-smooth.

– $\mathcal{C}^k_0(\mathbb{R}^d)$—resp. $\mathcal{C}^k_b(\mathbb{R}^d)$—is the subset of $\mathcal{C}_0(\Omega)$ —resp. $\mathcal{C}^c_b(\mathbb{R}^d)$— of functions f that are $\mathcal{C}^k$-smooth and such that each derivative $\partial^\alpha f \in \mathcal{C}_0(\mathbb{R}^d)$ –resp. $\partial^\alpha f \in \mathcal{C}_b(\mathbb{R}^d)$.

– $\mathcal{S}(\mathbb{R}^d)$ is the Schwarz class of functions $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that, for every $\alpha, \beta \in \mathbb{N}^d$,

$$\sup_{x\in\mathbb{R}^d} (1+|x^\beta|)|\partial^\alpha f(x)| < +\infty.$$

For $(\Omega, \mathcal{B}, \mu)$, a measure space and $1 \leq p \leq +\infty$

– $\mathcal{L}^0(\Omega) = \mathcal{L}^0(\Omega, \mathcal{B}, \mu)$ be the set of complex valued measurable functions on Ω.

– $L^p(\Omega)$ is the subset of $\mathcal{L}^0(\Omega)$ consisting of functions f such that

- when $1 \leq p < +\infty$, $\|f\|_p = \left(\int_\Omega |f(x)|^p \,\mathrm{d}x\right)^{1/p} < +\infty$;
- when $p = +\infty$, $\|f\|_\infty = \text{ess-sup}\,|f| < +\infty$.

– For $1 \leq p < +\infty$, the weak-L^p space $L^p_w(\Omega)$ is the set of measurable functions such that there exists a constant C for which, for every $\lambda > 0$,

$$|\{x \in \Omega\,:\, |f(x)| > \lambda\}| \leq \frac{C^p}{\lambda^p}.$$

The infimum over all possible C's is denoted by $\|f\|_{L^p_w}$.

– For $f \in L^1_{loc}(\mathbb{R}^d)$ and Q a cube, we write

$$f_Q = \frac{1}{|Q|}\int_Q f(x)\,\mathrm{d}x$$

for its mean over Q. The BMO-norm of f is the quantity

$$\|f\|_{BMO} = \sup_Q \frac{1}{|Q|}\int_Q |f(x) - f_Q|\,\mathrm{d}x$$

and the BMO space is the space of functions, *modulo constants* such that $\|f\|_{BMO} < +\infty$.

2.2. Transforms. The Fourier transform is normalized as

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2i\pi\langle x,\xi\rangle}\,\mathrm{d}x$$

and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}[f](\xi) = \int_{\mathbb{R}^d} f(x)e^{2i\pi\langle x,\xi\rangle}\,\mathrm{d}x.$$

– The Poisson kernel of the upper half spacs $\mathbb{R}_+^{d+1}$ is defined by

$$P_d(x,t) = c_d \frac{t}{(t^2+|x|^2)^{(d+1)/2}} \quad x \in \mathbb{R}^d, t>0$$

where $c_d = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}}$. The Poisson integral of a function f on $\mathbb{R}^d$ is given by

$$u(x,t) = \int_{\mathbb{R}^d} f(y) P_d(x-y,t)\,\mathrm{d}x.$$

– the Conjugate Poisson Integral of f on $\mathbb{R}_+^2$ is defined by

$$Q[f](x,t) = Q_t * f(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{x-y}{(x-y)^2+t^2} f(y)\,\mathrm{d}y.$$

– the Cauchy Transform of f on $\mathbb{C}\setminus\mathbb{R}$ is defined by

$$C[f](z) = \frac{1}{2i\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y+it}\,\mathrm{d}y.$$

The Hilbert transform is denoted by H and defined on $L^2(\mathbb{R})$ by $Hf = \mathcal{F}^{-1}[-i\,\mathrm{sign}(\cdot)\widehat{f}]$. Alternatively, we may define it as

$$Hf(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|y|>\varepsilon} \frac{f(x-y)}{y}\,\mathrm{d}y.$$

The principal value distribution associated to $1/x$ is defined on $\mathcal{C}_c^\infty(\mathbb{R})$ by

$$\left\langle vp\frac{1}{x},\varphi\right\rangle = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{\varphi(x)}{x}\,\mathrm{d}x.$$

The centered Hardy-Littlewood Maximal Function by

$$M[f](x) = \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(u)|\,\mathrm{d}u.$$

The uncentered Hardy-Littlewood Maximal Function

$$\mathcal{M}[f](x) = \sup_{r>0}\sup_{y\in B(x,r)}\frac{1}{|B(y,r)|}\int_{B(y,r)}|f(u)|\,\mathrm{d}u.$$

When balls are replaced by cubes, we denote the associated centered Hardy-Littlewood Maximal Function by

$$M^{\square}[f](x) = \sup_{r>0}\frac{1}{|Q(x,r)|}\int_{B(x,r)}|f(u)|\,\mathrm{d}u;$$

while the uncentered Hardy-Littlewood Maximal Function is given by

$$\mathcal{M}^{\square}[f](x) = \sup_{r>0}\sup_{y\in Q(x,r)}\frac{1}{|Q(y,r)|}\int_{B(y,r)}|f(u)|\,\mathrm{d}u.$$

The Dyadic Maximal function

$$M^d[f](x) = \sup_{x\in Q\in\mathcal{D}}\frac{1}{|Q|}\int_Q|f(u)|\,\mathrm{d}u;$$

where the supremum is taken over all dyadic cubes $Q\in\mathcal{D}$ that contain x.

The Sharp Maximal function is given by

$$M^\sharp f(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(x)-f_Q|\,\mathrm{d}x$$

where the supremum is taken over all cubes containing x.

Background

A major task in the investigation of Partial Differential Equations is to show that such a PDE admits a solution, to be able to construct it and to understand how it depends on the various parameters/data that enter it. There are a number of stategies that can be followed to accomplish this that can be informally summerized as follows:

– *Write down an* **explicit formula** *for the solution in terms of the given data.* Such a forumla usually takes the form of a (linear) operator T sending data to the solution. This may be seen as the most natural version but is unfortunately only available in very special cases. Further, such a formula may be rather complicated, so that it may still be difficult to describe the qualitative behavior of a solution from the formula.

Fortunately, other powerful methods have been described. Let us focus on two of them:

– *Approximate the original PDE by a sequence of simpler ones and show that the solution of those approximate problems converge to a solution of the original one.* PDEs are posed in spaces of functions, and those spaces are of infinite dimension. The crux of this strategy usually lies in carefully choosing finite dimensional approximating problems that can be solved explicitly (or numerically) and that still share important crucial features with the original problem.

– *Deform the original problem and let the deformation go to 0.* The idea is that if one can connect the given problem continuously with a simpler problem that one is able to solve, then one should be able to solve the original problem. Of course, the continuation of solutions requires careful analysis.

As a central object in this course, we will consider the following PDEs on an open connected bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary $\partial\Omega$. The reader may restrict his attention to the unit euclidean ball $B(0,1)$ with boundary $\mathbb{S}^{d-1}$ or the upper half-space

$$\mathbb{R}^{d+1}_+ = \{(x_1,\ldots,x_d,t) \int \mathbb{R}^d \times \mathbb{R}^*_+\}.$$

– the *Laplace Equation* $\Delta u = 0$ where

$$\Delta u = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$$

- the *Poisson Equation* $\Delta u = f$.

To Δ we associate the *fundamental solution*

$$\Gamma(x,y) = \begin{cases} \dfrac{1}{2\pi}\log|x-y| & \text{when } d=2 \\ \dfrac{1}{d(2-d)\omega_d}|x-y|^{2-d} & \text{when } d\geq 3 \end{cases}$$

where ω_d is the volume of the unit ball $B(0,1)$ in $\mathbb{R}^d$. We then have the following:

THEOREM 0.1 (Green Representation Formula). *Let Ω be a smooth domain in $\mathbb{R}^d$. Let $u \in \mathcal{C}^2(\overline{\Omega})$. Then, for every $y \in \Omega$,*

$$u(y) = \int_{\partial\Omega} u(\zeta)\frac{\partial}{\partial n_\zeta}\Gamma(\zeta,y) - \Gamma(\zeta,y)\frac{\partial}{\partial n}u(\zeta)\,d\sigma(\zeta) + \int_\Omega \Gamma(x,y)\Delta u(x)\,dx \tag{2.1}$$

where $\dfrac{\partial}{\partial n}$ is the exterior normal derivative on $\partial\Omega$ and $d\sigma$ the surface measure on $\partial\Omega$.

From this, one sees that a (regular) solution of the Poisson Equation is fully determined by its boundary data u restricted to $\partial\Omega$ and $\dfrac{\partial u}{\partial n}$. One may also ask if the converse is true, *i.e.* whether arbitrary boundary data determines u. This fact is not true and actually, only one of u and $\dfrac{\partial u}{\partial n}$ can be imposed on Ω.

To do so, one introduces a *Green function* for Ω which is a function G defined for $x \neq y \in \overline{\Omega}$ such that

- $G(x,y) = 0$ for $x \in \partial\Omega$;
- for every $y \in \Omega$, the function $h(x) = G(x,y) - \Gamma(x,y)$ is harmonic $\Delta h = 0$ in Ω.

Alternatively, we may define the Green function as follows

Definition 0.2. Let Ω be a $\mathcal{C}^1$ domain in $\mathbb{R}^d$ and assume that for every $y \in \Omega$ there exists a function $\Phi_y \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that

(i) $\Delta\Phi_y(x) = 0$ for all $x \in \Omega$,
(ii) $\Phi_y(x) = \Gamma(x,y)$ for every $x \in \partial\Omega$.

Then $G(x,y) = \Gamma(x,y) - \Phi_y(x)$ is the *Green function* of Ω.

We assume that such a function exists (which is true here) and apply the *Second Green Formula*:

$$\int_\Omega u(x)\Delta v(x) - v(x)\Delta u(x)\,\mathrm{d}x = \int_{\partial\Omega}\left(u(\zeta)\frac{\partial v}{\partial n}(\zeta) - v(\zeta)\frac{\partial u}{\partial n}(\zeta)\right)\mathrm{d}\sigma(\zeta) \tag{2.2}$$

(the minus sign comes from the convention that we differentiate with respect to the outer normal) to $v = -h$

$$\int_\Omega \Gamma(x,y)\Delta u(x)\,\mathrm{d}x = \int_\Omega G(x,y)\Delta u(x)\,\mathrm{d}x + \int_{\partial\Omega}\left(u(\zeta)\frac{\partial G}{\partial n}(\zeta,y) - u(\zeta)\frac{\partial \Gamma}{\partial n}(\zeta,y) + \Gamma(\zeta,y)\frac{\partial u}{\partial n}(\zeta)\right)\mathrm{d}\sigma(\zeta)$$

Adding the result to (2.1), we obtain

Theorem 0.3 (Poisson Representation Formula). *Let Ω be a smooth domain in $\mathbb{R}^d$ and G be a Green function for Ω. Let $u \in \mathcal{C}^2(\overline{\Omega})$. Then, for every $y \in \Omega$,*

$$u(y) = \int_{\partial\Omega} u(\zeta)\frac{\partial}{\partial n_\zeta}G(\zeta,y)\,d\sigma(\zeta) + \int_\Omega \Gamma(x,y)\Delta u(x)\,dx. \tag{2.3}$$

In particular, this shows that a solution $u \in \mathcal{C}^2(\overline{\Omega})$ of the Poisson Equation $\Delta u = f$ in Ω, is uniquely determined by its boundary data $u = \varphi$ on $\partial\Omega$ via

$$u(y) = \int_{\partial\Omega} \varphi(x)\frac{\partial}{\partial n_x}\Gamma(x,y)\,\mathrm{d}\sigma(x) + \int_\Omega \Gamma(x,y)f(x)\,\mathrm{d}x. \tag{2.4}$$

This raises several questions:

– Does this formula make sense and is it really valid: can one extend it to more general f and φ, does it provide a solution of $\Delta u = f$ and is $u = \varphi$ on $\partial\Omega$ in some sense?

– Can φ or f be recovered from u, what conditions should be imposed on u for this to be the case.

– Can one give weaker meanings to $\Delta u = f$ (solution in the sense of distriutions) and to $u = \varphi$ on $\partial\Omega$ ($u(x) \to \varphi(\zeta)$ when $x \to \zeta$).

– How do changes in f or φ affect u? Does u depend continuously on such changes?...

All those questions can be rephrased in terms of properties of the operators

$$\varphi \to \int_{\partial\Omega} u(x)\frac{\partial}{\partial n_x}\Gamma(x,y)\,\mathrm{d}\sigma(x)$$

and

$$f \to \int_\Omega \Gamma(x,y)f(x)\,\mathrm{d}x$$

in particular of continuity of those operators.

The aim of these notes is to provide some of the tools that may allow to do this. Those tools, like often in mathematics, can be usefull in many other fields, ranging from number theory to medical imaging, but those aspects will not be developped here.

Before going to the main topic, let us look how the computations work on $\mathbb{R}^{d+1}_+$. An alternative approach to the determination of the Poisson kernel is through the Green Kernel and its normal derivative, as was explained in the derivation of Formula (2.3) in the introductory section.

What we first need to do is to determine a (the) Green function for $\mathbb{R}^{d+1}_+$. For sake of simplicity, we will only do this for $d\geq 2$ so that the fundamental solution of the Laplace operator on $\mathbb{R}^{d+1}$ is then given by

$$\Gamma(x,y)=-\frac{1}{(d-1)\omega_d}\frac{1}{|x-y|^{d-1}}$$

where $\omega_d=\sigma_d(\mathbb{S}^d)$ is the surface measure of the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$. We can write $\Gamma(x,y)=N(y-x)$ where $N(u)=-\dfrac{1}{(d-1)\omega_d}\dfrac{1}{|u|^{d-1}}$ is the Newton potential.

Fix $x\in\mathbb{R}^{d+1}_+$ and let $\Psi_x(u)=N(u-x)$ is almost the function Φ_x we are looking for since $\Psi_x(u)=N(u-x)$ for $u\in\partial\mathbb{R}^{d+1}_+$ and $\Delta\Psi_x(u)=0$ in

$$\mathbb{R}^{d+1}_-=\{(x_1,\ldots,x_d,t)\,:\,(x_1,\ldots,x_d)\in\mathbb{R}^d,t<0\}.$$

To correct this, let us introduce the following notation: to $x=(x_1,\ldots,x_d,t)\in\mathbb{R}^{d+1}_+$ associate $\bar x=(x_1,\ldots,x_d,-t)\in\mathbb{R}^{d+1}_-$ its reflection trough $\partial\mathbb{R}^{d+1}_+=\partial\mathbb{R}^{d+1}_-$. Define $\Phi_x(u)=N(u-\bar x)=\Gamma(u,\bar x)$ and notice that $\Delta\Phi_x(u)=0$ (since Γ is a fundamental solution of Δ and $\bar x\notin\mathbb{R}^{d+1}_+$) and that $|x-u|-|\bar x-u|$ when $u=(u_1,\ldots,u_d,0)\in\partial\mathbb{R}^{d+1}_+$ so that $\Phi_x(u)=N(u-\bar x)=N(u-x)=\Gamma(x,u)$ for those $u's$. We have thus proven the following:

Lemma 0.4. *The* Green function for $\mathbb{R}^{d+1}_+$ *is given by*

$$G(x,y)=N(y-x)-N(y-\bar x)=\Gamma(x,y)-\Gamma(\bar x,y)=\frac{1}{(d-1)\omega_d}\left(\frac{1}{|\bar x-y|^{d-1}}-\frac{1}{|x-y|^{d-1}}\right)$$

where for $x=(x_1,\ldots,x_d,t)\in\mathbb{R}^{d+1}_+$, $\bar x=(x_1,\ldots,x_d,-t)\in\mathbb{R}^{d+1}_-$ *is its reflection trough* $\partial\mathbb{R}^{d+1}_+=\partial\mathbb{R}^{d+1}_-$.

Let us introduce some notation. We will write $x=(x_1,\ldots,x_d,t)=(x',t)\in\mathbb{R}^{d+1}_+$ so that $\bar x=(x',-t)$ and in the same way $y=(y_1,\ldots,y_d,s)=(y,s)\in\mathbb{R}^{d+1}_+$. Now, in view of Formula (2.3), if $\Delta u=0$ in $\mathbb{R}^{d+1}_+$ and $u(x_1,\ldots,x_d,0)=f(x_1,\ldots,x_d)$ then

$$u(y',s)=\int_{\mathbb{R}^d}f(x')\frac{\partial}{\partial t}G\big((x',0),(y',s)\big)\,\mathrm{d}x'.$$

The function $\dfrac{\partial}{\partial t}G\big((x',0),(y',s)\big)$ is called the Poisson kernel of $\mathbb{R}^{d+1}_+$. It is given by (note that the *exterior* normal derivative is $-\partial_t$)

$$\begin{aligned}
P\big((x',0),(y',s)\big)&=-\frac{\partial}{\partial t}G\big((x',t),(y',s)\big)\Big|_{t=0}=-\frac{1}{(d-1)\omega_d}\frac{\partial}{\partial t}\left(\frac{1}{|\bar x-y|^{d-1}}-\frac{1}{|x-y|^{d-1}}\right)\Big|_{t=0}\\
&=-\frac{1}{(d-1)\omega_d}\frac{\partial}{\partial t}\left(\frac{1}{\big(|x'-y'|^2+(t+s)^2\big)^{\frac{d-1}{2}}}-\frac{1}{\big(|x'-y'|^2+(t-s)^2\big)^{\frac{d-1}{2}}}\right)\Bigg|_{t=0}\\
&=\frac{1}{\omega_d}\left(\frac{t+s}{\big(|x'-y'|^2+(t+s)^2\big)^{\frac{d-1}{2}}}-\frac{t-s}{\big(|x'-y'|^2+(t-s)^2\big)^{\frac{d+1}{2}}}\right)\Bigg|_{t=0}\\
&=\frac{2}{\omega_d}\frac{s}{\big(|x'-y'|^2+s^2\big)^{\frac{d+1}{2}}}
\end{aligned}$$

The case $d=1$ is similar but the Newton potential is now given by $N(u)=-\dfrac{1}{2\pi}\ln|u|$.

DEFINITION 0.5. The *Poisson kernel* of the upper half spacs $\mathbb{R}^{d+1}_+$ is defined by

$$P_d(x,t) = c_d \frac{t}{(t^2 + |x|^2)^{(d+1)/2}} \quad x \in \mathbb{R}^d, t > 0$$

where $c_d = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}}$.

Under good regularity properties of u, we then have that $\Delta u = 0$ on $\mathbb{R}^{d+1}_+$ implies

$$u(x,t) = \int_{\mathbb{R}^d} f(y) P_d(x-y,t)\,\mathrm{d}x.$$

CHAPTER 1

Some complements on complex analysis

The aim of this section is to provide some complements to the first year course on complex analysis.

1. The maximum principle

THEOREM 1.1 (Maximum Principle). *Let Ω be a bounded open connected domain and f be a continuous function on $\overline{\Omega}$ that is holomorphic on Ω. If $|f|$ reaches its maximum at some point $z_0 \in \Omega$ then f is constant. Therefore*

$$\sup_{\overline{\Omega}} |f| = \sup_{\partial\Omega} |f|.$$

There is a trivial proof that uses only the power series:

TRIVIAL PROOF. First note that $\overline{\Omega}$ and $\partial\Omega$ are compact sets, so that there is a $z_0 \in \overline{\Omega}$ such that $\sup_{\overline{\Omega}} |f| = |f(z_0)|$. We want to show that $z_0 \notin \Omega$ unless f is constant

Now let $\zeta \in \Omega$ and let f be non constant. As f is holomorphic, there is a smallest $m \geq 1$ such that $a := \dfrac{f^{(m)}(\zeta)}{m!} \neq 0$. Now if $|z|$ is small enough, $\zeta + z \in \Omega$ and we can write the Taylor expension as $f(\zeta + z) = f(\zeta) + az^m + o(z^m)$.

First notice that if $f(\zeta) = 0$ then $|f(\zeta+z)| = |a||z|^m + o(|z|^m) > 0$ for $|z|$ small enough so that f has no local maximum at ζ. Otherwise, $|f(\zeta + z)|^2 = |f(\zeta)|^2 + 2\Re(\overline{f(\zeta)}az^m) + o(|z|^m)$. Write $\overline{f(\zeta)}a = \rho e^{-i\varphi}$ and $z = re^{i\theta}$ then

$$|f(\zeta + re^{i\theta})|^2 = |f(\zeta)|^2 + 2\rho r^m \cos(m\theta - \varphi) + o(r^m) > |f(\zeta)|^2$$

if $-\pi/2 < m\theta - \varphi < \pi/2$ and $r > 0$ is small enough. So f has no local maximum at ζ. □

The following proof applies to every sub-harmonic function:

PROOF. We will use that $u = \log|f| = \dfrac{1}{2}\log|f|^2$ is *sub-harmonic i.e.* for every $x \in \Omega$ and every $r > 0$ such that $B(x,r) \subset \Omega$

$$u(x) \leq \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)\,\mathrm{d}y$$

Now assume that there exists $x_0 \in \Omega$ such that

$$u(x_0) = M := \sup_{\Omega} u(y)$$

and let $F = \{x \in \Omega\ :\ u(x) = M\}$. As u is continuous on $\bar{\Omega}$, then F is relatively closed in Ω (*i.e.* $F = \mathcal{F} \cap \Omega$ with $\mathcal{F}$ closed in $\mathcal{C}$, *e.g.* $\mathcal{F} = \{x \in \Omega\ :\ u(x) = M\}$).

On the other hand, if $x \in F$ and r is small enough for the mean value property to hold, we have

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} \big(u(y) - u(x)\big)\,\mathrm{d}y = \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)\,\mathrm{d}y - u(x) \geq 0.$$

But u reaches its maximum at $x \in F$ so $u(y) - u(x) \leq 0$ thus $u(y) = u(x) = M$ on $B(x,r)$. Thus $B(x,r) \subset F$ which is therefore also open. As $x_0 \in F$, F is open, closed and non-empty in Ω which is connected, thus $F = \Omega$ and $u = M$ on Ω. □

This result is no longer true if Ω is not bounded. For instance, consider

$$\Omega = \left\{z \in \mathbb{C} : -\frac{\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2}\right\}$$

and $f(z) = e^{e^z}$. Then for x real, $f(x \pm i\frac{\pi}{2}) = e^{ie^x}$ is bounded but of course $f(x) = e^{e^x}$ is not. The key here is that this functions growth very fast. When growth is moderate, Phragmèn-Lindelöf principles show that some form of the maximum principle still holds.

2. The Phragmèn-Lindelöf principle

THEOREM 1.2 (Phragmèn-Lindelöf). *Let* $\Omega = \left\{z \in \mathbb{C} : -\frac{\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2}\right\}$ *and* f *be a continous function on* $\overline{\Omega}$ *that is holomorphic over* Ω. *Assume that there are constansts* $\alpha < 1$ *and* $A < \infty$ *such that, for every* $z = x + iy \in \Omega$,

$$|f(x + iy)| < \exp\bigl(A \exp(\alpha x)\bigr)$$

and that, for every $x \in \mathbb{R}$

$$|f(x \pm i\frac{\pi}{2})| \leq 1.$$

PROOF. The proof consists in introducing a barrier function which will allow us to apply the maximum modulus principle. To do so, choose β such that $\alpha < \beta < 1$. Then, for $\varepsilon > 0$, define

$$h_\varepsilon(z) = \exp\bigl(-\varepsilon(e^{\beta z} + e^{-\beta z})\bigr).$$

The first observation is that, if $z = x + iy$ with $|y| \leq \dfrac{\pi}{2}$, then

$$\Re(e^{\beta z} + e^{-\beta z}) = (e^{\beta x} + e^{-\beta x}) \cos \beta y \geq \cos\left(\beta\frac{\pi}{2}\right)(e^{\beta x} + e^{-\beta x})$$

and that $\delta := \cos\left(\beta\frac{\pi}{2}\right) > 0$. It follows that

$$|h_\varepsilon(z)| \leq \exp\bigl(-\varepsilon\delta(e^{\beta x} + e^{-\beta x})\bigr) < 1.$$

But then $|fh_\varepsilon| \leq 1$ on $\partial\Omega$ while

$$|f(z)h_\varepsilon(z)| \leq \exp\bigl(Ae^{\alpha|x|} - \varepsilon\delta(e^{\beta x} + e^{-\beta x})\bigr) < 1.$$

Since $\beta > \alpha$, $Ae^{\alpha|x|} - \varepsilon\delta(e^{\beta x} + e^{-\beta x}) \to -\infty$ when $x \to \pm\infty$ so that there is an x_0 such that, for $|x| \geq x_0$, $|f(z)h_\varepsilon(z)| \leq 1$.

On the other hand, applying the maximum modulus principle to fh_ε on the rectangle $\{-x_0 \leq x \leq x_0, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\}$ we get that $|f(z)h_\varepsilon(z)| \leq 1$ on this rectangle as well. In summary, $|fh_\varepsilon| \leq 1$ on Ω, regardless of which $\varepsilon > 0$ we have chosen.

Now fix $z \in \Omega$ and notice that $h_\varepsilon(z) \to 1$ when $\varepsilon \to 0$ so that $|f(z)| = \lim |f(z)h_\varepsilon(z)| \leq 1$ as claimed. □

We will now elaborate on this idea. Consider a bounded region Ω with smooth boundary $\partial\Omega$. Consider an holomorphic function f, $u = |f|$. Assume that u is continuous on $\partial\Omega$ so that there is a bound M of u on $\partial\Omega$, $|u(z)| \leq M$. Then the maximum principle states that u is bounded by M on all of Ω.

Assume now that $\partial\Omega$ splits into two parts $\partial\Omega = \Gamma_- \cup \Gamma_+$ and that there are $M_- \mathcal{L} M_+$ such that $|u| \leq M_-$ on Γ_- and $|u| \leq M_+$ on Γ_+. The maximum principle states that $|u| \leq M_+$ on all of Ω but one should expect that $|u(z)|$ is much smaller (near to M_-) when z approaches Γ_-.

EXAMPLE 1.3. This idea can be made more precise when $\Omega = \mathbb{D} = D(0,1)$ the unit disc and that $\Gamma_- = \{e^{i\theta} : 0 < \theta < \theta_0\}$ is an arc. In this case, we can use the Poisson integral: let $P(z,\theta) = \dfrac{1}{2\pi}\dfrac{1 - |z|^2}{|z - e^{i\theta}|^2}$ be the Poisson kernel of the disc then

$$u(z) = \int_0^{2\pi} u(e^{i\theta}) P(z,\theta)\, \mathrm{d}\theta.$$

The only property we need is that $\mu_z(\theta) := P(z,\theta)\,\mathrm{d}\theta$ is a probability measure so that

$$\begin{aligned} |u(z)| &\leq \left|\int_0^{\theta_0} u(e^{i\theta})P(z,\theta)\,\mathrm{d}\theta\right| + \left|\int_{\theta_0}^{2\pi} u(e^{i\theta})P(z,\theta)\,\mathrm{d}\theta\right| \\ &\leq \int_0^{\theta_0} |u(e^{i\theta})|P(z,\theta)\,\mathrm{d}\theta + \int_{\theta_0}^{2\pi} |u(e^{i\theta})|P(z,\theta)\,\mathrm{d}\theta \\ &\leq M_-\mu_z([0,\theta_0]) + M_+\mu_z([\theta_0,2\pi]). \end{aligned}$$

As $\mu_z([0,\theta_0]) + \mu_z([\theta_0,2\pi]) = 1$ this bound is smaller than M_+ and further $\mu_z([\theta_0,2\pi]) \to 0$ when $z \to e^{i\theta}$ with $0 < \theta < \theta_0$. This is precisely the expected behavior.

This example is a bit specific in the sense that we have an explicit expression of the Poisson kernel. The role of this kernel is the following

– Suppose we can find a function $h : \overline{\Omega} \to \mathbb{R}$ that is harmonic on Ω and equal to 0 on Γ_- and 1 on Γ_+. We also require that h is continuous on $\Omega \setminus \partial\Gamma_-$ (in the previous example $\partial\Gamma_- = \{0,\theta_0\}$, in general, we will assume that $\partial\Gamma_-$ is finite). In the disc example, $h(z) = \displaystyle\int_{\theta_0}^{2\pi} P(z,\theta)\,\mathrm{d}\theta$.

– We then consider $v(z) = M_- + (M_+ - M_-)h(z)$. This function is harmonic on Ω, continuous on $\Omega \setminus \partial\Gamma_-$ and $u(z) \leq v(z)$ on $\partial\Omega \setminus \partial\Gamma_-$.

– From the maximum principle, $u(z) \leq v(z)$ on Ω.

The argument works even if u is only sub-harmonic, in particular if $u = \log|f|$ with f holomorphic (which is harmonic if f is not zero).

Example 1.4. A second example which is important for us is the case of an annulus $\Omega = \{z \in \mathbb{C} : R_- < |z| < R_+\}$. Then $\log|z|$ is harmonic on Ω, continuous on $\overline{\Omega}$. In particular

$$h(z) = \frac{\log|z| - \log R_-}{\log R_+ - \log R_-}$$

satisfies $h(z) = 0$ for $z \in \Gamma_- := \{R_-e^{i\theta} : 0 \leq \theta \leq 2\pi\}$ and $h(z) = 1$ for $z \in \Gamma_+ := \{R_+e^{i\theta} : 0 \leq \theta \leq 2\pi\}$.

The above principle shows that if $u : \overline{\Omega} \to \mathbb{R}$ is (sub)-harmonic on Ω, continuous on $\overline{\Omega}$, with $u \leq M_-$ on Γ_- and $u \leq M_+$ on Γ_+ then, if $z \in \Omega$, we may wrire $\log|z| \leq \lambda\log R_- + (1-\lambda)\log R_+$ with $0 < \lambda < 1$ and then

$$\begin{aligned} u(z) &\leq M_- + (M_+ - M_-)h(z) = M_- + (M_+ - M_-)\frac{\lambda\log R_- + (1-\lambda)\log R_+ - \log R_-}{\log R_+ - \log R_-} \\ &= \lambda M_- + (1-\lambda)M_+. \end{aligned}$$

When applied to $u(z) = \log|f|$ with f holomorphic on Ω, continuous on $\overline{\Omega}$, we obtain the following:

Theorem 1.5 (Hadamard's three circle theorem). *Let f be an holomorphic function on*

$$\Omega\{z \in \mathbb{C} : R_- < |z| < R_+\},$$

continuous on $\overline{\Omega}$. For $r \in [R_-, R_+]$, let $M(r) = \sup_{|z|=r}|f(z)|$, then $M(r)$ is log-convex that is if $r_1 < r < r_2$, we write $\log r = \lambda\log r_1 + (1-\lambda)\log r_2$ then

$$M(r) \leq \lambda M(r_1) + (1-\lambda)M(r_2).$$

Example 1.6. We will also use a modification of Hadamard's three circle theorem:

Theorem 1.7 (Hadamard's Three Line Theorem). *Consider the strip $\Sigma = \{z \int \mathbb{C} : 0 < \Re(z) < 1\}$ and let F be an holomorphic function on Σ that is continuous and bounded on $\bar{\Sigma}$ with*

$$|F(it)| \leq M_0 \quad \textit{and} \quad |F(1+it)| \leq M_1 \qquad t \in \mathbb{R}.$$

Then for every $0 < \theta < 1$ and $t \in \mathbb{R}$,

$$|F(\theta + it)| \leq M_0^{1-\theta}M_1^{\theta}.$$

Proof. We introduce two auxilary functions on $\bar{\Sigma}$

$$G(z) = \frac{F(z)}{M_0^{1-z}M_1^z} \quad \text{and} \quad G_n(z) = G(z)e^{(z^2-1)/n}.$$

We will write $z = x + iy$ with $0 \leq x \leq 1$. First $|M_0^{1-z}M_1^z| = M_0^{1-x}M_1^x$ is bounded below in $\bar{\Sigma}$ so G is bounded by some M in $\bar{\Sigma}$. Further

$$|G_n(z)| = |G(z)|e^{(x^2-1-y^2)/n} \leq Me^{-y^2/n}$$

It follows that there is a y_n such that $|G_n(z)| \leq 1$ when $|y| \geq y_n$. On the other hand, for $x = 0$, $|G_n(iy)| = |G(iy)|e^{-(1+y^2)/n} \leq \dfrac{|F(iy)|}{M_1}e^{-1/n} \leq 1$ and $|G_n(1+iy)| = |G(1+iy)|e^{-y^2/n} \leq \dfrac{|F(1+iy)|}{M_1} \leq 1$. By the Maximum Principle, we also have $|G_n| \leq 1$ on $\{x+iy \,:\, 0 \leq x \leq 1,\ |y| \leq y_n\}$ so that $|G_n| \leq 1$ on $\bar{\Sigma}$.

But then for fixed $z = x+iy$, $|G(z)| = |G_n(z)||e^{-(z^2-1)/n}| \leq e^{(1-x^2+y^2)/n}$ and, letting $n \to +\infty$, $|G(z)| \leq 1$. In other words, $|F(z)| \leq |M_0^{1-z}M_1^z| = M_0^{1-x}M_1^x$ as claimed. □

EXAMPLE 1.8. We now consider Ω to be the half-disc $\Omega = \{z \in \mathbb{C} \,:\, |z| < R,\ \mathrm{Im}\, z > 0\}$. The boundary is composed of two pieces, the segment $\Gamma_- = [-R, R]$ and $\Gamma_+ = \{Re^{i\theta} \,: 0 \leq \theta \leq \pi\}$.

Recall that $\arg z = \mathrm{Im} \log z$ for $z \in \mathbb{C} \setminus (-\infty, 0]$ is a harmonic function. We consider the function $h_R(z) = \frac{2}{\pi}\big(\arg(z+R) + \pi - \arg(z-R)\big)$ which is then harmonic on Ω and continuous on $\overline{\Omega}$. This function has a geometric interpretation consider the triangle T with vertices R, z and $-R$, then $\arg(z+R)$ is the angle at $-R$ and $\pi - \arg(z-R)$ is the angle at R. In particular, both are 0 if $z \in \Gamma_- = [-R, R]$ so that $h_R(z) = 0$ for those z's. On the other hand, if $z \in \Gamma_+$, the angle at z in T is $\pi/2$ so that the sum of the two other angles is $\pi - \pi/2 = \pi/2$ so that $h_R(z) = 1$ for those z's. A further consequence, is that if we fix z and let $R \to +\infty$ then the sinuses of the two angles are $O(R^{-1})$ so that $h_R(z) = O(R^{-1})$ when $R \to +\infty$.

We are now in position to prove the following:

THEOREM 1.9 (Phragmène-Lindelöf for a half-plane). *Let f be holomorphic on the half plane $\mathbb{H} = \{z \in \mathbb{C} \,:\, \mathrm{Im}(z) > 0\}$, continuous on $\overline{\mathbb{H}}$ and bounded on the real line $|f(x)| \leq M$ for $x \in \mathbb{R}$. Define $M(R) = \sup\{|f(z)| \,:\, |z| = R,\ \mathrm{Im}\, z \geq 0\}$ and assume that $\displaystyle\lim_{R\to+\infty} \frac{1}{R} \log M(R) = 0$ then $|f| \leq M$ on $\mathbb{H}$.*

REMARK 1.10. Up to rotating and translating f, the half plane $\mathbb{H}$ can be replaced by any half-plane

$$\{z \in \mathbb{C} \,:\, \mathrm{Im}(e^{i\theta}z) > \alpha\}.$$

PROOF. First, up to replacing f by f/M, we may assume that $M = 1$. Then, for $z \in \mathbb{H}$, take $R > |z|$, so that

$$\log|f(z)| \leq \log 1 + (\log M(R) - \log 1)h_R(z) = h_R(z) \log M(R).$$

Finally, as $h_R(z) = O(R^{-1})$ and $R^{-1}\log M(R) \to 0$, it is enough to let $R \to +\infty$ to get $\log|f(z)| \leq 0$ in $\mathbb{H}$, that is $|f(z)| \leq 1$ as claimed. □

We can now prove the following:

COROLLARY 1.11 (Phragmène-Lindelöf for a sector). *Let $\alpha > 1$, $\theta_0 \in [0, 2\pi]$ and $S_{\theta_0,\alpha} = \{re^{i\theta} \,:\, r > 0,\ |\theta - \theta_0| < \frac{\pi}{2\alpha}\}$ be a sector of opening $\dfrac{\pi}{\alpha}$.*

Let f be a function on $\overline{S_{\theta_0,\alpha}}$ that is holomorphic on $S_{\theta_0,\alpha}$, continuous on $\overline{S_{\theta_0,\alpha}}$ and such that $|f(z)| \leq Ce^{|z|^\beta}$ for some $C > 0$ and $0 < \beta < \alpha$. If $|f(z)| \leq M(1+|z|)^N$ on $\partial S_{\theta_0,\alpha}$ then $|f(z)| \leq \kappa_\alpha^N M(1+|z|)^N$ on $S_{\theta_0,\alpha}$, where κ_α is a constant that depends continuously on α.

For future use, note that for $\alpha \geq 2$, that is a sector of opening $\leq \pi/2$, we can take $\kappa_\alpha = 2^{1/4}$.

PROOF. Up to replacing f by $f(e^{-i\theta_0}z)$, we may assume that $\theta_0 = 0$. We write $S_\alpha = S_{0,\alpha}$. Further, note that there is a constant κ_α such that

$$1 \leq \frac{1+|z|}{|1+z|} \leq \kappa_\alpha$$

for every $z \in S_\alpha$. A precise value of κ_α is not needed and we may just notice that if $|\theta| < \pi/2\alpha$, and $z = re^{i\theta}$,

$$(1+|z|)^2 = 1 + r^2 + 2r \leq \frac{1}{\cos\frac{\pi}{2\alpha}}(1 + r^2 + 2r\cos\theta) = \frac{1}{\cos\frac{\pi}{2\alpha}}|1+z|^2$$

that is $\kappa_\alpha = \left(\cos\frac{\pi}{2\alpha}\right)^{1/2}$.

As $1+z$ does not vanish on S_α, we may then consider

$$g(z) = \frac{f(z)}{M(1+z)^N}$$

and notice that g is holomorphic on S_α, bounded by 1 on ∂S_α and still satisfies $|g(z)| \leq Ce^{|z|^\beta}$ on S_α. Once we show the case $N = 0$ we conclude that $|g| \leq 1$ on S_α. But then $|f(z)| = M|1+z|^N|g(z)| \leq M\kappa_\alpha^{-N}(1+|z|)^N$ on S_α so that the general case follows.

So, from now on, we only consider the case $N = 0$ and, up to replacing f by f/M, we assume that $|f| \leq 1$ on ∂S_α.

We then consider the function $z \to z^{1/\alpha} = e^{\frac{1}{\alpha}\log z}$ which is holomorphic on $\mathbb{H} = \{\Re z > 0\}$ and continuous on $\overline{\mathbb{H}}$. Further, it is a bijective mapping $\mathbb{H}$ (resp. $\overline{\mathbb{H}}$) to S_α (resp. $\overline{S_\alpha}$).

It follows that $h(z) = f(z^{1/\alpha})$ is holomorphic on $\mathbb{H}$ and continuous $\overline{\mathbb{H}}$. Further $|h| \leq 1$ on $\partial\mathbb{H}$. Finally, $|h(z)| \leq Ce^{|z|^{\beta/\alpha}}$ so that, $\frac{1}{r}\sup_{|z|=r}\log|h(z)| \leq \frac{1}{r}\log C + r^{1-\beta/\alpha} \to 0$ when $r \to +\infty$. Applying the half-plane Phragmén-Lindelöf principle to h, we get that $|h| \leq 1$ and then that $|f| \leq 1$. $\square$

CHAPTER 2

L^p spaces, weak L^p spaces and interpolation

1. L^p spaces

We assume that content of this section is known to students following this course.

1.1. Definition. Let $1 \leq p < +\infty$ a *real* number, $(\Omega, \mathcal{B}, \mu)$ a σ-finite measure space. We define

$$L^p(\Omega, \mu) = \left\{ f : \Omega \to \mathbb{C},\ f\ \mu - \text{measurable},\ \int_\Omega |f(x)|^p \, \mathrm{d}\mu(x) < +\infty \right\}$$

and endow it with the "norm"

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, \mathrm{d}\mu(x) \right)^{\frac{1}{p}}.$$

For $p = +\infty$,we define

$$L^\infty(\Omega, \mu) = \{ f : \Omega \to \mathbb{C},\ f\ \mu - \text{mesurable, il existe } K > 0 \text{ telle que } |f(x)| \leq K,\ \mu - p.p. \}$$

and endow it with the "norm"

$$\|f\|_\infty = \inf\{K\ \ |f(x)| \leq K\ \mu - a.e.\}.$$

We almost have a normed vector space in the sense that

(i) For $f \in L^p(\Omega, \mu)$, we have $\|f\|_p \geq 0$ and $\|f\|_p = 0$ if and only if $f = 0$ μ-almost everywhere.
(ii) For $f \in L^p(\Omega, \mu)$ and $\lambda \in \mathbb{C}$, we have $\lambda f \in L^p(\Omega, \mu)$ and $\|\lambda f\|_p = |\lambda| \|f\|_p$.
(iii) For $f, g \in L^p(\Omega, \mu)$, $f + g \in L^p(\Omega, \mu)$ et $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Remark 2.1. It is important to keep in mind that the case $p = 2$, $L^2(\Omega, \mu)$ is a Hilbert space and that the norm is associated to the scalar product given by

$$\langle f, g \rangle_{L^2(\Omega,\mu)} = \int_\Omega f(x) \overline{g(x)} \, \mathrm{d}\mu(x).$$

Sketch of proof. For (i), one uses the fact that a non-negative function with 0 integral vanishes a.e. while (ii) is obvious.

On the other hand (iii) is trivial when $p = 1$ and $p = +\infty$ while the case $p = 2$ follows from Cauchy-Schwarz. The general case will be treated below and is more subtle. However, let us show that $L^p(\Omega, \mu)$ spces are vector spaces:

$$|f + g|^p \leq (|f| + |g|)^p = 2^p \left(\frac{|f| + |g|}{2} \right)^p \leq 2^{p-1} (|f|^p + |g|^p)$$

since $x \mapsto x^p$ is a convex function. In particular, if $f, g \in L^p(\Omega, \mu)$ then $f + g \in L^p(\Omega, \mu)$. □

In order to obtain a normed vector space, we will identify two functions f, g if $f = g$ a.e. or, in more rigorous terms, quotient the L^p spae by the equivalence relation $f \sim g$ if $f - g = 0$ a.e.

1.2. Hölder et Minkowski. The first inequality we will prove extends Cauchy-Schwarz and plays a key role in analysis.

THEOREM 2.2 (Hölder's Inequality). *Let $(\Omega,\mathcal{B},\mu)$ be a measure space. Let $1\le p,p'\le+\infty$ be such that $\dfrac{1}{p}+\dfrac{1}{p'}=1$ (with the convention that $p'=+\infty$ when $p=1$ and vice versa). Let $f\in L^p(\Omega,\mu)$ and $g\in L^{p'}(\Omega,\mu)$, then $fg\in L^1(\Omega,\mu)$ with*

$$\left|\int_\Omega f(x)g(x)\,d\mu(x)\right|\le\int_\Omega |f(x)g(x)|\,d\mu(x)\le\left(\int_\Omega |f(x)|^p\,d\mu(x)\right)^{\frac1p}\left(\int_\Omega |g(x)|^{p'}\,d\mu(x)\right)^{\frac{1}{p'}}.$$

Moreover,

— equality holds in the first inequality if and only if there is a $\theta\in\mathbb{R}$ such that $f(x)g(x)=e^{i\theta}|f(x)g(x)|$.

— if $f\neq 0$ equality holds in the second inequality if and only if there is a real $\lambda\ge0$ such that

(i) *for $1<p<+\infty$, $|g(x)|=\lambda|f(x)|^{p-1}$ μ-a.e.;*

(ii) *for $p=1$, $|g(x)|\le\lambda$ μ-a.e. and $|g(x)|=\lambda$ for μ-almost every x such that $f(x)\neq0$;*

(iii) *for $p=+\infty$, $|f(x)|\le\lambda$ μ-a.e. and $|f(x)|=\lambda$ for μ-almost every x such that $g(x)\neq0$.*

PROOF. The first inequality is the triangular inequality for integrals and is left to the reader. For the second one, the cases $f=0$ and $g=0$ are obvious and excluded. The cases $p=1$ (thus $p'=+\infty$) and $p=+\infty$ (thus $p'=1$) are straightforward. We thus assume that $1<p<\infty$ (so that que $1<p'<\infty$) and $f,g\neq0$. We can then introduce

$$u=\left(\frac{|f|}{\|f\|_p}\right)^p\quad\text{and}\quad v=\left(\frac{|g|}{\|g\|_{p'}}\right)^{p'}.$$

As log is concave, we get that, for $0<\alpha<1$, $u^\alpha v^{1-\alpha}\le\alpha u+(1-\alpha)v$. In particular, for $\alpha=1/p$, we have

$$\frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_{p'}}\le\frac1p\frac{|f|^p}{\|f\|_p^p}+\frac{1}{p'}\frac{|g|^{p'}}{\|g\|_{p'}^{p'}}.$$

Integrating with respect to μ, the result follows.

The equality case uses strict concavity and is left to the reader. □

Hölder's inequality is thus a convexity inequality. Another important convexity inequality is the following:

THEOREM 2.3 (Jensen's Inequality). *Let $(\Omega,\mathcal{B},\mu)$ be a* finite *measure space. Let $J:\mathbb{R}\to\mathbb{R}$ be a convex function. For $f\in L^1(\Omega,\mu)$, write*

$$\langle f\rangle=\frac{1}{\mu(\Omega)}\int_\Omega f(x)\,d\mu(x)$$

for its mean *over Ω. Then*

(i) *$[J\circ f]_-$, the negative part of $J\circ f$ is in $L^1(\Omega,\mu)$, thus $\displaystyle\int_\Omega J\circ f(x)\,d\mu(x)$ is well defined (possibly $=+\infty$);*

(ii) *$J(\langle f\rangle)\le\langle J\circ f\rangle$, that is*

$$J\left(\frac{1}{\mu(\Omega)}\int_\Omega f(x)\,d\mu(x)\right)\le\frac{1}{\mu(\Omega)}\int_\Omega J\big(f(x)\big)\,d\mu(x).$$

PROOF. As J is convex and, for sake of simplicity, we assume that J is $\mathcal{C}^1$, for $a,t\in\mathbb{R}$,

$$J(t)\ge J(a)+J'(a)(t-a).$$

Taking $t=f(x)$ and $a=\langle f\rangle$, this implies

$$J\big(f(x)\big)_+-J\big(f(x)\big)_-=J\big(f(x)\big)\ge J(\langle f\rangle)+J'(\langle f\rangle)f(x)-J'(\langle f\rangle)\langle f\rangle. \tag{1.5}$$

In particular, if x is such that $J\big(f(x)\big)_-\neq0$ then $J\big(f(x)\big)_+=0$, and

$$\begin{aligned}0\le J\big(f(x)\big)_- &\le -J'(\langle f\rangle)f(x)+J'(\langle f\rangle)\langle f\rangle-J(\langle f\rangle)\\ &\le |J'(\langle f\rangle)||f(x)|+|J'(\langle f\rangle)\langle f\rangle-J(\langle f\rangle)|.\end{aligned}$$

As $f\in L^1$, $|J'(\langle f\rangle)||f(x)|\in L^1$ et μ being *finite*, constants are integrable, thus $|J'(\langle f\rangle)\langle f\rangle-J(\langle f\rangle)|\in L^1$.

Next, integrate (1.5) to get

$$\frac{1}{\mu(\Omega)}\int_\Omega J\big(f(x)\big)\,\mathrm{d}\mu(x) \geq \frac{1}{\mu(\Omega)}\int_\Omega J(\langle f\rangle)\,\mathrm{d}\mu(x) + \frac{J'(\langle f\rangle)}{\mu(\Omega)}\int_\Omega f(x)-\langle f\rangle\,\mathrm{d}\mu(x).$$

But

$$\frac{1}{\mu(\Omega)}\int_\Omega f(x)-\langle f\rangle\,\mathrm{d}\mu(x)=0.$$

Jensen's Inequality follows. □

The smoothness requirement for J can be removed since an inequality of the form $J(t)\geq J(a)+c(t-a)$ is still valid.

Note that Hölder's Inequality can be deduced from Jensen's Inequality:

Second proof of Hölder. Up to replacing f, g by $|f|$, $|g|$, assume that $f,g\geq 0$. Again, the cases $p=1$ et $p=+\infty$ are obvious so that we restrict attention to $1<p<+\infty$.

Let $\Omega'=\{x\in\Omega\,:\,g(x)>0\}$. Then

$$\int_\Omega f^p(x)\,\mathrm{d}\mu(x)=\int_{\Omega'} f^p(x)\,\mathrm{d}\mu(x)+\int_{\Omega\setminus\Omega'} f^p(x)\,\mathrm{d}\mu(x)\geq\int_{\Omega'} f^p(x)\,\mathrm{d}\mu(x)$$

while

$$\int_\Omega f(x)g(x)\,\mathrm{d}\mu(x)=\int_{\Omega'} f(x)g(x)\,\mathrm{d}\mu(x)\quad\text{et}\quad\int_\Omega g(x)^{p'}\,\mathrm{d}\mu(x)=\int_{\Omega'} g(x)^{p'}\,\mathrm{d}\mu(x).$$

It is thus enough to prove Hölder for Ω' replacing Ω, that is to assume that g does not vanish over Ω.

We can now define a new measure $\mathrm{d}\nu(x)=g(x)^{p'}\,\mathrm{d}\mu(x)$ and introduce the function $F(x)=f(x)g(x)^{-p'/p}$. Note that

$$\nu(\Omega)=\int_\Omega 1\,\mathrm{d}\nu(x)=\int_\Omega g(x)^{p'}\,\mathrm{d}\mu(x)$$

so that ν is *finite*. Moreover

$$\begin{aligned}\frac{1}{\nu(\Omega)}\int_\Omega F(x)\,\mathrm{d}\nu(x) &= \frac{1}{\displaystyle\int_\Omega g(x)^{p'}\,\mathrm{d}\mu(x)}\int_\Omega f(x)g(x)^{-p'/p}g(x)^{p'}\,\mathrm{d}\nu(x)\\ &= \frac{\displaystyle\int_\Omega f(x)g(x)\,\mathrm{d}\mu(x)}{\displaystyle\int_\Omega g(x)^{p'}\,\mathrm{d}\mu(x)}\end{aligned}$$

since $-\dfrac{p'}{p}+p'=p'\left(1-\dfrac{1}{p}\right)=1$. Finally, Jensen's Inequality with $J(t)=|t|^p$ implies

$$\left(\frac{\displaystyle\int_\Omega f(x)g(x)\,\mathrm{d}\mu(x)}{\displaystyle\int_\Omega g(x)^{p'}\,\mathrm{d}\mu(x)}\right)^p\leq\frac{\displaystyle\int_\Omega f(x)^p g(x)^{-p'}g(x)^{p'}\,\mathrm{d}\mu(x)}{\displaystyle\int_\Omega g(x)^{p'}\,\mathrm{d}\mu(x)}$$

as expected. □

Theorem 2.4 (Minkowski's Inequality). *Let $(\Omega,\mathcal{B},\mu)$ and $(\Gamma,\tilde{\mathcal{B}},\gamma)$ be two measure spaces and $1\leq p<+\infty$. Then for every f $\gamma\otimes\mu$-measurable,*

$$\left(\int_\Gamma\left|\int_\Omega f(x,y)\,d\mu(y)\right|^p d\gamma(x)\right)^{\frac{1}{p}}\leq\int_\Omega\left(\int_\Gamma|f(x,y)|^p\,d\gamma(x)\right)^{\frac{1}{p}}d\mu(y). \tag{1.6}$$

Equality holds if and only if f is of the form $f(x,y)=\alpha(x)\beta(y)$.

In other words

$$\left\|x\to\int_\Omega|f(x,y)|\,\mathrm{d}\mu(y)\right\|_p\leq\int_\Omega\|x\to f(x,y)\|_p\,\mathrm{d}\mu(y).$$

This extends the easy triangular inequality

$$\left|\int_\Omega f(t)\,\mathrm{d}\mu(t)\right| \leq \int_\Omega |f(t)|\,\mathrm{d}\mu(t)$$

which corresponds to the particular case where Γ has a single element.

PROOF. We may assume that $f \geq 0$ with $f > 0$ on a set of positive mesure and that the right hand side of (1.6) is finite.

Let $f_n = f\chi_{E_n}$ with $E_n = F_n \cap \{(x,y) \in \Gamma \times \Omega : |f(x,y)| \leq n\}$ where F_n is an increasing family of sets of finite measure in $\Gamma \times \Omega$ that cover $\Gamma \times \Omega$: $\bigcup F_n = \Gamma \times \Omega$. For f_n, the left hand side of (1.6) is

$$\left(\int_\Gamma \left(\int_\Omega |f_n(x,y)|\,\mathrm{d}\mu(y)\right)^p \mathrm{d}\gamma(x)\right)^{\frac{1}{p}}$$

which is finite.

Further, monotone convergence shows that this quantity converges to

$$\left(\int_\Gamma \left(\int_\Omega |f(x,y)|\,\mathrm{d}\mu(y)\right)^p \mathrm{d}\gamma(x)\right)^{\frac{1}{p}}.$$

We may thus also assume that this is finite. In particular, we may define

$$H(x) = \int_\Omega |f(x,y)|\,\mathrm{d}\mu(y)$$

which is then finite a.e.

From Fubini (Tonneli),

$$\begin{aligned}\int_\Gamma H(x)^p\,\mathrm{d}\gamma(x) &= \int_\Gamma \left(\int_\Omega f(x,y)\,\mathrm{d}\mu(y)\right) H(x)^{p-1}\,\mathrm{d}\gamma(x)\\ &= \int_\Omega \int_\Gamma f(x,y)H(x)^{p-1}\,\mathrm{d}\gamma(x)\,\mathrm{d}\mu(y).\end{aligned}$$

From, Hölder $(1/p + 1/p' = 1)$ we get

$$\begin{aligned}\int_\Gamma f(x,y)H(x)^{p-1}\,\mathrm{d}\gamma(x) &\leq \left(\int_\Gamma f(x,y)^p\,\mathrm{d}\gamma(x)\right)^{1/p}\left(\int_\Gamma H(x)^{(p-1)p'}\,\mathrm{d}\gamma(x)\right)^{1/p'}\\ &= \left(\int_\Gamma f(x,y)^p\,\mathrm{d}\gamma(x)\right)^{1/p}\left(\int_\Gamma H(x)^p\,\mathrm{d}\gamma(x)\right)^{1-1/p}.\end{aligned}$$

Therefore

$$\int_\Gamma H(x)^p\,\mathrm{d}\gamma(x) \leq \int_\Omega \left(\int_\Gamma f(x,y)^p\,\mathrm{d}\gamma(x)\right)^{1/p}\mathrm{d}\mu(y)\left(\int_\Gamma H(x)^p\,\mathrm{d}\gamma(x)\right)^{1-1/p}.$$

As we assumed that $\displaystyle\int_\Gamma H(x)^p\,\mathrm{d}\gamma(x) \neq 0, +\infty$, we can divide both sides by

$$\left(\int_\Gamma H(x)^p\,\mathrm{d}\gamma(x)\right)^{1-1/p}$$

to get the result. □

COROLLARY 2.5. *Let $(\Omega, \mathcal{B}, \mu)$ be a measure space. Let $1 \leq p \leq +\infty$ and $f, g \in L^p(\Omega, \mu)$. Then*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

with equality if and only if $g = \lambda f$ for some $\lambda \geq 0$.

PROOF. Let $\Gamma = \{1,2\}$ endowed with the counting measure. Define F on $\Gamma \times \Omega$ by $F(1,y) = f(y)$, $F(2,y) = g(y)$. Minkowski reduces to the desired inequality. □

1.3. Completness of L^p spaces. The aim of this section is to prove that L^p is a Banach space. Before this, let us adapt dominated convergence to convergence in L^p:

LEMMA 2.6 (L^p-dominated convergence). *Let $(\Omega, \mathcal{B}, \mu)$ be a σ-finite measure space. Let $1 \leq p < +\infty$.*

Let (f_k) be a sequence in $L^p(\Omega, \mu)$ and f, F be two functions in $L^p(\Omega, \mu)$. Assume that

(i) *for every k, and μ-almost every $x \in \Omega$, $|f_k(x)| \leq F(x)$*
(ii) *for μ-almost every $x \in \Omega$, $f_k(x) \to f(x)$ when $k \to +\infty$. In particular, $|f(x)| \leq F(x)$ μ-a.e.*

Then $f_k \to f$ in $L^p(\Omega, \mu)$ i.e. $\|f_k - f\|_p \to 0$.

PROOF. We have to prove that

$$\int_\Omega |f_k(x) - f(x)|^p \, \mathrm{d}\mu(x) \to 0.$$

But Condition (ii) implies that $|f_k(x) - f(x)|^p \to 0$ μ-a.e.

Condition (ii) implies that

$$|f_k(x) - f(x)|^p \leq (|f_k(x)| + |f(x)|)^p \leq \big(2F(x)\big)^p \in L^1$$

since $F \in L^p$. We can thus apply the dominated convergence theorem to obtain the result. □

THEOREM 2.7 (L^p is complete). *Let $(\Omega, \mathcal{B}, \mu)$ be a σ-finite measure space. Let $1 \leq p \leq +\infty$. Then $L^p(\Omega, \mu)$ is complete (and thus a Banach space).*

More precisely, if (f_k) is a Cauchy sequence in $L^p(\Omega, \mu)$, then the exists a sub-sequence $(f_{k_j})_j$ and F in $L^p(\Omega, \mu)$ such that

(i) *for $j \geq 1$, $|f_{k_j}(x)| \leq F(x)$ and μ-almost every $x \in \Omega$;*
(ii) *for μ-almost every $x \in \Omega$, $f_{k_j}(x) \to f(x)$ when $j \to +\infty$.*

PROOF. We will concentrate on the case $1 \leq p < +\infty$. The case $p = +\infty$ follows mainly from the completness of $\mathbb{C}$ and is left as an exercice.

As noted in the above lemma, the second part of the theorem implies that every Cauchy sequence in L^p has a convergent sub-sequence. But a Cauchy sequence with a convergent sub-sequence is convergent.

The proof of the second part of the theorem is rather classical.

First, there exists i_1 such that, if $n \geq i_1$, $\|f_{i_1} - f_n\|_p \leq 1/2$ ($\varepsilon = 1/2$ in the definition of a Cauchy sequence). There exists $i_2 > i_1$ such that, if $n \geq i_2$, $\|f_{i_2} - f_n\|_p \leq 1/2^2$... This way, we inductively define $i_k > i_{k-1}$ such that, if $n \geq i_k$, $\|f_{i_k} - f_n\|_p \leq 1/2^k$.

Consider the *non-decreasing positive* sequence defined by

$$F_l(x) = |f_{i_1}(x)| + \sum_{k=1}^{l} |f_{i_{k+1}}(x) - f_{i_k}(x)|.$$

The triangular inequality yields

$$\|F_l\|_p \leq \|f_{i_1}\|_p + \sum_{k=1}^{l} \left\|f_{i_{k+1}} - f_{i_k}\right\|_p \leq \|f_{i_1}\|_p + \sum_{k=1}^{+\infty} 2^{-k} = 1 + \|f_{i_1}\|_p < +\infty.$$

The monotone convergence theorem implies that F_l converges almost everywhere to a function $F \in L^p$. In particular, $F(x)$ is finite for μ-almost every $x \in \Omega$. For such an x, the series

$$f_{i_1}(x) + \sum_{k=1}^{l} \big(f_{i_{k+1}}(x) - f_{i_k}(x)\big)$$

is absolutely convergent, thus convergent. But this is a telescopic sequence:

$$f_{i_1}(x) + \sum_{k=1}^{l} \big(f_{i_{k+1}}(x) - f_{i_k}(x)\big) = f_{i_{l+1}}(x).$$

We have thus shown that $f_{i_{l+1}}$ is convergent and, with the triangular inequality, $|f_{i_{l+1}}| \leq F_l \leq F$ which completes the proof. □

1.4. The projection Theorem. Projections play an essential role in Hilbert spaces. It turns out that a version of the projection theorem is still valid in L^p:

THEOREM 2.8. *Let $1 \leq p < +\infty$ and let E be a closed vector space in $L^p(\Omega, \mu)$. For $f \in L^p(\Omega, \mu)$, let us write $d(f, E) = \inf_{g \in E} \|f - g\|_p$. Then there exists g_0 such that $d(f, E) = \|f - g_0\|_p$.*

REMARK 2.9. Not that, if $\|g\|_p > 2\|f\|_p$ then

$$\|f - g\|_p \geq \|g\|_p - \|f\|_p > \|f\|_p = \|f - 0\|_p \geq d(f, E)$$

since $0 \in E$. Therefore $d(f, E) = \inf\{\|f - g\|_p : g \in E, \|g\|_p \leq 2\|f\|_p\}$.

If E is finite dimensional, the set $\{g \in E, \|g\|_p \leq 2\|f\|_p\}$ being bounded and closed, is compact. As $g \to \|f - g\|_p$ is continuous, the existence of g_0 follows.

In infinite dimensions, this argument is no longer valid.

PROOF WHEN $p \geq 2$. When $p = 2$ this follows from the parallelogram identity

$$\|u - v\|_2^2 + \|u + v\|_2^2 = 2\|u\|_2^2 + 2\|v\|_2^2.$$

Take $g_n \in E$ a sequence such that $\|f - g_n\|_2 \to d(f, E)$. Then the parallelogram identity applied to $u = \frac{f - g_m}{2}$, $v = \frac{f - g_n}{2}$ gives

$$\|g_n - g_m\|_2^2 = 4\left(\frac{1}{2}\|f - g_m\|_2^2 + \frac{1}{2}\|f - g_n\|_2^2 - \left\|\frac{g_n + g_m}{2} - f\right\|_2^2\right).$$

As $\frac{g_n + g_m}{2} \in E$, $\left\|\frac{g_n + g_m}{2} - f\right\|_2 \geq d(f, E)$ thus

$$\|g_n - g_m\|_2^2 \leq 2(\|f - g_m\|_2^2 - d(f, E)^2 + \|f - g_n\|_2^2 - d(f, E)^2)$$

from which one gets that (g_n) is a Cauchy sequence. Thus (g_n) is convergent and as E is closed, the limit $g_0 \in E$. By continuity of the norm $\|f - g_n\|_2 \to \|f - g_0\|_2$ which is then the g_0 we were looking for.

When $p > 2$, the parallelogram identity is no longer valid. However, it is valid pointwise: if $f, g \in L^p(\Omega, \mu)$ and $x \in \Omega$ then

$$|f(x) - g(x)|^2 + |f(x) + g(x)|^2 = 2|f(x)|^2 + 2|g(x)|^2.$$

As $p > 2$, $r = p/2 > 1$. But, for $a, b > 0$

$$a^r + b^r \leq (a + b)^r \leq 2^{r-1}(a^r + b^r). \tag{1.7}$$

From this, we get

$$\begin{aligned}
|f(x) - g(x)|^p + |f(x) + g(x)|^p &= \big(|f(x) - g(x)|^2\big)^r + \big(|f(x) + g(x)|^2\big)^r \\
&\leq \big(|f(x) - g(x)|^2 + |f(x) + g(x)|^2\big)^r \\
&= 2^r\big(|f(x)|^2 + |g(x)|^2\big)^r \leq 2^{2r-1}\big(|f(x)|^{2r} + |g(x)|^{2r}\big) \\
&= 2^{p-1}\big(|f(x)|^p + |g(x)|^p\big).
\end{aligned}$$

Integrating with respect to μ, we get

$$\|f - g\|_p^p + \|f + g\|_p^p \leq 2^{p-1}\big(\|f\|_p^p + \|g\|_p^p\big).$$

The remaining of the proof is exactly the same: take a sequence $g_n \in E$ such that $\|g_n - f\|_p \to d(f, E)$ and apply the inequality with f replaced by $f - g_n$ and g by $f - g_m$. We obtain

$$\begin{aligned}
\|g_n - g_m\|_p^p &\leq 2^{p-1}\big(\|f - g_n\|_p^p + \|f - g_m\|_p^p\big) - \|2f - g_n - g_m\|_p^p \\
&\leq 2^{p-1}\big(\|f - g_n\|_p^p + \|f - g_m\|_p^p - 2d(f, E)\big).
\end{aligned}$$

We then deduce that g_n is a Cauchy sequence, thus converges. As E is closed, the limit is in E and is the desired value. □

PROOF OF (1.7). Let us rewrite the inequality $a^r + b^r \leq (a+b)^r$ in the form $1 + (b/a)^r \leq (1+b/a)^r$ that is, setting $t = b/a$, $1+t^r \leq (1+t)^r$ for all $t > 0$. For $t \geq 0$ let $f(t) = (1+t)^r - (1+t^r)$. Clearly $f(0) = 0$ and $f'(t) = r((1+t)^{r-1} - t^{r-1}) \geq 0$ since $r \geq 1$ thus s^{r-1} is increasing.

The other inequality uses convexity of $t \to t^r$:

$$(a+b)^r = 2^r \left(\frac{a+b}{2}\right)^r \leq 2^r \frac{a^r + b^r}{2}$$

which is the expected inequality. □

The proof for $p < 2$ is more involved and requires the use of Hammer's inequality

$$\big|\|f+g\|_p + \|f-g\|_p\big|^p + \big|\|f+g\|_p - \|f-g\|_p\big|^p \leq 2^p\big(\|f\|_p^p + \|g\|_p^p\big).$$

As we won't use the projection theorem in that case, we will not develop the proof here.

1.5. Duality. Thanks to Hölder's inequality, it is easy to construct continous linear functionals on $L^p(\Omega,\mu)$. Indeed,

LEMMA 2.10. *Let $1 \leq p \leq +\infty$ and let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Let $g \in L^{p'}(\Omega,\mu)$ and define*

$$\Phi_g(f) = \int_\Omega f(x)g(x)\,d\mu(x).$$

Then Φ_g is a continous linear functional on $L^p(\Omega,\mu)$. Moreover

$$\|\Phi_g\| := \sup_{\|f\|_p \leq 1} \int_\Omega f(x)g(x)\,d\mu(x) = \|g\|_{p'}.$$

PROOF. Hölder's inequality directly shows continuity with $\|\Phi_g\| \leq \|g\|_{p'}$ while the equality follows from the equality case in Hölder's inequality. □

The key result of this section is the following converse of this lemma:

THEOREM 2.11. *Let $1 \leq p < +\infty$ and let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\Phi \in (L^p)'$ i.e. a bounded linear functional on $L^p(\Omega,\mu)$. Then there exists a unique $g \in L^{p'}(\Omega,\mu)$ such that $\Phi = \Phi_g$, that is*

$$\Phi(f) = \int_\Omega f(x)g(x)\,\mathrm{d}\mu(x)$$

for every $f \in L^p(\Omega,\mu)$.

REMARK 2.12. It is important to notice that the result is false for $p = +\infty$. The dual of $L^\infty(\Omega,\mu)$ is much more difficult to describe and is out of scope of this course.

PROOF OF UNIQUENESS. The uniqueness is easy to prove: assume that $g_1, g_2 \in L^{p'}$ are such that $\Phi_{g_1} = \Phi_{g_2}$ then, if $g = g_1 - g_2$, for every $f \in L^p$, $\Phi_g(f) = 0$.

If $p > 1$, then $p' < +\infty$, take $f(x) = \begin{cases} |g(x)|^{p'-2}\overline{g(x)} & \text{if } g(x) \neq 0 \\ 0 & \text{if } g(x) = 0 \end{cases}$. First $|f|^p = (|g|^{p'-1})^p = |g|^{p'}$ since $p = \frac{p'}{p'-1}$ when $\frac{1}{p} + \frac{1}{p'} = 1$. Thus $f \in L^p$. Next,

$$0 = \Phi_g(f) = \int_\Omega f(x)g(x)\,\mathrm{d}\mu(x) = \int_\Omega |g(x)|^{p'-2}\overline{g(x)}g(x)\,\mathrm{d}\mu(x) = \|g\|_{p'}^{p'}$$

thus $g = 0$ as claimed.

If $p = 1$, a slight modification is needed. Write $\Omega = \bigcup_{n\geq1} \Omega_n$ with $\mu(\Omega_n) < +\infty$ and $g(x) = e^{i\theta(x)}|g(x)|$. Then $f_n = e^{-i\theta}\mathbf{\Omega_n} \in L^1$ and

$$0 = \Phi_g(f_n) = \int_\Omega f_n(x)g(x)\,\mathrm{d}\mu(x) = \int_{\Omega_n} |g(x)|\,\mathrm{d}\mu(x).$$

It follows that $g = 0$ μ-almost everywhere on Ω_n *i.e.* there is an $E_n \subset \Omega_n$ such that $g = 0$ on $\Omega_n \setminus E_n$. Thus $g = 0$ on $\bigcup_{n\geq1} \Omega_n \setminus \bigcup_{n\geq1} E_n = \Omega \setminus \bigcup_{n\geq1} E_n$. As $\bigcup_{n\geq1} E_n$ is a countable union of sets of mesure 0, it has measure 0 thus $g = 0$ μ-almost everywhere. □

Recall that $L^2(\Omega,\mu)$ is a Hilbert space so that the theorem follows from the more general theorem by Riesz. It turns out that the case $1 \leq p < 2$ can be deduced from it.

PROOF IN THE CASE $1 \leq p < 2$. First let p' be the dual index, $\frac{1}{p}+\frac{1}{p'}=1$ and note that $p'>2$. Let r,s be given by $\frac{p}{2}+\frac{1}{s}=1$ *i.e.* $s=\frac{2}{2-p}$ and $r=ps$. Note that r, s have been chosen so that Hölder's inequality implies

$$\int_\Omega |f(x)|^p|g(x)|^p\,\mathrm{d}\mu(x) \leq \left(\int_\Omega |f(x)|^2\,\mathrm{d}\mu(x)\right)^{p/2}\left(\int_\Omega |g(x)|^{ps}\,\mathrm{d}\mu(x)\right)^{1/s} = \|f\|_2^p\|g\|_r^p. \tag{1.8}$$

Write $\Omega=\bigcup_{n\geq 2}\Omega_n$ with $\mu(\Omega_n)<+\infty$ and the Ω_n being disjoint. Let us define w through

$$w(x)=\sum_{n\geq 1}\alpha_n\mathbf{1}_{\Omega_n}$$

where the $\alpha_n>0$ are chosen so that

(i) for every n, $\alpha_n>0$ and $\alpha_{n+1}\leq\alpha_n$,
(ii) $\|w\|_r^r=\sum_{n\geq 1}\alpha_n^r\mu(\Omega_n)<+\infty$.

It follows from (1.8) that, for every $f\in L^2(\Omega,\mu)$, $fw\in L^p(\Omega,\mu)$ with $\|fw\|_p\leq\|w\|_r\|f\|_2$. In other words, the operator $T_w: L^2\to L^p$ defined by $T_wf=wf$ is bounded.

Now, let $\Phi\in(L^p)'$, that is let Φ be a bounded linear functional on $L^p(\Omega,\mu)$. It follows that ΦT_w is a bounded linear functional on $L^2(\Omega,\mu)$. According to Riesz's theorem, there exists $G\in L^2(\Omega,\mu)$ such that $\Phi T_w=\Phi_G$: *for every* $f\in L^2(\Omega,\mu)$,

$$\Phi T_w f=\Phi(fw)=\int_\Omega f(x)G(x)\,\mathrm{d}\mu(x).$$

Now consider the set $\mathcal{S}=\{\varphi\in L^p(\Omega,\mu)\ :\ \varphi/w\in L^2(\Omega,\mu)\}$. Note that $\mathcal{S}$ is dense in $L^p(\Omega,\mu)$. Indeed, if $f\in L^p(\Omega,\mu)$ and $\varepsilon>0$, there exists N such that, writing $\Phi_N=\bigcup_{n\leq N}\Omega_n$ $f_N=f\mathbf{1}_{\Phi_N}\mathbf{1}_{|f|\leq N}$, then $\|f-f_N\|_p\leq\varepsilon$ (note that $f_N\to f$ a.e. and that $|f_N|\leq f$ so that $f_N\to f$ in L^p). Further, for $x\in\Phi_N$, there is an $n\leq N$ such that $x\in\Omega_n$. Then $w(x)=\alpha_n\geq\alpha_N$ since the α_n have been chosen as a decreasing sequence. It follows that

$$\frac{|f_N(x)|}{w(x)}\leq\begin{cases}0 & \text{if } x\notin\Phi_N\\ \frac{N}{\alpha_N} & \text{if } x\in\Phi_N\end{cases}.$$

Thus f_N/w is bounded with support of finite measure and is thus in $L^2(\Omega,\mu)$ *i.e.* $f_N\in\mathcal{S}$.

Now, for $\varphi\in\mathcal{S}$, we can write $\varphi=fw$ with $f=\varphi/w\in L^2$. Therefore

$$\Phi(\varphi)=\Phi(fw)=\int_\Omega f(x)G(x)\,\mathrm{d}\mu(x)=\int_\Omega\varphi(x)\frac{G(x)}{w(x)}\,\mathrm{d}\mu(x)=\Phi_g(\varphi)$$

with $g:=G/w$. If we are able to prove that $g\in L^{p'}(\Omega,\mu)$, then Φ_g is a continuous linear functional on L^p as well. Therefore $\Phi=\Phi_g$ is an equality between two continuous functionals on L^p on the dense set $\mathcal{S}$ of L^p. This equality is then true on all of L^p, which is what we wanted to prove.

It remains to prove that $g\in L^{p'}(\Omega,\mu)$. We need to distinguish two cases.

First consider the case $1<p<2$. Consider $\varphi_N=\overline{g}|g|^{p-2}\mathbf{1}_{|g|\leq N}\mathbf{1}_{\Phi_N}$ and observe that $|\varphi_N|=|g|^{p-1}\mathbf{1}_{|g|\leq N}\mathbf{1}_{\Phi_N}$. In particular φ_N is bounded and has support of finite measure thus $\varphi_n\in L^p(\Omega,\mu)$ and on its support $w\geq\alpha_N$ so that $|\varphi_N/w|\leq|\varphi_N|/\alpha_N\in L^2(\Omega,\mu)$. In other words, $\varphi_N\in\mathcal{S}$. But then

$$\Phi(\varphi_N)=\Phi_g(\varphi_N)=\int_\Omega\varphi_N(x)g(x)\,\mathrm{d}\mu(x)=\int_\Omega|g(x)|^{p'}\mathbf{1}_{|g|\leq N}(x)\mathbf{1}_{\Phi_N}(x)\,\mathrm{d}\mu(x).$$

On the other hand, Φ is continuous on $L^p(\Omega,\mu)$ thus, for all φ, $|\Phi(\varphi)|\leq\|\Phi\|\|\varphi\|_p$, in particular

$$\begin{aligned}|\Phi(\varphi_N)| &\leq \|\Phi\|\|\varphi_N\|_p=\|\Phi\|\left(\int_\Omega|g|^{p(p-1)}(x)\mathbf{1}_{|g|\leq N}(x)\mathbf{1}_{\Phi_N}(x)\,\mathrm{d}\mu(x)\right)^{1/p}\\ &= \|\Phi\|\left(\int_\Omega|g|^{p'}(x)\mathbf{1}_{|g|\leq N}(x)\mathbf{1}_{\Phi_N}(x)\,\mathrm{d}\mu(x)\right)^{1/p}.\end{aligned}$$

Combining both identities shows that, for every N,

$$\left(\int_\Omega|g|^{p'}(x)\mathbf{1}_{|g|\leq N}(x)\mathbf{1}_{\Phi_N}(x)\,\mathrm{d}\mu(x)\right)^{1/p'}\leq C$$

Letting N go to infinity and applying Beppo-Levi's Lemma, we get $\|g\|_{p'} \leq C$ so that $g \in L^{p'}(\Omega,\mu)$ as expected.

When $p = 1$ the argument needs to be modified. We write $g = e^{i\theta}|g|$ and consider $\varphi_N = e^{-i\theta}\mathbf{1}_{|g|>\|\Phi\|+1/N}\mathbf{1}_{\Phi_N}$. As previously, $\varphi_N \in \mathcal{S}$. But then

$$\begin{aligned}\Phi(\varphi_N) = \Phi_g(\varphi_N) = \int_\Omega \varphi_N(x)g(x)\,\mathrm{d}\mu(x) &= \int_\Omega |g(x)|\mathbf{1}_{|g|>\|\Phi\|+1/N}\mathbf{1}_{\Phi_N}\,\mathrm{d}\mu(x)\\ &\geq (\|\Phi\|+1/N)|\{|g|>\|\Phi\|+1/N\}\cap\Phi_N|.\end{aligned}$$

On the other hand

$$\begin{aligned}|\Phi(\varphi_N)| \leq \|\Phi\|\|\varphi_N\|_1 &= \|\Phi\|\int_\Omega \mathbf{1}_{|g|>\|\Phi\|+1/N}\mathbf{1}_{\Phi_N}\,\mathrm{d}\mu(x)\\ &= \|\Phi\||\{|g|>\|\Phi\|+1/N\}\cap\Phi_N|.\end{aligned}$$

Combining both, we get that $|\{|g|>\|\Phi\|+1/N\}\cap\Phi_N| = 0$. Finally, As $\{|g|>\|\Phi\|\} = \bigcup_{N\geq 1}\{|g|>\|\Phi\|+1/N\}\cap\Phi_N$ we get that $|g| \leq \|\Phi\|$ almost everywhere. □

PROOF USING THE PROJECTION THEOREM WHEN $1<p<\infty$. Let Φ be a continuous linear functional on $L^p(\Omega,\mu)$. We are looking for $g \in L^{p'}(\Omega,\mu)$ such that $\Phi = \Phi_g$. We can assume that Φ is not identically zero (otherwise take $g=0$) so that there is an $f \in L^p(\Omega,\mu)$ with $L(f) \neq 0$. Up to replacing f by $f/L(f)$ we can assume that $L(f) = 1$.

Let $E = \ker\Phi = \Phi^{-1}(0)$ and note that E is a closed linear subspace of $L^p(\Omega,\mu)$. Therefore, there exists $g_0 \in E$ such that $\|f-g_0\|_p = d(f,E)$. Note that $L(f-g_0) = L(f)-L(g_0) = 1-0 = 0$ and that $\|f-g_0\|_p = \|(f-g_0)-0\|_p = d(f,E)$. Up to replacing f by $f-g_0$ we can assume that 0 is a projection of f on E: $L(f) = 1$ and $\|f\|_p = d(f,E)$, that is, for all $g \in E$, $\|f\|_p \leq \|f-g\|_p$.

Now fix $w \in E$ and consider the function φ defined on $(-1,1)\times\Omega$ by $\varphi(t,x) = |f(x)-tg(x)|^p$ and let Φ be defined on $\mathbb{R}$ by

$$\Phi(t) = \int_\Omega \varphi(t,x)\,\mathrm{d}x = \|f-tg\|_p^p.$$

First, observe that

– as $tg \in E$, $\Phi(t) = \|f-tg\|_p^p \geq \|f\|_p^p = \Phi(0)$. Thus Φ has a minimum at 0.

– φ is continuous in t. Moreover,

$$\varphi(t,x) = \big(|f(x)-tg(x)|^2\big)^{p/2} = \big(|f(x)|^2 + t^2|g(x)|^2 + 2t\Re\overline{f(x)}g(x)\big)^{p/2}$$

thus

$$\begin{aligned}\frac{\partial\varphi}{\partial t} &= \frac{p}{2}\big(|f(x)-tg(x)|^2\big)^{p/2-1}\big(2t|g(x)|^2 + 2\Re\overline{f(x)}g(x)\big)\\ &= p|f(x)-tg(x)|^{p-2}\big(t|g(x)|^2 + \Re\overline{f(x)}g(x)\big)\end{aligned}$$

— for $|t| \leq 1$ and $x \in \Omega$,

$$\begin{aligned}|\varphi(t,x)| &= |f(x)-tg(x)|^p = 2^p\left|\frac{f(x)-tg(x)}{2}\right|^p \leq 2^p\left(\frac{|f(x)|+|g(x)|}{2}\right)^p\\ &\leq 2^{p-1}(|f(x)|^p + |g(x)|^p).\end{aligned}$$

Lebesgue's theorem on continuity of integrals then shows that Φ is continuous.

— for $|t| \leq 1$ and $x \in \Omega$,

$$\begin{aligned}\left|\frac{\partial\varphi}{\partial t}\right| &\leq p(|f(x)|+|g(x)|)^{p-2}\big(2|g(x)|^2 + |f(x)|^2\big)\\ &\leq p2^{p-3}(|f(x)|^{p-2} + |g(x)|^{p-2})\big(2|g(x)|^2 + |f(x)|^2\big)\\ &\leq p2^{p-2}(|f(x)|^p + |g(x)|^p + |f(x)|^{p-2}|g(x)|^2 + |g(x)|^{p-2}|f(x)|^2).\end{aligned}$$

As $f,g \in L^p$, $|f(x)|^p + |g(x)|^p$ is integrable. Further if $q = \dfrac{p}{2}$ and q' is given by $\dfrac{1}{q} + \dfrac{1}{q'} = 1$ then $q' = \dfrac{q}{q-1} = \dfrac{p}{p-2}$ and Hölders inequality with these exponents gives

$$\int_\Omega |f(x)|^{p-2}|g(x)|^2\,\mathrm{d}\mu(x) \leq \left(\int_\Omega |f(x)|^p\,\mathrm{d}\mu(x)\right)^{(p-2)/p}\left(\int_\Omega |g(x)|^p\,\mathrm{d}\mu(x)\right)^{2/p}$$

thus $|f(x)|^{p-2}|g(x)|^2$ is also integrable. The same is true for $|g(x)|^{p-2}|f(x)|^2$.

We can thus apply Lebegue's derivation theorem and see that Φ is differentiable on $(-1,1)$ and

$$\Phi'(t)=\int_\Omega \frac{\partial\varphi}{\partial t}(t,x)\,\mathrm{d}\mu(x).$$

In particular,

$$\Phi'(0)=p\int_\Omega |f(x)|^{p-2}\Re\overline{f(x)}g(x)\,\mathrm{d}\mu(x).$$

As Φ has a minimum at 0, we get $\Phi'(0)=0$ that is, for every $g\in L^p$, with $L(g)=0$,

$$\Re\int_\Omega |f(x)|^{p-2}\overline{f(x)}g(x)\,\mathrm{d}\mu(x)=0.$$

Note that, if $g\in L^p$, with $L(g)=0$ then $ig\in L^p$ and $L(ig)=0$ so that

$$\Re i\int_\Omega |f(x)|^{p-2}\overline{f(x)}g(x)\,\mathrm{d}\mu(x)=0.$$

Finally, define $\tilde f$ by $\tilde f(x)=|f(x)|^{p-2}\overline{f(x)}$ and note that $|\tilde f|^{p'}=|f|^{(p-1)p'}=|f|^p$ so that $\tilde f\in L^{p'}$ with $\left\|\tilde f\right\|_{p'}=\|f\|_p$. We have proved that *for every $g\in L^p(\Omega,\mu)$ with $L(g)=0$,*

$$\int_\Omega \tilde f(x)g(x)\,\mathrm{d}\mu(x)=0.$$

In otherwords, if $L(g)=0$ then $\Phi_{\tilde f}(g)=0$.

Now let $h\in L^p$ and consider $g=h-L(h)f\in L^p$. Note that $L(g)=L(h)-L(h)L(f)=0$ since $L(f)=1$ and that $\Phi_{\tilde f}(f)=\|f\|_p^p$. Therefore $\Phi_{\tilde f}(g)=0$. But

$$0=\Phi_{\tilde f}(g)=\Phi_{\tilde f}\big(h-L(h)f\big)=\Phi_{\tilde f}(h)-L(h)\Phi_{\tilde f}(f)=\Phi_{\tilde f}(h)-L(h)\|f\|_p^p.$$

As $L(f)=1$, $f\neq 0$ thus $\|f\|_p^p\neq 0$ and we conclude that

$$L(h)=\frac{1}{\|f\|_p^p}\Phi_{\tilde f}(h)=\Phi_{\tilde f/\|f\|_p^p}(h)$$

which is the expected result. □

2. Weak L^p spaces

2.1. The distribution function.

DEFINITION 2.13. Let $(\Omega,\mathcal{B},\mu)$ be a measure space and $f:\Omega\to\mathbb{C}$ be measurable. The *distribution function* of f is the function $d_f:]0,+\infty)\to\mathbb{R}^+$ defined by

$$d_f(\lambda)=\mu\big(\{x\in\Omega\,:\,|f(x)|>\lambda\}\big).$$

We will say that f *vanishes at infinity* if $d_f(\lambda)<+\infty$ for every $\lambda>0$.

Let us introduce

$$D_f(\lambda)=\{y\in\Omega\,:\,|f(y)|>\lambda\}.$$

LEMMA 2.14 (Layer cake representation). *Let $(\Omega,\mathcal{B},\mu)$ be a measure space and $f:\Omega\to\mathbb{C}$ be measurable.*

$$|f(x)|=\int_0^{+\infty}\mathbf{1}_{\{y\in\Omega:\,|f(y)|>\lambda\}}\,d\lambda \tag{2.9}$$

so that

$$\int_\Omega |f(x)|\,d\mu(x)=\int_0^{+\infty}d_f(\lambda)\,d\lambda \tag{2.10}$$

PROOF. For (2.9), it is enough to notice that $\mathbf{1}_{\{y\in\Omega:\,|f(y)|>\lambda\}}(x)=1$ when $\lambda\in[0,|f(x)|[$ and is zero otherwise. Applying Fubini, (2.10) follows. □

Some further important properties of d_f are summarized in the following proposition:

PROPOSITION 2.15. *Let $(\Omega,\mathcal{B},\mu)$ be a measure space and $f,g,f_n:\Omega\to\mathbb{C}$ be measurable. Then the following hold:*

(i) $\lambda \mapsto d_f(\lambda)$ *is decreasing and right-continuous;*
(ii) *if* $|f| \leq |g|$ *then* $d_f \leq d_g$. *In particular if* $f = g$ *a.e. then* $d_f = d_g$*;*
(iii) *if for a.e.* $x \in \Omega$, $|f_n(x)|$ *increases and converges to* $|f(x)|$ *then* $d_{f_n} \to d_f$*;*

PROOF. Obviously, if $\lambda \leq \lambda'$ then $D_f(\lambda') \subset D_f(\lambda)$ so that $d_f(\lambda') \leq d_f(\lambda)$. To see that d_f is right semi-continuous, let λ_n be decreasing with $\lambda_n \to \lambda \geq 0$. Then $D_f(\lambda_n) \subset D_f(\lambda)$ with $D_f(\lambda) = \bigcup_{n\geq 1} D_f(\lambda_n)$ thus

$$\lim d_f(\lambda_n) = \lim \mu\big(D_f(\lambda_n)\big) = \mu\left(\bigcup_{n\geq 1} D_f(\lambda_n)\right) = \mu\big(D_f(\lambda)\big) = d_f(\lambda).$$

As d_f is decreasing, this shows right-continuity.

If $|f| \leq |g|$ then $D_f(\lambda) \subset D_g(\lambda)$ (up to a negligible set) thus $d_f(\lambda) \leq d_g(\lambda)$.

Finally, $D_{f_n}(\lambda)$ is an increasing family of sets such that

$$\bigcup_{n\geq 1} D_{f_n}(\lambda) = D_f(\lambda)$$

(up to a set of measure 0) so that we conclude as in the first part. □

EXERCICE 2.16. Let $(X, \mathcal{B}, \mu)$ be a measure space and $f : X \to \mathbb{R}^+$ be a *simple* function, that is f can be written in the form

$$f = \sum_{j=1}^{k} c_j \mathbf{1}_{E_j}$$

with $E_j \in \mathcal{B}$ two-by-two disjoint and $c_1 \leq c_2 \leq \cdots \leq c_k$. Compute d_f.

EXERCICE 2.17. Let $(X, \mathcal{B}, \mu)$ be a measure space and $f : X \to \mathbb{C}$ be a measurable function vanishing at infinity. Prove that $d_f(s) \to 0$ when $s \to +\infty$ and that

$$\lim_{t\to s^-} d_f(t) = d_f(s) + \mu(\{x : |f(x)| = s\}.$$

2.2. Weak L^p spaces. Before introducing weak L^p spaces, let us first prove the following lemma that shows how the distribution function behaves for L^p functions:

LEMMA 2.18. *Let* $(X, \mathcal{B}, \mu)$ *be a measure space,* $f : X \to \mathbb{C}$ *be a measurable function vanishing at infinity and* $1 \leq p < +\infty$. *Then*

(i) $\|f\|_p^p = p \displaystyle\int_0^{+\infty} s^{p-1} d_f(s)\,\mathrm{d}s$*;*
(ii) *if* $f \in L^p$ *then* $d_f(s) \leq \dfrac{\|f\|_p^p}{s^p}$.

PROOF. For the first identity, apply the Layer-Cake Representation to $|f|^p$ to get

$$|f(x)|^p = \int_0^{+\infty} \mathbf{1}_{\{|f|^p > \lambda\}}\,\mathrm{d}\lambda = p\int_0^{+\infty} \mathbf{1}_{\{|f|^p > s^p\}} s^{p-1}\,\mathrm{d}s$$

with the change of variable $\lambda = s^p$. Now $\{|f|^p > s^p\} = \{|f| > s\} = D_f(s)$ so that, integrating over X with respect to μ and applying Fubini, we get

$$\|f\|_p^p = \int_X |f(x)|^p\,\mathrm{d}\mu(x) = p\int_0^{+\infty}\int_X \mathbf{1}_{D_f(s)}\,\mathrm{d}\mu(x) s^{p-1}\,\mathrm{d}s = p\int_0^{+\infty} d_f(s) s^{p-1}\,\mathrm{d}s$$

as claimed.

The second assertion is just Markov's inequality:

$$\begin{aligned} d_f(s) &= \int_X \mathbf{1}_{\{|f|>s\}}(x)\,\mathrm{d}\mu(x) = \int_X \mathbf{1}_{\{|f|^p>s^p\}}(x)\,\mathrm{d}\mu(x) \\ &\leq \int_X \frac{|f(x)|^p}{s^p}\mathbf{1}_{\{|f|^p>s^p\}}(x)\,\mathrm{d}\mu(x) \leq \frac{1}{s^p}\int_X |f(x)|^p\,\mathrm{d}\mu(x) \end{aligned}$$

as claimed. □

We can now introduce the weak L^p spaces L^p_w as follows:

DEFINITION 2.19. Let $(X,\mathcal{B},\mu)$ be a measure space, $f : X \to \mathbb{C}$ be a measurable function vanishing at infinity and $1 \le p < +\infty$. We will say that $f \in L^p_w$, the *weak L^p-space*, if there exists a constant $C > 0$ such that $d_f(s) \le \dfrac{C^p}{s^p}$ for all $s > 0$.

The smallest such C is called the weak-L^p norm *i.e.*

$$\|f\|_{L^p_w} = \inf\left\{C > 0 : d_f(s) \le \frac{C^p}{s^p}\right\} = \sup\{s d_f(s)^{1/p} : s > 0\}. \tag{2.11}$$

When $p = +\infty$, $L^\infty_w = L^\infty$.

EXERCICE 2.20.
(1) Show that if $\lambda \in \mathbb{C}\setminus\{0\}$, then $d_{\lambda f}(s) = d_f(s/|\lambda|)$. Conclude that the two expressions in (2.11) are indeed equal.
(2) Let $f(x) = 1/x^\alpha$ $\alpha > 0$. Determine for which $p \ge 1$ is $f \in L^p_w(0,1)$ and for which p is $f \in L^p_w(1,+\infty)$.
(3) Let $f(x) = |x|^{-pd}$ on $\mathbb{R}^d$. Show that $f \in L^p_w(\mathbb{R}^d)$ and compute its norm.

As for usual L^p spaces, we will identify two functions if they are equal almost everywhere, that is we start with the space of functions and then quotient it by the equivalence relation $f \sim g$ if $f = g$ a.e. One easily checks that $d_f(s) = d_g(s)$ if $f \sim g$ so that this operation is legitimate.

LEMMA 2.21. *For $1 \le p < +\infty$, $L^p_w \subset L^p$ and $\|f\|_{L^p_w}$ is a quasi-norm.*

PROOF. The first assertion is already given in Lemma 2.18.

First, if $\|f\|_{L^p_w} = 0$ then, for every $s > 0$, $d_f(s) = 0$, *i.e.* $\{|f| > s\}$ is negligible. But then $\{f \ne 0\} = \bigcup_{n\in\mathbb{N}}\{|f| > 1/n\}$ is also negligible and $f = 0$ a.e.

Next, since $d_{\lambda f}(s) = d_f(s/|\lambda|)$, a straightforward computation shows that $\|\lambda f\|_{L^p_w} = |\lambda|\,\|f\|_{L^p_w}$.

We conclude by noticing that if $|f(x)+g(x)| > s$ then at least one of $|f(x)|$ or $|g(x)| > s/2$ (otherwise $|f(x)+g(x)| \le |f(x)| + |g(x)| \le s$). This implies that

$$\{x : |f(x)+g(x)| > s\} \subset \{x : |f(x)| > s/2\} \cup \{x : |g(x)| > s/2\}$$

therefore

$$d_{f+g}(s) \le d_f(s/2) + d_g(s/2).$$

Now $(a+b)^{1/p} \le a^{1/p} + b^{1/p}$ when $a, b \ge 0$ and $p \ge 1$ (factor out a and notice that $(1+t)^{1/p} \le 1 + t^{1/p}$ for $t > 0$ by differentiating). We conclude that

$$s d_{f+g}(s)^{1/p} \le 2^{1/p}\left(\frac{s^p}{2} d_f(s/2)\right)^{1/p} + 2^{1/p}\left(\frac{s^p}{2} d_g(s/2)\right)^{1/p}.$$

Taking proper supprema we get $\|f+g\|_{L^p_w} \le 2^{1/p}\big(\|f\|_{L^p_w} + \|g\|_{L^p_w}\big)$. □

REMARK 2.22. With this definition, the weak-L^p spaces are not normed spaces. However, there exists a norm that we will denote by $\|f\|_{L^{p,\infty}}$ which is equivalent to $\|f\|_{L^p_w}$.

To define this norm, we first need to introduce the decreasing rearrangement of f:

$$f^*(t) = \inf\{s > 0 : d_f(s) \le t\}$$

which is defined on $[0,+\infty)$. If the measure μ is non-atomic *i.e.* $\mu(\{x\}) = 0$ for all $x \in X$ we define

$$f^{**}(u) = \frac{1}{u}\int_0^p f^*(t)\,\mathrm{d}t$$

and then

$$\|f\|_{L^{p,\infty}} = \sup_{u>0} u^{1/p} f^{**}(u) = \sup\left\{\frac{1}{\mu(E)^{1-\frac{1}{p}}}\int_E |f(x)|\,\mathrm{d}\mu(x) \ :\ E \in \mathcal{B}\right\}.$$

In the general case, the first identity is still valid if we define f^{**} by

$$f^{**}(u) = \sup\left\{\frac{1}{\mu(E)}\int_E |f(x)|\,\mathrm{d}\mu(x) \ :\ E \in \mathcal{B} : \mu(E) \ge u\right\}$$

when $u < \mu(X)$ and

$$f^{**}(u) = \frac{1}{u}\int_X |f(x)|\,\mathrm{d}\mu(x)$$

when $u \geq \mu(X)$.

The space introduced this way is called the Lorentz space $L^{p,\infty}$ and more general spaces Lorentz space $L^{p,q}$ can be defined via

$$\|f\|_{L^{p,q}} = \left(\int_0^{+\infty} \left(u^{1/p} f^{**}(u)\right)^q \,\mathrm{d}u\right)^{1/q}$$

when $q < +\infty$. The reader may check that $L^{p,p} = L^p$.

We will not require any knowledge on Lorentz space in this course.

We can now introduce convergence in weak-L^p spaces in the usual way: $f_n \to f$ in L^p_w in $\|f_n - f\|_{L^p_w} \to 0$. Let us compare this convergence to two other convergences:

LEMMA 2.23. *Let $1 \leq p < +\infty$ and $(X, \mathcal{B}, \mu)$ be a measure space.*

(i) *If $f_n \to f$ in L^p then $f_n \to f$ in L^p_w.*
(ii) *If $f_n \to f$ in L^p_w then $f_n \to f$ in measure that is, for every $\varepsilon > 0$, there exists n_0 such that, if $n > n_0$ then*

$$\mu(\{x \,:\, |f_n(x) - f(x)| \geq \varepsilon\}) \leq \varepsilon.$$

PROOF. The first assertion follows from Lemma 2.18:

$$s^p d_{f_n - f}(s) \leq \|f_n - f\|_p^p$$

so that

$$\|f_n - f\|_{L^p_w} \leq \|f_n - f\|_p^p$$

which shows the desired implication.

For the second one, given $\varepsilon > 0$ there is an N such that for every $n \geq N$

$$\|f_n - f\|_{L^p_w} := \left(\sup_{s>0} s^p d_{f_n - f}(s)\right)^{1/p} \leq \varepsilon^{1/p+1}.$$

Taking $s = \varepsilon$ gives

$$\varepsilon^p \mu(\{x \,:\, |f_n(x) - f(x)| > \varepsilon\}) \leq \varepsilon^{1+p}$$

which gives the result after simplification by ε. □

We will not establish that weak L^p spaces are complete (this requires to use the Lorentz spaces *i.e.* the norm and not the quasi-norm). However, we will establish a weak version of completness. The first result is also the key step in establishing completeness of L^p spaces.

THEOREM 2.24 (Riesz). *Let $(X, \mathcal{B}, \mu)$ be a measure space and f_n, f be complex valued measurable functions on X. Assume that $f_n \to f$ in measure, then there exists a subsequence (f_{n_k}) such that $f_{n_k} \to f$ almost everywhere.*

PROOF. We choose n_k inductively such that $n_k > n_{k-1}$ and such that the set

$$A_k := \{x \,:\, |f_{n_k}(x) - f(x)| > 2^{-k}\}$$

satisfies $\mu(A_k) \leq 2^{-k}$. Then

$$\mu\left(\bigcup_{k=m}^{+\infty} A_k\right) \leq \sum_{k=m}^{+\infty} \mu(A_k) \leq \sum_{k=m}^{+\infty} 2^{-k} = 2^{-m+1}.$$

In particular

$$\mu\left(\bigcup_{k=1}^{+\infty} A_k\right) \leq 1 < +\infty.$$

It follows that

$$\mu\left(\bigcap_{m=1}^{+\infty} \bigcup_{k=m}^{+\infty} A_k\right) = 0$$

and this contains the set of all x's such that $f_{n_k}(x)$ does not converge to $f(x)$. □

An essentially similar argument allows to prove the following:

THEOREM 2.25 (Riesz). *Let $(X,\mathcal{B},\mu)$ be a measure space and f_n be complex valued measurable functions on X. Assume that f_n is Cauchy in measure, then there exists a subsequence (f_{n_k}) and a function f such that $f_{n_k}\to f$ almost everywhere.*

PROOF. This time the n_k's are chosen inductively such that $n_{k+1}>n_k$ and

$$A_k:=\{x\,:\,|f_{n_k}(x)-f_{n_{k+1}}(x)|>2^{-k}\}$$

satisfies $\mu(A_k)\leq 2^{-k}$). As previously

$$\mu\left(\bigcap_{m=1}^{+\infty}\bigcup_{k=m}^{+\infty}A_k\right)=0.$$

Now fix m, take $x\notin\displaystyle\bigcup_{k=m}^{+\infty}A_k$ and $j>i>i_0>m$ (i_0 large enough) then

$$|f_{n_j}(x)-f_{n_i}(x)|\leq\sum_{k=i}^{j-1}|f_{n_{k+1}}(x)-f_{n_k}(x)|\leq\sum_{k=i}^{j-1}2^{-k}\leq 2^{-i+1}\leq 2^{-i_0+1}.$$

It follows that $f_{n_i}(x)$ is Cauchy for every $x\in\left(\displaystyle\bigcup_{k=m}^{+\infty}A_k\right)$ and thus has a limit $\varphi_m(x)$.

We now define $f(x)=\lim f_{n_i}(x)$ when $x\notin\displaystyle\bigcap_{m=1}^{+\infty}\bigcup_{k=m}^{+\infty}A_k$ and 0 otherwise so that $f_{n_i}\to f$ a.e. □

EXERCICE 2.26. Show that convergence in measure, as defined above, is equivalent to the fact that, for every $\varepsilon>0$, $\mu(\{|f_n-f|>\varepsilon\})\to 0$.

2.3. First glimpse at interpolation. In this section, we will show that if a function is in two weak L^p-spaces then it is in all L^p spaces "between" them. This is a first step towards interpolation of operators were we will state that if an operator is bounded from L^p to L^q (weak or strong) for two different couples of (p,q)'s then it is also bounded for intermediate couples.

Let us start with functions:

PROPOSITION 2.27. *Let $(X,\mathcal{B},\mu)$ be a measure space and $1\leq p_0<p_1<+\infty$. Let $f\in L^{p_0}_w\cap L^{p_1}_w$. Then, for every $p_0<p<p_1$, $f\in L^p$ with*

$$\|f\|_p^p\leq\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)\|f\|_{L^{p_0}_w}^{p_0\frac{p_1-p}{p_1-p_0}}\|f\|_{L^{p_1}_w}^{p_1\frac{p-p_0}{p_1-p_0}}.$$

PROOF. The hypothesis is that if $C_0>\|f\|_{L^{p_0}_w}^{p_0}$ and $C_1>\|f\|_{L^{p_1}_w}^{p_1}$ then

$$d_f(s)\leq\frac{C_0}{s^{p_0}}\quad\text{and}\quad d_f(s)\leq\frac{C_1}{s^{p_1}}\tag{2.12}$$

and we want to estimate that

$$\|f\|_p^p=p\int_0^{+\infty}s^{p-1}d_f(s)\,\mathrm{d}s.$$

The first of the two estimates (2.12) is better for s near 0 while the second one is better for s near $+\infty$. The idea is then simple, cut the integral at some $\lambda>0$, use the best estimate on each piece and then optimise over λ.

First

$$\int_0^\lambda s^{p-1}d_f(s)\,\mathrm{d}s\leq C_0\int_0^\lambda s^{p-p_0-1}\,\mathrm{d}s=\frac{C_0}{p-p_0}\lambda^{p-p_0}$$

while

$$\int_\lambda^{+\infty}s^{p-1}d_f(s)\,\mathrm{d}s\leq C_1\int_\lambda^{+\infty}s^{p-p_1-1}\,\mathrm{d}s=\frac{C_1}{p_1-p}\lambda^{p-p_1}.$$

Note that the hypothesis $p_0<p<p_1$ guaranties that both integrals converge. It follows that for every $\lambda>0$, every $C_0>\|f\|_{L^{p_0}_w}^{p_0}$ and every $C_1>\|f\|_{L^{p_1}_w}^{p_1}$

$$\|f\|_p^p\leq\frac{C_0p}{p-p_0}\lambda^{p-p_0}+\frac{C_1p}{p_1-p}\lambda^{p-p_1}.$$

So we first optimise in C_0, C_1 to get

$$\|f\|_p^p \le p\left(\frac{\|f\|_{L_w^{p_0}}^{p_0}}{p-p_0}\lambda^{p-p_0} + \frac{\|f\|_{L_w^{p_1}}^{p_1}}{p_1-p}\lambda^{p-p_1}\right).$$

Now the right hand side goes to $+\infty$ when $\lambda\to 0$ and when $\lambda\to+\infty$ so there is a λ for which this quantity is minimal. To find this λ consider a function of the form

$$\varphi(t) = \frac{a}{\alpha}t^\alpha + \frac{b}{\beta t^\beta}$$

with $a,b,\alpha,\beta>0$. Then $\varphi'(t) = at^{\alpha-1} - bt^{-\beta-1}$ and thus $\varphi'(t)=0$ when $t=t_0 := \left(\dfrac{b}{a}\right)^{\frac{1}{\alpha+\beta}}$ and thus

$$\min\varphi(t) = \varphi(t_0) = \frac{a}{\alpha}\left(\frac{b}{a}\right)^{\frac{\alpha}{\alpha+\beta}} + \frac{b}{\beta}\left(\frac{a}{b}\right)^{\frac{\beta}{\alpha+\beta}} = \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)a^{\frac{\beta}{\alpha+\beta}}b^{\frac{\alpha}{\alpha+\beta}}$$

Now $\alpha = p-p_0$, $\beta = p_1-p$, so $\alpha+\beta = p_1-p_0$ we get

$$\|f\|_p^p \le p\left(\frac{1}{p-p_0}+\frac{1}{p_1-p}\right)\|f\|_{L_w^{p_0}}^{p_0\frac{p_1-p}{p_1-p_0}}\|f\|_{L_w^{p_1}}^{p_1\frac{p-p_0}{p_1-p_0}}$$

as claimed. □

Note that the case $p_1=+\infty$ is simpler as $d_f(s)=0$ when $s>\|f\|_\infty$. We leave this case as an exercice.

2.4. Real interpolation. Before switching to operators, we will need to introduce some vocabulary.

DEFINITION 2.28. We say that V is *closed under truncation* if for every $f\in V$ and every $0\le r<s\le+\infty$, the function $f\mathbf{1}_{\{r\le|f|\le s\}}$ still belongs to V.

Let T be a mapping $V\to\mathcal{L}^0(Y)$. We say that T is *sub-linear* if, for every $f,g\in V$ and every $\lambda\in\mathbb{C}$,

$$|T(f+g)|\le|T(f)|+|T(g)| \quad\text{and}\quad |T(\lambda f)| = |\lambda|\,|T(f)|.$$

We say that T is of *strong* (p,q)*-type* if there exists a constant $C_{p,q}$ such that for every $f\in L^p(X)\cap V$, $T(f)\in L^q(Y)$ with

$$\|T(f)\|_{L^q(Y)} \le C_{p,q}\|f\|_{L^p(X)}. \tag{2.13}$$

We say that T is of *weak* (p,q)*-type* if there exists a constant $C_{p,q}$ such that for every $f\in L^p(X)\cap V$, $T(f)\in L^q_w(Y)$ with

$$\|T(f)\|_{L^q_w(Y)} \le C_{p,q}\|f\|_{L^p(X)} \tag{2.14}$$

that is, for every $s>0$,

$$\nu(\{y\,:\,|T(f)(y)|>s\}) \le \frac{\big(C_{p,q}\|f\|_{L^p(X)}\big)^q}{s^q}.$$

Of course, strong (p,q)-type implies weak (p,q)-type but the converse is false.

An example of a vector space that is closed under truncation is the set $\mathcal{S}=\mathcal{S}(X,\mathcal{B},\nu)$ of simple functions *i.e.* of functions of the form $f=\sum_{j=1}^k c_j\mathbf{1}_{S_j}$ where the S_j's are disjoint and $c_j\in\mathbb{C}$ (it is easy to check that the truncation of a simple function is still a simple function).

THEOREM 2.29 (Marcienkiewiz). *Let $(X,\mathcal{B},\mu)$, $(Y,\tilde{\mathcal{B}},\nu)$ be two measure spaces and let $1\le p_0,p_1,q_0,q_1\le+\infty$. Let V be a subspace of $L^{p_0}(\mu)+L^{p_1}(\mu)$ closed by truncation. Let $T:V\to L^0(Y,\tilde{\mathcal{B}},\nu)$ be a sublinear operator that is of weak type (p_0,q_0) and (p_1,q_1) with*

– $\|T(f)\|_{L_w^{q_0}}\le C_0\|f\|_{p_0}$ *for all $f\in L^{p_0}(\mu)\cap V$;*

– $\|T(f)\|_{L_w^{q_1}}\le C_1\|f\|_{p_1}$ *for all $f\in L^{p_1}(\mu)\cap V$.*

Let $0<\theta<1$ and p,q be defined by

$$\frac{1}{p}=\frac{\theta}{p_0}+\frac{1-\theta}{p_1} \quad\textit{and}\quad \frac{1}{q}=\frac{\theta}{q_0}+\frac{1-\theta}{q_1}.$$

Then T is of strong (p,q) with

$$\|T(f)\|_{L^q}\leq C(p_0,p_1,q_0,q_1,\theta)C_0^\theta C_1^{1-\theta}\|f\|_p$$

for every $f\in V\cap L^p(\mu)$.

PROOF. For (much more) simplicity, we will only consider the case $q_0=p_0$ and $q_1=p_1$ so that $q=p$. We will assume that $p_0<p_1$ and it will be more convenient to write $p_-=p_0$ and $p_+=p_1$. Also, we only consider the case $p_+<+\infty$. The case $q_+=p_+=+\infty$ is left as an exercice. We have

$$\nu(\{y\,:\,T(f)(y)>u\})\leq\left(\frac{C_\pm\|f\|_{p_\pm}}{u}\right)^{p_\pm}$$

for all $u>0$ and $f\in V\cap L^{p_\pm}$.

Let $f\in V\cap L^p(\mu)$, fix $t>0$ and write $f=f_-+f_+$ with $f_-=f\mathbf{1}_{\{|f|\geq t\}}$ and $f_+=f\mathbf{1}_{\{|f|<t\}}$. As V is closed under truncation, $f_-\in V$ and as V is a vector space, $f_+=f-f_-\in V$. As $p\in(p_-,p_+)$ we have

$$\int_X|f_+(x)|^{p_+}\,\mathrm{d}x=\int_{\{|f|<t\}}|f(x)|^{p_+}\,\mathrm{d}x\leq t^{p_+-p}\int_{\{|f|<t\}}|f(x)|^p\,\mathrm{d}x\leq t^{p_+-p}\|f\|_p^p<+\infty$$

while

$$\int_X|f_-(x)|^{p_-}\,\mathrm{d}x=\frac{t^{p-p_-}}{t^{p-p_-}}\int_{\{|f|\geq t\}}|f(x)|^{p_-}\,\mathrm{d}x\leq\frac{1}{t^{p-p_-}}\int_{\{|f|\geq t\}}|f(x)|^p\,\mathrm{d}x\leq\frac{\|f\|_p^p}{t^{p-p_-}}<+\infty.$$

Further, by sub-linearity $|T(f)|\leq|T(f_-)|+|T(f_+)|$ so that

$$\{y\,:\,|T(f)(y)|>t\}\subset\{y\,:\,|T(f_-)(y)|>s/2\}\cup\{y\,:\,|T(f_+)(y)|>s/2\}$$

and thus

$$\begin{aligned}\nu(\{y\,:\,|T(f)(y)|>t\})&\leq\nu(\{y\,:\,|T(f_-)(y)|>t/2\})+\nu(\{y\,:\,|T(f_+)(y)|>t/2\})\\&\leq\left(\frac{2C_-}{t}\|f_2\|^{p_-}\right)^{p_-}+\left(\frac{2C_+}{t}\|f_1\|^{p_+}\right)^{p_+}\\&=(2C_-)^{p_-}t^{-p_-}\|f\mathbf{1}_{\{|f|\geq t\}}\|_{p_-}^{p_-}+(2C_+)^{p_+}t^{-p_+}\|f\mathbf{1}_{\{|f|<t\}}\|_{p_+}^{p_+}.\end{aligned}$$

We then conclude writing

$$\begin{aligned}\|T(f)\|_p^p&=p\int_0^{+\infty}t^{p-1}\nu(\{y\,:\,|T(f)(y)|>t\})\,\mathrm{d}t\\&\leq p(2C_-)^{p_-}\int_0^{+\infty}t^{p-1-p_-}\int_X\mathbf{1}_{\{|f|\geq t\}}(x)|f(x)|^{p_-}\,\mathrm{d}\mu(x)\,\mathrm{d}t\\&\quad+p(2C_+)^{p_+}\int_0^{+\infty}t^{p-1-p_+}\int_X\mathbf{1}_{\{|f|<t\}}(x)|f(x)|^{p_+}\,\mathrm{d}\mu(x)\,\mathrm{d}t\\&=p(2C_-)^{p_-}\int_X|f(x)|^{p_-}\int_0^{|f(x)|}t^{p-1-p_-}\,\mathrm{d}t\,\mathrm{d}\mu(x)\\&\quad+p(2C_+)^{p_+}\int_X|f(x)|^{p_+}\int_{|f(x)|}^{+\infty}t^{p-1-p_+}\,\mathrm{d}t\,\mathrm{d}\mu(x)\\&=p\left(\frac{(2C_-)^{p_-}}{p-p_-}+\frac{(2C_+)^{p_+}}{p-p_+}\right)int_X|f(x)|^p\,\mathrm{d}\mu(x)\end{aligned}$$

with Fubini and a simple computation. □

It should be noted that the constant $p\left(\frac{(2C_-)^{p_-}}{p-p_-}+\frac{(2C_+)^{p_+}}{p-p_+}\right)$ obtained in this computation explodes when $p\to p_\pm$ so that if T were of strong $(p_\pm,p_\pm)$-type those constants would most likely not be very good. The aim of complex interpolation is precisely to cover this case with better constants.

EXERCICE 2.30. Prove the following result:

LEMMA 2.31 (Kolmogorov). *Let T be an operator of weak type $(1,1)$ and $0<\nu<1$. Then for every $E\subset\mathbb{R}^d$ with $0<|E|<+\infty$ and every $f\in L^1(\mathbb{R}^d)$,*

$$\int_E |Tf(x)|^\nu \, \mathrm{d}x \le |E|^{1-\nu}\|f\|^\nu_{L^1(\mathbb{R}^d)}$$

where C is aconstant that depends on d and ν only.

Hint: Write the integral in the left hand side in terms of level sets $\{x\,:\,|Tf(x)|>\lambda\}$.

2.5. Complex interpolation.

THEOREM 2.32 (Riesz-Thorin). *Let $(X,\mathcal{B},\mu)$, $(Y,\tilde{\mathcal{B}},\nu)$ be two measure spaces and let $1\le p_0,p_1,q_0,q_1\le+\infty$. Let $T\,:\,S\to L^0(Y)$ be a* linear *operator and assume that T is of strong (p_0,q_0) and (p_1,q_1) type with*

$$\|T(f)\|_{L^{q_0}}\le C_0\|f\|_{L^{p_0}}$$

for every $f\in\mathcal{S}\cap L^{p_0}(\mu)$ and

$$\|T(f)\|_{L^{q_1}}\le C_1\|f\|_{L^{p_1}}$$

for every $f\in\mathcal{S}\cap L^{p_1}(\mu)$. Let $\theta\in[0,1]$ and define p,q via

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\quad \textit{and}\quad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}.$$

Then for every $f\in\mathcal{S}\cap L^{p_0}(\mu)$ and

$$\|T(f)\|_{L^q}\le C_0^{1-\theta}C_1^\theta\|f\|_{L^p}.$$

In particular, T extends to a bounded linear mapping from $L^p\to L^q$.

PROOF. The cases $\theta=0$ and $\theta=1$ are the hypothesis so they do not require a proof and we can assume that $0<\theta<1$. In particular, $q\neq 1$ and, if we define q' to be the dual index $\dfrac{1}{q}+\dfrac{1}{q'}=1$ we also have $q'\neq+\infty$. In particular, $\mathcal{S}$ is dense in $L^{q'}$ and

$$\|T(f)\|_{L^q}=\sup\left\{\left|\int_Y T(f)(y)g(y)\,\mathrm{d}\nu(y)\right|\,:\,g\in\mathcal{S},\ \|g\|_{L^{q'}(\nu)}=1\right\}.$$

We thus need to prove that, if f,g are simple functions with

$$\|f\|_{L^p(\mu)}=\|g\|_{L^{q'}(\nu)}=1$$

then

$$\left|\int_Y T(f)(y)g(y)\,\mathrm{d}\nu(y)\right|\le C_0^{1-\theta}C_1^\theta.$$

We write

$$f=\sum_{j=1}^m c_j\mathbf{1}_{E_j}\quad\text{and}\quad g=\sum_{k=1}^n d_k\mathbf{1}_{F_k}$$

with $c_j,d_j\in\mathbb{C}\setminus\{0\}$, $E_j\in\mathcal{B}$ pairwise disjoint with $0<\mu(E_j)<+\infty$, $F_k\in\tilde{\mathcal{B}}$ pairwise disjoint with $0<\nu(F_k)<+\infty$ and

$$\sum_{j=1}^m|c_j|^p\mu(E_j)=\sum_{k=1}^n|d_k|^{q'}\nu(F_k)=1.$$

Note that

$$\int_Y T(f)(u)g(u)\,\mathrm{d}\nu(u)=\sum_{k=1}^n\sum_{j=1}^m c_jd_kT(\mathbf{1}_{E_j})(y)\mathbf{1}_{F_k}(y)\,\mathrm{d}\nu(y).$$

We now write the c_j,d_k's in polar coordinares $c_j=|c_j|e^{i\theta_j}$ and $d_k=|d_k|e^{i\varphi_k}$. Define the following functions on $\Sigma=\{z\in\mathbb{C}\,:0<\Re(z)<1\}$.

$$p(z)=(1-z)\frac{p}{p_0}+z\frac{p}{p_1}\quad,\quad q(z)=(1-z)\frac{q'}{q'_0}+z\frac{q'}{q'_1}$$

and

$$f_z=\sum_{j=1}^m|c_j|^{p(z)}e^{i\theta_j}\mathbf{1}_{E_j}\quad,\quad g_z=\sum_{k=1}^n|d_k|^{q(z)}e^{i\varphi(j)}\mathbf{1}_{F_k}.$$

The first observation is the following: if $z = iy$, $y \in \mathbb{R}$ then $\|f_z\|_{L^{p_0}} = 1$ and $\|g_z\|_{L^{q_0'}} = 1$. The two identites are proved in the same way, so let us only prove the first one

– if $p_0 = +\infty$, then $p(iy) = iy\dfrac{p}{p_1}$ is purely imaginary so that

$$|f_z| = \sum_{j=1}^m \mathbf{1}_{E_j}$$

since the E_j's are pairwise disjoint and as $\nu(E_j) > 0$, $\|f_z\|_{L^\infty} = 1$.

– if $p_0 < +\infty$, then $\Re\big(p(iy)\big) = \dfrac{p}{p_0}$ so that

$$|f_z| = \sum_{j=1}^m |c_j|^{p/p_0}\mathbf{1}_{E_j} \quad \text{thus} \quad |f_z|^{p_0} = \sum_{j=1}^m |c_j|^p \mathbf{1}_{E_j}$$

and therefore

$$\int_X |f_z(u)|^{p_0}\,\mathrm{d}\mu(u) = \sum_{j=1}^m |c_j|^p \mu(E_j) = 1.$$

Note that exchanging z by $1-z$ amounts to exchanging p_0 with p_1 and q_0' with q_1' so that we also have $\|f_z\|_{L^{p_1}} = \|g_z\|_{L^{q_1'}} = 1$ when $z = 1+iy$.

Next, as the functions f_z, g_z are simple functions, we may define

$$F(z) = \int_Y T(f_z)(u)g_z(u)\,\mathrm{d}\nu(u) = \sum_{k=1}^n\sum_{j=1}^m |c_j|^{p(z)}|d_k|^{q(z)}e^{i\theta_j}e^{i\varphi(j)}T(\mathbf{1}_{E_j})(u)\mathbf{1}_{F_k}(u)\,\mathrm{d}\nu(u).$$

Clearly F is holomorphic on Σ and continuous on $\bar{\Sigma}$.

Moreover, using the fact that T has strong type (p_0, q_0) and Hölder, we get

$$|F(iy)| = \left|\int_Y T(f_{iy})(u)g_{iy}(u)\,\mathrm{d}\nu(u)\right| \leq \|T(f_{iy})\|_{L^{q_0}}\|g_z\|_{L^{q_0'}} \leq C_0\|f_{iy}\|_{L^{p_0}}\|g_z\|_{L^{q_0'}} = C_0$$

and, in a similar way, $|F(1+iy)| \leq C_1$. From Hadamard's Three Line Theorem, $|F(\theta+iy)| \leq C_0^{1-\theta}C_1^\theta$.

We will now only consider the case $y = 0$ *i.e.* $z = \theta$ and notice that $p(\theta) = (1-\theta)\dfrac{p}{p_0} + \theta\frac{p}{p_1} = 1$ and $q(\theta) = 1$ so that $f_\theta = f$, $g_\theta = g$. It follows that

$$\left|\int_Y T(f)(y)g(y)\,\mathrm{d}\nu(y)\right| = |F(\theta)| \leq C_0^{1-\theta}C_1^\theta$$

which is the claimed identity. □

For a stricking application, see the Hausdorff-Young in the section on Fourier analysis.

CHAPTER 3

Convolution

Multi-index notation

Before starting this section, we will introduce the multi-index notation:

A multi-index is a vector with integer coordinates: $\alpha=(\alpha_1,\ldots,\alpha_d)\in\mathbb{N}^d$. If $\beta=(\beta_1,\ldots,\beta_d)\in\mathbb{N}^d$, we will say that $\beta\leq\alpha$ if $\beta_j\leq\alpha_j$ for all $j\in\{1,\ldots,d\}$.

The length of a multi-index α is the sum of its coordinates: $|\alpha|=\alpha_1+\cdots+\alpha_d$.

We will write $\alpha!=\alpha_1!\cdots\alpha_d!$, and the binomial coefficient for $\beta\leq\alpha$

$$\binom{\alpha}{\beta}=\frac{\alpha!}{\beta!(\alpha-\beta)!}=\binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_d}{\beta_d}.$$

For $x=(x_1,\ldots,x_d)\in\mathbb{R}^d$, we write $x^\alpha=x_1^{\alpha_1}\cdots x_d^{\alpha_d}$. For a function $f:\mathbb{R}^d\to\mathbb{C}$ we write

$$\partial^\alpha f=\frac{\partial^{\alpha_1}}{\partial_{x_1}^{\alpha_1}}\cdots\frac{\partial^{\alpha_d}}{\partial_{x_d}^{\alpha_d}}f.$$

With this notation, some classical one-variable formula are written in the same way for multivariate functions:

– Leibnitz formula

$$\partial^\alpha(fg)=\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\partial^\beta f\partial^{\alpha-\beta}g$$

– Taylor formula

$$f(x_0+h)=\sum_{|\alpha|\leq n}\partial^\alpha f(x_0)\frac{h^\alpha}{|\alpha|!}+o(h^N).$$

1. Definition and basic examples

DEFINITION 3.1. Let f,g be two functions on $\mathbb{R}^d$, we define the *convolution* of f and g as being the function on $\mathbb{R}^d$ given by

$$f*g(x)=\int_{\mathbb{R}^d}f(y)g(x-y)\,\mathrm{d}y. \tag{1.15}$$

Note that in the definition, we have said nothing about the existence of $f*g$. The aim of this chapter is precisely to give a meaning to $f*g$. However, there are a few basic examples for which this is easy:

EXAMPLE 3.2. Let $f=\mathbf{1}_{[a,b]}$, $g=\mathbf{1}_{[c,d]}$.

First, the change of variable $t=x-y$ shows that $f*g=g*f$. On may thus assume that $b-a>d-c$, that is, the lenght of $[a,b]$ is bigger than the length of $[c,d]$.

It is obvious that, for x fixed, $f(y)g(x-y)=\mathbf{1}_{I_x}(y)$ where I_x is an intersection of two intervals and is thus an interval. It follows that $f*g(x)=|I_x|$ the length of this interval. Next $g(x-y)=1$ is and only if $c\leq x-y\leq d$ that is $y\in[x-d,x-c]$ so that $I_x=[a,b]\cap[x-d,x-c]$. The length of this interval is clearly a piecewise affine function since $[a,b]$ is fixed and we "slide" a second interval $[-d,-c]$ at constant speed, *i.e.* the second interval is $[-d,-c]+x$.

It is enough to find the nodes and determine the length at those nodes.

There are 5 cases:

– the interval $[-d,-c]+x$ is entirely on the left of $[a,b]$ (up to the end point), that is $-c+x\leq a$ *i.e.* $x\leq a+c$. In this case $f*g(x)=|I_x|=0$.

– the interval $[-d,-c]+x$ overlaps $[a,b]$ on the left side: $-d+x\leq a\leq -c+x$ *i.e.* $a+c\leq x\leq a+d$. In this case $I_x=[a,-c+x]$ and $f*g(x)=|I_x|=x-(a+c)$.

– the interval $[-d,-c]+x$ is entirely inside $[a,b]$: $a\leq -d+x\leq -c+x\leq b$ *i.e.* $a+d\leq x\leq b+c$. In this case $I_x=[-d,-c]+x$ and $f*g(x)=|I_x|=d-c$

– the interval $[-d,-c]+x$ overlaps $[a,b]$ on the right side: $-d+x\leq b\leq -c+x$ *i.e.* $b+c\leq x\leq b+d$. In this case $I_x=[-d+x,b]$ and $f*g(x)=|I_x|=b+d-x$.

– the interval $[-d,-c]+x$ is entirely on the left of $[a,b]$ (up to the end point), that is $b\leq -d+x$ *i.e.* $x\geq b+d$ and in this case again $f*g(x)=0$.

We strongly advise the reader to draw the 5 cases and the graph of $f*g$. Once this is done, one can note for future use that $f*g$ is continuous and compactly supported with support $[a,b]+[c,d]=\{x+y, x\in[a,b], y\in[c,d]\}=[a+c,b+d]$.

EXAMPLE 3.3. Assume that f,g are tensors:

$$f(x_1,\ldots,x_d)=f_1(x_1)\cdots f_d(x_d)\quad\text{and}\quad g(x_1,\ldots,x_d)=g_1(x_1)\cdots g_d(x_d).$$

Then if f_j*g_j are defined by (1.15), so if $f*g$ and

$$f*g(x_1,\ldots,x_d)=f_1*g_1(x_1)\cdots f_d*g_d(x_d).$$

An example of this are characteristic functions of cubes $Q=\prod_{j=1}^d I_j$ with I_j intervals, then $\mathbf{1}_Q(x_1,\ldots,x_d)=\mathbf{1}_{I_1}(x_1)\cdots\mathbf{1}_{I_d}(x_d)$. This allows to compute $\mathbf{1}_Q*\mathbf{1}_{Q'}$ when Q,Q' are cubes and shows that this function is continuous.

LEMMA 3.4. *Let $f,g\in\mathcal{C}_c(\mathbb{R}^d)$, the space of compactly supported continuous functions. Then $f*g\in\mathcal{C}_c(\mathbb{R}^d)$ and $f*g=g*f$.*

Morevoer, if $g\in\mathcal{C}_c^n(\mathbb{R}^d)$, then $f*g\in\mathcal{C}_c^n(\mathbb{R}^d)$ and for all $\alpha\in\mathbb{N}^d$, with $|\alpha|\leq n$, $\partial^\alpha(f*g)=f*(\partial^\alpha g)=(\partial^\alpha g)*f$.

Note that $\partial^\alpha(f*g)=(\partial^\alpha g)*f$ implies that, if $g\in\mathcal{C}_c^n(\mathbb{R}^d)$ then $f*g$ is of class $\mathcal{C}^{n+m}$ and $\partial^{\alpha+\beta}(f*g)=(\partial^\alpha f)*(\partial^\beta g)$ as long as $|\alpha|\leq m$, $|\beta|\leq n$.

PROOF. We will only prove the result in one variable, the proof for several variables is similar. Consider $F(x,t)=f(t)g(x-t)$. Then

(1) F is continuous in t so that $f*g(x)=\int_{\mathbb{R}}F(x,t)\,\mathrm{d}t$ is well defined. Further, the change of varible $s=x-t$ shows that $f*g=g*f$.

(2) Write I (resp. J) for an interval containing the support of f (resp. of g). As f,g are continuous with compact support, they are bounded, so we can take $C\geq\|f\|_\infty,\|g\|_\infty$. But then $|F(x,t)|\leq C^2\mathbf{1}_I(t)\mathbf{1}_J(x-t)$. It follows that

$$|f*g(x)|\leq C^2\int_{\mathbb{R}}\mathbf{1}_I(t)\mathbf{1}_J(x-t)\,\mathrm{d}t=C^2\mathbf{1}_I*\mathbf{1}_J(x).$$

The later one having compact support, $f*g$ has compact support. Further its support is included in $I+J=\{x+y, x\in I, y\in J\}$.

(3) Fix a bounded interval $K\subset\mathbb{R}$ and note that if $x\in K$ and $g(x-t)\neq 0$ then $t\in x-J\subset K-J=\{k-j, k\in K, j\in J\}$ (a bounded interval). It follows that $|F(x,t)|\leq C^2\mathbf{1}_I(t)\mathbf{1}_{K-J}(t)\in L^1(\mathbb{R})$. As $x\to F(x,t)$ is continuous for all t, Lebesgue's continuity theorem shows that $f*g$ is continuous on K and K is arbitrary.

The last part follows the same path noting that $\partial_x^\alpha F(x,t)=f(t)\partial^\alpha g(x-t)$ and then the same reasoning shows that this is bounded by an L^1 function independent of $x\in K$. It remains to apply Lebsgue's derivation theorem. □

2. Convolution between L^p and its dual space

THEOREM 3.5. *Let $1\leq p\leq+\infty$ and p' be such that $\dfrac{1}{p}+\dfrac{1}{p'}=1$. Let $f\in L^p(\mathbb{R}^d)$ and $g\in L^{p'}(\mathbb{R}^d)$ then*

$$f*g(x)=\int_{\mathbb{R}^d}f(t)g(x-t)\,dt \tag{2.16}$$

is well defined for every $x\in\mathbb{R}^d$. The mapping $(f,g)\to f*g$ is bilinear and continuous $L^p\times L^{p'}\to L^\infty$ with $\|f*g\|_\infty\leq\|f\|_p\|g\|_{p'}$.

Moreover, if $1<p<+\infty$ $f*g\in\mathcal{C}_0(\mathbb{R}^d)$ so that $(f,g)\to f*g$ is a bounded bilinear mapping $L^p\times L^{p'}\to\mathcal{C}_0$.

Recall that $\mathcal{C}_0(\mathbb{R}^d)$ is the space of continuous functions on $\mathbb{R}^d$ that go to 0 at infinity.

PROOF. First, if $g \in L^{p'}$ then $g_x : t \to g(x-t)$ is also in $L^{p'}$. Hölder's inequality then shows that $fg_x \in L^1$ thus $f*g$ is well defined through (2.16). Further, Hölder shows that $\|f*g\|_\infty \le \|f\|_p \|g\|_{p'}$. As $(f;g) \to f*g$ is clearly bilinear, it follows that $(f,g) \to f*g$ is a bounded bilinear mapping $L^p \times L^{p'} \to L^\infty$.

The key observation is that $\mathcal{C}_0$ is a closed subspace of L^∞. Indeed, if (f_k) is a sequence of elements of $\mathcal{C}_0$ that converges to some f in the L^∞-norm (*i.e.* uniformly) then

– the limit f is continuous (uniform limits of continuous functions are continuous);

– for $\varepsilon > 0$ there exists n such that $\|f - f_n\|_\infty \le \varepsilon$. But then, there exists K such that, if $\|x\| \ge K$, $|f_n(x)| \le \varepsilon$. Finally, for those x's, $|f(x)| \le |f_n(x)| + \|f-f_n\|_\infty \le 2\varepsilon$, so $f(x) \to 0$ when $\|x\| \to +\infty$.

In conclusion $f \in \mathcal{C}_0$ and $\mathcal{C}_0$ is closed in L^∞.

Now, Example 3.3 shows that, if f, g are characteristic functions of cubes, $f*g$ is continuous compactly supported. By bilinearity, if f, g are step functions, that is, finite linear combinations of characteristic functions of cubes, then $f*g \in \mathcal{C}_c(\mathbb{R}^d) \subset \mathcal{C}_0(\mathbb{R}^d)$.

Finally, let $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p'}(\mathbb{R}^d)$. As $p \ne +\infty$, there exists a sequence (f_k) of step functions such that $f_k \to f$ in $L^p(\mathbb{R}^d)$. As $p \ne 1$ we also have $p' \ne +\infty$, so there exists a sequence (g_k) of step functions such that $g_k \to f$ in $L^p(\mathbb{R}^d)$.

But then

$$\begin{aligned}\|f*g - f_k*g_k\|_\infty &= \|(f-f_k)*g + f_k(g-g_k)\|_\infty \le \|(f-f_k)*g\|_\infty + \|f_k(g-g_k)\|_\infty \\ &\le \|f-f_k\|_p \|g\|_{p'} + \|f_k\|_p \|g-g_k\|_{p'} \to 0\end{aligned}$$

since $\|f-f_k\|_p, \|g-g_k\|_{p'} \to 0$ and $\|f_k\|_p$ is bounded since f_k is convergent. □

3. Convolution of L^1 with itself

We want to make sense of

$$f*g(x) = \int_{\mathbb{R}^d} f(y)g(x-y)\,\mathrm{d}y. \tag{3.17}$$

This is possible as a Lebesgue integral when $\int_{\mathbb{R}^d} |f(y)g(x-y)|\,\mathrm{d}y$ is finite. But note that, integrating this quantity in the x variable, we obtain, with Fubini

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y)g(x-y)|\,\mathrm{d}y\,\mathrm{d}x &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y)g(x-y)|\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} |f(y)| \left(\int_{\mathbb{R}^d} |g(x-y)|\,\mathrm{d}x\right)\mathrm{d}y \\ &= \int_{\mathbb{R}^d} |f(y)| \left(\int_{\mathbb{R}^d} |g(t)|\,\mathrm{d}t\right)\mathrm{d}y = \|f\|_{L^1(\mathbb{R}^d)}\|g\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

It follows that, if $f, g \in L^1(\mathbb{R}^d)$ then

$$\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |f(y)g(x-y)|\,\mathrm{d}y\right)\mathrm{d}x < +\infty$$

but then, for almost every x, $\int_{\mathbb{R}^d} |f(y)g(x-y)|\,\mathrm{d}y$ is finite. It follows that (3.17) is well defined for almost every x. Moreover, the resulting function is in $L^1(\mathbb{R}^d)$. Let us summarize this:

PROPOSITION 3.6. *Let $f, g \in L^1(\mathbb{R}^d)$ then*

$$f*g(x) = \int_{\mathbb{R}^d} f(y)g(x-y)\,dy$$

is well defined for almost every $x \in \mathbb{R}^d$. Moreover, the mapping $(f,g) \to f*g$ is a bounded bilinear mapping $L^1(\mathbb{R}^d) \times L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ and

$$\|f*g\|_1 \le \||f|*|g|\|_1 = \|f\|_1\|g\|_1.$$

4. Extension principle

In this course, we will use the following general principle:
— X and Y are Banach spaces and $\mathcal{D}$ is a dense (vectorial) subspace of X;
— T is a linear mapping $\mathcal{D} \to Y$;
— T is bounded on $\mathcal{D}$, that is, there exists $C \geq 0$ such that, for all $x \in \mathcal{D}$, $\|Tx\|_Y \leq C\|x\|_X$.

Then T extends into a bounded linear mapping $\tilde{T} : X \to Y$ with same norm: for all $x \in \mathcal{D}$, $\tilde{T}x = Tx$ and for all $x \in X$, $\left\|\tilde{T}x\right\|_Y \leq C\|x\|_X$.

Of course, we then write $\tilde{T} = T$.

PROOF. Let us first extend T and then show it is linear bounded:

Let $x \in X$. From the density of $\mathcal{D}$ in X, there exists a sequence $(x_n)_n \subset \mathcal{D}$ that converges to x in X. In particular, it is a Cauchy sequence. Let us show that $(Tx_n)_n$ is also a Cauchy sequence. Indeed, let $\varepsilon > 0$, there exists $N \geq 0$ such that, if $p, q \geq N$, then $\|x_p - x_q\|_X \leq \varepsilon$. But then, as $x_p, x_q \in \mathcal{D}$ and T is linear on $\mathcal{D}$,

$$\|Tx_p - Tx_q\|_Y = \|T(x_p - x_q)\|_Y \leq C\|x_p - x_q\|_X \leq C\epsilon$$

since T is bounded on $\mathcal{D}$. Now, as $(Tx_n)_n$ is Cauchy in Y, a Banach space, $(Tx_n)_n$ has a limit that we denote by a.

We would of course like to call $a = Tx$. To do so, we need to show that, if $(y_n)_n$ is an other sequence of elements of $\mathcal{D}$ that converges to x in X, then Ty_n also converges to a. But, as $x_n, y_n \in \mathcal{D}$ and T is linear on $\mathcal{D}$,

$$\|Tx_n - Ty_n\|_Y = \|T(x_n - y_n)\|_Y \leq C\|x_n - y_n\|_X \to C\|x - x\| = 0$$

since the norm is a continuous mapping. We thus write $a = \tilde{T}x$.

Further, if $x \in \mathcal{D}$ the sequence $x_n = x$ converges to x so that $Tx = Tx_n \to \tilde{T}x$ and $\tilde{T}$ is an extension of T from $\mathcal{D}$ to X. We will thus denote $\tilde{T} = T$.

Let us now show that T is linear: let $x, y \in X$ and $\lambda, \mu \in \mathbb{K}$. By density, there exist sequences $(x_n), (y_n)$ in $\mathcal{D}$ that converge respectively to x and y. But then $\lambda x_n + \mu y_n \to \lambda x + \mu y$ so $T(\lambda x_n + \mu y_n) \to T(\lambda x + \mu y)$. On the other hand, as T is linear on $\mathcal{D}$, $T(\lambda x_n + \mu y_n) = \lambda T x_n + \mu T y_n \to \lambda Tx + \mu Ty$, so

$$T(\lambda x + \mu y) = \lambda Tx + \mu Ty.$$

Finally, if $x \in X$ and $(x_n)_n \subset \mathcal{D}$ converges to x, then $Tx_n \to Tx$ in Y and $\|Tx_n\|_Y \leq C\|x_n\|_X$. As norms are continuous, $\|Tx\|_Y \leq C\|x\|_X$. So T is a bounded linear mapping □

Let us illustrate this:

THEOREM 3.7. *Let $f \in L^1(\mathbb{R}^d)$ and $1 \leq p \leq +\infty$. Then the mapping $T_f : g \to f * g$ extends from $\mathcal{C}_c(\mathbb{R}^d) \to L^\infty$ to a mapping $L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$.*

Moreover, this mapping commutes with the translations τ_a.

Recall that $\tau_a g(x) = g(x - a)$.

PROOF. Note that we have already seen that $f * g$ is well defined when $f \in L^1$ and $g \in L^\infty$. What we have to prove is that there is a $C > 0$ such that, for all $g \in \mathcal{C}_c(\mathbb{R}^d)$, $\|f * g\|_{L^p(\mathbb{R}^d)} \leq C\|g\|_{L^p(\mathbb{R}^d)}$.

But this follows from Minkowski's inequality:

$$\left\|\int_{\mathbb{R}^d} f(t)g(\cdot - t)\,\mathrm{d}t\right\|_p \leq \int_{\mathbb{R}^d} |f(t)|\|g(\cdot - t)\|_p\,\mathrm{d}t = \|f\|_1\|g\|_p.$$

Finally, when $p \neq +\infty$, $g \in \mathcal{C}_c(\mathbb{R}^d)$

$$\begin{aligned} T_f\tau_a g(x) = f * (\tau_a g)(x) = \int_{\mathbb{R}^d} f(t)g(x - t - a)\,\mathrm{d}t \\ = \int_{\mathbb{R}^d} f(t)g\big((x - a) - t\big)\,\mathrm{d}t = f * g(x - a) = \tau_a T_f g(x). \end{aligned}$$

Thus $T_f\tau_a = \tau_a T_f$ holds on the dense subspace $\mathcal{C}_c(\mathbb{R}^d)$ of $L^p(\mathbb{R}^d)$ and T_f, τ_a are continuous linear mappings on L^p so the conclusion follows.

When $p = +\infty$, we can directly take $g \in L^\infty$ in the above computation. □

The extension principle works exactly the same way for bilinear mappings:

— X_1, X_2 and Y are Banach spaces and $\mathcal{D}_1$ (resp. $\mathcal{D}_2$) is a dense (vectorial) subspace of X_1 (resp. X_2);

— T is a bilinear mapping $\mathcal{D}_1 \times \mathcal{D}_2 \to Y$;

— T is bounded on $\mathcal{D}_1 \times \mathcal{D}_2$, that is, there exists $C \geq 0$ such that, for all $x \in \mathcal{D}$, $\|T(x_1, x_2)\|_Y \leq C\|x_1\|_{X_1}\|x_2\|_{X_2}$.

Then T extends into a bounded bilinear mapping $\tilde{T} : X_1 \times X_2 \to Y$ with same norm: for all $(x_1, x_2) \in \mathcal{D}_1 \times \mathcal{D}_2$, $\tilde{T}(x_1, x_2) = T(x_1, x_2)$ and for all $(x_1, x_2) \in X_1 \times X_2$, $\left\|\tilde{T}(x_1, x_2)\right\|_Y \leq C\|x_1\|_{X_1}\|x_2\|_{X_2}$.

Of course, we then write $\tilde{T} = T$.

5. Young's inequality

5.1. Young's Inequality in L^p. We would now like to extend the convolution to a bilinear mapping from $\mathcal{C}_c(\mathbb{R}^d) \times \mathcal{C}_c(\mathbb{R}^d) \to \mathcal{C}_c(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$. For this to be possible, one needs to have a constant $C > 0$ such that the inequality

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}. \tag{5.18}$$

To start, we will use a simple but common trick to check for which p, q, r this is possible:

Fix $f, g \in \mathcal{C}_c(\mathbb{R}^d) \setminus \{0\}$ and $f, g \geq 0$ so that $f * g \in \mathcal{C}_c(\mathbb{R}^d) \setminus \{0\}$ as well. Take a parameter $\lambda > 0$ and define $f_\lambda(x) = f(\lambda x)$, $g_\lambda(x) = g(\lambda x)$ then, changing variable $s = \lambda x$

$$f_\lambda * g_\lambda(x) = \int_{\mathbb{R}^d} f(\lambda t) g\big(\lambda(x-t)\big)\,\mathrm{d}t = \lambda^{-d}\int_{\mathbb{R}^d} f(s)g(\lambda x - s)\,\mathrm{d}s = \lambda^{-d} f * g(\lambda x).$$

On the other hand

$$\|f_\lambda\|_{L^p(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f(\lambda t)|^p\,\mathrm{d}t\right)^{1/p} = \left(\lambda^{-d}\int_{\mathbb{R}^d} |f(s)|^p\,\mathrm{d}s\right)^{1/p} = \lambda^{-d/p}\|f\|_{L^p(\mathbb{R}^d)}.$$

The same way, we have

$$\|g_\lambda\|_{L^q(\mathbb{R}^d)} = \lambda^{-d/q}\|g\|_{L^q(\mathbb{R}^d)} \quad \text{and} \quad \|f_\lambda * g_\lambda\|_{L^r(\mathbb{R}^d)} = \lambda^{-d(1+1/r)}\|f * g\|_{L^r(\mathbb{R}^d)}.$$

Thus, if we replace f, g by f_λ, g_λ in (5.18), then

$$0 < \frac{\|f * g\|_{L^r(\mathbb{R}^d)}}{C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}} \leq \lambda^{d\left(1+\frac{1}{r}-\frac{1}{p}-\frac{1}{q}\right)}.$$

Letting $\lambda \to 0$, this implies that the power of λ be ≤ 0 while letting $\lambda \to +\infty$, this implies that the power of λ be ≥ 0. We have thus shown that (5.18) implies $\dfrac{1}{p} + \dfrac{1}{q} = 1 + \dfrac{1}{r}$. In other words, the conditions on p, q, r in the following theorem are necessary. We will now show that this condition is also sufficient.

THEOREM 3.8 (Young's Inequality). *Young's Inequality!L^p spaces Let $1 \leq p, q, r \leq +\infty$ be three real numbers such that $\dfrac{1}{p} + \dfrac{1}{q} = 1 + \dfrac{1}{r}$. Then, for all $f, g \in \mathcal{C}_c(\mathbb{R}^d)$,*

$$\|f * g\|_r \leq \|f\|_p\|g\|_q. \tag{5.19}$$

It follows that the mapping $(f, g) \to f * g$ extends from $\mathcal{C}_c(\mathbb{R}^d) \times \mathcal{C}_c(\mathbb{R}^d) \to \mathcal{C}_c(\mathbb{R}^d)$ into a bounded bilinear mapping $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$.

Further $f * g = g * f$.

PROOF. We only have to prove (5.19).

Note that several particular cases have already be proven: when $r = +\infty$, then $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and this is (part of) Theorem 3.5.

When $r=1$ then $\dfrac{1}{p}+\dfrac{1}{q}=2$. As $p,q\geq 1$ this implies $p=q=1$ and Young's inequality is Proposition 3.6. More generally, the case $p=1$ was treated in Theorem 3.5 and, by symmetry $f*g=g*f$, so is the case $q=1$. Note finally that if $p=+\infty$, then as $1\leq q,r\leq+\infty$ then $1+1/r=1/q$ implies $q=1$ and $r=+\infty$ which is already covered. The same holds when $q=+\infty$.

We can then assume that $1<p,q,r<+\infty$. We define r' to be the dual index of r, $\dfrac{1}{r}+\dfrac{1}{r'}=1$, that is $r'=\dfrac{r}{r-1}$. Note that $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$ implies $r>p,q$ so that $0<\dfrac{p}{r},\dfrac{q}{r},1-\dfrac{p}{r},1-\dfrac{q}{r}<1$.

We will use the following fact which comes from the duality of L^r-$L^{r'}$ (actually from Hölder's Inequality) in the following way: *if* $\varphi\in L^r$ *then*

$$\|\varphi\|_r=\sup\left\{\int_{\mathbb{R}^d}\varphi(x)\psi(x)\,dx\ :\ \psi\in L^{r'},\|\psi\|_{r'}=1\right\}.$$

But now, if $f,g\in\mathcal{C}_c(\mathbb{R}^d)$, then $f*g\in\mathcal{C}_c(\mathbb{R}^d)\subset L^r(\mathbb{R}^d)$. Let $h\in L^{r'}$, we want to bound

$$I(f,g,h)=\int_{\mathbb{R}^d}f*g(x)h(x)\,\mathrm{d}x.$$

Obvously

$$|I(f,g,h)|\leq\int_{\mathbb{R}^d}|f*g(x)||h(x)|\,\mathrm{d}x\leq\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|f|(t)|g|(x-t)|h|(x)\,\mathrm{d}x\,\mathrm{d}t=I(|f|,|g|,|h|)$$

with Fubini. We may thus replace f,g,h with $|f|,|g|,|h|$, that is, we can now assume that $f,g,h\geq 0$. We have to prove that $I(f,g,h)\leq\|f\|_p\|g\|_q\|h\|_{r'}$.

Note that, as $f,g,h\geq 0$, we may apply Fubini and get

$$I(f,g,h)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}f(t)g(x-t)h(x)\,\mathrm{d}x\,\mathrm{d}t.$$

To bound this quantity we will first isolate h and apply Hölder's $L^r-L^{r'}$ inequality. To do so, write $f(t)g(x-t)h(x)=F_1(x,t)F_2(x,t)$ with

$$F_1(x,t)=f(t)^{p/r}g(x-t)^{q/r}\quad\text{and}\quad F_2(x,t)=f(t)^{1-p/r}g(x-t)^{1-q/r}h(x)$$

so that

$$I(f,g,h)\leq\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}F_1(x,t)^r\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}F_2(x,t)^{r'}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{r'}}. \tag{5.20}$$

Note that $F_1(x,t)^r,F_2(x,t)^{r'}\geq 0$ so that we will be able to change the order of integration.

The first of these two integrals is rather simple to bound: using Fubini, we first integrate with respect to x,

$$\begin{aligned}\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}F_1(x,t)^r\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{r}}&=\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}f(t)^pg(x-t)^q\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{r}}\\&=\left(\int_{\mathbb{R}^d}g(x)^q\,\mathrm{d}x\right)^{\frac{1}{q}\frac{q}{r}}\left(\int_{\mathbb{R}^d}f(t)^p\,\mathrm{d}t\right)^{\frac{1}{p}\frac{p}{r}}\\&=\|f\|_p^{\frac{p}{r}}\|g\|_q^{\frac{q}{r}}.\end{aligned}\tag{5.21}$$

The second term is more involved. First

$$\begin{aligned}\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}F_2(x,t)^{r'}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{r'}}&=\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}f(t)^{(1-p/r)r'}g(x-t)^{(1-q/r)r'}h(x)^{r'}\,\mathrm{d}t\,\mathrm{d}x\right)^{\frac{1}{r'}}\\&\leq\left(\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}f(t)^{(1-p/r)r'}g(x-t)^{(1-q/r)r'}\,\mathrm{d}t\right)^{\frac{1}{r'}}\left(\int_{\mathbb{R}^d}h(x)^{r'}\,\mathrm{d}x\right)^{\frac{1}{r'}}\\&=\left\|f^{(1-p/r)r'}*g^{(1-q/r)r'}\right\|_\infty^{1/r'}\|h\|_{r'}.\end{aligned}\tag{5.22}$$

We next introduce a parameter s to be determined soon and s' its dual index $\frac{1}{s}+\frac{1}{s'}=1$. Then from Theorem 3.5 we know that

$$\left\|f^{(1-p/r)r'} * g^{(1-q/r)r'}\right\|_\infty \leq \left\|f^{(1-p/r)r'}\right\|_s \left\|g^{(1-q/r)r'}\right\|_{s'}. \tag{5.23}$$

As we want an estimate with $\|f\|_p$ this leads to the choice $s\left(1-\frac{p}{r}\right)r'=p$. As $r'=\frac{r}{r-1}$ we thus have $\left(1-\frac{p}{r}\right)r'=\frac{r-p}{r-1}$ so that we chose

$$s=\frac{r-1}{r-p}p.$$

Remember that $r>p>1$ so $p<s<+\infty$. The dual index is then

$$s'=\frac{s}{s-1}=\frac{(r-1)p}{r(p-1)}=\frac{p'}{r'}$$

thus

$$\left(1-\frac{q}{r}\right)r's'=\left(1-\frac{q}{r}\right)p'=\left(1-\frac{q}{r}\right)p'.$$

But, multiplying $1+\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$ by q and rewriting it gives $1-\frac{q}{r}=q\left(1-\frac{1}{p}\right)=\frac{q}{p'}$. Finally

$$\left(1-\frac{q}{r}\right)r's'=q.$$

The choice of s then implies that

$$\left\|f^{(1-p/r)r'}\right\|_s=\left(\int_{\mathbb{R}^d} f(x)^{(1-p/r)r's}\,\mathrm{d}x\right)^{\frac{1}{s}}=\left(\int_{\mathbb{R}^d} f(x)^p\,\mathrm{d}x\right)^{\frac{1}{s}}=\|f\|_p^{p/s}$$

while

$$\left\|g^{(1-q/r)r'}\right\|_{s'}=\left(\int_{\mathbb{R}^d} g(x)^{(1-q/r)r's'}\,\mathrm{d}x\right)^{\frac{1}{s'}}=\left(\int_{\mathbb{R}^d} g(x)^q\,\mathrm{d}x\right)^{\frac{1}{s'}}=\|g\|_p^{q/s'}.$$

Injecting this into (5.23), we get

$$\left\|f^{(1-p/r)r'} * g^{(1-q/r)r'}\right\|_\infty \leq \|f\|_p^{p/s}\|g\|_p^{q/s'}.$$

From this, (5.22) reduces to

$$\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F_2(x,t)^{r'}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{r'}} \leq \|f\|_p^{p/r's}\|g\|_p^{q/r's'}\|h\|_{r'}.$$

Finally, with (5.21), we get that (5.20) reduces to

$$I(f,g,h)\leq \|f\|_p^{\frac{p}{r}+\frac{p}{r's}}\|g\|_p^{\frac{q}{r}+\frac{q}{r's'}}\|h\|_{r'}.$$

It remains to notice that

$$\frac{1}{r}+\frac{1}{r'}\frac{1}{s}=\frac{1}{r}+\frac{r-1}{r}\frac{r-p}{(r-1)p}=\frac{p+r-p}{rp}=\frac{1}{p}$$

and that $\frac{1}{r}+\frac{1}{r'}\frac{1}{s'}=\frac{1}{q}$. In conclusion we have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(t)g(x-t)h(x)\,\mathrm{d}x\,\mathrm{d}t\leq \|f\|_p\|g\|_q\|h\|_{r'}$$

for all $h\in L^{r'}$. It follows that, for all $f,g\in\mathcal{C}_c(\mathbb{R}^d)$,

$$\|f*g\|_r\leq\|f\|_p\|g\|_q.$$

The extension principle then shows that $f*g$ can be defined on $L^p\times L^q$. □

Remark 3.9. If $\dfrac{1}{p}+\dfrac{1}{q}=1+\dfrac{1}{r}>1$ then $f*g$ is a priori *not* defined by $\displaystyle\int_{\mathbb{R}^d} f(t)g(x-t)\,\mathrm{d}t$.

One needs to approximate f and/or g by a sequence of functions that converges to f and g in L^p and L^q respectively and for which the above definition makes sense.

To do so, write $f_k = f\mathbf{1}_{|f|\leq k}$ so that $f_k\to f$ in L^p. Further, $f_k\in L^s$ for every $s\geq p$. But $\dfrac{1}{p}+\dfrac{1}{q}=1+\dfrac{1}{r}$ can be rewritten as $\dfrac{1}{p}-\dfrac{1}{r}=1-\dfrac{1}{q}=\dfrac{1}{q'}$ so that $q'>p$. In particular, $f_k\in L^{q'}$. But then $f_k*g(x)=\displaystyle\int_{\mathbb{R}^d} f_k(t)g(x-t)\,\mathrm{d}t$. As $f_k*g\to f*g$ in L^r we conclude that

$$f*g(x)=\lim_{k\to+\infty}\int_{\mathbb{R}^d} f(t)\mathbf{1}_{|f|\leq k}(t)g(x-t)\,\mathrm{d}t.$$

5.2. Young's Inequality in weak-L^p.

Theorem 3.10 (Young's Inequality in weak L^p spaces). *Young's Inequality!weak L^p spaces Let $1<p,q,r<+\infty$ be such that $\frac{1}{p}+\dfrac{1}{q}=1+\dfrac{1}{r}$. Let $f\in L^p(\mathbb{R}^d)$, $g\in L^q_w(\mathbb{R}^d)$ then $f*g$ exists a.e. and $f*g\in L^r_w(\mathbb{R}^d)$ with*

$$\|f*g\|_{L^r_w}\leq C_{p,q,r}\|f\|_{L^p}\|g\|_{L^q_w}$$

where $C_{p,q,r}$ is a constant depending only on p,q,r.

Proof. It is enough to consider $f,g\geq 0$.

The proof is based on properly splitting $g=g_1+g_2$ with $g_1=g\mathbf{1}_{g\leq M}$ and $g_2=g\mathbf{1}_{g>M}$. Then

$$d_{g_1}(\alpha)=\begin{cases}0 & \text{if } \alpha\geq M\\ d_g(\alpha)-d_g(M) & \text{otherwise}\end{cases}\quad\text{and}\quad d_{g_2}(\alpha)=\begin{cases}d_g(\alpha) & \text{if } \alpha\geq M\\ d_g(M) & \text{otherwise}\end{cases}.$$

As $f*g=f*g_1+f*g_2$ is a sum of two non-negative functions,

$$\{x\,:\,f*g(x)>\alpha\}\subset\{x\,:\,f*g_1(x)>\alpha/2\}\cup\{x\,:\,f*g_2(x)>\alpha/2\}$$

thus

$$d_{f*g}(\alpha)\leq d_{f*g_1}(\alpha/2)+d_{f*g_2}(\alpha/2).$$

It remains to estimate each of d_{f*g_1}, d_{f*g_2}. We will fix α and chose M depending on α.

First, as g_1 is the small part of $g\in L^q$ it will be in every L^s, $s>q$:

$$\begin{aligned}\int_{\mathbb{R}^d} g_1(x)^s\,\mathrm{d}x &= s\int_0^{+\infty}\alpha^{s-1}d_{g_1}(\alpha)\,\mathrm{d}\alpha\\ &= s\int_0^M \alpha^{s-1}\big(d_g(\alpha)-d_g(M)\big)\,\mathrm{d}\alpha\\ &\leq s\int_0^M \alpha^{s-q-1}\|g\|^r_{L^q_w}\,\mathrm{d}\alpha - M^s d_g(M)\\ &= \frac{s}{s-q}M^{s-q}\|g\|^r_{L^q_w}-M^s d_g(M)\leq\frac{s}{s-q}M^{s-q}\|g\|^r_{L^q_w}\end{aligned}$$

Further, as g_2 is the large part of $g\in L^q$ it will be in every $L^{\tilde{s}}$, $\tilde{s}<q$:

$$\begin{aligned}\int_{\mathbb{R}^d} g_2(x)^{\tilde{s}}\,\mathrm{d}x &= \tilde{s}\int_0^{+\infty}\alpha^{\tilde{s}-1}d_{g_2}(\alpha)\,\mathrm{d}\alpha\\ &= \tilde{s}\int_0^M \alpha^{\tilde{s}-1}d_g(M)\,\mathrm{d}\alpha+\tilde{s}\int_M^{+\infty}\alpha^{\tilde{s}-1}d_g(\alpha)\,\mathrm{d}\alpha\\ &\leq M^{\tilde{s}}d_g(M)+\tilde{s}\int_M^{+\infty}\alpha^{\tilde{s}-q-1}\|g\|^q_{L^q_w}\,\mathrm{d}\alpha\\ &= M^{\tilde{s}}d_g(M)+\frac{\tilde{s}}{q-\tilde{s}}M^{\tilde{s}-q}\|g\|^q_{L^q_w}\\ &\leq \frac{q}{q-\tilde{s}}M^{\tilde{s}-q}\|g\|^q_{L^q_w}.\end{aligned}$$

Since $\frac{1}{q}=\frac{1}{p'}+\frac{1}{r}$ we have $1<q<p'$ so that we can chose $\tilde{s}=1$ and $s=p'$. We then apply Hölder's inequality (the "trivial" case of Young) to obtain that

$$|f*g_1(x)|\leq\|f\|_{L^p}\|g_1\|_{L^{p'}}\leq\left(\frac{p'}{p'-q}M^{p'-q}\|g\|_{L^q_w}^r\right)^{1/p'}\|f\|_{L^p}.$$

We can now chose M to be small enough for

$$\left(\frac{p'}{p'-q}M^{p'-q}\|g\|_{L^q_w}^r\right)^{1/p'}\|f\|_{L^p}=\frac{\alpha}{2}$$

which will imply that $d_{f*g_1}(\alpha/2)=0$. In other words, we chose

$$M=C(p,q)\alpha^{\frac{p'}{p'-q}}\|f\|_{L^p}^{-\frac{p'}{p'-q}}\|g\|_{L^q_w}^{-\frac{r}{p'-q}}$$

where $C(p,q)$ is a constant depending on p (p' actually) and q.

Next, the choice $\tilde{s}=1$ shows that $g_2\in L^1$ and Young's inequality implies that $f*g_2\in L^p$ with

$$\|f*g_2\|_{L^p}\leq\|f\|_{L^p}\|g_2\|_{L^1}\leq\frac{q}{q-\tilde{s}}\|f\|_{L^p}M^{\tilde{s}-q}\|g\|_{L^q_w}^q.$$

But then

$$\begin{aligned}d_{f*g}(\alpha)&\leq d_{f*g_2}(\alpha/2)\leq(2\|f*g_2\|_{L^p}/\alpha)^p\\&\leq\left(2\frac{q}{q-\tilde{s}}\|f\|_{L^p}M^{\tilde{s}-q}\|g\|_{L^q_w}^q\right)^p\alpha^{-p}\\&\leq C(p,q,r)\frac{\|f\|_{L^p}^r\|g\|_{L^q_w}^r}{\alpha^r}\end{aligned}$$

where $C(p,q,r)$ is a constant depending on p,q,r. □

EXERCICE 3.11. Prove the result in the case $p=1$.

One can actually prove a little better. Fix $g\in L^q_w$. Take $1<p_1<p<p_2<+\infty$ and define r_1,r_2 via $\frac{1}{p_i}+\frac{1}{q}=1+\frac{1}{r_i}$. We have just shown that, if , the operator $T_g:f\to f*g$ is of weak type (p_i,r_i). Define $0<\theta<1$ via $\frac{1}{p}=\frac{\theta}{p_1}+\frac{1-\theta}{p_2}$ and $\frac{1}{r}=\frac{\theta}{r_1}+\frac{1-\theta}{r_2}$ then p,r are related by

$$\frac{1}{p}+\frac{1}{q}=\frac{\theta}{p_1}+\frac{1-\theta}{p_2}+\frac{\theta+1-\theta}{q}=\theta\left(1+\frac{1}{r_1}\right)+(1-\theta)\left(1+\frac{1}{r_2}\right)=1+\frac{1}{r}.$$

Using Marcinkiewicz interpolation, T extends to a bounded operator $L^p\to L^r$, that is:

COROLLARY 3.12 (Young's Inequality in weak L^p spaces). *Young's Inequality!weak L^p spaces Let $1<p,q,r<+\infty$ be such that $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$. Let $f\in L^p(\mathbb{R}^d)$, $g\in L^q_w(\mathbb{R}^d)$ then $f*g$ exists a.e. and $f*g\in L^r(\mathbb{R}^d)$ with*

$$\|f*g\|_{L^r}\leq C_{p,q,r}\|f\|_{L^p}\|g\|_{L^q_w}$$

where $C_{p,q,r}$ is a constant depending only on p,q,r.

REMARK 3.13. Young's inequality fails in some of the end points:

– If $q=1$ and $1\leq p=r\leq\infty$, one can consider $f=\mathbf{1}_{[0,1]}$ and $g(x)=|x|^{-1}$ then $f*g=+\infty$ on $[0,1]$.

– If $r=+\infty$ and $1<q=p'<+\infty$, consider $f=(|x|^{1/p}\log|x|)^{-1}\mathbf{1}_{|x|\geq2}$ and $g=|x|^{-1/q}$ then $f*g=+\infty$ on $[-1,1]$.

DEFINITION 3.14. Given $0<\alpha<d$ and $f\in\mathcal{S}(\mathbb{R}^d)$. The *Riesz Potential* of f is defined by

$$I_\alpha(f)(x)=\int_{\mathbb{R}^d}\frac{f(y)}{|x-y|^\alpha}\,\mathrm{d}y.$$

Corollary 3.15 (Hardy-Littlewood-Sobolev Inequalities). *Let* $0<\alpha<d$, $1<p<d/\alpha$ *and* $r=\dfrac{pd}{d-\alpha p}$. *Then there exists a constant* $C=C(d,\alpha,p)$ *such that, for every* $f\in L^p(\mathbb{R}^d)$,

$$\|I_\alpha(f)\|_{L^r(\mathbb{R}^d)}\leq C\|f\|_{L^p(\mathbb{R}^d)}.$$

Equivalently, if $\frac{1}{p}+\frac{\alpha}{d}+\frac{1}{r'}=1$ *then there exists a constant* $C=C(p,\alpha,d)$ *such that, for every* $f\in L^p(\mathbb{R}^d)$ *and every* $h\in L^{r'}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}f(x)|x-y|^{-\alpha}h(y)\,dx\,dy\leq C\|f\|_{L^p}\|h\|_{L^{r'}}.$$

Proof. Consider the function g defined on $\mathbb{R}^d$ by $g(x)=|x|^{-d+\alpha}$ and notice that $g\in L_w^{d/(d-\alpha)}(\mathbb{R}^d)$. It remains to apply Young's Inequality with $1<p<d/\alpha$, $q=d/(d-\alpha)$ and to notice that $1<q<+\infty$, while $1+\dfrac{1}{r}=\frac{1}{p}+\dfrac{d-\alpha}{d}$ gives precisely $r=\dfrac{p\alpha}{d-\alpha p}$ and that the condition $1<p<d/\alpha$ is then equivalent to $1<r,p<+\infty$.

The second inequality follows the first one by duality. □

6. Regularization

6.1. Spaces of smooth functions: $\mathcal{C}_c^\infty(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$. Spaces of smooth functions will play a key role in the sequel. The first space we consider is the following:

$$\mathcal{C}_c^\infty(\mathbb{R}^d)=\{f\in\mathcal{C}^\infty(\mathbb{R}^d)\ :\exists R>0 \text{ s.t. } f(x)=0 \text{ if } \|x\|\geq R\}$$

the space of smooth functions with compact support.

One may wonder if such functions actually exist so let us start by giving an example:

Example 3.16. Let g be defined on $\mathbb{R}$ by $g(x)=\begin{cases}0 & \text{if } x\leq 0\\ e^{-1/x} & \text{if } x>0\end{cases}$. Then g is clearly $\mathcal{C}^\infty$ on $\mathbb{R}\setminus\{0\}$. Moreover, for every k, there exists a polynomial P_k such that $g^{(k)}(x)=\dfrac{P_k(x)}{x^{2k}}g(x)$ when $x\neq 0$.

Indeed, the formula is clearly true for $k=0$. For $k=1$, $g'(x)=0$ when $x\leq 0$ while $g'(x)=-\dfrac{1}{x^2}e^{-1/x}$ so that the formula is also true for $k=1$. Assuming $g^{(k)}$ is of that form up to some rank $k\geq 1$ we get

$$g^{(k+1)}(x)=\frac{P_k'(x)}{x^{2k}}g(x)-\frac{2kP_k(x)}{x^{2k+1}}g(x)\frac{P_k(x)}{x^{2k}}g'(x)=\frac{x^2P_k'(x)-(2kx+1)P_k(x)}{x^{2k+2}}g(x)$$

and if P_k is a polynomial, so is $P_{k+1}(x):=x^2P_k'(x)-(2kx+1)P_k(x)$.

Alternatively, one may also show that $g^{(k)}(x)=Q_k(1/x)g(x)$ with Q_k a polynomial.

Next, it is clear that g is continuous at 0. Assuming g is of class $\mathcal{C}^{k-1}$ on $\mathbb{R}$, as $g^{(k)}(x)=\dfrac{P_k(x)}{x^{2k}}e^{-1/x}$ we get that $g^{(k)}(x)\to 0$ when $x\to 0^+$ and as $g^{(k)}(x)=0$ when $x<0$ we also get that $g^{(k)}(x)\to 0$ when $x\to 0^-$. It follows that $g^{(k)}$ extends by continuity at 0 so that $g^{(k-1)}$ is of class $\mathcal{C}^1$, thus g is of class $\mathcal{C}^k$.

Finally define f through $f(x)=g(1-\|x-a\|^2/\eta^2)$ and note that g is clearly $\mathcal{C}^\infty$ (taking the euclidean norm) and that $f(x)=0$ when $1-\|x-a\|^2/\eta^2\leq 0$ that is, when $|x-a\|\geq\eta$. Thus f is $\mathcal{C}^\infty$ supported in the ball $B(a,\eta)$.

Example 3.17. We still consider g defined on $\mathbb{R}$ by $g(x)=\begin{cases}0 & \text{if } x\leq 0\\ e^{-1/x} & \text{if } x>0\end{cases}$. Next, we define

$$h(x)=\frac{g(x)}{g(x)+g(1-x)}=\begin{cases}0 & \text{for } x\leq 0\\ \frac{e^{-1/x}}{e^{-1/x}+e^{-1/(1-x)}} & \text{for } 0<x<1\\ 1 & \text{for } x\geq 1\end{cases}.$$

As $g(x)+g(1-x)\neq 0$ for all x, clearly h is of class $\mathcal{C}^\infty$ on $\mathbb{R}$.

Next, we define $B(x) = h(2+x)h(2-x)$ which is clearly $\mathcal{C}^\infty$. Further, for $|x| \geq 2$, one of $2+x, 2-x$ is ≤ 0 so $B(x) = 0$. For $|x| \leq 1$, both $2+x, 2-x$ are ≥ 1 so that $h(2+x) = h(2-x) = 1$ and $B(x) = 1$. Finally as $0 \leq g \leq 1$, $0 \leq B \leq 1$. It follows that:

The function B is a smooth bump function:

— *B is $\mathcal{C}^\infty$ with support $[-2,2]$ and,*

— *$B(x) = 1$ for $x \in [-1,1]$ and $0 \leq B \leq 1$.*

Note that given $a < b < c < d$ there exists a function $B \in \mathcal{C}^\infty$ such that $B = 1$ on $[b,c]$, $B = 0$ outside $[a,b]$ and $0 \leq b \leq 1$. To do so, one choses $B(x) = h(\alpha + \beta x)h(\gamma - \delta x)$ with $\beta, \delta \geq 0$, $\gamma - \delta d = \alpha + \beta a = 0$ and $\alpha + \beta b = \gamma - \delta c = 1$. The choice is thus

$$\alpha = \frac{-a}{b-a}, \quad \beta = \frac{1}{b-a}, \quad \gamma = \frac{d}{d-c}, \quad \delta = \frac{1}{d-c}.$$

Note that one may tensor such functions: $B(x_1, \ldots, x_d) = \prod_{i=1}^d B_i(x_i)$. Then, if Q_1, Q_2 are two cubes with the closure of Q_1 in the interior of Q_2 (so that the boundaries don't touch) then there exists $B \in \mathcal{C}^\infty$ such that $B(x) = 1$ on Q_1, $B(x) = 0$ outside Q_2 and $0 \leq B \leq 1$.

It should be noted that once we have an element of $\mathcal{C}_c^\infty$, we get many others:

LEMMA 3.18. *Let $\varphi \in L^1(\mathbb{R}^d)$ and $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ then $\varphi * f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and, if φ is compactly supported then so if $\varphi * f \in \mathcal{C}_c(\mathbb{R}^d)$.*

We will define the support of $\varphi \in L^1(\mathbb{R}^d)$ in a precise way later on, here we simply mean that there is an $R > 0$ such that $\varphi(x) = 0$ whenever $\|x\| \geq R$.

PROOF. Indeed, if $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ then f is bounded so that $\varphi * f(x) = \displaystyle\int_{\mathbb{R}^d} \varphi(t) f(x-t)\,\mathrm{d}t$. Set $F(x,t) = \varphi(t) f(x-t)$ and note that, for t fixed, $x \to F(x,t)$ is $\mathcal{C}^\infty$ (unless $|\varphi(t)| = +\infty$ so this is true for almost every t). Further for every $\alpha \in \mathbb{N}^d$, $\partial_x^\alpha F(x,t) = \varphi(t)\partial^\alpha f(x-t)$. But $\partial^\alpha f$ is continuous with compact support so that it is bounded $|\partial^\alpha f(u)| \leq C_\alpha$ thus $|\partial_x^\alpha F(x,t)| \leq C_\alpha |\varphi(t)| \in L^1(\mathbb{R}^d)$. Lebesgue's derivation theorem then implies that $\varphi * f$ is of class $\mathcal{C}^\infty$ with $\partial^\alpha(\varphi * f) = \varphi * \partial^\alpha f$.

Finally, if φ and f are both compactly supported, there is an R such that, if $|t| \geq R$ and $|u| \geq R$ then $\varphi(t) = 0$ and $f(u) = 0$. But then, if $|x| \geq 2R$ and $|t| \leq R$, $|x-t| \geq R$. It follows that, when $|x| \geq 2R$, $F(x,t) = \varphi(t) f(x-t) = 0$ for all $t \in \mathbb{R}^d$ thus

$$\varphi * f(x) = \int_{\mathbb{R}^d} F(x,t)\,\mathrm{d}t = 0$$

and the proof is complete. □

Although $\mathcal{C}_c(\mathbb{R}^d)$ is a large class (we will even see that it is dense in every $L^p(\mathbb{R}^d)$ space with $p < +\infty$), this class is too small to contain a function like the Gaussian. We will thus define a larger class that has almost the same property. To do so, for $\alpha, \beta \in \mathbb{N}^d$ and $f : \mathbb{R}^d \to \mathbb{C}$, let

$$p_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f(x)|.$$

DEFINITION 3.19. The *Schwarz class* is the set

$$\mathcal{S}(\mathbb{R}^d) = \{f \in \mathcal{C}^\infty(\mathbb{R}^d) : \forall \alpha, \beta \in \mathbb{N}^d, p_{\alpha,\beta}(f) < +\infty\}.$$

The Schwarz class is thus the space of all smooth functions such that all derivatives have fast decrease at infinity (*i.e.* faster than any polynomial). The class is not empty as obviously $\mathcal{C}_c^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$.

EXAMPLE 3.20. Let f be a Gaussian on $\mathbb{R}^d$, $f(x) = e^{-a\|x\|^2}$, $a > 0$ (the norm is the Euclidean norm). Then $f \in \mathcal{S}(\mathbb{R}^d)$.

For simplicity, we will show this for $d = 1$ and $a = 1/2$ so $f(x) = e^{-x^2/2}$. Then, for every k, there exists a polynomial P_k such that $f^{(k)}(x) = P_k(x)e^{-x^2/2}$. This is clear since $P_0 = 1$ and, by induction, $f^{(k+1)}(x) = \big(P_k'(x) - xP_k(x)\big)e^{-x^2/2}$ and $P_{k+1} = P_k'(x) - xP_k(x)$ is a polynomial if P_k is. Finally, $x^N P_k(x)e^{-x^2/2}$ is clearly bounded.

It should be noted that the choice of $p_{\alpha,\beta}$ to define $\mathcal{S}(\mathbb{R}^d)$ is somewhat arbitrary. We may as well take m,n two integers and define

$$\tilde{p}_{m,n}(f)=\sup_{x\in\mathbb{R}^d}(1+|x|^2)^m\sum_{|\beta|\leq n}\left|\frac{\partial^\beta f}{\partial x^\beta}f(x)\right|.$$

Then if we notice that $(1+|x|^2)^m$ is a polynomial of degree $2m$

$$(1+|x|^2)^m=\sum_{|\alpha|\leq 2m}c_\alpha x^\alpha$$

and $C=\max|c_\alpha|$ then

$$\tilde{p}_{m,n}(f)\leq\sum_{|\alpha|\leq 2m}|c_\alpha|\sum_{|\beta|\leq n}\sup_{x\in\mathbb{R}^d}|x^\alpha\partial^\beta f(x)|\leq C\sum_{|\alpha|\leq 2m}\sum_{|\beta|\leq n}p_{\alpha,\beta}(f).$$

On the other hand,

$$|x^\alpha|=|x_1|^{\alpha_1}\cdots|x_d|^{\alpha_d}\leq\|x\|_\infty^{|\alpha|}\leq|x|^{|\alpha|}\leq(1+|x|^2)^{|\alpha|}.$$

For the last inequality, one checks separately the cases $|x|\leq 1$ and $|x|\geq 1$. But then

$$p_{\alpha,\beta}(f)\leq\tilde{p}_{|\alpha|,|\beta|}(f).$$

It follows that

$$\mathcal{S}(\mathbb{R}^d)=\{f\in\mathcal{C}^\infty(\mathbb{R}^d)\,:\forall m,n\in\mathbb{N},\tilde{p}_{m,n}(f)<+\infty\}.$$

This change of "semi-norm" is sometimes convenient, for instance for the following lemma

LEMMA 3.21. *For every $1\leq p\leq\infty$, $\mathcal{S}(\mathbb{R}^d)\subset L^p(\mathbb{R}^d)$.*

PROOF. The lemma is trivial when $p=+\infty$ since $\tilde{p}_{0,0}(f)=\|f\|_\infty$.
For other p's we will use the fact that, integrating in polar coordinates

$$\int_{\mathbb{R}^d}\frac{\mathrm{d}x}{(1+|x|^2)^\kappa}=\int_{\mathbb{S}^{d-1}}\int_0^{+\infty}\frac{r^{d-1}}{(1+r^2)^\kappa}\,\mathrm{d}r\,\mathrm{d}\sigma_{d-1}(\theta)$$
$$=\sigma_{d-1}(\mathbb{S}^{d-1})\int_0^{+\infty}\frac{r^{d-1}}{(1+r^2)^\kappa}\,\mathrm{d}r<+\infty$$

if $2\kappa>d$. It follows that, if

$$\int_{\mathbb{R}^d}|f(x)|^p\,\mathrm{d}x=\int_{\mathbb{R}^d}|(1+|x|^2)^d f(x)|^p\frac{\mathrm{d}x}{(1+|x|^2)^{dp}}\leq\tilde{p}_{d,0}(f)\int_{\mathbb{R}^d}\frac{\mathrm{d}x}{(1+|x|^2)^{dp}}<+\infty.$$

□

It is now easy to prove the following that we leave as an exercice

PROPOSITION 3.22. *Let $\alpha\in\mathbb{N}^d$, $\lambda,\mu\in\mathbb{C}$, $T\in GL(\mathbb{R}^d)$ an invertible linear transformation. Then*

- *if $f,g\in\mathcal{C}_c^\infty(\mathbb{R}^d)$ so is $\lambda f+\mu g$, $f\circ T$, fg, $x^\alpha f$, $\partial^\alpha f$;*
- *if $f,g\in\mathcal{S}(\mathbb{R}^d)$ so is $\lambda f+\mu g$, $f\circ T$, fg, $x^\alpha f$, $\partial^\alpha f$.*

Let us now extend Lemma 3.18 which shows that we can add $f*g$ to the above list.

LEMMA 3.23. *Let $1\leq p\leq\infty$, $\varphi\in L^p(\mathbb{R}^d)$ and $f\in\mathcal{S}(\mathbb{R}^d)$ then $\varphi*f\in\mathcal{C}^\infty(\mathbb{R}^d)$. Further if, for every $\alpha\in\mathbb{N}^d$, $t^\alpha\varphi\in L^p(\mathbb{R}^d)$ then $\varphi*f\in\mathcal{S}(\mathbb{R}^d)$.*

The second part of the lemma is satisfied if φ is compactly supported or if $\varphi\in\mathcal{S}(\mathbb{R}^d)$.

PROOF. The general scheme of proof is the same as for Lemma 3.18. Note that, as $\mathcal{S}(\mathbb{R}^d)\subset L^{p'}(\mathbb{R}^d)$, $1/p+1/p'=1$, we have $\varphi*f\in L^\infty(\mathbb{R}^d)$ and

$$\varphi*f(x)=\int_{\mathbb{R}^d}\varphi(t)f(x-t)\,\mathrm{d}t.$$

For $p=1$, there is nothing to change: we again define $F(t,x)=\varphi(t)f(x-t)$ and, for every $\alpha\in\mathbb{N}^d$ $\partial_x^\alpha F(t,x)=\varphi(t)\partial^\alpha f(x-t)$ so that $|\partial_x^\alpha F(t,x)|\leq p_{\alpha,0}(f)|\varphi(t)|\in L^1(\mathbb{R}^d)$. By Lebesgue's Differentiation Theorem, $\varphi*f$ is of class $\mathcal{C}^\infty$ with $\partial^\alpha(\varphi*f)=\varphi*(\partial^\alpha f)$.

For $p > 1$, this can not work and we need to use the fact that f has some extra decrease that can compensate the fact that $\varphi \notin L^1$. First, note that it is enough to show that $\varphi * f$ is of class $\mathcal{C}^\infty$ on the ball $B(0,R)$ with R arbitrary. So assume that $|x| \leq R$

$$\partial_x^\alpha F(t,x) = \varphi(t)\partial^\alpha f(x-t) = \frac{\varphi(t)}{(1+|t|^2)^d}\frac{(1+|t|^2)^d}{(1+|x-t|^2)^d}(1+|x-t|^2)^d\partial^\alpha f(x-t).$$

First, as $(1+|t|^2)^{-d} \in L^{p'}(\mathbb{R}^d)$ (it is in all $L^q(\mathbb{R}^d)$ spaces, $q \geq 1$) and $\varphi \in L^p$, Hölder's inequality shows that $\Phi(t) := \dfrac{|\varphi(t)|}{(1+|t|^2)^d} \in L^1(\mathbb{R}^d)$.

Next $(1+|x-t|^2)^d|\partial^\alpha f(x-t)| \leq \tilde{p}_{d,|\alpha|}(f)$.

Finally if $|t| \geq 2R$, and $|x| \leq R$, $|x-t| \geq |t| - |x| \geq |t| - R \geq |t|/2$ so that

$$\frac{(1+|t|^2)^d}{(1+|x-t|^2)^d} \leq \left(\frac{1+|t|^2}{1+|t|^2/4}\right)^d \leq 4^d$$

while for $|t| \leq 2R$,

$$\frac{(1+|t|^2)^d}{(1+|x-t|^2)^d} \leq (1+2R)^d.$$

Assuming $R \geq 2$, we get that this bound also holds for $|t| \geq 2R$ and finally

$$|\partial_x^\alpha F(t,x)| \leq \tilde{p}_{d,|\alpha|}(f)(1+2R)^d\Phi(t) \in L^1(\mathbb{R}).$$

By Lebesgue's Differentiation Theorem, $\varphi * f$ is of class $\mathcal{C}^\infty$ with $\partial^\alpha(\varphi * f) = \varphi * (\partial^\alpha f)$ on $B(0,R)$ and as R is arbitrary, the same holds on $\mathbb{R}^d$.

It remains to prove that, for all α, β, $x^\alpha\partial^\beta(\varphi * f) = x^\alpha\varphi * (\partial^\beta f)$ is bounded. As $f \in \mathcal{S}(\mathbb{R}^d)$ implies that $\partial^\beta f \in \mathcal{S}(\mathbb{R}^d)$, it is enough to consider the case $\beta = 0$. But now, define $M_i\psi(t) = t_i\psi(t)$, then

$$\begin{aligned} x_i\varphi * f(x) = \int_{\mathbb{R}^d}\varphi(t)x_i f(x-t)\,\mathrm{d}t &= \int_{\mathbb{R}^d}\varphi(t)(x_i - t_i)f(x-t)\,\mathrm{d}t + \int_{\mathbb{R}^d}t_i\varphi(t)f(x-t)\,\mathrm{d}t \\ &= \varphi * M_i f + M_i\varphi * f \end{aligned}$$

which is bounded since $M_i\varphi \in L^p(\mathbb{R}^d)$ and $M_i f \in \mathcal{S}(\mathbb{R}^d)$. An induction on the length of α then shows that, for every $\alpha \in \mathbb{N}^d$, $x^\alpha\varphi * f$ is bounded. □

REMARK 3.24. A careful examination of the above proofs shows that, for $\varphi \in L^p(\mathbb{R}^d)$ and $f \in \mathcal{C}^k(\mathbb{R}^d)$ such that for every α with $|\alpha| \leq k$ there is a $\kappa > 0$ such that $(1+|t|^2)^{-\kappa} \in L^{p'}$ (i.e. $2\kappa p' > d$) and $(1+|t|^2)^\kappa\partial^\alpha f \in L^\infty$, we have $\varphi * f \in \mathcal{C}^k$.

6.2. Regularization by convolution.

THEOREM 3.25 (Approximation of unity). *Let $1 \leq p < +\infty$ and $j \in \mathcal{S}(\mathbb{R}^d)$ be such that $j \geq 0$ and $\int_{\mathbb{R}^d} j(x)\,dx = 1$. For $s > 0$, denote by j_s the function defined by $j_s(t) = s^{-d}j(t/s)$.*

Then, for every $\varphi \in L^p(\mathbb{R}^d)$, $\varphi * j_s \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $\varphi * j_s \to \varphi$ in L^p when $s \to 0$.

For $p = +\infty$, L^∞ has to be replaced by $\mathcal{C}_0(\mathbb{R}^d)$: for every $\varphi \in \mathcal{C}_0(\mathbb{R}^d)$, $\varphi * j_s \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $\varphi * j_s \to \varphi$ uniformly when $s \to 0$.

PROOF. We will only give the proof for $1 \leq p < +\infty$. We leave to the reader the case $p = +\infty$. The only thing that one needs to use is the fact that functions in $\mathcal{C}_0(\mathbb{R}^d)$ are uniformly continuous.

Let us first note that $j_s \in \mathcal{S}(\mathbb{R}^d)$ and that

$$\int_{\mathbb{R}^d} j_s(t)\,\mathrm{d}t = \int_{\mathbb{R}^d} j(t/s)\,s^{-d}\mathrm{d}t = \int_{\mathbb{R}^d} j(r)\,\mathrm{d}r = 1$$

with a change of variable $r = t/s$. In particular, $\varphi * j_s \in \mathcal{C}^\infty(\mathbb{R}^d)$.

Next, $j_s \in L^{p'}(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{p'} = 1$, so that

$$\varphi * j_s(x) = \int_{\mathbb{R}^d} j_s(t)\varphi(x-t)\,\mathrm{d}t.$$

But then

$$\begin{aligned}\varphi(x) - \varphi * j_s(x) &= f(x)\int_{\mathbb{R}^d} j_s(t)\,\mathrm{d}t - \int_{\mathbb{R}^d} j_s(t)\varphi(x-t)\,\mathrm{d}t \\ &= \int_{\mathbb{R}^d} j_s(t)\big(\varphi(x) - \varphi(x-t)\big)\,\mathrm{d}t.\end{aligned}$$

From Minkowski's inequality we deduce that

$$\|\varphi - \varphi * j_s\|_p \le \int_{\mathbb{R}^d} j_s(t)\|\varphi - \tau_t\varphi\|_p\,\mathrm{d}t.$$

Now fix $\varepsilon > 0$. As $p < +\infty$, we have seen that $\|\varphi - \tau_t\varphi\|_p \to 0$ when $t \to 0$ so that there exists $\eta > 0$ such that, if $|t| < \eta$, $\|\varphi - \tau_t\varphi\|_p \le \varepsilon$. When $|t| \ge \eta$ we can simply use that $\|\varphi - \tau_t\varphi\|_p \le 2\|\varphi\|_p$ We then write

$$\begin{aligned}\|\varphi - \varphi * j_s\|_p &\le \int_{|t|\le\eta} j_s(t)\|\varphi - \tau_t\varphi\|_p\,\mathrm{d}t + \int_{|t|\ge\eta} j_s(t)\|\varphi - \tau_t\varphi\|_p\,\mathrm{d}t \\ &\le \varepsilon\int_{\mathbb{R}^d} j_s(t)\,\mathrm{d}t + 2\|\varphi\|_p\int_{|t|\ge\eta} j_s(t)\,\mathrm{d}t.\end{aligned}$$

It remains to recall that $\int_{\mathbb{R}^d} j_s(t)\,\mathrm{d}t = 1$ and to notice that

$$\int_{|t|\ge\eta} j_s(t)\,\mathrm{d}t = \int_{|t|\ge\eta} s^{-d}j(t/s)\,\mathrm{d}t = \int_{r\ge\eta/s} j(r)\,\mathrm{d}r \to 0$$

when $s \to 0$. In particular, there is an η' such that, if $s < \eta'$, $2\|\varphi\|_p \displaystyle\int_{|t|\ge\eta} j_s(t)\,\mathrm{d}t \le \varepsilon$ and then $\|\varphi - \varphi * j_s\|_p \le 2\varepsilon$. □

Again, the hypothesis can be weakened without changing the proof. To do so, we may assume that $(j_s)_{s\ge0}$ is a family of $L^1(\mathbb{R}^d)$ functions such that

(1) there is a constant $C > 0$ such that, for all $s > 0$,

$$\int_{\mathbb{R}^d} j_s(x)\,\mathrm{d}x = 1 \quad \text{and} \quad \int_{\mathbb{R}^d} |j_s(x)|\,\mathrm{d}x \le C.$$

(2) For every $\eta > 0$, $\int_{|x|\ge\eta} |j_s(x)|\,\mathrm{d}x \to 0$ when $s \to 0$.

Such a family is called an *approximation of the identity* (and sometimes a *mollifier*).

Corollary 3.26. *The space $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is dense in every $L^p(\mathbb{R}^d)$ space with $1 \le p < +\infty$ and thus so is every space containing it like $\mathcal{C}_c(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$.*

Proof. Let $f \in L^p(\mathbb{R}^d)$ and $\varepsilon > 0$. Let $j \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and $j_s(t) = s^{-d}j(t/s)$ First, for R large enough $\|f - f\mathbf{1}_{|x|\le R}\| \le \varepsilon$. Next there exists s such that $\|f\mathbf{1}_{|x|\le R} - (f\mathbf{1}_{|x|\le R}) * j_s\| \le \varepsilon$. But then $\|f - (f\mathbf{1}_{|x|\le R}) * j_s\| \le 2\varepsilon$ and $(f\mathbf{1}_{|x|\le R}) * j_s \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. □

Remark 3.27. One has to be careful with the density of $\mathcal{C}_c(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$. The proof given here relies on approximation of unity. This in turn relies on the fact that translations are continuous.

We have proven this last fact by first proving it for characteristic functions of cubes, from which we deduced the fact for simple step functions. Then we concluded that translations are continuous by density of step functions in L^p. Our proof is thus not circular.

It turns out that it is simpler to prove that translations are continuous by first proving this fact for functions in $\mathcal{C}_c(\mathbb{R}^d)$ and then using the density of this last step. The approximation of unity theorem then allows to prove that $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is dense in L^p, but the density of $\mathcal{C}_c(\mathbb{R}^d)$ then needs a different proof.

CHAPTER 4

Some Fourier analysis

The aim of this chapter is to recall some facts about Fourier analysis and complex analysis that are needed in this course.

1. Fourier Transforms

1.1. The L^1-theory.

DEFINITION 4.1. For $f \in L^1(\mathbb{R}^d)$ we define the *Fourier transform* of f, and denote it either by $\hat{f}$ or $\mathcal{F}f$, the function defined on $\mathbb{R}$ by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2i\pi\langle x,\xi\rangle}\,\mathrm{d}x.$$

Let us start with a fundamental example:

EXAMPLE 4.2. Let $a < b \in \mathbb{R}$ and $f = \mathbf{1}_{[a,b]}$. Then if $\xi \neq 0$,

$$\begin{aligned}
\hat{f}(\xi) &= \int_a^b e^{-2i\pi x\xi}\,\mathrm{d}x = \frac{-1}{2i\pi\xi}\left(e^{-2i\pi b\xi} - e^{-2i\pi a\xi}\right)\\
&= \frac{e^{2i\pi\frac{a+b}{2}\xi}}{\pi\xi}\,\frac{e^{2i\pi\frac{b-a}{2}\xi} - e^{-2i\pi\frac{b-a}{2}\xi}}{2i}\\
&= e^{2i\pi\frac{a+b}{2}\xi}\,\frac{\sin\pi(b-a)\xi}{\pi\xi}.
\end{aligned}$$

When $\xi = 0$, $\hat{f}(\xi) = \displaystyle\int_a^b \mathrm{d}x = b - a$.

It is convenient to introduce the function $\operatorname{sinc} t = \begin{cases} 1 & \text{if } t = 0\\ \dfrac{\sin t}{t} & \text{if } t \neq 0\end{cases}$. Note that this is an analytic function.

If we write $c = \frac{a+b}{2}$ for the center of the interval $[a,b]$ and ℓ for its length, $\ell = 2r$ then

$$\hat{f}(\xi) = \ell e^{2i\pi c\xi}\operatorname{sinc}\pi\ell\xi = 2re^{2i\pi c\xi}\operatorname{sinc}2\pi r\xi.$$

Let us now notice that, if f is a tensor function $f(x_1,\ldots,x_d) = \displaystyle\prod_{j=1}^d f_j(x_j)$, then so is $\hat{f}$: $\hat{f}(\xi_1,\ldots,\xi_d) = \displaystyle\prod_{j=1}^d \hat{f}_j(\xi_j)$. This follows directly from Fubini's Theorem and the fact that $e^{-2i\pi\langle x,\xi\rangle} = e^{-2i\pi\sum_{j=1}^d x_j\xi_j} = \displaystyle\prod_{j=1}^d e^{-2i\pi x_j\xi_j}$.

Now, for $Q = \displaystyle\prod_{j=1}^d [a_j,b_j]$ a cube, write $\ell_j = b_j - a_j$ for its side length, $|Q| = \displaystyle\prod_{j=1}^d \ell_j$ for its volume, $c = \left(\frac{a_1+b_1}{2},\ldots,\frac{a_d+b_d}{2}\right)$ for its center of gravity. Let $f(x) = \mathbf{1}_Q(x) = \displaystyle\prod_{j=1}^d \mathbf{1}_{[a_j,b_j]}(x_j)$ then

$$\hat{f}(\xi) = |Q|\,e^{2i\pi\langle c,\xi\rangle}\prod_{j=1}^d \operatorname{sinc}\pi\ell_j\xi_j.$$

Note for future use that $\hat{f} \in \mathcal{C}_0(\mathbb{R}^d)$.

Let us now start detailing properties of the Fourier transform. First, it is well defined. Indeed, let $F(x,\xi) = f(x)e^{-2i\pi\langle x,\xi\rangle}$. Then, for x fixed, $\xi \to F(x,\xi)$ is continuous. Moreover, $|F(x,\xi)| = |f(x)| \in L^1(\mathbb{R}^d)$, it follows that $\hat{f}(\xi) = \int_{\mathbb{R}^d} F(x,\xi)\,\mathrm{d}x$ is well defined and continuous. Further,

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}^d} |F(x,\xi)|\,\mathrm{d}x = \int_{\mathbb{R}^d} |f(x)|\,\mathrm{d}x = \|f\|_{L^1(\mathbb{R}^d)}.$$

As $f \to \hat{f}$ is clearly linear, this shows that this mapping is bounded $L^1(\mathbb{R}^d) \to \mathcal{C}_b(\mathbb{R}^d)$, the space of bounded continuous functions on $\mathbb{R}^d$. Actually, a bit more is true:

THEOREM 4.3 (Riemann-Lebesgue Lemma). *The Fourier transform $\mathcal{F}$ is a bounded linear mapping $L^1(\mathbb{R}^d) \to \mathcal{C}_0(\mathbb{R}^d)$ with $\|\mathcal{F}f\|_\infty \leq \|f\|_1$.*

PROOF. We have already seen that $\mathcal{F}$ is a bounded linear mapping $L^1(\mathbb{R}^d) \to \mathcal{C}_b(\mathbb{R}^d)$ with $\|\mathcal{F}f\|_\infty \leq \|f\|_1$. It remains to prove that $\mathcal{F}f \in \mathcal{C}_0(\mathbb{R}^d)$ when $f \in L^1(\mathbb{R}^d)$.

This is indeed the case when $f = \mathbf{1}_Q$, Q a cube, thus also when f is a (finite) linear combination of such functions, that is, when f is a step function. But step functions are dense. Thus, if $f \in L^1(\mathbb{R}^d)$, there exists a sequence f_k of step functions, such that $\|f_k - f\|_{L^1} \to 0$ when $k \to \infty$. But then

$$\|\mathcal{F}f - \mathcal{F}f_k\|_\infty = \|\mathcal{F}(f - f_k)\|_\infty \leq \|f - f_k\|_1 \to 0.$$

In other words, $\mathcal{F}f_k \to \mathcal{F}f$ in $\mathcal{C}_b(\mathbb{R}^d)$. As $\mathcal{F}f_k \in \mathcal{C}_0(\mathbb{R}^d)$ which is closed in $\mathcal{C}_b(\mathbb{R}^d)$ (see the chapter on convolutions for a proof), we get that $\mathcal{F}f \in \mathcal{C}_0(\mathbb{R}^d)$. □

A SECOND PROOF. There is an alternative proof of the fact that $\hat{f}(\xi) \to 0$ when $\xi \to \pm\infty$. First note that $-1 = e^{-i\pi} = e^{-2i\pi\left\langle \xi/2|\xi|^2,\xi\right\rangle}$ thus

$$\begin{aligned} 2\hat{f}(\xi) &= \int_{\mathbb{R}^x} f(x)e^{-2i\pi\langle x,\xi\rangle}\,\mathrm{d}x - \int_{\mathbb{R}^d} f(txe^{-2i\pi\left\langle \xi/2|\xi|^2,\xi\right\rangle}e^{-2i\pi\langle x,\xi\rangle}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} f(x)e^{-2i\pi\langle x,\xi\rangle}\,\mathrm{d}t - \int_{\mathbb{R}^d} f(x)e^{-2i\pi\left\langle x+\xi/2|\xi|^2,\xi\right\rangle}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \left[f(x) - f\left(x - \frac{\xi}{2|\xi|^2}\right)\right] e^{-2i\pi\langle x,\xi\rangle}\,\mathrm{d}x. \end{aligned}$$

In other words, $\hat{f}(\xi) = \dfrac{1}{2}\mathcal{F}[f - \tau_{\xi/2|\xi|^2}f](\xi)$. It follows that $|\hat{f}(\xi)| \leq \left\|f - \tau_{\xi/2|\xi|^2}f\right\|_1$. Now letting $|\xi| \to \infty$ and using the continuity of $a \to \tau_a f$ from $\mathbb{R}^d \to L^1(\mathbb{R}^d)$ shows that $|\hat{f}(\xi)| \to 0$.

Recall that this continuity required the same density argument. □

Let us now list the main properties of the Fourier transform. To do so, we need to introduce some notation. For $a, \omega \in \mathbb{R}^d$, $\lambda > 0$, $T \in GL_n(\mathbb{R}^d)$ (a $d \times d$ invertible matrix) and f a function on $\mathbb{R}^d$, we define new functions on $\mathbb{R}^d$

$$\tau_a f(x) = f(x-a),\ M_\omega f(x) = e^{-2i\pi\langle\omega,x\rangle}f(x),\ \delta_\lambda f(x) = f(\lambda x),\ \Delta_T f(x) = f(T^{-1}x).$$

Note that $\tau_a, M_\omega, \delta_\lambda, \Delta_T$ are continuous linear mappings $L^p \to L^p$ for every p.

PROPOSITION 4.4. *Assume that $f \in L^1(\mathbb{R}^d)$ then*

– $\mathcal{F}[\tau_a f] = M_a\mathcal{F}[f]$, $\mathcal{F}[M_\omega f] = \tau_{-\omega}\mathcal{F}[f]$,

– $\mathcal{F}[\delta_\lambda f] = \lambda^{-d}\mathcal{F}[\delta_{1/\lambda}f]$ *and more generally* $\mathcal{F}[\Delta_T f] = |\det T|\Delta_{[T^{-1}]^t}\mathcal{F}[f]$.

– *If $\xi_j f \in L^1(\mathbb{R}^d)$ then $\hat{f}$ admits a continuous partial derivative in the ξ_j direction with* $\dfrac{\partial\hat{f}}{\partial\xi_j}(\xi) = -2i\pi\mathcal{F}[x_j f](\xi)$.

– *If f is $\mathcal{C}^1$ with* $\dfrac{\partial f}{\partial x_j} \in L^1(\mathbb{R}^d)$, *then* $\mathcal{F}\left[\dfrac{\partial f}{\partial x_j}\right](\xi) = 2i\pi\xi_j\mathcal{F}[f]\xi)$.

– *If $f, g \in L^1(\mathbb{R}^d)$ then* $\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g]$.

PROOF. The first 4 follow from a simple change of variable
– changing variable $y = x - a$,

$$\mathcal{F}[\tau_a f](\xi) = \int_{\mathbb{R}^d} f(x-a)e^{-2i\pi\langle x,\xi\rangle}\,\mathrm{d}x = \int_{\mathbb{R}^d} f(y)e^{-2i\pi\langle y+a,\xi\rangle}\,\mathrm{d}y$$
$$= e^{-2i\pi\langle a,\xi\rangle}\int_{\mathbb{R}^d} f(y)e^{-2i\pi\langle y,\xi\rangle}\,\mathrm{d}y = e^{-2i\pi\langle a,\xi\rangle}\hat{f}(\xi).$$

– the next one is even easier

$$\mathcal{F}[M_\omega f](\xi) = \int_{\mathbb{R}^d} f(x)e^{-2i\pi\langle \omega,\xi\rangle}e^{-2i\pi\langle x,\xi\rangle}\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)e^{-2i\pi\langle x+\omega,\xi\rangle}\,\mathrm{d}x = \hat{f}(\xi+\omega).$$

– changing variable $y = \lambda x$,

$$\mathcal{F}[\delta_\lambda f](\xi) = \int_{\mathbb{R}^d} f(\lambda x)e^{-2i\pi\langle x,\xi\rangle}\,\mathrm{d}x = \lambda^{-d}\int_{\mathbb{R}^d} f(y)e^{-2i\pi\langle y/\lambda,\xi\rangle}\,\mathrm{d}y$$
$$= \lambda^{-d}\int_{\mathbb{R}^d} f(y)e^{-2i\pi\langle y,\xi/\lambda\rangle}\,\mathrm{d}y = \lambda^{-d}\hat{f}(\xi/\lambda).$$

It is a particular case of the following:
– changing variable $y = T^{-1}x$, $x = Ty$

$$\mathcal{F}[\Delta_T f](\xi) = \int_{\mathbb{R}^d} f(T^{-1}x)e^{-2i\pi\langle x,\xi\rangle}\,\mathrm{d}x = |\det T|\int_{\mathbb{R}^d} f(y)e^{-2i\pi\langle Ty,\xi\rangle}\,\mathrm{d}y$$
$$= |\det T|\int_{\mathbb{R}^d} f(y)e^{-2i\pi\langle y,T^t\xi\rangle}\,\mathrm{d}y = |\det T|\hat{f}(T^t\xi).$$

–The next two ones are slightly more subtle. First assume that $x_j f \in L^1(\mathbb{R}^d)$ and consider again $F(x,\xi) = f(x)e^{-2i\pi\langle x,\xi\rangle}$. Then, for x fixed, $\xi \to F(x,\xi)$ is of class $\mathcal{C}^1$, $|F(x,\xi)| = |f(x)| \in L^1(\mathbb{R}^d)$ and

$$\left|\frac{\partial F}{\partial \xi_j}(x,\xi)\right| = \left|-2i\pi x_j f(x)e^{-2i\pi\langle x,\xi\rangle}\right| = 2\pi|x_j f| \in L^1(\mathbb{R}^d).$$

It follows that $\hat{f}(\xi) = \displaystyle\int_{\mathbb{R}^d} F(x,\xi)\,\mathrm{d}x$ is differentiable with respect to ξ_j with

$$\frac{\partial \hat{f}}{\partial \xi_j}(\xi) = \int_{\mathbb{R}^d}\frac{\partial F}{\partial \xi_j}(x,\xi) = \int_{\mathbb{R}^d} -2i\pi x_j f(x)e^{-2i\pi\langle x,\xi\rangle}\,\mathrm{d}x = \mathcal{F}[-2i\pi x_j f](\xi).$$

– Now, assume that $f \in \mathcal{C}^1$, $f, \dfrac{\partial f}{\partial \xi_j} \in L^1$. To simplify notation, we will take $j = 1$. Note that, from Fubini's Theorem,

$$\int_{\mathbb{R}^d} |f(x)|\,\mathrm{d}x = \int_{\mathbb{R}^{d-1}}\left(\int_{\mathbb{R}} |f(x_1,x_2,\ldots,x_d)|\,\mathrm{d}x_1\right)\mathrm{d}x_2\cdots\mathrm{d}x_d < +\infty$$

so that $\displaystyle\int_{\mathbb{R}} |f(x_1,x_2,\ldots,x_d)|\,\mathrm{d}x_1 < +\infty$ for almost every $(x_2,\ldots,x_d)$. The same is true with $\dfrac{\partial f}{\partial \xi_j}$ replacing f. If two properties hold almost everywhere, they jointly hold almost everywhere. We may thus take an $(x_2,\ldots,x_d)$ such that

$$\int_{\mathbb{R}} |f(x_1,x_2,\ldots,x_d)|\,\mathrm{d}x_1 < +\infty \quad\text{and}\quad \int_{\mathbb{R}}\left|\frac{\partial f}{\partial \xi_j}(x_1,x_2,\ldots,x_d)\right|\mathrm{d}x_1 < +\infty$$

and almost every $(x_2,\ldots,x_d)$ is like that. The fundamental theorem of calculus then shows that

$$f(x_1,x_2,\ldots,x_d) = f(0,x_2,\ldots,x_d) + \int_0^{x_1}\frac{\partial f}{\partial \xi_j}(t,x_2,\ldots,x_d)\mathrm{d}t$$
$$\to f(0,x_2,\ldots,x_d) + \int_0^{\pm\infty}\frac{\partial f}{\partial \xi_j}(t,x_2,\ldots,x_d)\mathrm{d}t$$

when $x_1 \to \pm\infty$. Thus $f(x_1,x_2,\ldots,x_d)$ has a limit in $\pm\infty$. But then $\displaystyle\int_{\mathbb{R}} |f(x_1,x_2,\ldots,x_d)|\,\mathrm{d}x_1 < +\infty$ implies that this limit is zero.

Next, write $x,\xi\in\mathbb{R}^d$ as $x=(x_1,\bar x)$, $\xi=(\xi_1,\bar\xi)$ with $\bar x,\bar\xi\in\mathbb{R}^{d-1}$. Integrating by parts,

$$\begin{aligned}\int_{\mathbb{R}}\frac{\partial f}{\partial\xi_1}(x_1,\bar x)e^{-2i\pi\langle x,\xi\rangle}\mathrm{d}x_1 &= \int_{\mathbb{R}}\frac{\partial f}{\partial\xi_1}(x_1,\bar x)e^{-2i\pi x_1\xi_1}\mathrm{d}x_1 e^{-2i\pi\left\langle\bar x,\bar\xi\right\rangle}\\ &= e^{-2i\pi\left\langle\bar x,\bar\xi\right\rangle}\left[f(x_1,\bar x)e^{-2i\pi x_1\xi_1}\right]_{-\infty}^{+\infty}\\ &\quad +2i\pi\xi_1\int_{\mathbb{R}}f(x_1,\bar x)e^{-2i\pi x_1\xi_1}\mathrm{d}x_1 e^{-2i\pi\left\langle\bar x,\bar\xi\right\rangle}\\ &= 2i\pi\xi_1\int_{\mathbb{R}}f(x_1,\bar x)e^{-2i\pi\langle x,\xi\rangle}\,\mathrm{d}x_1.\end{aligned}$$

It remains to integrate with respect to the $d-1$ remaining variables and to use Fubini.

The last property is a direct consequence of Fubini and the change of variable $u=x-y$

$$\begin{aligned}\mathcal{F}[f*g](\xi) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}f(y)g(x-y)\,\mathrm{d}y\,e^{-2i\pi\langle x,\xi\rangle}\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d}f(y)\int_{\mathbb{R}^d}g(x-y)e^{-2i\pi\langle x,\xi\rangle}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{\mathbb{R}^d}f(y)\int_{\mathbb{R}^d}g(u)e^{-2i\pi\langle u+y,\xi\rangle}\,\mathrm{d}u\,\mathrm{d}y\\ &= \int_{\mathbb{R}^d}f(y)\int_{\mathbb{R}^d}g(u)e^{-2i\pi\langle u,\xi\rangle}\,\mathrm{d}u\,e^{-2i\pi\langle y,\xi\rangle}\,\mathrm{d}y\\ &= \int_{\mathbb{R}^d}f(y)\hat g(\xi)\,e^{-2i\pi\langle y,\xi\rangle}\,\mathrm{d}y=\hat h(\xi)\hat g(\xi)\end{aligned}$$

as claimed. □

We can now give as a second example the case of the Gaussian:

EXAMPLE 4.5. Let f be the Gaussian defined for $x\in\mathbb{R}$ by $f(x)=e^{-\pi x^2}$, then $\hat f(\xi)=e^{-\pi\xi^2}$.

Indeed, first note that $\hat f(0)=\displaystyle\int_{\mathbb{R}}e^{-\pi x^2}\,\mathrm{d}x$. But then, using Fubini in the first line and changing to polar coordinates:

$$\begin{aligned}\hat f(0)^2 &= \int_{\mathbb{R}}e^{-\pi x^2}\,\mathrm{d}x\int_{\mathbb{R}}e^{-\pi y^2}\,\mathrm{d}y=\int_{\mathbb{R}^2}e^{-\pi(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^{+\infty}\int_0^{2\pi}e^{-\pi r^2}\,\mathrm{d}\theta r\,\mathrm{d}r\\ &= \int_0^{+\infty}2\pi re^{-\pi r^2}\,\mathrm{d}r=[-e^{-\pi r^2}]_0^{+\infty}=1.\end{aligned}$$

As $\hat f(0)$ is the integral of a positive function, $\hat f(0)\geq 0$ thus $\hat f(0)=1$.

Next, note that f satisfies the differential equation $f'=-2\pi xf$ thus $\mathcal{F}[f']=-2\pi\mathcal{F}[xf]$. As clearly f is $\mathcal{C}^1$ with $f,xf,f'\in L^1$ we can use the above properties: $\hat f'=-2i\pi\mathcal{F}[xf]$ $\mathcal{F}[f']=2i\pi\xi\hat f$. It follows that $\hat f$ satisfies the differential equation $(\hat f)'=-2\pi\xi\hat f$ which is the same equation as the one satisfied by the Gaussian. Thus $\hat f=cf$. Comparing values at 0, we get $\hat f=f$.

In higher dimensions, we immediately get that, if $\gamma(x)=e^{-\pi|x|^2}$ then $\hat\gamma(\xi)=e^{-\pi|\xi|^2}$. The result is more general

EXAMPLE 4.6. Now, let A be a positive definite symetric matrix and define f on $\mathbb{R}^d$ through $f(x)=e^{-\pi\langle Ax,x\rangle}$. Then $\hat f(\xi)=\det(A)^{-1/2}e^{-\pi\left\langle A^{-1}x,x\right\rangle}$.

Indeed, as A is a real sumetric matrix, it is diagonalizable in an orthonormal matrix, $A=P\Delta P^t$ with Δ a diagonal martrix and P an orthogonal matrix. Write $\Delta=\mathrm{diag}\,(\lambda_1,\ldots,\lambda_d)$. As A is positive definite, the λ_j's are >0 thus we can write $\lambda_j=\mu_j^2$. Then define $B=P\mathrm{diag}\,(\mu_1,\ldots,\mu_d)P^t$ and notice that $B^t=B$ and that $A=B^2=B^tB$. It follows that $\langle Ax,x\rangle=\langle B^tBx,x\rangle=|Bx|^2$. As the μ_j's are >0, B is invertible thus $f(x)=\gamma(Bx)$. It follows that $f\in L^1(\mathbb{R}^d)$ and that

$\hat f(x) = |\det B^{-1}|\gamma(B^{-1}x)$. But $B^{-1} = P\mathrm{diag}\,(1/\mu_1,\dots,1/\mu_d)P^t$ is symetric with $(B^{-1})^tB^{-1} = (B^{-1})^2 = A^{-1}$ thus $|\det B^{-1}| = \det(A)^{-1/2}$ and

$$|B^{-1}x|^2 = \langle B^{-1}x, B^{-1}x\rangle = \langle (B^{-1})^tB^{-1}x, x\rangle = \langle A^{-1}x, x\rangle$$

It follows that $\hat f(\xi) = \det(A)^{-1/2}e^{-\pi\langle A^{-1}x,x\rangle}$.

1.2. The inversion formula and the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$. We are now going to show that the Fourier transform can be inverted and that it is (almost) its own inverse. To do so, let us start with the following simple observation:

Assume that $f, g \in L^1(\mathbb{R}^d)$, then $\hat f, \hat g \in \mathcal{C}_0(\mathbb{R}^d)$ so that $f\hat g$ and $g\hat f$ are both integrable. But as

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x)g(y)|\,\mathrm{d}y\,\mathrm{d}x = \|f\|_{L^1}\|g\|_{L^1} < +\infty,$$

Fubini's theorem shows that

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)\hat g(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(y)e^{-2i\pi\langle x,y\rangle}\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} g(y)\int_{\mathbb{R}^d} f(x)e^{-2i\pi\langle y,x\rangle}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^d} g(y)\hat f(y)\,\mathrm{d}y.\end{aligned} \tag{1.24}$$

Let us now replace g by $M_{-\omega}g$ so that $\hat g$ is replaced by $\tau_\omega\hat g$. We get

$$\int_{\mathbb{R}^d} f(x)\hat g(x-\omega)\,\mathrm{d}x = \int_{\mathbb{R}^d} g(y)\hat f(y)e^{2i\pi\langle\omega,y\rangle}\,\mathrm{d}y. \tag{1.25}$$

The right hand side looks like a convolution and is indeed $g * \hat f$ when g is even. Let us take as an example $g(y) = e^{-\pi|\lambda y|^2}$ so that $\hat g(x) = \lambda^{-d}e^{-\pi|x/\lambda|^2}$. Write $\gamma_\lambda(x) = \lambda^{-d}e^{-\pi|x/\lambda|^2}$. Then (1.25) reads

$$f * \gamma_\lambda(\omega) = \int_{\mathbb{R}^d} e^{-\pi|\lambda y|^2}\hat f(y)e^{2i\pi\langle\omega,y\rangle}\,\mathrm{d}y. \tag{1.26}$$

Now, since $\gamma \in \mathcal{S}(\mathbb{R}^d)$, according to Theorem 3.25, $f * \gamma_\lambda \to f$ in $L^1(\mathbb{R}^d)$. In particular, if $f_1, f_2 \in L^1(\mathbb{R}^d)$ are such that $\hat f_1 = \hat f_2$ then $f_1 * \gamma_\lambda(\omega) = f_2 * \gamma_\lambda(\omega)$. Letting $\lambda \to 0$ shows that $f_1 = f_2$. In other words, the Fourier transform is one-to-one.

What about the right hand side? Note that $e^{-\pi|\lambda y|^2}\hat f(y)e^{2i\pi\langle\omega,y\rangle} \to \hat f(y)e^{2i\pi\langle\omega,y\rangle}$ when $\lambda \to 0$. Further, as $|e^{-\pi|\lambda y|^2}\hat f(y)e^{2i\pi\langle\omega,y\rangle}| = |e^{-\pi|\lambda y|^2}\hat f(y)| \le |\hat f(y)|$, if $\hat f \in L^1(\mathbb{R}^d)$, we can use dominated convergence and obtain the following theorem:

THEOREM 4.7 (Fourier inversion formula). *The Fourier transform is one-to-one $L^1(\mathbb{R}^d) \to \mathcal{C}_0(\mathbb{R}^d)$. Let $f \in L^1(\mathbb{R}^d)$ be such that $\hat f \in L^1(\mathbb{R}^d)$, then $f \in \mathcal{C}_0(\mathbb{R}^d)$ and*

$$f(x) = \int_{\mathbb{R}^d} \hat f(\xi)e^{2i\pi\langle\xi,x\rangle}\,\mathrm{d}\xi.$$

PROOF. We have not fully proven the above theorem, we have only shown that the inversion formula is valid in $L^1(\mathbb{R}^d)$. The observation is that the right hand side is $\mathcal{F}[\hat f](-x)$. As $\hat f \in L^1(\mathbb{R}^d)$, Riemann-Lebesgue's lemma implies that the right hand side is in $\mathcal{C}_0$. Now $f*\gamma_\lambda \to f$ in L^1 thus has a subsequence that converges almost-everywhere, thus f is almost everywhere equal to $\mathcal{F}[\hat f](-x)$ i.e. is in the same class as a $\mathcal{C}_0$ function. Our convention is that we chose f to be this $\mathcal{C}_0$ function. □

The Fourier inversion theorem shows that the Fourier transform is almost its own inverse, this explains the very symetric properties we have already observed in Proposition 4.4.

REMARK 4.8. If $f = \mathbf{1}_{[-1,1]}$ then $\hat f = \mathrm{sinc}\,2\pi t \notin L^1(\mathbb{R})$. It follows that $\int_{\mathbb{R}} \hat f(\xi)e^{2i\pi\xi x}\,\mathrm{d}\xi$ does not make sense. We will see below that

$$\lim_{R,S\to+\infty}\int_{-R}^{S} \hat f(\xi)e^{2i\pi\xi x}\,\mathrm{d}\xi \to \mathbf{1}_{[-1,1]}(x)$$

in L^2. Actually,

$$\lim_{R\to+\infty}\int_{-R}^{R} \hat f(\xi)e^{2i\pi\xi x}\,\mathrm{d}\xi \to \mathbf{1}_{[-1,1]}(x)$$

is valid pointwise, excepted at the jumps ± 1. Note that we now integrate over a symetric interval.

REMARK 4.9. It is important to have in mind that the Fourier transform is *not* a bijection $L^1(\mathbb{R}^d) \to \mathcal{C}_0(\mathbb{R}^d)$ as there are functions in $\mathcal{C}_0(\mathbb{R}^d)$ that are not Fourier transforms of L^1 functions.

Now, let $f \in \mathcal{S}(\mathbb{R}^d)$. For every $\alpha \in \mathbb{N}^d$, $x^\alpha f \in \mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$. It follows from Proposition 4.4 that $\hat{f} \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $\partial^\alpha \hat{f} = (-2i\pi)^{|\alpha|}\mathcal{F}[x^\alpha f]$. Further as $x^\alpha f \in \mathcal{S}$, for every $\beta \in \mathbb{N}^d$, $\partial^\beta(x^\alpha f) \in \mathcal{S} \subset L^1(\mathbb{R}^d)$. Applying again Proposition 4.4 we obtain that $x^\beta \partial^\alpha \hat{f} = (-2i\pi)^{|\alpha|-|\beta|}\mathcal{F}[\partial^\beta(x^\alpha f)]$. But then, Riemann-Lebesgue's Lemma implies that $\mathcal{F}[\partial^\beta(x^\alpha f)]$ is in $\mathcal{C}_0$, in particular, it is bounded. We have just shown that, $\hat{f} \in \mathcal{C}^\infty(\mathbb{R}^d)$ and that, for every $\alpha, \beta \in \mathbb{N}^d$, $x^\beta \partial^\alpha \hat{f}$ is bounded, that is, that $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$.

Finally, as $\mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$, the Fourier inversion theorem applies to every $f \in \mathcal{S}(\mathbb{R}^d)$ and for such an f, $f(x) = \mathcal{F}[\hat{f}](-x)$. Writing $Z\hat{f}(y) = \hat{f}(-y)$ and noticing that $Z\hat{f} \in \mathcal{S}(\mathbb{R}^d)$ and that $\mathcal{F}[\hat{f}](-x) = \mathcal{F}[Zf](x)$, we see that every $f \in \mathcal{S}(\mathbb{R}^d)$ is the Fourier transform of a function in the Schwartz class. We have thus shown the following:

THEOREM 4.10. *The Fourier transform is a bijection $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. The inverse map is given by $\mathcal{F}^{-1}[f](\xi) = \mathcal{F}[f](-\xi)$.*

1.3. The L^2-theory, Hausdorff-Young. Our aim here is to extend the Fourier transform to other L^p spaces. Let us recall that if $f, g \in \mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$ then

$$\int_{\mathbb{R}^d} f(x)\hat{g}(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} g(y)\hat{f}(y)\,\mathrm{d}y.$$

Now let $h \in \mathcal{S}(\mathbb{R}^d)$, then $\bar{h} \in \mathcal{S}(\mathbb{R}^d)$ and the Fourier inversion Formula reads

$$\bar{h}(x) = \overline{\int_{\mathbb{R}^d} \hat{h}(y)e^{2i\pi\langle y,x\rangle}\,\mathrm{d}y} = \int_{\mathbb{R}^d} \overline{\hat{h}(y)}e^{-2i\pi\langle y,x\rangle}\,\mathrm{d}y = \mathcal{F}[\overline{\hat{h}(y)}].$$

We now replace g by $\overline{\hat{h}(y)} \in \mathcal{S}(\mathbb{R}^d)$ in the above formula. We thus obtain

$$\int_{\mathbb{R}^d} f(x)\overline{h(x)}\,\mathrm{d}x = \int_{\mathbb{R}^d} \hat{f}(y)\overline{\hat{h}(y)}\,\mathrm{d}y, \qquad f, h \in \mathcal{S}(\mathbb{R}^d).$$

In particular, taking $h = f$, we get $\|\mathcal{F}[f]\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}$ for every $f \in \mathcal{S}(\mathbb{R}^d)$. As $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, we can apply the Banach extension principle. It follows that $\mathcal{F}$ extends to a continuous linear mapping $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. Further the mapping $\mathcal{F}^{-1}(f)(x) = \mathcal{F}(f)(-x)$ also extends from $\mathcal{S}(\mathbb{R}^d)$ to a continuous linear mapping $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. As $\mathcal{F}^{-1}\big[\mathcal{F}[f]\big] = \mathcal{F}\big[\mathcal{F}^{-1}[f]\big] = f$ for all $f \in \mathcal{S}(\mathbb{R}^d)$, by density of $\mathcal{S}(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$, this identity stays true for $f \in L^2(\mathbb{R}^d)$. In particular, $\mathcal{F}$ is a bijection $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ and its inverse map is $\mathcal{F}^{-1}$.

Finally, the mappings $\tau_a, M_\omega, \delta_\lambda, \Delta_T$ are all continuous on $L^2(\mathbb{R}^d)$, so the corresponding properties in Proposition 4.4 stay true in $L^2(\mathbb{R}^d)$.

In summary

THEOREM 4.11. *The Fourier transform extends into a continuous linear mapping $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ and the extended map is a bijection. The mapping is an isometry and satisfies*

- *Plancherel's identity: for all $f \in L^2(\mathbb{R}^d)$*

$$\int_{\mathbb{R}^d} |f(x)|^2\,dx = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2\,d\xi.$$

- *Parseval's identity: for all $f, g \in L^2(\mathbb{R}^d)$*

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)}\,dx = \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{g(\xi)}\,d\xi.$$

Further, the identities $\mathcal{F}[\tau_a f] = M_a\mathcal{F}[f]$, $\mathcal{F}[M_\omega f] = \tau_{-\omega}\mathcal{F}[f]$, $\mathcal{F}[\delta_\lambda f] = \lambda^{-d}\mathcal{F}[\delta_{1/\lambda} f]$ and $\mathcal{F}[\Delta_T f] = |\det T|\Delta_{[T^{-1}]^t}\mathcal{F}[f]$ are all valid for $f \in L^2(\mathbb{R}^d)$.

Let us note that the convolution identity $\widehat{f * g} = \hat{f}\hat{g}$ does not extend to $f, g \in L^2(\mathbb{R}^d)$ as in this case $f * g \in \mathcal{C}_0(\mathbb{R}^d)$ and $\widehat{f * g}$ has to be understood in the sense of distributions.

It is then a direct consequence of interpolation theory that the Fourier transform also extends to a bounded linear operator from $L^p \to L^{p'}$ with $1 \le p \le 2$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. It should be noted that this result is false when $p > 2$, but we will not prove this here.

THEOREM 4.12 (Hausdorff-Young Inequality). *Let $1 \leq p \leq 2$ and p' be the dual index $\frac{1}{p}+\frac{1}{p'}=1$. Then the Fourier transform $\mathcal{F}$ extends to a continuous operator from $L^p \to L^{p'}$ with*

$$\|\mathcal{F}[f]\|_{p'} \leq \|f\|_p.$$

PROOF. It is enough to notice that the result is true for $p=1$ (Riemann-Lebesgue) and $p=2$ (Parseval) and that if $1<p<2$, there exists $\theta \in (0,1)$ such that $\frac{1}{p}=\frac{1-\theta}{1}+\frac{\theta}{2}$ and that the dual index is given by

$$\frac{1}{p'}=\frac{\theta}{2}=\frac{1-\theta}{\infty}+\frac{\theta}{2}$$

with the convention $1/\infty=0$. □

EXAMPLE 4.13. Let $a>0$ and define f on $\mathbb{R}$ as $e_a^+(t)=\mathbf{1}_{[0,+\infty)}e^{-at}$ and $e_a^-(t)=\mathbf{1}_{(-\infty,0]}e^{at}$. Note that $e_a^\pm \in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ so that its Fourier transform is given by

$$\hat{e}_a^+(\xi)=\int_0^{+\infty} e^{-(a+2i\pi\xi)t}\,\mathrm{d}t=\frac{1}{a+2i\pi\xi}$$

while

$$\hat{e}_a^-(\xi)=\int_{-\infty}^{0} e^{(a-2i\pi\xi)t}\,\mathrm{d}t=\frac{1}{a-2i\pi\xi}.$$

Let $c_a^\pm$ be defined on $\mathbb{R}$ by $c_a^\pm(x)=\dfrac{1}{a\pm 2i\pi x}$. Note that $c_a^\pm \in L^2$ but *not* in L^1 so that it has an L^2-Fourier transform but not an L^1-Fourier transform. Never the less $c_a^\pm=\mathcal{F}[e_a^\pm]$ in L^1-sense thus also in the L^2-sense. Thus, the Fourier inversion theorem gives $\mathcal{F}[c_a^\pm](\xi)=\mathcal{F}\big[\mathcal{F}[e_a^\pm]\big](\xi)=\mathcal{F}^{-1}\big[\mathcal{F}[e_a^\pm]\big](-\xi)=e_a^\pm(-\xi)=e_a^\mp(\xi)$. This has to be understood in the L^2 sense, in particular, equalities hold only almost everywhere.

One may notice that $e_a^\pm$ is not continuous so that, according to Riemann-Lebesgue, they are not Fourier transforms of L^1 functions. Note however that

$$\widehat{e_a^- + e_a^+}(\xi)=\hat{e}_a^-(\xi)+\hat{e}_a^+(\xi)=\frac{1}{a-2i\pi\xi}+\frac{1}{a+2i\pi\xi}=\frac{2a}{a^2+(2\pi\xi)^2}$$

is an L^1-function. Thus Fourier inversion applies and we get the following expressions

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-a|t|}e^{-2i\pi t\xi}\,\mathrm{d}t &= \frac{2a}{a^2+(2\pi\xi)^2}\\ \int_{-\infty}^{+\infty}\frac{2a}{a^2+(2\pi\xi)^2}e^{2i\pi t\xi}\,\mathrm{d}\xi &= e^{-a|t|}\end{aligned}$$

In particular, taking $a=2\pi$ and using parity, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-2\pi|t|}e^{2i\pi t\xi}\,\mathrm{d}t &= \frac{1}{\pi}\frac{1}{1+\xi^2},\\ \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{1}{1+\xi^2}e^{-2i\pi t\xi}\,\mathrm{d}\xi &= e^{-2\pi|t|}.\end{aligned} \tag{1.27}$$

EXAMPLE 4.14. **An example of a function in $\mathcal{C}_0$ that is not a Fourier transform of an L^1 function.**

Let us define f on $\mathbb{R}$ by $f(t)=\dfrac{\mathrm{sgn}(t)}{1+|t|}$. Note that $f\in L^2(\mathbb{R})$ but $f\notin L^1(\mathbb{R})$. The Fourier transform of f can thus not be calculated via $\int f(t)e^{-2i\pi t\xi}\,dt$ but only as an L^2 limit. To carry out this limit, we will need the following identity

$$\frac{1}{1+|t|}=\int_0^{+\infty} e^{-(1+|t|)x}\,\mathrm{d}x.$$

Using Fubini's Theorem, we see that

$$\begin{aligned}\int_{-R}^{R}\frac{\operatorname{sgn}(t)}{1+|t|}e^{-2i\pi t\xi}\,\mathrm{d}t &= \int_{-R}^{R}\int_0^{+\infty}\operatorname{sgn}(t)e^{-(1+|t|)x}\,\mathrm{d}xe^{-2i\pi t\xi}\,\mathrm{d}t\\ &= \int_0^{+\infty}\int_{-R}^{R}\operatorname{sgn}(t)e^{-(1+|t|)x}e^{-2i\pi t\xi}\,\mathrm{d}t\,\mathrm{d}x\\ &= \int_0^{+\infty}e^{-x}\int_{-R}^{R}\operatorname{sgn}(t)e^{-|t|x}e^{-2i\pi t\xi}\,\mathrm{d}t\,\mathrm{d}x \qquad (1.28)\end{aligned}$$

To see that one is allowed to apply Fubini's theorem, one writes $|\operatorname{sgn}(t)e^{-(1+|t|)x}e^{-2i\pi t\xi}| = e^{-(1+|t|)x} \leq e^{-x} \in L^1([-R,R]\times\mathbb{R},\mathrm{d}t\,\mathrm{d}x)$. But now, if $\xi\neq 0$, (or $x\neq 0$)

$$\begin{aligned}\int_{-R}^{R}e^{-|t|x}e^{-2i\pi t\xi}\,\mathrm{d}t = -\int_{-R}^{0}e^{t(x-2i\pi\xi)}\,\mathrm{d}t+\int_0^R e^{-t(x+2i\pi\xi)}\,\mathrm{d}t\\ = \left[-\frac{e^{t(x-2i\pi\xi)}}{x-2i\pi\xi}\right]_{-R}^{0}+\left[-\frac{e^{-t(x+2i\pi\xi)}}{x+2i\pi\xi}\right]_0^R\\ = \frac{-1+e^{-R(x-2i\pi\xi)}}{x-2i\pi\xi}+\frac{1-e^{-R(x+2i\pi\xi)}}{x+2i\pi\xi} = \frac{-4i\pi\xi}{x^2+(2\pi\xi)^2}+\frac{e^{-R(x-2i\pi\xi)}}{x-2i\pi\xi}-\frac{e^{-R(x+2i\pi\xi)}}{x+2i\pi\xi}.\end{aligned}$$

Inserting this into (1.28) gives

$$\int_{-R}^{R}\frac{e^{-2i\pi t\xi}}{1+|t|}\,\mathrm{d}t = 4i\pi\xi\int_0^{+\infty}\frac{e^{-x}}{x^2+(2\pi\xi)^2}\,\mathrm{d}x+\int_0^{+\infty}\left(\frac{e^{-R(x-2i\pi\xi)}}{x-2i\pi\xi}-\frac{e^{-R(x+2i\pi\xi)}}{x+2i\pi\xi}\right)e^{-x}\,\mathrm{d}x.$$

But, if $x>0$

$$\frac{e^{-R(x-2i\pi\xi)}}{x-2i\pi\xi}-\frac{e^{-R(x+2i\pi\xi)}}{x+2i\pi\xi}\to 0$$

when $R\to+\infty$ while, if $\xi\neq 0$,

$$\begin{aligned}\left|\left(\frac{e^{-R(x-2i\pi\xi)}}{x-2i\pi\xi}-\frac{e^{-R(x+2i\pi\xi)}}{x+2i\pi\xi}\right)e^{-x}\right| &\leq \left(\frac{e^{-Rx}}{|x-2i\pi\xi|}+\frac{e^{-Rx}}{|x+2i\pi\xi|}\right)e^{-x}\\ &\leq \frac{e^{-x}}{\pi\xi}\in L^1(\mathbb{R}).\end{aligned}$$

We may thus apply domintated convergence and obtain that, for $\xi\neq 0$

$$\int_0^{+\infty}\left(\frac{e^{-R(x-2i\pi\xi)}}{x-2i\pi\xi}-\frac{e^{-R(x+2i\pi\xi)}}{x+2i\pi\xi}\right)e^{-x}\,\mathrm{d}x\to 0$$

when $R\to+\infty$ and thus

$$\lim_{R\to+\infty}\int_{-R}^{R}\frac{e^{-2i\pi t\xi}}{1+|t|}\,\mathrm{d}t = 4i\pi\xi\int_0^{+\infty}\frac{e^{-x}}{x^2+(2\pi\xi)^2}\,\mathrm{d}x.$$

But, the L^2-limit of this integral (seen as a function of ξ) is the Fourier transform of f. It follows that, for almost every ξ,

$$\hat{f}(\xi) = 4i\pi\xi\int_0^{+\infty}\frac{e^{-x}}{x^2+(2\pi\xi)^2}\,\mathrm{d}x = 2i\operatorname{sgn}(\xi)\int_0^{+\infty}\frac{e^{-2\pi|\xi|u}}{u^2+1}\,\mathrm{d}u$$

with a change of variable $x=2\pi|\xi|u$.

One may observe that this function is continuous except at 0 where it has a jump discontinuity and goes to 0 at infinity. This follows immediately from Lebesgue's theorem: if we write $F(\xi,u) = \dfrac{e^{-2\pi|\xi|u}}{u^2+1}$ then

– $|F(\xi,u)| = \left|\dfrac{e^{-2\pi|\xi|u}}{u^2+1}\right| \leq \dfrac{1}{1+u^2}\in L^1(\mathbb{R}^+)$;

– if we fix u, $\xi\to F(\xi,u)$ thus $\xi\to\int_0^{+\infty}F(\xi,u)\,\mathrm{d}u$ is continuous over $\mathbb{R}$. In particular $\displaystyle\int_0^{+\infty}F(\xi,u)\,\mathrm{d}u = \int_0^{+\infty}\frac{1}{1+u^2}\,\mathrm{d}u = \frac{\pi}{2}$

– if we fix $u>0$, $F(\xi,u)\to 0$ when $\xi\to\pm\infty$. Thus $\int_0^{+\infty}F(\xi,u)\,\mathrm{d}u\to 0$ as well.

Thus $\hat{f}(\xi) = 2i\,\mathrm{sgn}(\xi)\int_0^{+\infty} F(\xi,u)\,\mathrm{d}u$ has the properties we just announced with $\hat{f}(0^+) = -\hat{f}(0^-) = i\pi$.

One should also note that $\lim_{R\to+\infty}\int_{-R}^{R} f(t)\,\mathrm{d}t = 0$ since f is odd.

Let us now erase the jump discontinuities with the help of the previous example. Let $g = f + i\pi(c_1^+ - c_1^-) = \frac{\mathrm{sgn}(t)}{1+|t|} + \frac{4\pi^2 t}{1+(2\pi t)^2}$. Note that $g(t)\sim 3\,\mathrm{sgn}(t)/t$ in $\pm\infty$ so that $g\in L^2(\mathbb{R})$ but not in $L^1(\mathbb{R})$. By linerity $\hat{g} = \hat{f} - i\pi e_1^+ + i\pi e_1^-$. All three functions $\hat{f}, e_1^+, e_1^-$ are continuous outside 0 and $\hat{f}(0^\pm) = \pm i\pi$, $e_1^\pm(0^\mp) = 0$, $e_1^\pm(0^\pm) = 1$. Thus the jump discontinuities cancels.

2. Computation of the Poisson kernel in the upper half space $\mathbb{R}^{d+1}_+$

In this section, we will consider $\Omega = \mathbb{R}^{d+1}_+ = \{(x,t) : x\in\mathbb{R}^d,\ t>0\}$ and its boundary $\partial\Omega = \mathbb{R}^d$ (identified with the set of vectors of the form $(x,0)$, $x\in\mathbb{R}^d$). The Laplace operator on $\mathbb{R}^{d+1}_+$ is the operator

$$\Delta = \sum_{j=1}^{d}\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial t^2}.$$

Recall that a function $f\in\mathcal{C}^2(\mathbb{R}^{d+1}_+)$ is said to be *harmonic* if $\Delta f = 0$. We will be dealing with the *Dirichlet Problem*, $u\in\mathcal{C}^2(\mathbb{R}^{d+1}_+)\cap\mathcal{C}(\overline{\mathbb{R}^{d+1}_+})$

$$\begin{cases}\Delta u = 0 & \text{in } \Omega\\ u(x,0) = f(x) & x\in\mathbb{R}^d\end{cases}.$$

Here we assume to start that $u\in\mathcal{S}(\mathbb{R}^{d+1}_+)$ and that $f\in\mathcal{S}(\mathbb{R}^d)$. Let us denote by $\widehat{u}(\xi,t)$ the Fourier transform of u in the x variable:

$$\widehat{u}(\xi,t) = \int_{\mathbb{R}^d} u(x,t)e^{-2i\pi\langle\xi,x\rangle}\,\mathrm{d}x.$$

This is well defined due to the hypothesis $u\in\mathcal{S}$ which also justifies the following computation: integration by parts shows that

$$\widehat{\partial^2_{x_j}u}(\xi,t) = (2i\pi\xi_j)^2\widehat{u}(\xi,t) = -(2\pi\xi_j)^2\widehat{u}(\xi,t)$$

while inverting differentiation and integration shows that

$$\widehat{\partial_t^2 u}(\xi,t) = \partial_t^2\widehat{u}(\xi,t).$$

We thus want $\partial_t^2\widehat{u}(\xi,t) + (2\pi|\xi|)^2\widehat{u}(\xi,t) = 0$ and $\widehat{u}(\xi,0) = \widehat{f}(\xi)$. Solving this ODE gives

$$\widehat{u}(\xi,t) = e^{-2\pi|\xi|t}A(\xi) + e^{2\pi|\xi|t}B(\xi).$$

Notice that we are appearently missing a boundary condition for unique determination of $\widehat{u}$. However, this is not the case since we assumed that $u\in\mathcal{S}(\mathbb{R}^{d+1}_+)$ so that, for fixed ξ, we require $\widehat{u}(\xi,t)\to 0$ when $t\to+\infty$ which requires $B(\xi)=0$. Then the condition $\widehat{u}(\xi,0) = \widehat{f}(\xi)$ shows that $A(\xi) = \widehat{f}(\xi)$ and

$$\widehat{u}(\xi,t) = e^{-2\pi|\xi|t}\widehat{f}(\xi).$$

It remains to recognize $e^{-2\pi|\xi|t}$ as a Fourier transform (in x) of a function $P_d(x,t)$ that we want to determine explicitely.

The first observation is that $e^{-2\pi|\xi|t} = \varphi(t\xi)$ with $\varphi(s) = e^{-2\pi|s|}$ so $P_d(x,t) = t^{-d}P_d(x/t,1)$ and it is enough to determine $P(u) = P_d(u,1)$ which has Fourier transform $\varphi(s) = e^{-2\pi|s|}$.

In dimension $d=1$, this has been done in (1.27) which shows that

$$P_1(x) = \frac{1}{\pi}\frac{1}{1+x^2}.$$

Let us prove the analogous fact in any dimension, namely that:

LEMMA 4.15. *The* Poisson kernel on $\mathbb{R}^{d+1}_+$ *is given by*

$$P_d(x,t) = \frac{c_d t}{(t+|x|^2)^{\frac{d+1}{2}}} \qquad (2.29)$$

where $c_d = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}$ *is constant such that* $\int_{\mathbb{R}^d} P_d(x)\,dx = 1$.

Proof of (2.29). We will of course only compute $P_d(x,1)$ and use scaling. To do so, we will use the following Subordination Formula that we will prove later:

$$e^{-2\pi|\xi|} = \int_0^{+\infty} e^{-\pi\xi^2/s}\frac{e^{-\pi s}}{\sqrt{s}}\,\mathrm{d}s. \tag{2.30}$$

From this Formula, we can compute the inverse Fourier transform of the function $\mathbb{R}^d \to \mathbb{R}$ given by $\xi \mapsto e^{-2\pi|\xi|}$

$$\begin{aligned}
\int_{\mathbb{R}^d} e^{-2\pi|\xi|}e^{2i\pi\langle\xi,x\rangle}\,\mathrm{d}\xi &= \int_{\mathbb{R}^d}\int_0^{+\infty} e^{-\pi|\xi|^2/s}\frac{e^{-\pi s}}{\sqrt{s}}\,\mathrm{d}s\, e^{2i\pi\langle\xi,x\rangle}\,\mathrm{d}\xi \\
&= \int_0^{+\infty}\int_{\mathbb{R}^d} e^{-\pi|\xi|^2/s}e^{i\langle\xi,x\rangle}\,\mathrm{d}\xi\,\frac{e^{-\pi s}}{\sqrt{s}}\,\mathrm{d}s \\
&= \int_0^{+\infty} s^{d/2}e^{-s\pi|x|^2}\,\frac{e^{-\pi s}}{\sqrt{s}}\,\mathrm{d}s \\
&= \int_0^{+\infty} s^{(d+1)/2}e^{-s\pi(1+|x|^2)}\,\frac{\mathrm{d}s}{s} \\
&= \frac{1}{\pi^{(d+1)/2}(1+|x|^2)^{(d+1)/2}}\int_0^{+\infty} t^{(d+1)/2}e^{-t}\,\frac{\mathrm{d}t}{t} = \frac{c_d}{(1+|x|^2)^{(d+1)/2}}
\end{aligned}$$

where we have used Fubini in the second line (show its validity as an exercice) and the fact that $e^{-pi|x|^2}$ is its own Fourier transform and the scaling property of the Fourier transform on $\mathbb{R}^d$. In the next to last line, we use the change of variable $t = \pi(1+|x|^2)s$. The constant is

$$c_d = \frac{1}{\pi^{(d+1)/2}}\int_0^{+\infty} t^{(d+1)/2}e^{-t}\,\frac{\mathrm{d}t}{t} = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}}$$

as claimed.

The constant c_d could be computed independently using that

$$\int_{\mathbb{R}^d} P_d(x)\,\mathrm{d}x = \widehat{P_d}(0) = e^{-|0|} = 1.$$

Thus, integrating in polar coordinates

$$\begin{aligned}
c_d^{-1} &= \int_{\mathbb{R}^d}\frac{\mathrm{d}x}{(1+|x|^2)^{(d+1)/2}} = \sigma_{d-1}(\mathbb{S}^{d-1})\int_0^{+\infty}\frac{r^{d-1}}{(1+r^2)^{(d+1)/2}}\,\mathrm{d}r \\
&= \sigma_{d-1}(\mathbb{S}^{d-1})\int_0^{\pi/2}(\sin\theta)^{d-1}\,\mathrm{d}\theta \qquad r = \tan\theta \\
&= \frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{1}{2}\frac{\Gamma(d/2)\Gamma(1/2)}{\Gamma\big((d+1)/2\big)} = \frac{\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}
\end{aligned}$$

using classical computations. □

Proof of the subordination formula (2.30). The proof is based on the computation of the Fourier transform of $e^{-2\pi|x|}$ in dimension 1 given in (1.27):

$$e^{-2\pi|t|} = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{1}{1+\xi^2}e^{-2i\pi t\xi}\,\mathrm{d}\xi.$$

An alternative way to prove this formula is to apply the theory of residues to $e^{itz}/(1+z^2)$.

Next, a simple change of variables shows that

$$\frac{1}{1+\xi^2} = \frac{\pi}{\pi(1+\xi^2)} = \pi\int_0^{+\infty} e^{-\pi(1+\xi^2)s}\,\mathrm{d}s$$

so that, with Fubini, for $\xi > 0$,

$$\begin{aligned} e^{-2\pi\xi} &= \int_{-\infty}^{+\infty}\int_0^{+\infty} e^{-\pi(1+\xi^2)s}\,\mathrm{d}s e^{-2i\pi t\xi}\,\mathrm{d}\xi \\ &= \int_0^{+\infty}\int_{-\infty}^{+\infty} e^{-\pi(1+\xi^2)s}e^{-2i\pi t\xi}\,\mathrm{d}\xi\,\mathrm{d}s \\ &= \int_0^{+\infty} e^{-\pi s}\int_{-\infty}^{+\infty} e^{-s\pi\xi^2}e^{-2i\pi t\xi}\,\mathrm{d}\xi\,\mathrm{d}s \\ &= \int_0^{+\infty} e^{-\pi s}e^{-\pi\xi^2/s}\frac{\mathrm{d}s}{\sqrt{s}} \end{aligned}$$

where we use the fact that $e^{-\pi\xi^2}$ is its own Fourier transform so that

$$\mathcal{F}[e^{-s\pi\xi^2}](t) = s^{-1/2}e^{-\pi\xi^2/s}.$$

The result for $\xi < 0$ is obtained by parity. □

Now that those computations have been done, let us be a bit more formal and extend the result we are looking for:

DEFINITION 4.16. The *Poisson kernel* of the upper half spacs $\mathbb{R}_+^{d+1}$ for is defined by

$$P(x,t) = c_d\frac{t}{(t^2+|x|^2)^{(d+1)/2}} \qquad x\in\mathbb{R}^d, t>0$$

where $c_d = \dfrac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}}$.

3. The Paley-Wiener theorems

We start with the following observation: if we fix x, the function $\xi \to e^{-2i\pi x\xi}$ is holomorphic, we may thus be willing to apply Lebesgue's holomorphy theorem to the Fourier transform

$$\widehat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)e^{-2i\pi x\xi}\,\mathrm{d}\xi$$

we need to bound $f(x)e^{-2i\pi x\xi}$ by an L^1-function that is independant of ξ in the domain of holomorphy. As $|f(x)e^{-2i\pi x\xi}| = |f(x)|e^{2\pi x \operatorname{Im}\xi}$, this will be easy when f is supported in $[-c,c]$. It turns out that it is a bit simpler to work with inverse Fourier transform.

THEOREM 4.17 (Paley-Wiener). *Let $A,c>0$ and $f\in L^2(\mathbb{R})$. The following are equivalent*

(1) *$f = F$ a.e. on $\mathbb{R}$ with F an holomorphic function over $\mathbb{C}$ such that $|F(z)| \leq Ae^{1\pi c|z|}$.*
(2) *$\widehat{f}$ is supported in $[-c,c]$.*

PROOF. Assume first that $\widehat{f}$ is supported in $[-c,c]$. Recall from Plancherel that $\widehat{f}\in L^2(\mathbb{R})$ so that Cauchy-Schwarz implies that $\widehat{f}\in L^1(\mathbb{R})$. Further, if $x\in[-c,c]$ and $|z|<\rho$, then $|\widehat{f}(\xi)e^{-2i\pi\xi z}| \leq |\widehat{f}(x)e^{2\pi c\rho}|$ It follows that the function F defined by

$$F(z) = \int_{-c}^{c}\widehat{f}(\xi)e^{-2i\pi\xi z}\,\mathrm{d}\xi$$

is holomorphic over the disc $D(0,\rho)$ with

$$|F(z)| \leq \int_{-c}^{c}|\widehat{f}(\xi)|e^{2\pi c\rho}\,\mathrm{d}\xi \leq \sqrt{2c}\|\widehat{f}\|_2 e^{2\pi c\rho}.$$

As ρ is arbitrary, we get that F is holomorphic over $\mathbb{C}$ and that $|F(z)| \leq \sqrt{2c}\|f\|_2 e^{2\pi c|z|}$. On the other hand, the L^2 Fourier inversion theorem shows that $f=F$ a.e. on $\mathbb{R}$.

Let us now show the converse. We will not distinguish between f and F, that is we assume that f extends to an entire function. For $\varepsilon>0$ define $f_\varepsilon(x) = f(x)e^{-2\pi\varepsilon|x|}$. Cauchy-Schwarz shows that $f_\varepsilon\in L^1(\mathbb{R})$. We are going to show that

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} f_\varepsilon(x)e^{-2i\pi\xi x}\,\mathrm{d}x = 0 \qquad \xi\in\mathbb{R},\ |\xi|>c. \tag{3.31}$$

Before we do so, let us see how we conclude from there. First, an easy application of dominated convergence shows that $\|f-f_\varepsilon\|_2 \to 0$ so that, by L^2-continuity of the Fourier transform $\widehat{f_\varepsilon} \to \widehat{f}$ in L^2. In particular, there is a sequence such that $\widehat{f_{\varepsilon_k}}(\xi) \to \widehat{f}(\xi)$ for almost every $\xi \in \mathbb{R}$. But (3.31) shows that $\widehat{f_{\varepsilon_k}}(\xi) \to 0$ for $|\xi| > c$ so that $\widehat{f} = 0$ a.e. on $\mathbb{R} \setminus [-c,c]$ and is thus supported in $[-c,c]$ as claimed.

For $\theta \in \mathbb{R}$ we now define the path $\Gamma_\theta = \{te^{i\theta} \ : \ t \geq 0\}$ and the half-plane $\Pi_{\theta,\eta} = \{z \in \mathbb{C} : \Re(ze^{i\theta}) > \eta\}$. Let

$$\Phi_\theta(z) = \int_{\Gamma_\theta} F(w)e^{-2\pi wz}\,\mathrm{d}w = e^{i\theta}\int_0^{+\infty} F(te^{i\theta})\exp(-2\pi zte^{i\theta})\,\mathrm{d}t.$$

But, if $\eta > c$ and $z \in \Pi_{\theta,\eta}$ then

$$|F(te^{i\theta})\exp(-2\pi zte^{i\theta})| \leq A\exp(-2\pi[\Re(ze^{i\theta}) - c]t) \leq Ae^{-2\pi(\eta-c)t} \in L^1(\mathbb{R})$$

and $z \to F(te^{i\theta})\exp(-zte^{i\theta})$ is holomorphic so that Φ_θ is holomorphic over $\Pi_{\theta,\eta}$. As $\eta > c$ is arbitrary, Φ_θ is holomorphic over the half-plane $\Pi_{\theta,c}$.

However, for $\theta = 0$ and $\theta = \pi$ more is true since

$$\Phi_0(z) = \int_0^{+\infty} f(t)e^{-2\pi zt}\,\mathrm{d}t$$

is holomorphic in the half-plane $\Re(z) > 0$ while

$$\Phi_\pi(z) = -\int_0^{+\infty} f(-t)e^{2\pi zt}\,\mathrm{d}t = -\int_{-\infty}^0 f(t)e^{-2\pi zt}, \mathrm{d}t$$

is holomorphic in the half-plane $\Re(z) < 0$. This follows from the fact that $f \in L^2(\mathbb{R})$. Indeed, $z \to f(t)e^{-2\pi zt}$ is holomorphic and if $\Re(z) > \alpha > 0$ then $|f(t)e^{-2\pi zt}| \leq |f(t)|e^{-2\pi\alpha t} \in L^1(\mathbb{R})$ with Cauchy-Schwarz so that Φ_0 is holomorphic in the half plane $\Re(z) > \alpha > 0$ and α is arbitrary. The argument for Φ_π is similar.

Now notice that

$$\begin{aligned}\int_{-\infty}^{+\infty} f_\varepsilon(x)e^{-2i\pi\xi x}\,\mathrm{d}x &= \int_{-\infty}^0 f(t)e^{-2\pi(-\varepsilon+i\xi)t}\,\mathrm{d}t + \int_0^{+\infty} f(t)e^{-2\pi(\varepsilon+i\xi)t}\,\mathrm{d}t \\ &= -\Phi_\pi(-\varepsilon+i\xi) + \Phi_0(\varepsilon+i\xi).\end{aligned}$$

We want to show that this quantity goes to 0 as $\varepsilon \to 0$ when $|\xi| > c$. To do so, we will show that Φ_θ and Φ_φ agree on $\Pi_{\theta,c} \cap \Pi_{\varphi,c}$, the intersection of their domain of definition *i.e.* they are analytic continuations of one another. In particular, if $\xi < -c$,

$$-\Phi_\pi(-\varepsilon+i\xi) + \Phi_0(\varepsilon+i\xi) = -\Phi_{\pi/2}(-\varepsilon+i\xi) + \Phi_{\pi/2}(\varepsilon+i\xi) \to 0 \quad \text{when } \varepsilon \to 0$$

by continuity. For $\xi > c$ we conclude with $\Phi_{-\pi/2}$ instead of $\Phi_{\pi/2}$.

Now take θ, φ with $0 < \theta - \varphi < \pi$ and put $\gamma = \dfrac{\theta+\varphi}{2}$. If $z = \rho e^{-i\gamma}$, then

$$\Re(ze^{i\varphi}) = \rho\cos\frac{\beta-\varphi}{2} = \Re(ze^{i\theta})$$

which shows that $z \in \Pi_{\varphi,c} \cap \Pi_{\theta,c}$ as soon as $\rho > c/\eta$ with $\eta = \cos\dfrac{\beta-\varphi}{2} > 0$.

Consider the arc $\Gamma_r = \{re^{i\alpha} \ : \varphi \leq \alpha \leq \theta\}$ and

$$\psi(r) = \int_{\Gamma_r} f(w)e^{-2\pi wz}\,\mathrm{d}w = r\int_\varphi^\theta f(re^{i\alpha})e^{-2\pi re^{i\alpha}z}e^{i\alpha}\,\mathrm{d}\alpha.$$

Note that for $\varphi \leq \alpha \leq \theta$

$$|e^{-2\pi re^{i\alpha}z}| = e^{-2\pi r\rho\cos(\alpha-\gamma)} \leq e^{-2\pi r\rho\eta}$$

thus

$$|\psi(r)| \leq r\int_\varphi^\theta e^{2\pi(c-\rho\eta)r}\,\mathrm{d}\alpha \leq (\theta-\varphi)re^{2\pi(c-\rho\eta)r}$$

so that $\psi(r) \to 0$ when $r \to +\infty$ as soon as $\rho > c/\eta$.

But then, for $\rho > c/\eta$, integrating $f(w)e^{-2\pi zw}$ along the segment $\{te^{i\varphi} : 0 \le t \le r\}$ then along Γ_r and then along the segment $\{te^{i\theta} : r \ge t \ge 0\}$ gives

$$0 = \int_0^r f(te^{i\varphi})e^{-2\pi te^{i\varphi}z}\, e^{i\varphi}\,\mathrm{d}t + \psi(r) + \int_r^0 f(te^{i\theta})e^{-2\pi te^{i\theta}z}\, e^{i\theta}\,\mathrm{d}t.$$

Letting $r \to +\infty$, we obtain $0 = \Phi_\varphi(z) - \Phi_\theta(z)$ provided $|z| = \rho > c/\eta$.

In other words, $\Phi_\varphi(z) = \Phi_\theta(z)$ on the half-line $\{z = \rho e^{-i\gamma} : \rho > c/\eta\}$ and therefore coincide on $\Pi_{\varphi,c} \cap \Pi_{\theta,c}$ as claimed and the proof is completed. □

CHAPTER 5

The maximal function

In this chapter, we will introduce the Hardy-Littlewood Maximal Function (in its centered and uncentered version) and investigate its main properties. Those functions play an important role in harmonic analysis as they control many operators appearing in harmonic analysis.

1. Maximal functions

Recall that for $A\subset\mathbb{R}^d$, we denote by $|A|$ its Lebesgue measure and that $B(x,r)$ is the (open) ball centered at x and of radius r, $B(x,r)=\{y\in\mathbb{R}^d\,:\,|x-y|<r\}$.

Definition 5.1. For $f\in L^1_{loc}(\mathbb{R}^d)$, we define the
– *(centered) Hardy-Littlewood Maximal Function* by

$$M[f](x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(u)|\,\mathrm{d}u;$$

– *(uncentered) Hardy-Littlewood Maximal Function* by

$$\mathcal{M}[f](x)=\sup_{r>0}\sup_{y\in B(x,r)}\frac{1}{|B(y,r)|}\int_{B(y,r)}|f(u)|\,\mathrm{d}u.$$

These are respectively the maximum of the averages of $|f|$ over all balls centered at x and over all balls containing x. Those functions are obviously well-defined for $f\in L^1_{loc}$, in particular, they are well defined for $f\in L^p(\mathbb{R}^d)$.

Here are some properties of M and $\mathcal{M}$:

Proposition 5.2. *The maximal functions satisfy the following properties*
(i) $M[f]=M[|f|]$ *and* $\mathcal{M}[f]=\mathcal{M}[|f|]$.
(ii) *If, for some* $x\in\mathbb{R}^d$, $M[f](x)=0$ *(resp.* $\mathcal{M}[f](x)=0$*) then* $f=0$.
(iii) M *and* $\mathcal{M}$ *are sub-linear.*
(iv) $\|M[f]\|_\infty\le\|f\|_\infty$ *and* $\|\mathcal{M}[f]\|_\infty\le\|f\|_\infty$.
(v) $M[f]\le\mathcal{M}[f]\le 2^dM[f]$.

Proof. (i) is obvious and for (ii) if $M[f](x)=0$ then, for every $r>0$, $\displaystyle\int_{B(x,r)}|f(u)|\,\mathrm{d}u=0$ thus $f=0$ a.e. on $B(x,r)$. As r is arbitrary, $f=0$ a.e. on $\mathbb{R}^d$.

Clearly

$$\begin{aligned}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(u)+g(u)|\,\mathrm{d}u &\le \frac{1}{|B(x,r)|}\int_{B(x,r)}|f(u)|\,\mathrm{d}u+\frac{1}{|B(x,r)|}\int_{B(x,r)}|g(u)|\,\mathrm{d}u\\ &\le M[f](x)+M[g](x)\end{aligned}$$

thus taking the suppremum over $r>0$ gives $M[f+g]\le M[f]+M[g]$. $M[\lambda f]=|\lambda|M[f]$ is obvious. The same argument works for $\mathcal{M}$.

(iv) is trivial as is $M[f]\le\mathcal{M}[f]$ (take $y=x$ in the definition of $\mathcal{M}$). On the other hand if $y\in B(x,r)$ then $B(y,r)\subset B(x,2r)$ thus

$$\frac{1}{|B(y,r)|}\int_{B(y,r)}|f(u)|\,\mathrm{d}u\le\frac{1}{|B(y,r)|}\int_{B(x,2r)}|f(u)|\,\mathrm{d}u=\frac{2^d}{|B(x,2r)|}\int_{B(x,2r)}|f(u)|\,\mathrm{d}u\le 2^dM[f](x).$$

Taking the supremum over all $y\in B(x,r)$ and then all $r>0$ gives $\mathcal{M}[f]\le 2^dM[f]$. □

The maximal functions have been defined with the balls associated to the Euclidean norm. The properties of $M,\mathcal{M}$ don't depend on this choice. To illustrate this, let us consider cubes instead of balls. Write $Q(x,r)=\{y\in\mathbb{R}^d\,:\,\max_i|x_i-y_i|<r\}$.

DEFINITION 5.3. For $f \in L^1_{loc}(\mathbb{R}^d)$, we define the
– *square Maximal Function* by

$$M^\square[f](x) = \sup_{r>0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(u)| \,\mathrm{d}u;$$

– *uncentered square Maximal Function* by

$$\mathcal{M}^\square[f](x) = \sup_{r>0} \sup_{y \in Q(x,r)} \frac{1}{|Q(y,r)|} \int_{B(y,r)} |f(u)| \,\mathrm{d}u.$$

LEMMA 5.4. *For $f \in L^1_{loc}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,*

$$2^{-d^2/2} M[f](x) \leq M^\square[f](x) \leq 2^{d^2/2} M[f](x)$$

and

$$2^{-d^2/2} \mathcal{M}[f](x) \leq \mathcal{M}^\square[f](x) \leq 2^{d^2/2} \mathcal{M}[f](x)$$

PROOF. Recall that $Q(x, 2^{-d/2}r) \subset B(x,r) \subset Q(x,r)$ thus

$$\begin{aligned}\frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(u)| \,\mathrm{d}u &\leq \frac{1}{|B(x,r)|} \int_{B(x,2^{d/2}r)} |f(u)| \,\mathrm{d}u \\ &= \frac{2^{d^2/2}}{|B(x,2^{d/2}r)|} \int_{B(x,2^{d/2}r)} |f(u)| \,\mathrm{d}u \leq 2^{d^2/2} M[f](x)\end{aligned}$$

thus, taking the supremum over r, $M^\square[f](x) \leq 2^{d^2/2} M[f](x)$. Conversely

$$\begin{aligned}\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(u)| \,\mathrm{d}u &\leq \frac{1}{|Q(x,2^{-d/2}r)|} \int_{Q(x,r)} |f(u)| \,\mathrm{d}u \\ &= \frac{2^{-d^2/2}}{|Q(x,r)|} \int_{Q(x,r)} |f(u)| \,\mathrm{d}u \leq 2^{-d^2/2} M^\square[f](x).\end{aligned}$$

The proof for the uncentered case is the same. □

EXERCICE 5.5. Let $f = \mathbf{1}_{[a,b]}$, show that

$$M[f](x) = \begin{cases} \dfrac{b-a}{2|x-b|} & \text{when } x \leq a \\ 1 & \text{when } a < x < b \\ \dfrac{b-a}{2|x-a|} & \text{when } x \geq b \end{cases} \quad \text{and} \quad \mathcal{M}[f](x) = \begin{cases} \dfrac{b-a}{|x-b|} & \text{when } x \leq a \\ 1 & \text{when } a < x < b \\ \dfrac{b-a}{|x-a|} & \text{when } x \geq b \end{cases}.$$

REMARK 5.6. Note that in this example, $M[f], \mathcal{M}[f] \notin L^1(\mathbb{R})$ though $f \in L^1(\mathbb{R})$. This is a general fact:

If in the supremum defining $M[f](x)$ we consider the average over the ball $B(x, |x|+R)$ and notice that this ball contains $B(0,R)$ then

$$M[f] \geq \frac{1}{|B(0,1)|(|x|+R)^d} \int_{B(0,R)} |f(u)| \,\mathrm{d}u.$$

Now, if $f \in L^1(\mathbb{R}^d)$ and $f \neq 0$, we chose R large enough to have $\displaystyle\int_{B(0,R)} |f(u)| \,\mathrm{d}u > 0$ then this inequality shows that $M[f] \notin L^1(\mathbb{R}^d)$. In particular, the Hardy-Littlewood maximal functions are *not of strong* $(1,1)$*-type*.

The main result of this section is the following which is more or less te best possible.

THEOREM 5.7 (Hardy-Littlewood). *Both M and $\mathcal{M}$ are of weak-type* $(1,1)$

$$|\{x \,:\, \mathcal{M}[f](x) > \alpha\}| \leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(u)| \,\mathrm{d}u$$

and of strong type (p,p) for every $1 < p \leq +\infty$.

The (trivial) case $p=+\infty$ was stated in the previous proposition and, once we establish that $M,\mathcal{M}$ are of weak-type $(1,1)$, Marcienkiewicz interpolation allows to conclude that they are of strong type (p,p). Also, in view of $M[f]\leq\mathcal{M}[f]\leq 2^dM[f]$, it is enough to consider the case of $\mathcal{M}$.

To do so, we will need a simple *covering lemma*:

LEMMA 5.8 (Covering lemma). *Let $B_1,\ldots,B_n$ be a collection of balls. Then there exists a sub-collection $B_{j_1},\ldots,B_{j_m}$ of* pairwise disjoint *balls such that*

$$\sum_{k=1}^m|B_{j_k}|=\left|\bigcup_{k=1}^m B_{j_k}\right|\geq 3^{-d}\left|\bigcup_{j=1}^n B_j\right|.$$

PROOF. Up to reordeing the balls, we can assume that $|B_1|\geq|B_2|\geq\cdots\geq|B_n|$ (in other words, we re-order the balls by decreasing radius).

We then go along the sequence of balls and keep only those balls that do not intersect any previous ball. In other words, we set $j_1=1$ then $j_2=\min\{k>j_1\,:B_k\cap B_{j_1}=\emptyset\}$ so that B_{j_2} is the first ball that does not intersect B_{j_1}. Next $j_3=\min\{k>j_2\,:B_k\cap(B_{j_1}\cup B_{j_2})=\emptyset\}$ so that B_{j_3} is the first ball that does not intersect B_{j_1} nor B_{j_2}...

By construction, this family of balls is disjoint. Now take any ball B_j in the orginal collection of balls. Either this is one of balls in the subcollection, $B_j=B_{j_\ell}$ or there is an ℓ such that $j_\ell<j<j_{\ell+1}$ which means that B_j intesects one of $B_{j_1}\cup\cdots\cup B_{j_\ell}$ (otherwise we would have $j_{\ell+1}=j$), say $B_{j_{\ell_0}}$. But, from the fact that the balls have been ordered with decreasing radius, the radius of B_j is smaller than the radius of $B_{j_{\ell_0}}$ and then $B_j\subset 3B_{j_{\ell_0}}$ (the ball with same center as $B_{j_{\ell_0}}$ but radius multiplied by 3). This implies that $\displaystyle\bigcup_{j=1}^n B_j\subset\bigcup_{k=1}^m 3B_{j_k}$ But then

$$\left|\bigcup_{j=1}^n B_j\right|\leq\left|\bigcup_{k=1}^m 3B_{j_k}\right|\leq\sum_{k=1}^m|3B_{j_k}|=3^d\sum_{k=1}^m|B_{j_k}|.$$

□

PROOF OF THE THEOREM. Let $\alpha>0$ and $E_\alpha=\{x\,:\,\mathcal{M}[f](x)>\alpha\}$ and $x\in E_\alpha$. Then there exists a ball $B(y,r)$ containing x and such that $\displaystyle\frac{1}{|B(y,r)|}\int_{B(y,r)}|f(u)|\,\mathrm{d}u>\alpha$. But this implies that if $z\in B(y,r)$, $\mathcal{M}[f](z)>\alpha$, since the ball $B(y,r)$ is one of the balls in the supremum defining $\mathcal{M}[f](z)$. It follows that $B(y,r)\subset E_\alpha$ which is therefore an open set.

Now let $K\subset E_\alpha$ be a compact set. By definition, for each $x\in E_\alpha$, there exists a ball B_x containing x and such that $\displaystyle\int_{B_x}|f(u)|\,\mathrm{d}u>\alpha|B_x|$. As $\{B_x\,:\,x\in K\}$ is a covering of K, we can extract a finite collection of balls $B_1,\ldots,B_n$ that still covers K. Apply the covering Lemma to this collection and get a subcollection $B_{j_1},\ldots,B_{j_m}$. Then each B_{j_k} satisfies $\displaystyle|B_{j_k}|\leq\frac{1}{\alpha}\int_{B_{j_k}}|f(u)|\,\mathrm{d}u$. Further

$$\begin{aligned}|K| &\leq \left|\bigcup_{j=1}^n B_j\right|\leq 3^d\sum_{k=1}^m|B_{j_k}|\leq\frac{3^d}{\alpha}\sum_{k=1}^m\int_{B_{j_k}}|f(u)|\,\mathrm{d}u\\ &= \frac{3^d}{\alpha}\int_{\bigcup_{k=1}^m B_{j_k}}|f(u)|\,\mathrm{d}u\leq\frac{3^d}{\alpha}\int_{\mathbb{R}^d}|f(u)|\,\mathrm{d}u\end{aligned}$$

where we have used the disjointness of the B_{j_k}'s in the last line. □

2. Control of other maximal functions

We will now consider an other family of maximal functions:

THEOREM 5.9. *Let k be a decreasing, non-negative, continuous function (except at finitely many points) on $[0,+\infty)$. Let $K(x)=k(|x|)$ be the radial function associated to k, and assume* *that $K \in L^1(\mathbb{R}^d)$. For $\varepsilon > 0$, define $K_\varepsilon = \varepsilon^{-d} K(x/\varepsilon)$ the dilates of K. Let*

$$M_k[f](x) = \sup_{\varepsilon > 0} K_\varepsilon * |f|(x)$$

be the maximal function associated to the dilates of K. Then

$$M_k[f](x) \leq \|K\|_{L^1(\mathbb{R}^d)} M[f](x). \tag{2.32}$$

In particular, $f \to M_k[f]$ is of weak-type $(1,1)$ and of strong type (p,p) for every $1 < p \leq +\infty$. Further $M_k[f]$ is finite almost everywhere when $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq +\infty$.

Proof. The second part of the theorem follows from (2.32).

Note that $K_\varepsilon * |f|(x)$ is well defined as the integral of a non-negative quantity when $f \in L^1_{loc}$.

First, let $k_j = k\mathbf{1}_{[0,j]}$ so that k_j is compactly supported and, for each x, $k_j(x) \to k(x)$ increasingly. If (2.32) holds for each k_j, then it is enough to pass to the limit to obtain (2.32) for k. We can thus assume that k is compactly supported. Further, it is enough to establish (2.32) for $x = 0$ and then apply the inequality with $f(t+x)$ instead of $f(t)$ to obtain the result in full generality.

For simplicity, we will now assume that k is $\mathcal{C}^1$-smooth (using Stieltjes integrals avoids this) and supported in $[0, R]$. Up to replacing f by $|f|$ wa may also assume that $f \geq 0$. We want to estimate

$$K_\varepsilon * f(0) = \int_{\mathbb{R}^d} f(x) K_\varepsilon(-x)\,\mathrm{d}x = \int_0^{+\infty} \varepsilon^{-d} k(r/\varepsilon) \int_{\mathbb{S}^{d-1}} f(r\zeta)\,\mathrm{d}\sigma(\zeta)\, r^{d-1}\,\mathrm{d}r.$$

Write $F(r) = \displaystyle\int_{\mathbb{S}^{d-1}} f(r\zeta)\,\mathrm{d}\sigma(\zeta)$ and note that this is well defined for almost every r. Define $G(r) = \displaystyle\int_0^r F(s) s^{d-1}\,\mathrm{d}s$ and note that $G(0) = 0$ and that, integrating in polar coordinates, (and using that $f = |f|$)

$$G(r) = \int_{B(0,r)} |f(u)|\,\mathrm{d}u = |B(0,r)| \frac{1}{|B(0,r)|} \int_{B(0,r)} |f(u)|\,\mathrm{d}u \leq |B(0,r)| M[f](0).$$

In particular, $G(r)$ is finite.

Then, as $k(R) = 0$, integrating by parts gives

$$\begin{aligned} K_\varepsilon * f(0) &= \int_0^{\varepsilon R} \varepsilon^{-d} k(r/\varepsilon) F(r)\, r^{d-1}\,\mathrm{d}r = \int_0^{\varepsilon R} \varepsilon^{-d} k(r/\varepsilon) G'(r)\,\mathrm{d}r \\ &= \int_0^{\varepsilon R} -\varepsilon^{-d-1} k'(r/\varepsilon) G(r)\,\mathrm{d}r \\ &\leq \int_0^{\varepsilon R} -\varepsilon^{-d-1} k'(r/\varepsilon) |B(0,r)|\,\mathrm{d}r\, M[f](0) \end{aligned} \tag{2.33}$$

where we have used that k decreases so that $-k' \geq 0$. It remains to notice that, changing variable $s = r/\varepsilon$ and then interating again by parts gives

$$\begin{aligned} \int_0^{\varepsilon R} -\varepsilon^{-d-1} k'(r/\varepsilon) |B(0,r)|\,\mathrm{d}r &= |B(0,1)| \int_0^{\varepsilon R} -\varepsilon^{-d-1} k'(r/\varepsilon) r^d\,\mathrm{d}r \\ &= |B(0,1)| \int_0^R -k'(s) s^d\,\mathrm{d}s \\ &= d|B(0,1)| \int_0^R k(s) s^{d-1}\,\mathrm{d}s = \|K\|_{L^1(\mathbb{R}^d)} \end{aligned}$$

where we have used that k is radial and supported in $[0, R]$.

Grouping (2.33) and (2.34) gives (2.32). □

We leave as an exercice to adapt the proof to the case where k is piecewise $\mathcal{C}^1$ with only finitely many jump discontinuities.

A careful reading will show that the crux of the proof is the following lemma that we state here with less regularity for the kernel.

LEMMA 5.10. *Let h be a non-negative decreasing function and define g on $\mathbb{R}^d$ by $g(x) = h(|x|)$ so that g is* radial. *Assume that $g \in L^1(\mathbb{R}^d)$. Let $v \in L^1(\mathbb{R}^d)$ be such that $|v(x)| \leq g(x)$. Then, for $u \in L^p(\mathbb{R}^d)$, $1 \leq p \leq +\infty$,*

$$\left|\int u(y)v(x-y)\,\mathrm{d}y\right| \leq \|g\|_1 M[u](x)$$

almost everywhere in $\mathbb{R}^d$.

It can be proved along the lnes above but we give a more direct approach which avoids Stiletjes measures:

PROOF. It is of course enough to prove

$$\int u(y)g(x-y)\,\mathrm{d}y \leq \int_{\mathbb{R}^d} g(y)\,\mathrm{d}y M[u](x). \tag{2.34}$$

with u non-negative.

We first notice that, if $h = \mathbf{1}_{[0,\rho)}$ then $g = \mathbf{1}_{B(0,\rho)}$ then $\|g\|_1 = |B(0,\rho)| = |B(x,\rho)|$ and this is the trivial inequality

$$\int_{B(x,\rho)} u(y)\,\mathrm{d}y \leq |B(x,\rho)| \sup_{s>0} \frac{1}{|B(x,s)|}\int_{B(x,s)} u(y)\,\mathrm{d}y. \tag{2.35}$$

Next, if h is a non-negative decreasing step function, then we can write

$$h(r) = \sum_{j=1}^{n} c_j \mathbf{1}_{[0,\rho_j]}$$

with $c_j \geq 0$ and $\rho_1 \leq \rho_2 \leq \cdots \leq \rho_n$. But then $g = \sum_{j=1}^{n} c_j \mathbf{1}_{B(0,\rho_j)}$ and in this case (2.34) is just a linear combination of (2.35). We conclude by noticing that every decreasing function can be approximated from below by an increasing sequence of simple step functions of this type and conclude by monotone convergence. □

3. Using maximal functions for almost everywhere convergence

3.1. General principle. Let us explain how maximal functions are used to obtain almost everywhere convergence.

The general setting is as follows: $(X, \mathcal{B}, \mu)$ and $(Y, \tilde{\mathcal{B}}, \nu)$ are two measure spaces and $1 \leq p, q < +\infty$.

For each $\varepsilon > 0$ we consider a linear operator $T_\varepsilon : L^p(\mu) \to L^0(Y)$ and to this family of operators we associate the maximal function

$$T_*[f](y) = \sup_{\varepsilon>0} |T_\varepsilon[f](y)|.$$

We assume that $T_*[f]$ is measurable.

Next we assume that there is some *dense* vector space $\mathcal{D} \subset L^p(\mu)$ and a *bounded* operator $T : L^p(\mu) \to L^q(\nu)$ such that, if $f \in \mathcal{D}$, $T(f) := \lim_{\varepsilon\to 0} T_\varepsilon(f)$ exists and is finite ν-almost everywhere. Note that this defines a linear operator on $\mathcal{D}$.

With thouse notations we have the following theorem:

THEOREM 5.11. *Assume that T_* is of weak-type (p, q) i.e. there exists $B > 0$ such that, for every $f \in L^p(\mu)$,*

$$\|T_*[f]\|_{L^q_w(\nu)} \leq B\|f\|_{L^p(\mu)}.$$

Then for every $f \in L^p(\mu)$,

$$T[f](y) = \lim_{\varepsilon\to 0} T_\varepsilon[f](y)$$

exists for ν-almost every $y \in Y$ and defines a linear mapping T on $L^p(\mu)$ (uniquely extending T from $\mathcal{D}$ to $L^p(\mu)$) such that, for every $f \in L^p(\mu)$,

$$\|T[f]\|_{L^q_w(\nu)} \leq B\|f\|_{L^p(\mu)}. \tag{3.36}$$

PROOF. Given $f \in L^p(\mu)$, we define its *oscillation* by

$$Osc[f](y) = \limsup_{\varepsilon\to 0} \limsup_{\eta\to 0} |T_\varepsilon[f](y) - T_\eta[f](y)|.$$

We would like to show that, for every $f \in L^p(\mu)$ and every $\delta > 0$,

$$\nu(\{y : Osc[f](y) > \delta\}) = 0. \tag{3.37}$$

Once this is done, we would have that, for almost every $y \in Y$, $T_\varepsilon[f](y)$ is a Cauchy-family so that $T[f](y) = \lim_{\varepsilon\to 0} T_\varepsilon[f](y)$ exists. This then defines an operator T on $L^p(\mu)$ that coincides with T on $\mathcal{D}$. Since then $|T[f]| \leq T_*[f]$, the of bound (3.36) follows.

We will use density of $\mathcal{D}$ to approximate O_f. Given $\eta > 0$, there exists $g \in \mathcal{D}$ such that $\|f - g\|_{L^p(\mu)} \leq \eta$. Further, if $T_\varepsilon[g](y)$ converges (which happens for almost every y), $Osc[g](y) = 0$ *i.e.* $Osc[g] = 0$ ν-a.e. Next, as the T_ε's are linear,

$$|T_\varepsilon[f](y) - T_\theta[f](y)| \leq |T_\varepsilon[f-g](y) - T_\theta[f-g](y)| + |T_\varepsilon[g](y) - T_\theta[g](y)|$$

thus, taking lim sup's,

$$Osc[f](y) \leq Osc[f-g](y) + Osc[g](y)$$

and we get that $Osc[f] \leq Osc[f-g]$ ν-a.e.

Finally, note that $Osc[f-g] \leq 2T_*[f-g]$ thus $\{y : Osc[f](y) > \delta\} \subset \{y : 2T_*[f-g] > \delta\}$ and therefore

$$\begin{aligned} \nu(\{y : Osc[f](y) > \delta\}) &\leq \nu(\{y : T_*[f-g] > \delta/2\}) \\ &\leq \left(\frac{2B\|f-g\|_{L^p(\mu)}}{\delta}\right)^q \\ &\leq \left(\frac{2B\eta}{\delta}\right)^q. \end{aligned}$$

As $\eta > 0$ was arbitrary, we can let $\eta \to 0$ and obtain that (3.37) as desired. □

3.2. Lebesgue's Differentiation Theorem. We will first use the (centered) maximal function. Consider , $K(x) = |B(0,1)|^{-1}\mathbf{1}_{(0,1)}$ (which is associated to the radial decreasing function $k(r) = |B(0,1)|^{-1}\mathbf{1}_{[0,1]}(r)$ that has only one jump) and note that

$$K_\varepsilon(x) := \varepsilon^{-d}K(x/\varepsilon) = \frac{1}{|B(0,\varepsilon)|}\mathbf{1}_{B(0,\varepsilon)}(x).$$

Let T_ε be defined by $T_\varepsilon[f] = K_\varepsilon * f$ that is

$$T_\varepsilon[f](y) = \frac{1}{|B(0,\varepsilon)|}\int_{\mathbb{R}^d} f(x)\mathbf{1}_{B(0,\varepsilon)}(y-x)\,\mathrm{d}x = \frac{1}{|B(0,\varepsilon)|}\int_{B(y,\varepsilon)} f(x)\,\mathrm{d}x$$

is the mean of f over the ball centered at y of radius ε. As a consequence $T_* = M$ is the centered Hardy-Littlewood Maximal Function. In particular, we already know that T_* is of weak-type $(1,1)$ and of strong-type (p,p).

Next, let $\mathcal{D} = \mathcal{C}_c(\mathbb{R}^d)$ the set of continuous compactly supported functions. This set is dense in $L^p(\mathbb{R}^d)$ for $1 \leq p < +\infty$. Further, if $f \in \mathcal{C}_c(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, then for every $\eta > 0$, there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ and $x \in B(y,\varepsilon)$ then $|f(x) - f(y)| \leq \eta$. It follows that

$$\begin{aligned} |T_\varepsilon[f](y) - f(y)| &= \left|\frac{1}{|B(0,\varepsilon)|}\int_{B(y,\varepsilon)} f(x)\,\mathrm{d}x - \frac{1}{|B(0,\varepsilon)|}\int_{B(y,\varepsilon)} f(y)\,\mathrm{d}x\right| \\ &\leq \frac{1}{|B(0,\varepsilon)|}\int_{B(y,\varepsilon)} |f(x) - f(y)|\,\mathrm{d}x \leq \eta. \end{aligned}$$

As a consequence, $T_\varepsilon[f](y) \to f(y)$ when $\varepsilon \to 0$ (for all y). This is exactly the setting of the previous section with $T = Id$ and we have proven the following:

THEOREM 5.12 (Lebesgue Differentiation). *Let $1 \leq p < +\infty$ and $f \in L^p(\mathbb{R}^d)$. Then, for almost every $y \in \mathbb{R}^d$,*

$$\frac{1}{|B(0,\varepsilon)|}\int_{B(y,\varepsilon)} f(x)\,dx \to f(y) \tag{3.38}$$

when $\varepsilon \to 0$.

REMARK 5.13. A point such that (3.38) holds is called a *Lebesgue point* of f.

Note that if $f = g$ a.e. then $\dfrac{1}{|B(0,\varepsilon)|}\displaystyle\int_{B(y,\varepsilon)} f(x)\,\mathrm{d}x = \dfrac{1}{|B(0,\varepsilon)|}\displaystyle\int_{B(y,\varepsilon)} g(x)\,\mathrm{d}x$. In particular, Lebesgue points do not depend on the particular element of the equivalence class (a.e.) of f one takes.

EXERCICE 5.14. Show that the theorem still holds if $f \in L^1_{loc}(\mathbb{R}^d)$.
Hint: Apply the theorem to $f\mathbf{1}_{B(0,n)}$ and show that this determines all Lebesgue points of f in $B(0,n)$.

3.3. Limits of convolutions.

PROPOSITION 5.15. *Let $\varphi \in L^1(\mathbb{R}^d)$, $m = \displaystyle\int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}x$ and, for $t > 0$, $\varphi_t(x) = t^{-d}\varphi(x/t)$. Let $1 \leq p < +\infty$ and $f \in L^p(\mathbb{R}^d)$. Then*

(i) *$\varphi_t * h \in L^p(\mathbb{R}^d)$ with $\|\varphi_t * h\| \leq \|\varphi\|_1\|h\|_p$ and $\varphi_t * h \to mh$ in $L^p(\mathbb{R}^d)$ when $t \to 0$.*
For $p = +\infty$ the same is true provided (say) f is continuous with compact support.

(ii) *Assume further that the* least radial majorant *of φ, $\Phi(x) := \sup_{|y|\geq x} |\varphi(y)|$ is in $L^1(\mathbb{R}^d)$ then $\varphi_t * h \to mf$ a.e. when $t \to 0$.*

PROOF. The inequality $\|\varphi_t * h\| \leq \|\varphi\|_1\|h\|_p$ is Young's inequality since $\|\varphi_t\|_1 = \|\varphi\|_1$. We then note that

$$\varphi_t * h(x) - mh(x) = \int_{\mathbb{R}^d} \varphi_t(y)\big(h(x-y) - h(x)\big)\,\mathrm{d}y = \int_{\mathbb{R}^d} \varphi(y)\big(h(x-ty) - h(x)\big)\,\mathrm{d}y$$

thus

$$\|\varphi_t * h - mh\|_p \leq \int_{\mathbb{R}^d} |\varphi(y)|\|\tau_{ty}h - h\|_p\,\mathrm{d}y.$$

As for all $y \neq 0$, $\|\tau_{ty}h - h\|_p \to 0$ when $t \to 0$ and $|\varphi(y)|\|\tau_{ty}h - h\|_p \leq 2\|h\|_p|\varphi(y)| \in L^1(\mathbb{R})$ we get the first statement from dominated convergence.

The second statement follows from the general principle and the fact that $|\varphi_t * h(x)| \leq |\|\Phi\|_1 M[f](x)$. The details are left to the reader. □

3.4. Harmonic functions on the upper half plane and Poisson integrals. In this section, we will consider $\Omega = \mathbb{R}^{d+1}_+ = \{(x,t)\ :\ x \in \mathbb{R}^d,\ t > 0\}$ and its boundary $\partial\Omega = \mathbb{R}^d$ (identified with the set of vectors of the form $(x,0)$, $x \in \mathbb{R}^d$). The Laplace operator on $\mathbb{R}^{d+1}_+$ is the operator

$$\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial t^2}.$$

A function $f \in \mathcal{C}^2(\mathbb{R}^{d+1}_+)$ is said to be *harmonic* if $\Delta f = 0$. We will be dealing with the *Dirichlet Problem*, $u \in \mathcal{S}(\bar\omega)$

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u(x,0) = f(x) & x \in \mathbb{R}^d \end{cases}.$$

Recall the following:

DEFINITION 5.16. The *Poisson kernel* of the upper half spacs $\mathbb{R}^{d+1}_+$ for is defined by

$$P(x,t) = c_d\frac{t}{(t^2 + |x|^2)^{(d+1)/2}} \qquad x \in \mathbb{R}^d, t > 0$$

where $c_d = \dfrac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}}$.

The key properties we need here are the following:
– $P(x,t) = t^{-d}P_d(x/t)$ where P_d is defined in (2.29);
– $P_d \in L^1(\mathbb{R}^d)$ and $\displaystyle\int_{\mathbb{R}^d} P_d(x)\,\mathrm{d}x = 1$;
– $\Delta P = 0$ where Δ is the Laplace operator on $\mathbb{R}^{d+1}_+$.

The first two properties are obvious or already proven. The last property is a direct computation (though we have already proven it via the Fourier transform)

$$\partial_t P(t,x) = c_d \frac{1}{(t^2+|x|^2)^{(d+1)/2}} - (d+1)c_d \frac{t^2}{(t^2+|x|^2)^{(d+3)/2}}$$

thus

$$\begin{aligned}\partial_t^2 P(t,x) &= -c_d(d+1)\frac{3t}{(t^2+|x|^2)^{(d+3)/2}} + c_d(d+1)(d+3)\frac{t^3}{(t^2+|x|^2)^{(d+5)/2}} \\ &= \frac{c_d(d+1)}{(t^2+|x|^2)^{(d+5)/2}} t(dt^2-3|x|^2)\end{aligned}$$

while

$$\partial_{x_j} P(t,x) = -c_d(d+1)\frac{x_j t}{(t^2+|x|^2)^{(d+3)/2}}$$

thus

$$\partial_{x_j}^2 P(t,x) = -c_d(d+1)\frac{t}{(t^2+|x|^2)^{(d+3)/2}} + c_d(d+1)(d+3)\frac{tx_j^2}{(t^2+|x|^2)^{(d+5)/2}}$$

and, summing from $j=1$ to d,

$$\begin{aligned}\sum_{j=1}^d \partial_{x_j}^2 P(t,x) &= -c_d d(d+1)\frac{t}{(t^2+|x|^2)^{(d+3)/2}} + c_d(d+1)(d+3)\frac{t|x|^2}{(t^2+|x|^2)^{(d+5)/2}} \\ &= \frac{c_d(d+1)}{(t^2+|x|^2)^{(d+5)/2}} t(-dt^2+3|x|^2)\end{aligned}$$

and the fact that $\Delta P = 0$ follows.

We can now prove the main result of this section:

THEOREM 5.17. *Let $1 \le p < +\infty$ and $f \in L^p(\mathbb{R}^d)$. Let*

$$u(x,t) = P(t,\cdot) * f(x) = \int_{\mathbb{R}^d} P(x-y,t) f(y)\, dy$$

be the Poisson extension *of f. Then $u \in \mathcal{C}^2(\mathbb{R}^{d+1}_+)$ with $\Delta u = 0$ in $\mathbb{R}^{d+1}_+$ and, for almost every $x \in \mathbb{R}^d$, $u(x,t) \to f(x)$ when $t \to 0$.*

PROOF. As $P(x,t) = t^{-d} P_d(x/t)$, for t fixed, $P(\cdots,t) \in L^1(\mathbb{R}^d)$ so that u is well defined. Further, the family $\{P(\cdot,t) : t > 0\}$ is an approximation of the identity so that $u(\cdots,t) \to f$ in $L^p(\mathbb{R}^d)$.

The previous computations show that $\partial_t P, \partial_t^2 P, \partial_{x_j} P, \partial_{x_j}^2 P \in L^2(\mathbb{R}^d)$ so that there is no difficulty in applying Lebesgue's theorem to show that $\Delta u = 0$.

Finally, the properties of P show that we are in the framework of Section 3.1 so that $u(\cdot,t) \to f$ a.e. □

REMARK 5.18. Note that this shows that if $f \in \mathcal{C}_c(\mathbb{R}^d)$ then $u(\cdots,t) \to f$ uniformly.

4. The Calderón-Zygmund decomposition

In the course on interpolation, we have already used several times a decomposition of a function $f \in L^1(\mathbb{R})$ in the form ($\lambda > 0$)

$$f(x) = f\mathbf{1}_{|f|\le\lambda} + f\mathbf{1}_{|f|>\lambda} := g + b.$$

The first piece, $g = f\mathbf{1}_{|f|\le\lambda}$ is the *good* part of f since $g \in L^1(\mathbb{R})$ and is also bounded,

$$\|g\|_1 \le \|f\|_1 \quad \text{and} \quad \|g\|_\infty \le \lambda.$$

The bad part $b = f\mathbf{1}_{|f|>\lambda}$ satisfies

$$\|b\|_1 \le \|f\|_1 \quad \text{and} \quad |\operatorname{supp} b| \le \frac{\|f\|_1}{\lambda}$$

with Markov's inequality.

For general measure spaces, this is the best one can hope for. But in $\mathbb{R}^d$ one can get a better decomposition of the function b. To do so, we will use dyadic cubes:

DEFINITION 5.19. A *dyadic cube* is a subset of $\mathbb{R}^d$ of the form

$$Q = Q_{k,m} := [2^k m_1, 2^k(m_1+1)[\times[2^k m_2, 2^k(m_2+1)[\times\cdots\times[2^k m_d, 2^k(m_d+1)[$$

where $k \in \mathbb{Z}$ and $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$.

We denote by $\mathcal{D}$ the set of dyadic cubes and define the *dyadic maximal function* a

$$M^d[f](x) = \sup_{x\in Q\in\mathcal{D}} \frac{1}{|Q|}\int_Q |f(y)|\,\mathrm{d}y.$$

One easily checks that $|Q_{k,m}| = 2^{kd}$ and that if Q, Q' are two diyadic cubes with $|Q| \leq |Q'|$ then either Q and Q' are disjoint or $Q \subset Q'$. The dyadic cubes thus enjoy a tree structure. We call k the generation of the cube and $\mathcal{D}_k$ the set of cubes of generation k and note that this is a covering of $\mathbb{R}^d$. Every cube Q of generation k can be divided into 2^d cubes of generation $k+1$ which we call the daughters of Q. In the opposite direction, there is a unique cube Q' of generation $k+1$ such that $Q \subset Q'$ and Q' is called the mother of Q.

We leave as an exercice to show that there are constants a, b, c (depending on the dimension d) such that for every $\lambda > 0$,

$$|\{x : M^d[f](x) > a\lambda\}| \leq |\{x : M[f](x) > \lambda\}| \leq b|\{x : M^d[f](x) > c\lambda\}|.$$

In particular, M^d is of weak-type $(1,1)$ and of strong type (p,p) for every $p > 1$.

THEOREM 5.20 (Calderón-Zygmund Decomposition). *Let $f \in L^1(\mathbb{R}^d)$ and $\alpha > 0$. Then there exists functions g and b such that*

(1) $f = g + b$;
(2) $\|g\|_1 \leq \|f\|_1$ *and* $\|g\|_\infty \leq 2^d\alpha$;
(3) *the function b may be written as* $b = \sum_j b_j$ *where*
   (a) *each b_j is supported in a dyadic cube Q_j;*
   (b) *if $j \neq k$, $Q_j \cap Q_k = \emptyset$;*
   (c) $\displaystyle\int_{Q_j} b_j(x)\,\mathrm{d}x = 0$;
   (d) $\|b_j\|_1 \leq 2^{d+1}\alpha|Q_j|$;
   (e) $\displaystyle\sum_j |Q_j| \leq \frac{1}{\alpha}\|f\|_1$.

This decomposition of f is called the Calderón-Zygmund Decomposition *of f at scale (or level) α.*

REMARK 5.21. The function g is called the *good* function as it is in every L^p space with $\|g\|_p^p \leq \|g\|_1\|g\|_\infty^{p-1} \leq 2^{d(p-1)}\alpha^{p-1}\|f\|_1$.

The function b is the *bad* part, it satifies $\|b\|_1 \leq \|f\|_1 + \|g\|_1 \leq 2\|f\|_1$ and contains the singularities of f but has *mean zero*. We can't expect any L^p-regularity for this function.

PROOF. We first chose the smallest k such that $2^{kd} \geq \frac{1}{\alpha}\|f\|_1$. Now for each cube $Q \in \mathcal{D}_k$, we consider its 2^d daugthers. Such a daughter $\tilde{Q}$ will be added to the set of selected cubes S_1 if

$$\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |f(x)|\,\mathrm{d}x > \alpha.$$

For each $Q \in \mathcal{D}_{k-1} \setminus S_1$, we repeat the operation and consider its daughters. Such a daughter $\tilde{Q}$ will be added to the set of selected cubes S_2 if

$$\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |f(x)|\,\mathrm{d}x > \alpha...$$

At each generation $k-j$, $j \geq 1$ we thus construct a set of selected cubes S_j such that if $\tilde{Q} \in S_j$, then

$$\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |f(x)|\,\mathrm{d}x > \alpha.$$

Next, notice that the set of selected cubes $\bigcup_{\ell\geq 1} S_\ell$ is countable and that two selected cubes are disjoint since once a cube is selected, none of its daughters can be selected and two cubes of the same generation are disjoint. We re-order the set of selected cubes as $\bigcup_{\ell\geq 1} S_\ell = \{Q_j\}_j$ which is precisely the set of disjoint cubes we are looking for. Note that this set may be empty in which case we set $b=0$ and $g=f$.

Now, for each j, we define

$$b_j = \left(f - \frac{1}{|Q_j|}\int_{Q_j} f(y)\,\mathrm{d}y\right)\mathbf{1}_{Q_j}$$

which is well defined since $f\in L^1(\mathbb{R}^d)$, is supported in Q_j and has clearly integral 0 (over Q_j which is its support thus over $\mathbb{R}^d$). We then set $b=\sum b_j$ (note that this sum is well defined since $\sum b_j(x)$ has at most one non-zero term since the b_j's have disjoint support) and $g=f-b$.

For each j, let $\tilde Q_j$ be the mother of Q_j. As $\tilde Q_j$ was *not* selected,

$$\frac{1}{|\tilde Q_j|}\int_{\tilde Q_j}|f(x)|\,\mathrm{d}x \leq \alpha.$$

It follows that

$$\frac{1}{|Q_j|}\int_{Q_j}|f(x)|\,\mathrm{d}x \leq \frac{1}{|Q_j|}\int_{\tilde Q_j}|f(x)|\,\mathrm{d}x = \frac{2^d}{|\tilde Q_j|}\int_{\tilde Q_j}|f(x)|\,\mathrm{d}x \leq 2^d\alpha.$$

But then

$$\int_{Q_j}|b_j(x)|\,\mathrm{d}x \leq \int_{Q_j}|f(x)|\,\mathrm{d}x + \int_{Q_j}\frac{1}{|Q_j|}\int_{Q_j}|f(y)|\,\mathrm{d}y \leq 2\int_{Q_j}|f(x)|\,\mathrm{d}x \leq 2^{d+1}\alpha|Q_j|.$$

On the other hand, as Q_j was selected,

$$|Q_j| \leq \frac{1}{\alpha}\int_{Q_j}|f(x)|\,\mathrm{d}x$$

therefore, using the fact that the Q_j's are disjoint,

$$\sum|Q_j| \leq \frac{1}{\alpha}\int_{\bigcup Q_j}|f(x)|\,\mathrm{d}x \leq \frac{1}{\alpha}\|f\|_1.$$

It remains to prove the estimate of g. We obviously have

$$g(x) = \begin{cases} f(x) & \text{for } x\in\mathbb{R}^d\setminus\bigcup Q_j \\ \frac{1}{|Q_j|}\int_{Q_j} f(y)\,\mathrm{d}y & \text{for } x\in Q_j \end{cases}.$$

Note that this is well defined as x can only belong to at most one Q_j. A direct consequence is that

$$\begin{aligned}
\int_{\mathbb{R}^d}|g(x)|\,\mathrm{d}x &= \int_{\mathbb{R}^d\setminus\bigcup Q_j}|g(x)|\,\mathrm{d}x + \sum_j\int_{Q_j}|g(x)|\,\mathrm{d}x \\
&\leq \int_{\mathbb{R}^d\setminus\bigcup Q_j}|f(x)|\,\mathrm{d}x + \sum_j\int_{Q_j}\frac{1}{|Q_j|}\int_{Q_j}|f(y)|\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d\setminus\bigcup Q_j}|f(x)|\,\mathrm{d}x + \sum_j\int_{Q_j}|f(y)|\,\mathrm{d}y = \|f\|_1.
\end{aligned}$$

Further, we have already shown that $\frac{1}{|Q_j|}\int_{Q_j}|f(y)|\,\mathrm{d}y \leq 2^d\alpha$ so that $|g|\leq 2^d\alpha$ on $\bigcup Q_j$. It remains to prove the same estimate on $R^d\setminus\bigcup Q_j$.

But, for each $j\geq 1$, there is a unique cube in $\mathcal{D}_{k-j}$ to which x belongs and this cube has *not* been selected. Call it $Q_x^{(j)}$ and note that

$$\left|\frac{1}{|Q_x^{(j)}|}\int_{Q_x^{(j)}} f(y)\,\mathrm{d}y\right| \leq \frac{1}{|Q_x^{(j)}|}\int_{Q_x^{(j)}}|f(y)|\,\mathrm{d}y \leq \alpha.$$

Further, the diameter of $Q_x^{(j)}$ goes to 0 as $j\to+\infty$ and each of these cubes contains x. So the intersection of its closures is reduced to a single point which can only be x. Using Lebesgue's

Differentiation Theorem, we conclude that, for almost every $x \in R^d \setminus \bigcup Q_j$,

$$f(x) = \lim_{j \to +\infty} \frac{1}{|Q_x^{(j)}|} \int_{Q_x^{(j)}} f(y)\,\mathrm{d}y$$

thus $|f(x)| \leq \alpha$. □

The fact that Lebesgue's Differentiation Theorem holds for dyadic cubes is a direct consequence of the weak-type $(1,1)$ property of M^d.

CHAPTER 6

The Hilbert and Newton transforms

1. The Hilbert transform

1.1. The conjugate Poisson kernel. In this section we will identify $\mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ with the upper half-plane $\mathbb{R}^2_+$ in the usual way. For $f \in L^p(\mathbb{R})$ $(1 < p < +\infty)$, real valued, the Poisson integral $U(x+it) = u(x,t) = P_t * f(x)$ is a real valued harmonic function on $\mathbb{C}^+$ and it is well known from your course on complex analysis that U is the real part of an holomorphic function, $U = \Re F$. We can thus write $F = U + iV$ with V harmonic and further V is unique up to a constant.

At least when $f \in \mathcal{S}(\mathbb{R})$, it is easy to explicitely determine V with the help of Fourier analysis. Indeed, $U(x+it) = P_t * f(x) = \mathcal{F}^{-1}[\widehat{P_t}\widehat{f}](x)$, that is

$$\begin{aligned} U(x+it) &= \int_{\mathbb{R}} e^{-2\pi t|\xi|}\widehat{f}(\xi)e^{2i\pi x\xi}\,\mathrm{d}\xi \\ &= \int_0^{+\infty} e^{2\pi\xi(x+it)}\widehat{f}(\xi)\,\mathrm{d}\xi + \int_{-\infty}^0 e^{2\pi\xi(x-it)}\widehat{f}(\xi)\,\mathrm{d}\xi. \end{aligned}$$

Now the first integral is holomorphic in $z = x+it$ while the second one is anti-holomorphic (of the form $G(\bar{z})$ with G holomoprhic) so that both are harmonic. One can thus chose

$$iV(x+it) = \int_0^{+\infty} e^{2i\pi\xi(x+it)}\widehat{f}(\xi)\,\mathrm{d}\xi - \int_{-\infty}^0 e^{2i\pi\xi(x-it)}\widehat{f}(\xi)\,\mathrm{d}\xi$$

which is harmonic and such that $U + iV$ is holomorphic. In other words

$$V(x+it) = \int_{\mathbb{R}} -i\,\mathrm{sign}(\xi)e^{-2\pi t|\xi|}\widehat{f}(\xi)e^{2i\pi x\xi}\,\mathrm{d}\xi = \int_{\mathbb{R}} -i\,\mathrm{sign}(t\xi)e^{-2\pi t|\xi|}\widehat{f}(\xi)e^{2i\pi x\xi}\,\mathrm{d}\xi.$$

Using Fourier inverstion again, $V(x+it) = Q_t * f(x)$ with $Q_t(x) = t^{-1}Q_1(x/t)$ and $\widehat{Q_1}(\xi) = -i\,\mathrm{sign}(\xi)e^{-2\pi|\xi|}$.

It is not difficult to compute the inverse Fourier transform of $\widehat{Q_1}$ and to obtain

$$\begin{aligned} \mathcal{F}^{-1}[\widehat{Q_1}](x) &= \int_{\mathbb{R}} -i\,\mathrm{sign}(\xi)e^{-2\pi|\xi|}e^{2i\pi x\xi}\,\mathrm{d}\xi \\ &= \left(\int_{-\infty}^0 e^{2\pi\xi(1+ix)}\,\mathrm{d}\xi - \int_0^{+\infty} e^{2\pi\xi(-1+ix)}\,\mathrm{d}\xi\right) \\ &= i\left(\frac{1}{1+ix} + \frac{1}{-1+ix}\right) = \frac{1}{\pi}\frac{x}{x^2+1}. \end{aligned}$$

One thus obtains an $L^2(\mathbb{R})$-function that is not in $L^1(\mathbb{R})$. However, L^2-Fourier inversion shows that the L^2-Fourier transform of $Q_1 = \dfrac{1}{\pi}\dfrac{x}{x^2+1}$ is $-i\,\mathrm{sign}(\xi)e^{-2\pi|\xi|}$.

We can now introduce the following:

Definition 6.1. The Poisson kernel on $\mathbb{R}^2_+$ is given by

$$P_t(x) = \frac{1}{\pi}\frac{t}{x^2+t^2}$$

and the *conjugate Poisson kernel* on $\mathbb{R}^2_+$ is given by

$$Q_t(x) = \frac{1}{\pi}\frac{x}{x^2+t^2}.$$

Observe the following facts:

– $P_t(x) = t^{-1}P_1(x/t)$ and $Q_t = t^{-1}Q_1(x/t)$;

– $P_1 \in L^1(\mathbb{R})$, $Q_1 \notin L^1(\mathbb{R})$ but $Q_1 \in L^q(\mathbb{R})$ for all $q > 1$;

– $\widehat{P_t}(\xi) = e^{-2\pi t|\xi|}$ (L^1-Fourier transform) and $\widehat{Q_t}(\xi) = -i\,\mathrm{sign}(\xi)e^{-2\pi t|\xi|}$ (L^2-Fourier transform).

From Young's inequality, we may thus define $Q_t * f$ when $f \in L^p(\mathbb{R})$, $1 \leq p < +\infty$ and obtain a function in $L^r(\mathbb{R})$ with $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q} - 1$. Note that the condition $\dfrac{1}{r} \geq 0$ requires to chose $\dfrac{1}{q} \geq 1 - \dfrac{1}{p} = \dfrac{1}{p'}$ that is $q \leq p'$.

Next, observe that

$$P_t + iQ_t = \frac{1}{\pi}\frac{t+ix}{x^2+t^2} = \frac{i}{\pi}\frac{x-it}{|x+it|^2} = \frac{i}{\pi}\frac{1}{x+it}$$

so that

$$P_t * f(x) + iQ_t * f(x) = \frac{i}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x+it-y}\,\mathrm{d}y$$

is indeed holomorphic in $z = x + it$, provided the integral converges.

Definition 6.2. For $f \in L^p(\mathbb{R})$, $1 \leq p < +\infty$, we define

– the *conjugate Poisson integral* of f on $\mathbb{R}^2_+$ by

$$Q[f](x,t) = Q_t * f(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{x-y}{(x-y)^2+t^2} f(y)\,\mathrm{d}y$$

– the *Cauchy-transform* of f on $\mathbb{C} \setminus \mathbb{R}$ by

$$C[f](z) = \frac{1}{2i\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y+it}\,\mathrm{d}y.$$

The above computations show that

$$C[f](x+it) = \begin{cases} \frac{1}{2}\big(P[f](x,t) + iQ[f](x,t)\big) & \text{when } t > 0 \\ \frac{1}{2}\big(-P[f](x,-t) + iQ[f](x,-t)\big) & \text{when } t < 0 \end{cases}.$$

We now want to study the convergence of $Q_t[f]$ when $t \to 0$. We can no longer apply the same theory as for the Poisson kernel since

$$\lim_{t\to 0} Q_t = \frac{1}{\pi x} \notin L^p(\mathbb{R})$$

for any p. Actually, it is not even a tempered distribution.

On the other hand $\lim_{t\to 0}\widehat{Q_t}(\xi) = -i\,\mathrm{sign}(\xi)$ so that, applying Parseval twice, if $f \in L^2(\mathbb{R})$ then $-i\,\mathrm{sign}(\xi)\widehat{f} \in L^2(\mathbb{R})$ and the operator

$$f \to \mathcal{F}^{-1}[-i\,\mathrm{sign}(\xi)\widehat{f}]$$

is bounded $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

This is the operator we are going to study now

1.2. The Hilbert transform.

Definition 6.3. The *Hilbert transform* is the operator $L^2(\mathbb{R})L^2(\mathbb{R})$ defined by

$$\widehat{Hf}(\xi) = -i\,\mathrm{sign}(\xi)\widehat{f}(\xi)$$

that is

$$Hf(x) = \lim_{R\to+\infty}\frac{1}{2\pi}\int_{-R}^{R} -i\,\mathrm{sign}(\xi)\widehat{f}(\xi)e^{ix\xi}\,\mathrm{d}\xi$$

where the limit is in the $L^2(\mathbb{R})$-sense.

Proposition 6.4. *The Hilbert transform has the following properties:*

(i) *For $f \in L^2(\mathbb{R})$, $Hf = \lim_{t\to 0} Q_t * f$.*

(ii) $\|Hf\|_2 = \|f\|_2$, $H^* = H^{-1} = -H$ *and* $H^2 = -I$.

(iii) $Q_t[f] = P_t[Hf]$ *and* $Q_t[f](x) \to Hf(x)$ *in* L^2 *and a.e. when* $t \to 0$.
(iv) *Let* $\tau_a f(x) = f(x-a)$ *the translations* ($a \in \mathbb{R}$), $\delta_s f(x) = f(sx)$ *the positive dilations* ($s > 0$) *and* $Rf(x) = f(-x)$ *the reflection. Then* H *commutes with the translations and positive dilations:* $H[\tau_a f] = \tau_a H[f]$, $H[\delta_s f] = \delta_s H[f]$ *and anti-commutes with the reflection* $RH[f] = -H[Rf]$.
(v) *If a bounded operator* $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *commutes with the translations and positive dilations and anti-commutes with the reflection, then it is a multiple of the Hilbert transform.*

Proof. For (i), we have already seen that $Q_t * f$ is well defined and in $L^2(\mathbb{R})$ when $f \in L^2(\mathbb{R})$ through Hausdorff-Young and that

$$\widehat{Q_t * f}(\xi) = -i\,\mathrm{sign}(\xi)e^{-2\pi t|\xi|}\widehat{f}(\xi).$$

Now $|-i\,\mathrm{sign}(\xi)e^{-2\pi t|\xi|}\widehat{f}(\xi)| \le |\widehat{f}(\xi)| \in L^2(\mathbb{R})$ and $-i\,\mathrm{sign}(\xi)e^{-2\pi t|\xi|}\widehat{f}(\xi) \to -i\,\mathrm{sign}(\xi)\widehat{f}(\xi)$ a.e. when $t \to 0$ so that dominated convergence immediately shows that $\widehat{Q_t * f} \to -i\,\mathrm{sign}(\xi)\widehat{f} = \widehat{Hf}$ in $L^2(\mathbb{R})$ when $t \to 0$. Using the continuity of the inverse Fourier transform (Parseval) we get that $Q_t * f \to Hf$ in $L^2(\mathbb{R})$ when $t \to 0$.

For (ii), the first two are just Parseval:

$$\|Hf\|_2^2 = \|\widehat{Hf}\|_2^2 = \|-i\,\mathrm{sign}(\xi)\widehat{f}\|_2^2 = \|\widehat{f}\|_2^2 = \|f\|_2^2$$

while

$$\begin{aligned}\langle Hf, g\rangle &= \langle \widehat{Hf}, \widehat{g}\rangle = \int_{\mathbb{R}} -i\,\mathrm{sign}(\xi)\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,\mathrm{d}\xi \int_{\mathbb{R}} \widehat{f}(\xi)\overline{i\,\mathrm{sign}(\xi)\widehat{g}(\xi)}\,\mathrm{d}\xi \\ &= \langle \widehat{f}, -\widehat{Hg}\rangle = \langle f, -Hg\rangle\end{aligned}$$

that is $H^* = -H$

On the other hand, $Hf = g$ if and only if $\widehat{Hf} = \widehat{g}$ that is $-i\,\mathrm{sign}(\xi)\widehat{f}(\xi) = \widehat{g}(\xi)$ or, equivalently $\widehat{f}(\xi) = i\,\mathrm{sign}(\xi)\widehat{g}(\xi) = -\widehat{Hg}$ that is $f = -Hg$. Thus $H^{-1} = -H$ as well. As H can be defined as a Fourier multiplier $\widehat{H[f]} = -i\,\mathrm{sign}(\xi)\widehat{f}$, we get

$$\widehat{H^2[f]} = -i\,\mathrm{sign}(\xi)\widehat{H[f]} = \big(-i\,\mathrm{sign}(\xi)\big)^2\widehat{f} = -\widehat{f}$$

thus $H^2 = -I$.

For (iii), we first notice that if $f \in L^2(\mathbb{R})$ then $Q_t[f] = P_t[Hf]$ coincide since both have as Fourier transform $-i\,\mathrm{sign}(\xi)e^{-2\pi t|\xi|}f(\xi)$. Then, as $Hf \in L^2(\mathbb{R})$, we have already shown that $P_t[Hf] \to Hf$ in L^2 and a.e. when $t \to 0$ so the result follows for $Q_t[f]$.

The last two are left as an exercice. One should first show that if a bounded operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ commutes with the translations then there exists $m \in L^\infty$ such that $\widehat{Tf} = m\widehat{f}$. □

We will now give a more direct definition of the Hilbert transform. One would of course like to define $Hf = \dfrac{1}{\pi x} * f$ but this is not possible since $\dfrac{1}{\pi x} \notin L^1(\mathbb{R})$ and even the more general Sobolev inequality does not apply.

To overcome this issue, let us introduce

Definition 6.5. The *principal value* distribution associated to $1/x$

$$\left\langle vp\frac{1}{x}, \varphi\right\rangle = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x, \qquad \varphi \in \mathcal{S}(\mathbb{R}).$$

The first step is to notice that, as $1/x$ is odd, this is well defined. Indeed, let χ be an even function supported in $[-2,2]$ and such that $\chi(x) = 1$ on $[-1,1]$. Then

$$\int_{|x|>\varepsilon} \frac{\chi(x)}{x}\,\mathrm{d}x = \int_{-2}^{-\varepsilon} \frac{\chi(x)}{x}\,\mathrm{d}x + \int_{\varepsilon}^{2} \frac{\chi(x)}{x}\,\mathrm{d}x = 0$$

after changing variable $x \to -x$ in the first integral. It follows that

$$\int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x = \int_{\varepsilon<|x|<2} \frac{\varphi(x) - \varphi(0)\chi(x)}{x}\,\mathrm{d}x + \int_{|x|>2} \frac{\varphi(x)}{x}\,\mathrm{d}x.$$

It remains to notice that

– the second integral is absolutely convergent when $\varphi\in\mathcal{S}(\mathbb{R})$ since $x^{-1}\varphi(x)=O(x^{-2})$ in this case.

– as φ is smooth, $\dfrac{\varphi(x)-\varphi(0)\chi(x)}{x}$ extends into a continuous function over $[-2,2]$ so that $\displaystyle\int_{-2}^{2}\frac{\varphi(x)-\varphi(0)\chi(x)}{x}\,\mathrm{d}x$ is absolutely convergent thus

$$\int_{\varepsilon<|x|<2}\frac{\varphi(x)-\varphi(0)\chi(x)}{x}\,\mathrm{d}x\to\int_{-2}^{2}\frac{\varphi(x)-\varphi(0)\chi(x)}{x}\,\mathrm{d}x$$

when $\varepsilon\to 0$.

It is not difficult to slightly modify this argument to show that $\operatorname{pv}\dfrac{1}{x}$ is a distribution of order 1 with

$$\left|\left\langle \operatorname{pv}\frac{1}{x},\varphi\right\rangle\right|\leq\sup_{x\in\mathbb{R}}|xf(x)|+4\sup_{x\in\mathbb{R}}|f'(x)|.$$

Next, we can fix $f\in\mathcal{S}(\mathbb{R})$, $x\in\mathbb{R}$ and apply this to $\varphi(y)=f(x-y)$. This shows that

$$\operatorname{pv}\frac{1}{x}*f(x):=\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{f(x-y)}{y}\,\mathrm{d}y$$

is well defined pointwise. Our aim is to show that this coincides with the Hilbert transform:

PROPOSITION 6.6. *For $f\in L^2(\mathbb{R})$, the limit*

$$\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{f(x-y)}{y}\,dy$$

exists both in $L^2(\mathbb{R})$ and almost everywhere and is equal to πHf. In other words, the Hilbert transform *can also be defined as*

$$Hf(x)=\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|y|>\varepsilon}\frac{f(x-y)}{y}\,dy$$

where the limit exists in $L^2(\mathbb{R})$ and almost everywhere.

PROOF. Let us introduce $h_t(x)=\dfrac{1}{\pi x}\mathbf{1}_{|x|>t}$ and note that $h_t(x)=t^{-1}h_1(x/t)$. Define $H_t[f]=h_t*f$ and observe that as $h_t\in L^2(\mathbb{R})$,

$$H_t[f](x)=\int_{|y|>t}\frac{1}{y}f(x-y)\,\mathrm{d}y=\int_{\mathbb{R}}h_t(y)f(x-y)\,\mathrm{d}y$$

is well defined for every $x\in\mathbb{R}$ when $f\in L^2(\mathbb{R})$.

We now consider $\Psi_t[f]=Q_t[f]-H_t[f]=\varphi_t*f$ with

$$\varphi_t(x)=Q_t(x)-h_t(x)=\frac{1}{t}\big(Q_1(x/t)-h_1(x/t)\big)=\frac{1}{t}\varphi_1(x/t)$$

where

$$\varphi_1(u)=\frac{1}{\pi}\left(\frac{x}{x^2+1}-\frac{1}{x}\mathbf{1}_{|x|>1}\right)=\begin{cases}\dfrac{1}{\pi}\dfrac{x}{x^2+1} & \text{when } |x|<1\\[2mm] \dfrac{1}{\pi}\dfrac{-1}{x(1+x^2)} & \text{when } |x|\geq 1\end{cases}.$$

One can then note that φ_1 is odd and has thus mean $m=0$. Further, its radial majorant is easily computed

$$\Phi(x)=\sup_{|y|\geq x}|\varphi(y)|=\begin{cases}\dfrac{1}{2\pi} & \text{if } |x|\leq 1\\ |\varphi(x)| & \text{if } |x|\geq 1\end{cases}$$

since $|\varphi|$ is continuous and increases from 0 to $1/2\pi$ on $[0,1]$ while it is decreasing after 1. In particular, $\Phi\in L^1(\mathbb{R})$. Applying Proposition 5.15, we get that $\Psi_t[f]\to 0$ in $L^2(\mathbb{R})$ and a.e. when $t\to 0$. But we already know that $Q_t[f]\to Hf$ both in L^2 and a.e. so that the same holds for $H_t[f]$. □

To conclude this section, we will show two further applications of the Hilbert transform, the first one connects the transform to complex analysis:

THEOREM 6.7 (Plemelj Formula). *Let $f \in \mathcal{C}^1(\mathbb{R})$ be such that $f(x) = O(x^{-1})$ when $x \to \pm\infty$, then*

$$\frac{1}{2i\pi}\lim_{\varepsilon\to 0}\int_{\mathbb{R}} \frac{f(y)}{y-(x\pm i\varepsilon)}\,\mathrm{d}y = \frac{\pm f(x) + iHf(x)}{2}.$$

Note that the conditions on f are just to make everything defined pointwise. Boundedness considerations and a density argument allow to extend this formula in various settings.

PROOF. First, a change of variable allows to only consider the case $x = 0$. Also, we only consider the + sign in front of ε. We thus want to prove that

$$\frac{1}{2i\pi}\lim_{\varepsilon\to 0}\int_{\mathbb{R}} \frac{f(y)}{y-i\varepsilon}\,\mathrm{d}y = \frac{f(0)+iHf(x)}{2} = \frac{f(0)}{2} + \lim_{\varepsilon\to 0}\frac{i}{2\pi}\int_{|y|>\varepsilon}\frac{f(y)}{y}\,\mathrm{d}y$$

that is (changing variable $y = \varepsilon w$ in the integrals and multplying by $2i\pi$)

$$\int_{\mathbb{R}} f(\varepsilon w)\left(\frac{1}{w-i} - \frac{\mathbf{1}_{|w|\geq 1}}{w}\right)\mathrm{d}y - i\pi f(0) \to 0.$$

Now let $\varphi(w) = \dfrac{1}{w-i} - \dfrac{\mathbf{1}_{|w|\geq 1}}{w}$ and note φ is bounded while for $|w| > 1$, $\varphi(w) = \frac{i}{w(w-i)}$ which is integrable at infinity, thus $\varphi \in L^1(\mathbb{R})$. Further, decomposing φ as a sum of an even and an odd integrable function

$$\varphi(w) = \frac{1}{w-i} - \frac{\mathbf{1}_{|w|\geq 1}}{w} == \frac{i}{w^2+1} + \frac{w}{w^2+1} - \frac{\mathbf{1}_{|w|\geq 1}}{w}$$

we get

$$\int_{\mathbb{R}} \varphi(w)\,\mathrm{d}w = i\int_{\mathbb{R}}\frac{\mathrm{d}w}{w^2+1} = i\pi.$$

It follows that

$$\int_{\mathbb{R}} f(\varepsilon w)\left(\frac{1}{w-i} - \frac{\mathbf{1}_{|w|\geq 1}}{w}\right)\mathrm{d}y - i\pi f(0) = \int_{\mathbb{R}}\big(f(\varepsilon w) - f(0)\big)\left(\frac{1}{w-i} - \frac{\mathbf{1}_{|w|\geq 1}}{w}\right)\mathrm{d}y.$$

As $f(\varepsilon w) - f(0) \to 0$ and is bounded by $2\|f\|_\infty < +\infty$ the conclusion follows from dominated convergence. □

Assume now that f further extends to an holomorphic function in the upper-half plane $\mathbb{C}^+$ with a decay bound $|f(z)| \leq \dfrac{C}{1+|z|}$. Cauchy's Formula then gives

$$\frac{1}{2i\pi}\int_{\mathbb{R}}\frac{f(y)}{y-(x+i\varepsilon)}\,\mathrm{d}y = f(x+i\varepsilon) \to f(x)$$

when $\varepsilon \to 0$ so that Plemelj's Formula shows that $f = \frac{f+iH}{2}$ *i.e.* $Hf = -if$. In particular, comparing real and imaginary parts, we get

$$\Re f = -H\,\mathrm{Im}\,f \quad \text{and} \quad \mathrm{Im}\,f = H\Re f.$$

Let us conclude this section with the link between Hilbert transform and Fourier integrals: define the modulation operator $M_a f(x) = e^{2i\pi a x}f(x)$. Note that $\widehat{M_a f}(\xi) = \widehat{f}(\xi - a)$.

Next observe that if $a < b$

$$\mathrm{sign}(\xi-a) - \mathrm{sign}(\xi-b) = \begin{cases} -1+1=0 & \text{if } \xi < a < b \\ 1-1=0 & \text{if } \xi > b > a \\ 1+1=2 & \text{if } a < \xi < b \end{cases} = 2\mathbf{1}_{(a,b)}(\xi)$$

so that

$$\begin{aligned}\int_a^b \widehat{f}(\xi)e^{2i\pi x\xi}\,\mathrm{d}\xi &= \frac{1}{2}\int_{\mathbb{R}}\mathrm{sign}(\xi-a)\widehat{f}(\xi)e^{2i\pi x\xi}\,\mathrm{d}\xi - \frac{1}{2}\int_{\mathbb{R}}\mathrm{sign}(\xi-b)\widehat{f}(\xi)e^{2i\pi x\xi}\,\mathrm{d}\xi \\ &= \frac{e^{2i\pi ax}}{2}\int_{\mathbb{R}}\mathrm{sign}(\xi)\widehat{f}(\xi+a)e^{2i\pi ix\xi}\,\mathrm{d}\xi - \frac{e^{2i\pi bx}}{2}\int_{\mathbb{R}}\mathrm{sign}(\xi)\widehat{f}(\xi+b)e^{2i\pi ix\xi}\,\mathrm{d}\xi \\ &= -\frac{e^{2i\pi ax}}{2i}\int_{\mathbb{R}} -i\,\mathrm{sign}(\xi)\widehat{M_{-a}f}(\xi)e^{2i\pi x\xi}\,\mathrm{d}\xi + \frac{e^{2i\pi bx}}{2i}\int_{\mathbb{R}} -i\,\mathrm{sign}(\xi)\widehat{M_{-b}f}(\xi)e^{2i\pi x\xi}\,\mathrm{d}\xi\end{aligned}$$

so that Fourier inversion gives

$$\int_a^b \widehat{f}(\xi)e^{2i\pi\xi}\,\mathrm{d}\xi = \frac{1}{2i}\big(M_bH[M_{-b}f] - M_aH[M_{-a}f]\big).$$

Note that if $f\in\mathcal{S}(\mathbb{R})$, then

$$\lim_{R\to+\infty}\int_{-R}^{R}\widehat{f}(\xi)e^{2i\pi x\xi}\,\mathrm{d}\xi = f(x)$$

by Fourier inversion. If we are able to show that H is bounded $L^p(\mathbb{R})\to L^p(\mathbb{R})$ for some $p\geq 1$ (we already know this for $p=2$ and we will do so for all $1<p<+\infty$) then the family of operators

$$S_R[f](x) = \frac{1}{2\pi}\int_{-R}^{R}\widehat{f}(\xi)e^{ix\xi}\,\mathrm{d}\xi\frac{1}{2i}\big(M_RH[M_{-R}f] - M_{-R}H[M_Rf]\big)$$

is uniformly bounded. It follows from Banach-Steinhaus that $S_R[f]\to f$ in $L^p(\mathbb{R})$.

A difficult example of Kolmogorov shows that this is false in $L^1(\mathbb{R})$ but it is more easy to show that H is not bounded on $L^1(\mathbb{R})$:

EXERCICE 6.8. Compute $H\mathbf{1}_{(a,b)}$.

EXERCICE 6.9. Show that if $f\in\mathcal{S}(\mathbb{R})$ then

$$xHf(x)\to\frac{1}{\pi}\int_{\mathbb{R}}f(x)\,\mathrm{d}x$$

when $x\to+\infty$.

Conclude that H is not a bounded operator $L^1(\mathbb{R})\to L^1(\mathbb{R})$.

2. Newton potential

We continue with an operator that appeared in the introduction:

DEFINITION 6.10. Let $d\geq 3$.

The *Newton potential* is the function defined on $\mathbb{R}^d\setminus\{0\}$ by

$$\Gamma(x) = \frac{c_d}{|x|^{d-2}}\qquad c_d = \frac{1}{d(2-d)|B(0,1)|}$$

and the *Newton potential of a function* $f\in\mathcal{S}(\mathbb{R}^d)$ is given by the convolution with Γ:

$$\Gamma[f](x) = \int_{\mathbb{R}^d}\Gamma(x-y)f(y)\,\mathrm{d}y.$$

Note that, for $x\neq 0$ and $j,k\in\{1,\ldots,d\}$

$$\frac{\partial\Gamma}{\partial x_j}(x) = -(d-2)c_d\frac{x_j}{|x|^d}$$

and if $k\neq j$

$$\frac{\partial^2\Gamma}{\partial x_k\partial x_j}(x) = d(d-2)c_d\frac{x_jx_k}{|x|^{d+2}}$$

while

$$\frac{\partial^2\Gamma}{\partial x_j^2}(x) = (d-2)c_d\left(\frac{dx_j^2}{|x|^{d+2}} - \frac{1}{|x|^d}\right)$$

in particular $\Delta\Gamma=0$. Note that $\dfrac{\partial\Gamma}{\partial x_j}$ is integrable so that Lebesgue's Theorem shows that

$$\frac{\partial\Gamma[f]}{\partial x_j}(x) = -(d-2)c_d\int_{\mathbb{R}^d}\frac{x_j-y_j}{|x-y|^d}f(y)\,\mathrm{d}y.$$

The argument does not work for the second derivative for which the following formula is only *formal*:

$$\frac{\partial^2\Gamma[f]}{\partial x_k\partial x_j}(x) = d(d-2)c_d\int_{\mathbb{R}^d}\frac{(x_j-y_j)(x_k-y_k)}{|x-y|^{d+2}}f(y)\,\mathrm{d}y.$$

Instead we are now going to prove the following:

THEOREM 6.11. *Let $f\in\mathcal{S}(\mathbb{R}^d)$ then, for every $x\in\mathbb{R}^d$ and every $1\leq j,k\leq d$,*

$$\frac{\partial^2\Gamma[f]}{\partial x_k\partial x_j}(x)=\left(2(d-2)c_d\int_{\mathbb{S}^{d-1}}y_jy_k\,\mathrm{d}\sigma_{d-1}(y)\right)f(x)\\ +d(d-2)c_d\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d\setminus B(x,\varepsilon)}\frac{(x_j-y_j)(x_k-y_k)}{|x-y|^{d+2}}f(y)\,\mathrm{d}y.$$

In particular, this limit exists for every x.

PROOF. Throughout the proof $f\in\mathcal{S}(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$ are fixed.

It is enough to prove the formula for $k=1$. To simplify notation, we write $\partial=\dfrac{\partial}{\partial x_1}$ and $\partial_j=\dfrac{\partial}{\partial x_j}$ Let $e=(1,0,\ldots,0)$ and notice that, for every $t\in\mathbb{R}$ and $R>2$,

$$\frac{\partial_j\Gamma[f](x+te)-\partial_j\Gamma[f](x)}{t}=\int_{\mathbb{R}^d}\frac{\partial_j\Gamma(x+te-y)-\partial_j\Gamma(x-y)}{t}f(y)\,\mathrm{d}y\\ =\int_{\mathbb{R}^d\setminus B(x,|t|R)}\left(\frac{\partial_j\Gamma(x-y+te)-\partial_j\Gamma(x-y)}{t}-\partial\partial_j\Gamma(x-y)\right)f(y)\,\mathrm{d}y\\ +\int_{\mathbb{R}^d\setminus B(x,|t|R)}\partial\partial_j\Gamma(x-y)f(y)\,\mathrm{d}y+\frac{1}{t}\int_{B(x,|t|R)}\partial_j\Gamma(x+te-y)f(y)\,\mathrm{d}y+\frac{1}{t}\int_{B(x,|t|R)}\partial_j\Gamma(x-y)f(y)\,\mathrm{d}y.$$

Note that the two first integrals are well defined when $f\in\mathcal{S}(\mathbb{R}^d)$ since Γ and its partial derivatives are bounded on the domain of integration.

Denote the 4 terms by $T^j_{R,t}$, $j=1,2,3,4$. The remaining of the proof consists in treating each of these terms in a specific claim. We will let $t\to 0$ and $R\to+\infty$. In some steps, those convergences can be independent and in some not. The simplest would be to chose $R=t^{-1/3}$ but this can make notation a bit heavy.

Claim 1. *We have* $\lim_{t\to 0}\lim_{R\to+\infty}T^1_{R,t}=0$.

Applying the mean value theorem twice, there are a $\theta_1,\theta_2\in[0,1]$ such that

$$\frac{\partial_j\Gamma(x-y+tey)-\partial_j\Gamma(x-y)}{t}-\partial\partial_j\Gamma(x-y)\\ =\partial\partial_j\Gamma(x-y+t\theta_1e)-\partial\partial_j\Gamma(x-y)=t\theta_1\partial^2\partial_j\Gamma(x-y+t\theta_1\theta_2e).$$

But, on one hand, there is a constant C such that $|\partial^2\partial_j\Gamma(u)|\leq C/|u|^{d+1}$, on the other hand if $R\geq 2$ then $|t\theta_1\theta_2e|\leq|t|\leq\frac{Rt}{2}$ thus, for $y\in\mathbb{R}^d\setminus B(x,|t|R)$ *i.e.* $|x-y|\geq|t|R$

$$|x-y+t\theta_1\theta_2e|\geq|x-y|-|t\theta_1\theta_2e|\geq|x-y|/2$$

so that

$$|\partial^2\partial_j\Gamma(x-y+t\theta_1\theta_2e)|\leq\frac{2^{d+1}C}{|x-y|^{d+1}}.$$

We have thus shown that

$$|T^1_{R,t}|\leq 2^{d+1}Ct\int_{\mathbb{R}^d\setminus B(x,|t|R)}\frac{|f(y)|}{|x-y|^{d+1}}\,\mathrm{d}y\leq 2^{d+1}C|t|\int_{\mathbb{R}^d}h(|x-y|)|f(y)|\,\mathrm{d}y$$

where $h(r)=\begin{cases}r^{-d-1} & \text{if } r>|t|R\\ (|t|R)^{-d-1} & \text{if } r\leq|t|R\end{cases}$. Note that h is decrasing and that $g(x)=h(|x|)$ satifies

$$\|g\|_1=\sigma(\mathbb{S}^{d-1})\int_0^{+\infty}h(r)r^{d-1}\,\mathrm{d}r=\sigma(\mathbb{S}^{d-1})\left(\frac{(|t|R)^d}{d(|t|R)^{d+1}}+\frac{1}{|t|R}\right)=\frac{\kappa_d}{|t|R}.$$

We then apply Lemma 5.10 to get

$$|T^1_{R,t}|\leq 2^{d+1}C|t|\|g\|_1M[f](x)=\frac{2^{d+1}C\kappa_d}{R}$$

which goes to zero when $t\to 0$ and $R\to+\infty$.

Claim 2. *We claim that*

$$T^2_{R,t} \to \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d \setminus B(x,\varepsilon)} \partial\partial_j \Gamma(x-y) f(y)\,\mathrm{d}y \tag{2.39}$$

when $t \to 0$, $R \to +\infty$ with $Rt \to 0$ (and the limit in (2.39) exist for every x).

We need the following symmetry observations: Let $\mathcal{D} = B(0,\rho) \setminus B(0,r)$ with $\rho > r$. then

$$\int_{\mathcal{D}} \frac{\partial^2\Gamma}{\partial x_k \partial x_j}(x)\,\mathrm{d}x = 0. \tag{2.40}$$

When $k \neq j$ this comes from the fact that $\mathcal{D}$ is symmetric with respect to the transformation $x_j \to -x_j$ while $\dfrac{\partial^2\Gamma}{\partial x_k \partial x_j}(x)$ is odd for this transform. When $k = j$, we use the fact that $\mathcal{D}$ is invariant under permutation of variables thus

$$\int_{\mathcal{D}} \frac{x_j^2}{|x|^{d+2}}\,\mathrm{d}x = \int_{\mathcal{D}} \frac{x_k^2}{|x|^{d+2}}\,\mathrm{d}x$$

so that

$$d\int_{\mathcal{D}} \frac{x_j^2}{|x|^{d+2}}\,\mathrm{d}x = \sum_{k=1}^{d} \int_{\mathcal{D}} \frac{x_k^2}{|x|^{d+2}}\,\mathrm{d}x = \int_{\mathcal{D}} \frac{|x|^2}{|x|^{d+2}}\,\mathrm{d}x.$$

The formula for $\dfrac{\partial^2\Gamma}{\partial x_k^2}$ sows that

$$\int_{\mathcal{D}} \frac{\partial^2\Gamma}{\partial x_k^2}(x)\,\mathrm{d}x = 0.$$

We then write, for $|t|R < 1$,

$$T^2_{R,t} = \int_{\mathbb{R}^d \setminus B(x,1)} \partial\partial_j \Gamma(x-y) f(y)\,\mathrm{d}y + \int_{B(x,1)\setminus B(x,R|t|)} \partial\partial_j \Gamma(x-y) f(y)\,\mathrm{d}y.$$

The first integral is well defined. For the second one, we use (2.40) to write

$$\int_{B(x,1)\setminus B(x,R|t|)} \partial\partial_j \Gamma(x-y) f(y)\,\mathrm{d}y = \int_{B(x,1)\setminus B(x,R|t|)} \partial\partial_j \Gamma(x-y)\big(f(y)-f(x)\big)\,\mathrm{d}y.$$

Since $f \in \mathcal{S}(\mathbb{R}^d)$, the mean value theorm gives

$$|f(x) - f(y)| \le \|\nabla f\|_1 |x-y|$$

thus with the estimate $|\partial^2\Gamma(x-y)| \le C|x-y|^{-d}$ we get that

$$|\partial\partial_j \Gamma(x-y) f(y)| \le C\frac{\|\nabla f\|_1}{|x-y|^{d-1}}$$

which is integrable aover $B(x,1)$. Thus

$$\int_{B(x,1)\setminus B(x,R|t|)} \partial\partial_j \Gamma(x-y) f(y)\,\mathrm{d}y$$

has a limit when $R \to +\infty$, $t \to 0$ with $Rt \to 0$ and

$$\lim_{t\to 0, R\to+\infty, Rt\to 0} T^2_{R,t} = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d \setminus B(x,\varepsilon)} \partial\partial_j \Gamma(x-y) f(y)\,\mathrm{d}y.$$

Claim 3. *We claim that*

$$\lim_{t\to 0} \lim_{R\to+\infty} T^3_{R,t} = \left((d-2)c_d \int_{\mathbb{S}^{d-1}} y_j y_k\,\mathrm{d}\sigma_{d-1}(y)\right) f(x).$$

We write

$$T^3_{R,t} = \frac{1}{t}\int_{B(x,|t|R)} \partial_j\Gamma(x+te-y)\big(f(y) - f(x+te)\big)\,\mathrm{d}y$$
$$+ \frac{f(x+te)}{t}\int_{B(x,|t|R)} \partial_j\Gamma(x+te-y)\,\mathrm{d}y := T^{31}_{R,t} + T^{32}_{R,t}.$$

Using the mean value theorem and the bound on $|\partial_j\Gamma(u)|\leq C|u|^{-d-1}$ we get

$$\begin{aligned}|T^{31}_{R,t}| &\leq \frac{1}{|t|}\int_{B(x,|t|R)} C\frac{\|\nabla f\|_1\,|y-(x+te)|}{|x+te-y|^{d-1}}\,\mathrm{d}y\\ &\leq \frac{C\|\nabla f\|_1}{|t|}\int_{B(x+te,2|t|R)}\frac{\mathrm{d}y}{|x+te-y|^{d-2}}\end{aligned}$$

since $B(x,|t|R)\subset B(x+te,2|t|R)$ as $R\geq 1$. One can then change variable $u=x+te-y$ and integrate in polar coordinates $u=r\zeta$ to get

$$\begin{aligned}|T^{31}_{R,t}| &\leq \frac{C\|\nabla f\|_1}{|t|}\sigma_{d-1}(\mathbb{S}^{d-1})\int_0^{2|t|R}\frac{r^{d-1}}{r^{d-2}}\,\mathrm{d}r\\ &= 2C\|\nabla f\|_1\sigma_{d-1}(\mathbb{S}^{d-1})R^2|t|\end{aligned}$$

which goes to zero when $t\to 0$, $R\to+\infty$ but $R^2t\to 0$.

For the second term, we notice that

$$\partial_j\Gamma(x+te-y):=\frac{\partial}{\partial x_j}\Gamma(x+te-y)=-\frac{\partial}{\partial y_j}\Gamma(x+te-y).$$

Then, integrating by parts (Green's Formula)

$$\begin{aligned}\frac{1}{t}\int_{B(x,|t|R)}\partial_j\Gamma(x+te-y)\,\mathrm{d}y &= -\frac{1}{t}\int_{B(x,|t|R)}\frac{\partial}{\partial y_j}\Gamma(x+te-y)\,\mathrm{d}y\\ &= -\frac{1}{t}\int_{\partial B(x,|t|R)}\Gamma(x+te-y)\nu_j\,\mathrm{d}y\\ &= -\frac{1}{t}(R|t|)^{d-1}\int_{\mathbb{S}^{d-1}}\Gamma(R|t|\zeta+te)\zeta_j\,\mathrm{d}\sigma_{d-1}(\zeta)\\ &= -R\,\mathrm{sign}(t)\int_{\mathbb{S}^{d-1}}\Gamma(\zeta+R^{-1}\,\mathrm{sign}(t)e)\zeta_j\,\mathrm{d}\sigma_{d-1}(\zeta)\end{aligned}$$

since Γ is homogeneous of degree $-d+2$. Next we use again a symmetry argument to get

$$\int_{\mathbb{S}^{d-1}}\Gamma(\zeta)\zeta_j\,\mathrm{d}\sigma_{d-1}(\zeta)=0$$

so that

$$\begin{aligned}\int_{B(x,|t|R)}\partial_j\Gamma(x+te-y)\,\mathrm{d}y &= -R\,\mathrm{sign}(t)\int_{\mathbb{S}^{d-1}}\big(\Gamma(\zeta+R^{-1}\,\mathrm{sign}(t)e)-\Gamma(\zeta)\big)\zeta_j\,\mathrm{d}\sigma_{d-1}(\zeta)\\ &\to -\int_{\mathbb{S}^{d-1}}\frac{\partial\Gamma}{\partial x_1}(\zeta)\zeta_j\,\mathrm{d}\sigma_{d-1}(\zeta)\\ &= c_d(d-2)\int_{\mathbb{S}^{d-1}}\zeta_1\zeta_j\,\mathrm{d}\sigma_{d-1}(\zeta)\end{aligned}$$

when $R\to+\infty$ (with the fact that $\partial\Gamma(\zeta)=-c_d(d-2)\zeta_1$ when $|\zeta|=1$).

The conclusion follows by letting $t\to 0$

The proof for $T^4_{R,t}$ is similar and we obtain the same limit. To make all steps work simultaneously, it is enough to take $R=t^{-1/3}$ and to let $t\to 0$. □

CHAPTER 7

Calderon-Zygmund operators

1. Definition

In this chapter we are interested in operators that are (at least formally) defined by a kernel K *i.e.* are of the form

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)\,\mathrm{d}y$$

and we are looking for conditions on K which garantee that T is of strong type (p,p). We are in particular looking for conditions that go beyond the Schur test and would cover the Hilbert transform and the Newton potential.

IWe will denote by $\Delta = \{(x,x) \in \mathbb{R}^d \times \mathbb{R}^d \,:\, x \in \mathbb{R}^d\}$.

DEFINITION 7.1. A function $K \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta)$ is called a *standard or singular kernel* if there is a constant $C_0 > 0$ and an α with $0 < \alpha \leq 1$ such that the following estimates hold:

(i) For every $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \Delta$, $|K(x,y)| \leq \dfrac{C}{|x-y|^d}$;

(ii) For every $x,y,z \in \mathbb{R}^d$ with $y,z \neq x$, and $|y-z| \leq \dfrac{1}{2}|x-y|$

$$|K(x,y) - K(x,z)| \leq C_0 \frac{|y-z|^\alpha}{|x-y|^{d+\alpha}}$$

and for every $x,y,z \in \mathbb{R}^d$ with $x,y \neq z$, and $|x-y| \leq \dfrac{1}{2}|x-z|$

$$|K(x,z) - K(y,z)| \leq C_0 \frac{|x-y|^\alpha}{|x-z|^{d+\alpha}}.$$

REMARK 7.2. The second property of a standard kernel is usually called the *smoothness* property of K as it is essentially a Hölder smoothness property. The factor $1/2$ appearing there is of mild interest as long as this factor is < 1.

To check that K is a standard kernel, it is usually more convenient to check that there is a constant C_1 such that

$$|\nabla_x K(x,y)|, |\nabla_y K(x,y)| \leq \frac{C_1}{|x-y|^{d+1}}.$$

Indeed, let $x,y,z \in \mathbb{R}^d$ with $y,z \neq x$ and $|y-z| \leq \dfrac{1}{2}|x-y|$ then, by the mean value theorem, there is a $\theta \in [y,z]$ such that

$$K(x,y) - K(x,z) = \langle \nabla_y K(x,\theta), y-z \rangle.$$

Thus

$$|K(x,y) - K(x,z)| = |\nabla_y K(x,\theta)|\,|y-z| \leq \frac{C_1}{|x-\theta|^{d+1}}|y-z|.$$

As $\theta \in [y,z]$, $|y-\theta| \leq |y-z| \leq \dfrac{1}{2}|x-y|$,

$$|x-\theta| \geq |x-y| - |y-\theta| \geq |x-y| - |y-z| \geq \frac{1}{2}|x-y|.$$

We conclude that

$$|K(x,y) - K(x,z)| \leq \frac{2^{d+1}C_1}{|x-y|^{d+1}}|y-z|.$$

The second estimate is similar.

Example 7.3. In dimension $d=1$, both $K(x,y)=\dfrac{1}{x-y}$ and $K(x,y)=\dfrac{1}{|x-y|}$ are singular kernels. If $\Omega:\mathbb{S}^{d-1}\to\mathbb{R}$ is Hölder continuous then

$$K(x,y)=\Omega\left(\frac{x-y}{|x-y|}\right)\frac{1}{|x-y|^d}$$

is a singular kernel on $\mathbb{R}^d\times\mathbb{R}^d$.

We leave this fact as an exercice.

These conditions are not sufficient for the operator T associated to K to be of strong $(2,2)$-type.

Definition 7.4. A *Calderón-Zygmund operator* is an operator T of strong type $(2,2)$ such that there exists a singular kernel K such that if $f\in L^2(\mathbb{R})$ is supported in a compact set E then for every $x\notin E$,

$$Tf(x)=\int_{\mathbb{R}^d}K(x,y)f(y)\,\mathrm{d}y.$$

Example 7.5. Let b be a bounded function on $\mathbb{R}^d$ and $Tf=bf$ then T is bounded $L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ and if $f\in L^2(\mathbb{R})$ is supported in a compact set E then for every $x\notin E$

$$Tf(x)=b(x)f(x)=0=\int_{\mathbb{R}^d}0f(y)\,\mathrm{d}y$$

so that T is a Calderón-Zygmund operator with kernel 0. In particular the kernel is not unique. However, a functional analytic argument allows to show that this is the only source of non-uniqueness.

In general, Calderón-Zygmund operators are just singular in the sense that K just fails to be integrable so the difficulty is to analyse the singularity correctly to see that some cancelation occurs.

Example 7.6. The Hilbert transform H is a Calderón-Zygmund operator with kernel $K(x,y)=\dfrac{1}{\pi}\dfrac{1}{x-y}$ since, we proved that H is of strong type $(2,2)$ and if $f\in L^2(\mathbb{R})$ is supported in a compact set E then for every $x\notin E$

$$Hf(x)=\lim_{\varepsilon\to0}\int_{|x-y|\geq\varepsilon}\frac{1}{\pi}\frac{1}{x-y}f(y)\,\mathrm{d}y=\int_{\mathbb{R}^d}\frac{1}{\pi}\frac{1}{x-y}f(y)\,\mathrm{d}y.$$

This follows from the fact that if $x\notin E$ then for ε small enough, $|x-y|\leq\varepsilon$ implies that $f(y)=0$.

Remark 7.7. A Calderón-Zygmund operator needs not be translation invariant, self-adjoint, dilation invariant,... However, if T is a Calderón-Zygmund operator with kernel K, then $\tau_{-a}T\tau_a$, $\delta_{1/\lambda}T\delta_\lambda$ and T^* are also Calderón-Zygmund operators. We leave as an exercice to determine the kernel in each case.

As the adjoint will play a role in the sequel, let us detail that case:

Lemma 7.8. *Let T be a Calderón-Zygmund operator and K be a standard kernel associated to it. Define its adjoint by*

$$\int_{\mathbb{R}^d}Tf(x)\overline{g(x)}\,\mathrm{d}x=\int_{\mathbb{R}^d}f(x)\overline{T^*g(x)}\,\mathrm{d}x$$

for every $f,g\in L^2(\mathbb{R}^d)$. Then T^* is also a Calderón-Zygmund operator with kernel $K^*(x,y)=\overline{K(y,x)}$.

Proof. That T^* is a well-defined bounded linear operator on $L^2(\mathbb{R}^d)$ is a standard fact from any course on Hilbert spaces. The only thing that needs to be shown is that T^* is associated to the kernel K^* as it is clear that K^* is also a standard kernel.

Now, let $f,g\in L^2(\mathbb{R}^d)$ have disjoint compact support. Then

$$\begin{aligned}\int_{\mathbb{R}^d}f(x)\overline{T^*g(x)}\,\mathrm{d}x &= \int_{\mathbb{R}^d}Tf(x)\overline{g(x)}\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K(x,y)f(y)\,\mathrm{d}y\overline{g(x)}\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d}f(y)\overline{\left(\int_{\mathbb{R}^d}\overline{K(x,y)}g(x)\,\mathrm{d}y\right)}\,\mathrm{d}x\end{aligned}$$

whith Fubini which is justified by the fact that $f(y)K(x,y)\overline{g(x)} \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$ since $f, g \in L^1$ and K is bounded over $\operatorname{supp} f \times \operatorname{supp} g$ (since those supports are compact and disjoint thus at a positive distance).

Now fix $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ with $\int \phi = 1$ and $z \notin \operatorname{supp} g$. For $\varepsilon > 0$ take $f = \varepsilon^{-d}\phi\big((x-z)/\varepsilon\big)$ which has disjoint support from g when ε is small enough and let $\varepsilon \to 0$ then,

$$\phi_\varepsilon * \overline{T^*g}(z) = \int_{\mathbb{R}^d} f(x)\overline{T^*g(x)}\,\mathrm{d}x = \int_{\mathbb{R}^d} f(y)\overline{\left(\int_{\mathbb{R}^d} \overline{K(x,y)}g(x)\,\mathrm{d}y\right)}\,\mathrm{d}x = \phi_\varepsilon * \overline{\left(\int_{\mathbb{R}^d} \overline{K(\cdot,y)}g(x)\,\mathrm{d}y\right)}(z).$$

Letting $\varepsilon \to 0$, by the approximation of unity theorem, we get

$$\overline{T^*g}(z) = \overline{\int_{\mathbb{R}^d} \overline{K(z,y)}g(x)\,\mathrm{d}y}$$

for almost every z. □

Remark 7.9. Some authors prefer defining Calderón-Zygmund operators with the help of distribution theory. Here we consider $W \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ and assume that W coincides with a standard kernel K on $\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta$ where $\Delta = \{(x,x) : x \in \mathbb{R}^d\}$ is the diagonal of $\mathbb{R}^d \times \mathbb{R}^d$. This means that, if $F \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ is supported away from Δ (i.e. $dist(\operatorname{supp} F, \Delta) > 0$) then

$$\langle W, F\rangle_{\mathcal{S}'(\mathbb{R}^d\times\mathbb{R}^d),\mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d)} = \iint_{\mathbb{R}^d\times\mathbb{R}^d} K(x,y)F(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Note that this integral is absolutely convergent and that more than one distribution W can coincide with K.

Next, we consider an operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. The Schwarz kernel theorem states that there exists $W_T \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$\langle Tf, \varphi\rangle_{\mathcal{S}'(\mathbb{R}^d),\mathcal{S}'(\mathbb{R}^d)} = \langle W_T, f \otimes \varphi\rangle_{\mathcal{S}'(\mathbb{R}^d\times\mathbb{R}^d),\mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d)}$$

for every $f, \varphi \in \mathcal{S}(\mathbb{R}^d)$. We then say that T is a Calderón-Zygmund operator if its kernel W_T coincides with a standard kernel of the diagonal and if T extends to a bounded operator $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

The last part means that there exists $C > 0$ such that, for every $f \in \mathcal{S}(\mathbb{R}^d)$, $Tf \in L^2(\mathbb{R}^d)$ (as a distribution) meaning that there is a $g \in L^2(\mathbb{R}^d)$ with $\|g\|_2 \le C\|f\|_2$ such that, for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$

$$\langle Tf, \varphi\rangle_{\mathcal{S}(\mathbb{R}^d),\mathcal{S}'(\mathbb{R}^d)} = \int_{\mathbb{R}^d} g(x)\varphi(x)\,\mathrm{d}x.$$

Of course, we identify $g = Tf$.

Under this definition, T is thus always defined on $\mathcal{S}(\mathbb{R}^d)$. Saying that T is of strong type (p,p), $1 < p < +\infty$, then means that the g above is also in $L^p(\mathbb{R}^d)$ with $\|g\|_p \le C_p\|f\|_p$. This in turn implies that T (uniquely) extends from an operator $\mathcal{S}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ to a bounded operator $L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$.

We will now turn to boundedness properties of Calderón-Zygmund operators.

2. Boundedness of Calderón-Zygmund operators

We will start with a simple lemma that shows that an operator with singular kernel maps fluctuating (zero-mean) localized functions into weakly localized functions:

Lemma 7.10. *Let K be a singular kernel and denote by C_0, α the parameters in Definition 7.1. Let $\varphi \in L^1(\mathbb{R}^d)$ be supported in a ball $B = B(x_0, r)$ with zero mean $\displaystyle\int_B \varphi(x)\,dx = 0$. For $x \notin 2B = B(x_0, 2r)$ define*

$$T\varphi(x) = \int_B K(x,y)\varphi(y)\,dy$$

then

$$|Tf(x)| \le \frac{C_0 r^\alpha}{|x - x_0|^{d+\alpha}} \int_B |f(y)|\,dy$$

so that

$$\|Tf\|_{L^1(\mathbb{R}^d\setminus 2B)} \le C\frac{C_0}{\alpha 2^\alpha}\|f\|_1$$

where C depends on the dimension, but not on f nor on K.

PROOF. Using the fact that f has mean 0, we write

$$Tf(x) = \int_B K(x,y)f(y)\,\mathrm{d}y = \int_B \big(K(x,y) - K(x,x_0)\big)f(y)\,\mathrm{d}y.$$

But $x \notin 2B$ and $x_0, y \in B$ thus $|x_0 - y| \leq r \leq \frac{1}{2}2r \leq \frac{1}{2}|x - x_0|$ so that

$$|K(x,y) - K(x,x_0)| \leq C_0 \frac{|y - x_0|^\alpha}{|x - x_0|^{d+\alpha}} \leq C_0 \frac{r^\alpha}{|x - x_0|^{d+\alpha}}.$$

We thus get the first estimate from the triangle inequality. But then, integrating in polar coordinates

$$\|Tf\|_{L^1(\mathbb{R}^d\setminus 2B)} \leq C_0 r^\alpha \int_B |f(y)|\,\mathrm{d}y \sigma_{d-1}(\mathbb{S}^{d-1}) \int_{2r}^{+\infty} \frac{t^{d-1}}{t^{d+\alpha}}\,\mathrm{d}t \tag{2.41}$$

$$= \frac{C_0}{\alpha 2^\alpha} \sigma_{d-1}(\mathbb{S}^{d-1}) \int_B |f(y)|\,\mathrm{d}y \tag{2.42}$$

as claimed. □

This lemma of course applies to Calderón-Zygmund operators but we did not need L^2-boundedness which will now play a key role:

THEOREM 7.11. *Let T be a Calderón-Zygmund operator, then T extends into an operator of weak-type* $(1,1)$.

PROOF. We take $f \in \mathcal{S}(\mathbb{R}^d)$ and $\lambda > 0$. We want to show that

$$|\{|Tf| > \lambda\}| \leq \frac{\|f\|_1}{\lambda}.$$

To do so, we will exploit the Calderón-Zygmund decomposition and the previous lemma. The parameter in the decomposition can be chosen to be $\alpha = \lambda$. We thus get a decomposition $f = g + b$, $b = \sum_i b_i$ with b_i supported in Q_i with $\int b_j = 0$. Further $\|g\|_2 \leq 2^d \lambda \|f\|_1$, $\|b_j\|_1 \leq 2^{d+1}\lambda |Q_j|$ and $\sum |Q_j| \leq \frac{\|f\|_1}{\lambda}$.

First $Tf = Tg + \sum Tb_i$ so that $|Tf| \leq |Tg| + \sum |Tb_i|$ and if this sum is $\geq \lambda$, then at least one of the terms is $\geq \lambda/2$ so that

$$|\{|Tf| > \lambda\}| \leq |\{|Tg| > \lambda/2\}| + |\{|\sum Tb_i| > \lambda/2\}|.$$

For the first term, we exploit the fact that $g \in L^2$ so that $Tg \in L^2$ and Chebichev's inequality:

$$|\{|Tg| > \lambda/2\}| \leq \frac{4}{\lambda^2}\|Tg\|_{L^2}^2 \leq \frac{4}{\lambda^2}\|T\|_{L^2\to L^2}^2 \|g\|_{L^2}^2 \leq \frac{C_1}{\lambda}\|f\|_1.$$

with $C_1 = 2^{d+2}\|T\|_{L^2\to L^2}^2$.

For the second term, we will start exploiting the previous lemma which states that, if

$$\|Tb_j\|_{L^1(\mathbb{R}^d\setminus 2Q_j)} \leq C\|b_j\|_1 \leq C_2 \lambda |Q_j|$$

where C_2 depends on the dimension and the parameters of the kernel associated to T. In particular

$$\|\sum Tb_j\|_{L^1(\mathbb{R}^d\setminus\bigcup 2Q_j)} \leq \sum \|Tb_j\|_{L^1(\mathbb{R}^d\setminus 2Q_j)} \leq C_2\lambda \sum |Q_j| \leq C_2\|f\|_1.$$

Therefore, using Markov's estimate

$$\left|\left\{x \in \mathbb{R}^d \setminus \bigcup 2Q_j \;:\; \left|\sum Tb_i(x)\right| > \lambda/2\right\}\right| \leq \frac{2}{\lambda}\left\|\sum Tb_j\right\|_{L^1(\mathbb{R}^d\setminus\bigcup 2Q_j)} \leq \frac{2C_2}{\lambda}\|f\|_1.$$

Now comes the big advantage of weak-type estimates of strong type estimates: we do not need to estimate $\sum Tb_i$ over the remaining set $\bigcup 2Q_j$, we only need to estimate the size of this set:

$$\left|\bigcup 2Q_j\right| \leq \sum |2Q_j| \leq 2^d \sum |Q_j| \leq \frac{2^d}{\lambda}\|f\|_1$$

so that finally

$$\left|\left\{|\sum Tb_i| > \lambda/2\right\}\right| \leq \frac{C_3}{\lambda}\|f\|_1$$

with $C_3 = 2C_2 + 2^d$.

Putting everything together, we get

$$|\{|Tf| > \lambda\}| \le \frac{C_1}{\lambda}\|f\|_1 + \frac{C_3}{\lambda}\|f\|_1$$

as expected. □

COROLLARY 7.12. *Let T be a Calderón-Zygmund operator and $1 < p < +\infty$, then T is of strong type (p,p).*

PROOF. Recall that, by definition, a Calderón-Zygmund operator is bounded on L^2. First, for $1 < p < 2$, using the fact that T is of weak-type $(1,1)$ and of strong type $(2,2)$, we get that T is of strong type (p,p) by interpolation.

Next, let $2 < p < +\infty$. If T is a Calderón-Zygmund operator then so is T^*. Thus T^* is of strong-type (p',p') thus T is of strong type (p,p). □

The case $p = +\infty$ is left-out for the moment and will require the introduction of a new function space, the space of Bounded Mean Oscillating functions

3. Truncated Calderón-Zygmund operators

3.1. Truncation of Calderón-Zygmund operators. Calderón-Zygmund operators are formaly defined as

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)\,\mathrm{d}y$$

and we would like to give a reasonable meaning to this definition. The example of the Hilbert transform suggest that we should look at the truncated version

$$T_\varepsilon f(x) = \int_{|x-y|\ge\varepsilon} K(x,y)f(y)\,\mathrm{d}y \tag{3.43}$$

and to let $\varepsilon \to 0$.

The first observation is that T_ε makes sense:

LEMMA 7.13. *Let K be a standard kernel and define T_ε via (3.43). Then $T_\varepsilon f$ is well defined and of strong type (p,∞) for every $1 \le p < +\infty$.*

PROOF. It is enough to apply Hölder's inequality

$$\begin{aligned}|T_\varepsilon(f)(x)| &\le \int_{\mathbb{R}^d\setminus B(x,\varepsilon)} |K(x,y)|\,|f(y)|\,\mathrm{d}y \le \int_{\mathbb{R}^d} \frac{C_0\mathbf{1}_{\mathbb{R}^d\setminus B(x,\varepsilon)}}{|x-y|^d}|f(y)|\,\mathrm{d}y \\ &\le \left\|\frac{C_0\mathbf{1}_{\mathbb{R}^d\setminus B(x,\varepsilon)}}{|x-y|^d}\right\|_{p'}\|f\|_p.\end{aligned}$$

When $p=1$, $p'=+\infty$ and $\dfrac{C_0\mathbf{1}_{\mathbb{R}^d\setminus B(x,\varepsilon)}}{|x-y|^d} \le C_0\varepsilon^{-d}$ so that $|T_\varepsilon(f)(x)| \le C_0\varepsilon^{-d}\|f\|_1$.

When $1 < p < +\infty$, $1 < p' < +\infty$ and we integrate in polar coordinates

$$\left\|\frac{C_0\mathbf{1}_{\mathbb{R}^d\setminus B(x,\varepsilon)}}{|x-y|^d}\right\|_{p'}^{p'} = \sigma_{d-1}(\mathbb{S}^{d-1})\int_\varepsilon^{+\infty}\frac{r^{d-1}\mathrm{d}r}{r^{dp'}} = \sigma_{d-1}(\mathbb{S}^{d-1})\int_\varepsilon^{+\infty}\frac{\mathrm{d}r}{r^{1+d(p'-1)}} < +\infty$$

and the conclusion follows. □

The kernel of T_ε is

$$K_\varepsilon(x,y) = K(x,y)\mathbf{1}_{\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d\,:\,|x-y|>\varepsilon\}}$$

which is no longer a standard kernel as the smoothness condition is no longer satisfied.

One can overcome this by introducing a smooth cutoff function. Let $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ be such that $0 \le \varphi \le 1$, φ is radial and $\varphi(x) = 0$ when $|x| \le 1$ and $\varphi(x) = 1$ when $|x| \ge 2$ and define

$$K_\varepsilon^\varphi(x,y) = K(x,y)\varphi\left(\frac{x-y}{\varepsilon}\right).$$

Then K_ε^φ is a standard kernel: let x,y,z be such that $x \ne y,z$ and $|y-z| \le \dfrac{1}{2}|x-y|$.

– As $|K(x,y)|\leq\dfrac{C_0}{|x-y|^d}$ and $0\leq\varphi\leq 1$, $|K_\varepsilon^\varphi(x,y)|\leq\dfrac{C_0}{|x-y|^d}$.

– Next recall that
$$|K(x,y)-K(x,z)|\leq\frac{C_0|y-z|}{|x-y|^{d+\alpha}}$$
and write
$$K_\varepsilon^\varphi(x,y)-K_\varepsilon^\varphi(x,z)=\big(K(x,y)-K(x,z)\big)\varphi\left(\frac{x-y}{\varepsilon}\right)+K(x,z)\left(\varphi\left(\frac{x-y}{\varepsilon}\right)-\varphi\left(\frac{x-z}{\varepsilon}\right)\right).$$
Again, as $0\leq\varphi\leq 1$ we immediately get that
$$\left|\big(K(x,y)-K(x,z)\big)\varphi\left(\frac{x-y}{\varepsilon}\right)\right|\leq\frac{C_0|y-z|^\alpha}{|x-y|^{d+\alpha}}.$$
On the other hand, from the Mean Value Theorem, there is a $\theta\in[y,z]$ such that
$$\varphi\left(\frac{x-y}{\varepsilon}\right)-\varphi\left(\frac{x-z}{\varepsilon}\right)=\frac{1}{\varepsilon}\left\langle\nabla\varphi\left(\frac{x-\theta}{\varepsilon}\right),y-z\right\rangle.$$
But φ is constant in $B(0,1)$ and in $\mathbb{R}^d\setminus B(0,2)$ so that $\nabla\varphi\left(\dfrac{x-\theta}{\varepsilon}\right)=0$ unless $1\leq\frac{|x-\theta|}{\varepsilon}\leq 2$. Further $\theta\in[y,z]$ *i.e.* $\theta=ty+(1-t)z$ for some $0\leq t\leq 1$ and $|y-z|\leq\dfrac{1}{2}|x-y|$ so that
$$|x-\theta|=|x-y+(1-t)(y-z)|\geq|x-y|-(1-t)|y-z|\geq\frac{1}{2}|x-y|.$$
We conclude that, if $\nabla\varphi\left(\dfrac{x-\theta}{\varepsilon}\right)\neq 0$
$$1\leq\frac{2\varepsilon}{|x-\theta|}\leq\frac{2^2\varepsilon}{|x-y|}.$$
It folows that
$$\left|\varphi\left(\frac{x-y}{\varepsilon}\right)-\varphi\left(\frac{x-z}{\varepsilon}\right)\right|\leq\frac{2^2}{|x-y|}\|\nabla\varphi\|_\infty|y-z|\leq 2^{1+\alpha}\|\nabla\varphi\|_\infty|y-z|^\alpha\frac{|y-z|^\alpha|x-y|^{1-\alpha}}{|x-y|}$$
using again that $|y-z|\leq\dfrac{1}{2}|x-y|$. In conclusion
$$\left|K(x,z)\left(\varphi\left(\frac{x-y}{\varepsilon}\right)-\varphi\left(\frac{x-z}{\varepsilon}\right)\right)\right|\leq 2^{1+\alpha}\|\nabla\varphi\|_\infty C_0\frac{|y-z|^\alpha}{|x-y|^{d+\alpha}}$$
as expected.

The second smoothness estimate is obtained the same way.

We can then define
$$T_\varepsilon^\varphi f(x)=\int_{\mathbb{R}^d}K_\varepsilon^\varphi(x,y)f(y)\,\mathrm{d}y.$$
We leave as an exercice to show that $T_\varepsilon^\varphi f(x)$ is well defined when $f\in L^p(\mathbb{R}^d)$.

PROPOSITION 7.14. *Let $1\leq p<+\infty$ and $f\in L^p(\mathbb{R}^d)$. With the notations above we have*
$$|T_\varepsilon^\varphi f(x)-T_\varepsilon f(x)|\leq CM[f](x)$$
where C is a constant depending on the dimension and the parameters of the kernel and on φ but not on ε.

In particular, T_ε is of of weak type $(1,1)$ (resp. strong type (p,p) for $1<p<+\infty$) if and only if T_ε^φ is.

PROOF. Write
$$T_\varepsilon^\varphi f(x)-T_\varepsilon f(x)=\int_{\mathbb{R}^d}\left(\mathbf{1}_{\mathbb{R}^d\setminus B(x,\varepsilon)}(y)-\varphi\left(\frac{x-y}{\varepsilon}\right)\right)K(x,y)f(y)\,\mathrm{d}t.$$
Note that
$$\left|\mathbf{1}_{\mathbb{R}^d\setminus B(x,\varepsilon)}(y)-\varphi\left(\frac{x-y}{\varepsilon}\right)\right|=\begin{cases}0\text{ and if }y\notin B(x,2\varepsilon)\\ \leq 1\qquad\qquad\qquad\text{if }y\in B(x,2\varepsilon)\setminus B(x,\varepsilon)\end{cases}.$$

It follows that

$$\begin{aligned}|T_\varepsilon^\varphi f(x)-T_\varepsilon f(x)| &\leq C_0\int_{B(x,2\varepsilon)\setminus B(x,\varepsilon)}\frac{|f(y)|}{|x-y|^d}\\ &\leq \frac{C_0}{\varepsilon^d}\int_{B(x,2\varepsilon)}|f(y)|\,\mathrm{d}y=\frac{C_0|B(0,2)|}{|B(0,2\varepsilon)|}\int_{B(x,2\varepsilon)}|f(y)|\,\mathrm{d}y\\ &\leq C_0|B(0,2)|M[f](x)\end{aligned}$$

as claimed. □

We now want to define Tf as $\lim T_\varepsilon f$. The first observation is that this can *not* be done pointwise. This can already be seen from the simple operator $Tf(x)=b(x)f(x)$, b bounded, which is a Calderón-Zygmund operator with kernel 0 so that $T_\varepsilon f(x)=0$ though $Tf\neq 0$.

The following lemma clears out the situation for pointwise limite.

LEMMA 7.15. *The limit* $\lim_{\varepsilon\to 0}T_\varepsilon f(x)$ *exists a.e. for every* $f\in\mathcal{S}(\mathbb{R}^d)$ *if and only if the limit*

$$\lim_{\varepsilon\to 0}\int_{\varepsilon\leq|x-y|\leq 1}K(x,y)\,\mathrm{d}y$$

exists almost everywhere.

PROOF. First, assume that $\lim_{\varepsilon\to 0}T_\varepsilon f(x)$ exists a.e. for every $f\in\mathcal{S}(\mathbb{R}^d)$. For f take a function $\in\mathcal{S}(\mathbb{R}^d)$ that is 1 in $B(0,2)$ and note that, if $|x|\leq 1$ and $|x-y|\leq 1$ then $|y|\leq 2$ so that

$$T_\varepsilon f(x)=\int_{\varepsilon\leq|x-y|\leq 1}K(x,y)\,\mathrm{d}y+\int_{|x-y|\geq 1}K(x,y)f(y)\,\mathrm{d}y.$$

The second integral is absolutely convergent and does not depend on ε so that

$$\lim_{\varepsilon\to 0}\int_{\varepsilon\leq|x-y|\leq 1}K(x,y)\,\mathrm{d}y=\lim_{\varepsilon\to 0}T_\varepsilon f(x)$$

exists almost everywhere in $B(0,1)$. We leave as an exercice to adapt the proof to show almost everywhere convergence in any ball $B(x_0,1)$.

Conversely, suppose that for some $x\in\mathbb{R}^d$,

$$L=\lim_{\varepsilon\to 0}\int_{\varepsilon\leq|x-y|\leq 1}K(x,y)\,\mathrm{d}y$$

exists. For $f\in\mathcal{S}(\mathbb{R}^d)$, write

$$\begin{aligned}T_\varepsilon f(x) &= \int_{\varepsilon\leq|x-y|\leq 1}K(x,y)f(y)\,\mathrm{d}y+\int_{|x-y|\geq 1}K(x,y)f(y)\,\mathrm{d}y\\ &= \int_{\varepsilon\leq|x-y|\leq 1}K(x,y)\big(f(y)-f(x)\big)\,\mathrm{d}y+f(x)\int_{\varepsilon\leq|x-y|\leq 1}K(x,y)\,\mathrm{d}y\\ &\quad+\int_{|x-y|\geq 1}K(x,y)f(y)\,\mathrm{d}y.\end{aligned}$$

The second term is a limit when $\varepsilon\to 0$ and the third one is an absolutely convergent integral that does not depend on ε. It remains to show that the first one has a limit. But from the mean value theorem and the growth estimate of K we get

$$\int_{|x-y|\leq 1}|K(x,y)|\,|f(x)-f(y)|\,\mathrm{d}y\leq\int_{|x-y|\leq 1}\frac{C_0}{|x-y|}\|\nabla f\|_\infty|x-y|\,\mathrm{d}y<+\infty$$

thus the integral $\displaystyle\int_{|x-y|\leq 1}K(x,y)f(y)\,\mathrm{d}y$ is absolutely convergent and

$$\int_{\varepsilon\leq|x-y|\leq 1}K(x,y)f(y)\,\mathrm{d}y\to\int_{|x-y|\leq 1}K(x,y)f(y)\,\mathrm{d}y$$

when $\varepsilon\to 0$. □

3.2. Maximal truncated Calderón-Zygmund operator.

DEFINITION 7.16. Let K be a standard kernel and define the *truncated Calderón-Zygmund operators* associated to K for $f \in \mathcal{S}(\mathbb{R}^d)$ by

$$T_\varepsilon f(x) = \int_{|x-y|\geq\varepsilon} K(x,y)f(y)\,\mathrm{d}y$$

and the *maximal truncated Calderón-Zygmund operator* associated to K for $f \in \mathcal{S}(\mathbb{R}^d)$ by

$$T_* f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|.$$

Note that T_ε is well defined so that T_* is also (but may take the value $+\infty$) and is a *sublinear* operator.

THEOREM 7.17. *Assume that T is a Calderón-Zygmund operator associated to a kernel K and let T_* be the* maximal truncated Calderón-Zygmund operator *associated to K.*
Then T_* is of weak type $(1,1)$ and of strong type (p,p) for $1 < p < +\infty$.

The proof of this theorem requires several steps. The first one is Kolmogorov's Lemma 2.31. The second one shows that T_* can be controlled by expressions involving maximal functions.

LEMMA 7.18. *Let T be a Calderón-Zygmund operator associated to a kernel K and $0 < \nu \leq 1$. For every $f \in \mathcal{C}_c(\mathbb{R}^d)$,*

$$|T_* f(x)| \leq C\Big[\big(M[|Tf|^\nu](x)\big)^{1/\nu} + Mf(x)\Big]$$

with a constant C that depends on K and d only.

PROOF. Let $x \in \mathbb{R}^d$, $\varepsilon > 0$ and $B = B(x,\varepsilon/2)$, $2B = B(x,\varepsilon)$. Write

$$f = f\mathbf{1}_{2B} + f(1-\mathbf{1}_{2B}) := f_1 + f_2.$$

As f is compactly supported, $f_2 \in L^2(\mathbb{R}^d)$ and f_2 is supported in $\mathbb{R}^d \setminus B(x,2\varepsilon)$ so that

$$Tf_2(x) = \int_{\mathbb{R}^d\setminus B(x,\varepsilon)} K(x,y)f(y)\,\mathrm{d}y = T_\varepsilon f(x).$$

Further, if $z \in B(x,\varepsilon/2)$ then $z \notin \operatorname{supp} f_2$ thus

$$\begin{aligned}
|Tf_2(x) - Tf_2(z)| &= \left|\int_{\mathbb{R}^d\setminus 2B} K(x,y)f_2(y)\,\mathrm{d}y - \int_{\mathbb{R}^d\setminus 2B} K(z,y)f_2(y)\,\mathrm{d}y\right| \\
&\leq \int_{\mathbb{R}^d\setminus 2B} |K(x,y)-K(z,y)|\,|f_2(y)|\,\mathrm{d}y \\
&\leq \int_{\mathbb{R}^d\setminus B(x,\varepsilon)} C_0\frac{|x-z|^\alpha}{|x-y|^{d+\alpha}}|f(y)|\,\mathrm{d}y
\end{aligned}$$

since $|f_2| \leq |f|$ and $|x-z| \leq \dfrac{\varepsilon}{2} \leq \frac{1}{2}|y-x|$ when $y \notin B(x,\varepsilon)$. But then

$$\begin{aligned}
|Tf_2(x) - Tf_2(z)| &\leq \frac{C_0}{2^\alpha} + \sum_{k=0}^{+\infty}\int_{y\in B(x,2^{k+1}\varepsilon)\setminus B(x,2^k\varepsilon)} \frac{\varepsilon^\alpha}{(2^k\varepsilon)^{(d+\alpha)}}|f(y)|\,\mathrm{d}y \\
&\leq \frac{C_0}{2^\alpha}\sum_{k=0}^{+\infty}\frac{1}{2^{k\alpha}(2^k\varepsilon)^d}\int_{B(x,2^{k+1}\varepsilon)\setminus B(x,2^k\varepsilon)}|f(y)|\,\mathrm{d}y.
\end{aligned}$$

Now notice that

$$\begin{aligned}
\frac{1}{(2^{(k+1)}\varepsilon)^d}\int_{B(x,2^{k+1}\varepsilon)\setminus B(x,2^k\varepsilon)}|f(y)|\,\mathrm{d}y &\leq 2^d|B(0,1)|\frac{1}{|B(x,2^{k+1}\varepsilon)|}\int_{B(x,2^{k+1}\varepsilon)}|f(y)|\,\mathrm{d}y \\
&\leq 2^d|B(0,1)|M[f](x)
\end{aligned}$$

so that

$$|Tf_2(x) - Tf_2(z)| \leq C_0 2^{d-\alpha}|B(0,1)|M[f](x)\sum_{k=0}^{+\infty}\frac{1}{2^{k\alpha}} = \frac{C_0|B(0,2)|}{2^\alpha - 1}M[f](x) := AM[f](x).$$

We thus get

$$|T_\varepsilon f(x)| = |Tf_2(x)| \le AM[f](x) + |Tf_2(z)| \le AM[f](x) + |Tf(z)| + |Tf_1(z)| \tag{3.44}$$

whenever $z \in B = B(x, \varepsilon/2)$.

As the right hand side does not depend on ε, it is enough to bound it by the desired quantity. To do so, we will now separate two cases.

Case 1. *The lemma when* $\nu = 1$.

If we had $T_\varepsilon f(x) = 0$ there would be nothing to prove so we assume that $T_\varepsilon f(x) \neq 0$ and take $0 < \lambda < |T_\varepsilon f(x)|$. We define

$$B_1 = \{z \in B : |Tf(z)| > \lambda/2\} \quad , \quad B_2 = \{z \in B : |Tf_1(z)| > \lambda/2\}$$

and

$$B_3 = \begin{cases} \emptyset & \text{if } M[f](x) < \lambda/3A \\ B & \text{if } M[f](x) \ge \lambda/3A \end{cases}.$$

Note that if $z \in B$ then either $z \in B_1$, $z \in B_2$ or (3.44) implies that $M[f](x) \ge \lambda/3A$ in which case $B_3 = B$. In any case, we have $B = B_1 \cup B_2 \cup B_3$.

However, Markov's inequality shows that

$$|B_1| \le \frac{2}{\lambda} \int_B |Tf(z)|\,\mathrm{d}z \le \frac{2|B|}{\lambda} M[Tf](x).$$

On the other hand, T is a Calderón-Zygmun operator, it is of weak-type $(1,1)$ so that

$$|B_2| \le \frac{C}{\lambda} \|f_1\|_1 \le \frac{2C|B|}{\lambda} M[f]$$

since $f_1 = f$ on $2B$ and $f_1 = 0$ on $\mathbb{R}^d \setminus 2B$. In particular, if $B_3 = \emptyset$, then

$$|B| \le |B_1| + |B_2| \le \frac{2|B|}{\lambda} M[Tf](x) + \frac{2C|B|}{\lambda} M[f]$$

which implies that

$$\lambda \le 2M[Tf](x) + 2CM[f](x).$$

A contrario, if $B_3 \neq \emptyset$ then

$$\lambda \le 3AM[f] \le \max(2C, 3A) M[f](x) + 2M[Tf](x)$$

and this inequality holds in all cases so that, taking the supremum over all λ

$$|T_\varepsilon f(x)| \le \max(2C, 3A] M[f](x) + 2M[Tf](x)$$

and it remains to take the supremum over all ε to establish the lemma in this case.

Case 2. *The lemma when* $0 < \nu < 1$.

We will use the following inequalities for $a, b, c \ge 0$ and $0 < \nu < 1$

– $(a+b+c)^\nu \le a^\nu + b^\nu + c^\nu$.

To see this, note first that $(1+t)^\nu \le 1 + t^\nu$ since both quantities are equal when $t = 0$ and the derivative of $(1+t)^\nu - (1+t^\nu)$ is $\nu \left(\dfrac{1}{(1+t)^{1-\nu}} - \dfrac{1}{t^{1-\nu}} \right) \le 0$ when $t \ge 0$. Then, factoring a or b one get $(a+b)^\nu \le a^\nu + b^\nu$ and iterating one gets the desired inequality.

– $(a+b+c)^{1/\nu} \le \kappa_\nu (a^{1/\nu} + b^{1/\nu} + c^{1/\nu})$

This time $t \to t^{1/\nu}$ is convex so that

$$(a+b)^{1/\nu} = 2^{1/\nu} \left(\frac{a+b}{2} \right)^{1/\nu} \le 2^{1/\nu} \frac{a^{1/\nu} + b^{1/\nu}}{2}$$

and then

$$(a+b+c)^{1/\nu} \le 2^{1/\nu - 1} \big(a^{1/\nu} + (b+c)^{1/\nu} \big) \le 2^{1/\nu - 1} \big(a^{1/\nu} + 2^{1/\nu - 1} b^{1/\nu} + 2^{1/\nu - 1} c^{1/\nu} \big)$$

which gives the result with $\kappa_\nu = 2^{2/\nu - 2}$.

Now, from (3.44), we deduce that

$$|T_\varepsilon f(x)|^\nu \le AM[f](x)^\nu + |Tf(z)|^\nu + |Tf_1(z)|^\nu$$

and, averaging over $z \in B$ we get

$$|T_\varepsilon f(x)|^\nu \leq AM[f](x)^\nu + \frac{1}{|B|}\int_B |Tf(z)|^\nu \,\mathrm{d}z + \frac{1}{|B|}\int_B |Tf_1(z)|^\nu \,\mathrm{d}z$$

thus

$$|T_\varepsilon f(x)| \leq K_\nu \left(M[f](x) + \left(\frac{1}{|B|}\int_B |Tf(z)|^\nu \,\mathrm{d}z\right)^{1/\nu} + \left(\frac{1}{|B|}\int_B |Tf_1(z)|^\nu \,\mathrm{d}z\right)^{1/\nu}\right)$$

with K_ν depending in A and ν. Next

$$\left(\frac{1}{|B|}\int_B |Tf(z)|^\nu \,\mathrm{d}z\right)^{1/\nu} \leq \big(M[|Tf|^\nu](x)\big)^{1/\nu}.$$

On the other hand, T begin of weak-type $(1,1)$ we may apply Kolmogorov's Lemma 2.31 to get

$$\left(\frac{1}{|B|}\int_B |Tf_1(z)|^\nu \,\mathrm{d}z\right)^{1/\nu} \leq \left(\frac{1}{|B|}C_T|B|^{1-\nu}\|f_1\|_1^\nu\right)^{1/\nu} = C_T^{1/\nu}\frac{1}{|B|}\int_B |f(z)|\,\mathrm{d}z \leq C_T^{1/\nu}M[f](x).$$

Gathering all estimates, we get the lemma in this case as well. □

For L^p-boundedness of T^*, we only need the case $\nu = 1$ sincenoth $f \to M[f]$ and $f \to M[|Tf|]$ are of strong type (p,p) (as T is). The remaining of the section consists in proving the T_* is also of weak-type $(1,1)$. We first need one more lemma.

Lemma 7.19. *Let T be a Calderón-Zygmune operator and let $0 < \nu < 1$. Then for every $\lambda > 0$, and every $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$,*

$$|\{x\,:\,\big(M[|Sf|^\nu](x)\big)^{1/\nu} > \lambda\}| \leq C\frac{\|f\|_1}{\lambda}$$

where C depends on the norm of S and d only.

To prove this, we will use the following maximal operator:

Definition 7.20. A dyadic interval is an interval of the form $I_{j,k} = [2^{-k}j, 2^{-k}(j+1)[$ and a dyadic cube is a set of the form $Q_{j,k} = \prod_{\ell=1}^d [2^{-k}j_\ell, 2^{-k}(j_\ell+1)[$. The set of all dyadic cubes is denoted by $\mathcal{D}$.

For $f \in L^1_{loc}(\mathbb{R}^d)$, we define the *Dyadic Maximal Function M_d* as

$$M^d[f](x) = \sup_{x\in Q\in\mathcal{D}} \frac{1}{|Q|}\int_Q |f(u)|\,\mathrm{d}u$$

where the supremum is taken over all dyadic cubes $Q \in \mathcal{D}$ that contain x.

This maximal function is comparable to M in the sense that

$$|\{x\,:\,M[\varphi](x) > c\lambda\}| \leq c'|\{x\,:\,M_d[\varphi](x) > \lambda\}|$$

for some constants c, c' depending only on the dimension. Note that they are not comparable pointwise as one may have $M_d[\varphi](x) = 0$ (e.g. away from the support of φ) while $M[\varphi](x) \neq 0$.

Proof. It is enough to prove the statement for M_d instead of M.

The second property is that there is a constant c_d depending on the dimension only such that for every $\varphi \in L^1(\mathbb{R}^d)$,

$$|\{x\,:\,M_d[\varphi](x) > \lambda\}| \leq c_d \int_{\{x\,:\,M_d[\varphi](x)>\lambda\}} |\varphi(x)|\,\mathrm{d}x.$$

We leave as an exercice to prove this statement using the Calderón-Zygmund decomposition.

We apply this estimate to $\varphi(x) = |Sf|^\nu(x)$ to get

$$\begin{aligned}|\{x\,:\,\big(M_d[|Sf|^\nu](x)\big)^{1/\nu} > \lambda\}| &= |\{x\,:\,M_d[|Sf|^\nu](x) > \lambda^\nu\}|\\ &\leq \frac{c_d}{\lambda^\nu}\int_{\{x\,:\,\left(M_d[|Sf|^\nu](x)\right)^{1/\nu}>\lambda\}} |Sf|^\nu(x)|\,\mathrm{d}x.\end{aligned}$$

But, for any $p > 1$ (exercice based on Hölder) $M_d[\varphi] \leq M[|\varphi|^p]^{1/p}$. In particular, if $p = q/\nu$ with $q > 1 > \nu$

$$\big(M_d[|Sf|^\nu](x)\big)^{1/\nu} \leq \big(M_d[|Sf|^q](x)\big)^{1/q}$$

thus

$$\begin{aligned}\left|\left\{x\,:\,\left(M_d[|Sf|^\nu](x)\right)^{1/\nu}>\lambda\right\}\right| &\leq \left|\left\{x\,:\,\left(M_d[|Sf|^q](x)\right)^{1/q}>\lambda\right\}\right|\\ &\leq \frac{c_d}{\lambda^q}\int_{\{x:\,\left(M_d[|Sf|^q](x)\right)^{1/q}>\lambda\}}|Sf|^q(x)\,\mathrm{d}x\\ &\leq \frac{c_d}{\lambda^q}\int_{\mathbb{R}^d}|Sf|^q(x)\,\mathrm{d}x\leq\frac{c_d}{\lambda^q}\|f\|_q<+\infty\end{aligned}$$

since S is of strong type (q,q) and $f\in L^1\cap L^\infty$ thus also in L^q.

This set being of finite measure, we can apply Kolmogorov's Lemma 2.31 to get

$$\begin{aligned}\left|\left\{x\,:\,\left(M_d[|Sf|^\nu](x)\right)^{1/\nu}>\lambda\right\}\right| &\leq \frac{c_d}{\lambda^\nu}\int_{\{x:\,\left(M_d[|Sf|^\nu](x)\right)^{1/\nu}>\lambda\}}|Sf(x)|^\nu\,\mathrm{d}x\\ &\leq \frac{c_d}{\lambda^\nu}|\{x\,:\,\left(M_d[|Sf|^\nu](x)\right)^{1/\nu}>\lambda\}|^{1-\nu}\|f\|_1^\nu\end{aligned}$$

which is the desired estimate

$$|\{x\,:\,\left(M_d[|Sf|^\nu](x)\right)^{1/\nu}>\lambda\}|\leq\frac{c_d^{1/\nu}}{\lambda}\|f\|_1$$

after simplification. □

The weak $(1,1)$ boundedness. Let $f\in\mathcal{C}_c^\infty(\mathbb{R}^d)$. From Lemma 7.18 we have

$$|\{x\,:\,|T_*f(x)|>\lambda\}|\leq|\{x\,:\,\left(M[|Tf|^\nu](x)\right)^{1/\nu}>\lambda/2C\}|+|\{x\,:\,Mf(x)>\lambda/2C\}|.$$

From the weak $(1,1)$ boundedness of the maximal function,

$$|\{x\,:\,Mf(x)>\lambda/2C\}|\leq\frac{C'}{\lambda}\|f\|_1$$

and from Lemma 7.19 we get that

$$|\{x\,:\,\left(M[|Tf|^\nu](x)\right)^{1/\nu}>\lambda/2C\}|\leq\frac{C'}{\lambda}\|f\|_1.$$

Putting everything together, we get the weak $(1,1)$ bound for $f\in\mathcal{C}_c^\infty(\mathbb{R}^d)$. We then get the general case by density. □

3.3. Extension to the vector valued setting. For future use, note that one may extend results from this section (without difficulty) to the Hilbert valued setting. To start with, let us fix a Hilbert space H. Then a function $f:\mathbb{R}^d\to H$ is said to be measurable if, for every $h\in H$, $x\to\langle f(x),h\rangle$ is measurable which implies that $x\to\|f(x)\|$ is measurable as well since $\|f(x)\|=\sup_{|h\|\leq1}\langle f(x),h\rangle$.

Then, for $1\leq p<+\infty$, $L^p(\mathbb{R}^d,H)$ is the space of all measurable functions such that $\|f\|\in L^p(\mathbb{R}^d)$ with

$$\|f\|_{L^p(\mathbb{R}^d,H)}=\left(\int_{\mathbb{R}^d}\|f(x)\|^p\,\mathrm{d}x\right)^{1/p}$$

with the usual adaptation wen $p=+\infty$.

It is not hard to check (using Riesz' Representation Theorem) that the dual of $L^p(\mathbb{R}^d,H)$ is $L^{p'}(\mathbb{R}^d,H)$ with $1/p+1/p'=1$. The Marcinkiewicz interpolation theorem and the Riesz-Thorin interpolation theorem go through in this setup as well.

Further, if $f\in L^1(\mathbb{R}^d,H)$, we can define is integral as follows: first, if $h\in H$, then $|\langle f(x),h\rangle|\leq\|f(x)\|_H\|h\|_H\in L^1(\mathbb{R}^d)$ so that we may define

$$I_f(h)=\int_{\mathbb{R}^d}\langle f(x),h\rangle\,\mathrm{d}x$$

and check that $h\to I_f(h)$ is linear with $|I_f(h)|\leq\|f\|_{L^1(\mathbb{R}^d,H)}\|h\|_H$. That is $I_f\in H'$ and from Riesz-representation, this means that there is a unique $\mathcal{I}_f\in H$ such that, for every $h\in H$, $I_f(h)=\langle\mathcal{I}_f,h\rangle$. One denotes $\mathcal{I}_f=\displaystyle\int_{\mathbb{R}^d}f(x)\,\mathrm{d}x$ and

$$\langle\int_{\mathbb{R}^d}f(x)\,\mathrm{d}x,h\rangle=\int_{\mathbb{R}^d}\langle f(x),h\rangle\,\mathrm{d}x\qquad\forall h\in H.$$

We can now extend the theory of Calderón-Zygmund operators to the vector valued setting. Let H_1, H_2 be two Hilbert spaces. A function $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathcal{B}(H_1, H_2)$ is a standard kernel if

(1) It is measurable *i.e.* for every $h \in H_1$, the function $Kh : \mathbb{R}^d \times \mathbb{R}^d \to H_2$ is measurable;

(2) If $x \neq y$ then $\|K(x,y)\|_{H_1 \to H_2} \leq \dfrac{C}{|x-y|^d}$;

(3) If $|y-z| \leq \dfrac{1}{2}|x-y|$ then

$$\|K(x,y) - K(x,z)\|_{H_1 \to H_2} \leq \frac{C|y-z|^\alpha}{|x-y|^{d+\alpha}}$$

and if $|x-y| \leq \dfrac{1}{2}|x-z|$ then

$$\|K(x,z) - K(y,z)\|_{H_1 \to H_2} \leq \frac{C|x-y|^\alpha}{|x-z|^{d+\alpha}}$$

where C, α are constants.

A linear operator $T : L^2(\mathbb{R}^d, H_1) \to L^2(\mathbb{R}^d, H_2)$ is then called a Calderón-Zygmund operator if it is bounded and if there exists a standard kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathcal{B}(H_1, H_2)$ such that, if $f \in L^2(\mathbb{R}^d, H_1)$ has compact support and $x \notin \operatorname{supp} f$, then

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y)\,\mathrm{d}y.$$

The following result is proven exactly the same way as in the complex valued case:

THEOREM 7.21. *If T is a Calderón-Zygmund operator, then T (has an extension that) is of weak type $(1,1)$*

$$|\{x \in \mathbb{R}^d \ : \ \|Tf(x)\|_{H_2} > \lambda\}| \leq C \frac{\|f\|_{L^1(\mathbb{R}^d, H_1)}}{\lambda}$$

and of strong type (p,p) for $1 < p < +\infty$: $\|Tf\|_{L^p(\mathbb{R}^d, H_2)} \leq C\|f\|_{L^p(\mathbb{R}^d, H_1)}$.

4. The space $BMO(\mathbb{R}^d)$

4.1. Singular integral operators on L^∞. So far, we have developped a fairly satisfactory theory of singular operators on $L^p \cap L^2$ for $1 < p < +\infty$ (Calderón-Zygmund operators were assumed to be continuous on L^2, we showed that they extend boundedly to $L^p \cap L^2$, $1 < p < +\infty$ and then extended them to all of L^p). The situation is more delicate for $f \in L^\infty$. To see why, let us investigate the formula

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y)\,\mathrm{d}y.$$

This formula is problematic for two reasons

– first it is singular when x is closed to y since $K(x,y) \approx |x-y|^{-n}$ is not integrable when $y \approx x$. To deal with this issue, one can try to localize f away from x.

– A second issue is that the integral does not make sense whe $y \to +\infty$ neither since $K(x,y) \approx |y|^{-n}$ in this case. When $f \in L^p$, $p < +\infty$ we dealt with this problem by using the Hölder inequality which allowed to gain some decrease $|y|^{-np'}$ at infinity. To over come this, one may look at two nearby points x_1, x_2 and consider (formally)

$$Tf(x_1) - Tf(x_2) = \int_{\mathbb{R}^d} \big(K(x_1,y) - K(x_2,y)\big) f(y)\,\mathrm{d}y.$$

Now when $K(x_1,y) - K(x_2,y) \approx |x_1-x_2|^\alpha |x_1-y|^{-n-\alpha} \approx |y|^{-n-\alpha}$ when $|y| \to \infty$. This is sufficient to ensure integrability at infinity in the integral.

To implement this heuristic, let T be a Calderón-Zygmund operator with kernel K. Fix $f \in L^\infty(\mathbb{R}^d)$, Q a cube with center c_Q, $Q^* = (1+2d^{1/2})Q$ then we split f into a local and global part

$$f = f\mathbf{1}_{Q^*} + f(1 - \mathbf{1}_{Q^*})$$

and, for $x \in Q$, define $\tilde{T}f(x)$ by

$$\tilde{T}f(x) = T[f\mathbf{1}_{Q^*}](x) + \int_{\mathbb{R}^d \setminus Q^*} \big(K(x,y) - K(c_Q,y)\big) f(y)\,\mathrm{d}y. \tag{4.45}$$

Several observations are to be made:

– the right-hand side of (4.45) is well defined since $f\mathbf{1}_{Q^*}$ is bounded with compact support thus in L^2 and T is bounded on L^2. On the other hand $|K(x,y)-K(c_Q,y)| \leq |x-c_Q|^\alpha|x-y|^{-d-\alpha}$ when $x \in Q$ and $y \notin Q^*$ (we enlarge Q to Q^* to be able to use the smoothness asumption on K here) thus the intergal in (4.45) converges when f is bounded.

– The operator $\tilde{T}$ is not well defined as an x can belong to several cubes. To overcome this, one may fix a family of cubes $\mathcal{Q} = \{Q_x\}_{x\in\mathbb{R}^d}$ attached to each x and define

$$T^{\mathcal{Q}}f(x) = T[f\mathbf{1}_{Q_x^*}](x) + \int_{\mathbb{R}^d\setminus Q_x^*} K(x,y) - K(c_Q,y)\big)f(y)\,\mathrm{d}y.$$

This would lead to cumbersome checking in the sequel and it is worth noting that two different families $\mathcal{Q},\tilde{\mathcal{Q}}$ would lead to two operators that differ by a constant only : $T^{\tilde{\mathcal{Q}}}f(x) = T^{\mathcal{Q}}f(x)+c_f$. We will thus rather use (4.45) and consider that $\tilde{T}f$ is defined modulo a constant:

Definition 7.22. On the set of functions on $\mathbb{R}^d$, we define the *equivalence class* $f \sim g$ if $f-g$ is a constant function. By (common) abuse of notation, we can consider an equivalence class $[f]$ and identify if with any of its elements f in which case we say that f is defined modulo constants. Note that $f=0$ modulo constants means that f is a constant function.

– When $f \in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ then $\tilde{T}f = Tf$. Note that the left-hand side is only defined modulo a constant.

We now introduce the BMO space which will play a key role soon.

Definition 7.23. For $f \in L^1_{loc}(\mathbb{R}^d)$ and Q a cube, we write

$$f_Q = \frac{1}{|Q|}\int_Q f(x)\,\mathrm{d}x$$

for its mean over Q. The BMO-norm of f is the quantity

$$\|f\|_{BMO} = \sup_Q \frac{1}{|Q|}\int_Q |f(x)-f_Q|\,\mathrm{d}x$$

where the supremum runs over all cubes in $\mathbb{R}^d$. The space $BMO(\mathbb{R}^d)$ is the space of all functions *modulo constants* such that $\|f\|_{BMO} < +\infty$.

It is clear that the space $BMO(\mathbb{R}^d)$ is only defined modulo constants. First, if $g = f+c$ then for every Q, $g_Q = f_Q + c$ so that $\|f\|_{BMO} = \|g\|_{BMO}$. Further, if f is such that $\|f\|_{BMO} = 0$ then for every cube Q $\displaystyle\int_Q |f(x)-f_Q|\,\mathrm{d}x = 0$ so that $f = f_Q$ over Q and f is constant over Q. In particular, f is constant over each $[-n,n]^d$ and letting $n \to +\infty$, we get that f is a constant, that is $f = 0$ modulo constants.

We leave the following proposition as an exercice

Proposition 7.24. (1) *$\|\lambda f\|_{BMO} = |\lambda|\,\|f\|_{BMO}$ and $\|f+g\|_{BMO} \leq \|f\|_{BMO}+\|g\|_{BMO}$;*

(2) *for $f \in L^1_{loc}(\mathbb{R}^d)$, let*

$$\|f\|_{BMO_c} = \sup_Q \inf_{\beta\in\mathbb{C}} \frac{1}{|Q|}\int_Q |f(x)-\beta|\,\mathrm{d}x.$$

Then $\|f\|_{BMO_c} \leq \|f\|_{BMO} \leq 2\|f\|_{BMO_c}$.

This means that, to show that $f \in BMO$ and to estimate $\|f\|_{BMO}$ it is enough to find a number A such that, for each cube Q, a complex number β_Q such that

$$\int_Q |f(x)-\beta_Q|\,\mathrm{d}x \leq A|Q|.$$

This then implies that $\|f\|_{BMO} \leq 2A$.

(3) *BMO is invariant under translations $\tau_a f(x) = f(x-a)$ and dilations $\delta_\lambda f(x) = f(\lambda x)$ with*

$$\|\tau_a f\|_{BMO} = \|\delta_\lambda f\|_{BMO} = \|f\|_{BMO}.$$

(4) *for* $f \in L^1_{loc}(\mathbb{R}^d)$*, and* B *a ball, let*

$$f_B = \frac{1}{|B|}\int_B f(x)\,\mathrm{d}x$$

be the mean of f *over* B*. The* $BMO^{\bigcirc}$*-norm of* f *is the quantities*

$$\|f\|_{BMO^{\bigcirc}} = \sup_B \frac{1}{|B|}\int_B |f(x) - f_Q|\,\mathrm{d}x$$

and

$$\|f\|_{BMO^{\bigcirc}_c} = \sup_B \inf_{a\in\mathbb{C}} \frac{1}{|B|}\int_B |f(x) - a|\,\mathrm{d}x.$$

The space $BMO^{\bigcirc}(\mathbb{R}^d)$ *is the space of all functions* modulo constants *such that* $\|f\|_{BMO^{\bigcirc}} < +\infty$*.*

Then $\|f\|_{BMO^{\bigcirc}_c} \leq \|f\|_{BMO^{\bigcirc}} \leq 2\|f\|_{BMO^{\bigcirc}_c}$*. Further* $BMO^{\bigcirc}\mathbb{R}^d) = BMO(\mathbb{R}^d)$ *and* $\|f\|_{BMO^{\bigcirc}}$ *is equivalent to* $\|f\|_{BMO}$*.*

(5) $L^\infty(\mathbb{R}^d) \subset BMO(\mathbb{R}^d)$ *with* $\|f\|_{BMO} \leq 2\|f\|_\infty$*.*

(6) $\log|x| \in BMO(\mathbb{R}^d)$*. However* $\mathbf{1}_{\mathbb{R}^+}\log x \notin BMO(\mathbb{R})$*.*

Hint: For $BMO = BMO^O$ use the sup inf definition of the norm and the smallest B containing Q (or vice versa) and note note that $|B| \approx |Q|$.

THEOREM 7.25. *Let* T *be a Calderón-Zygmund operator with kernel* K*. Then* $T : L^\infty(\mathbb{R}^d) \to BMO(\mathbb{R}^d)$ *continuously.*

PROOF. We need to show that if Q is a cube (with center c_Q) there exists β_Q such that

$$\frac{1}{|Q|}\int_Q |Tf(x) - \beta_Q|\,\mathrm{d}x \leq C\|f\|_\infty$$

(with C independent of f).

First, for $x \in Q$, we can write

$$Tf(x) = T[f\mathbf{1}_{Q^*}](x) + \int_{\mathbb{R}^d\setminus Q^*}\big(K(x,y) - K(c_Q,y)\big)f(y)\,\mathrm{d}y + \beta_Q$$

where β_Q is a constant depending Q (and which representative of T we have chosen).

Next, from Cauchy-Schwarz and the L^2-boundedness of T,

$$\begin{aligned}\frac{1}{|Q|}\int_Q |T[f\mathbf{1}_{Q^*}](x)|\,\mathrm{d}x &= \frac{1}{|Q|}\int_{\mathbb{R}^d}\mathbf{1}_Q(x)|T[f\mathbf{1}_{Q^*}](x)|\,\mathrm{d}x \leq \frac{1}{|Q|}\|T[f\mathbf{1}_{Q^*}]\|_{L^2(\mathbb{R}^d)}\|\mathbf{1}_Q\|_{L^2(\mathbb{R}^d)}\\ &\leq \frac{1}{|Q|}\|T\|_{L^2\to L^2}\|f\mathbf{1}_{Q^*}\|_2|Q|^{1/2} \leq frac1|Q|^{1/2}\|T\|_{L^2\to L^2}|Q^*|^{1/2}\|f\|_\infty\\ &\leq (1+2d^{1/2})^{d/2}\|T\|_{L^2\to L^2}\|f\|_\infty.\end{aligned}$$

On the other hand, as $|x - c_Q| \leq \dfrac{1}{2}|x-y|$,

$$\begin{aligned}\left|\int_{\mathbb{R}^d\setminus Q^*}\big(K(x,y) - K(c_Q,y)\big)f(y)\,\mathrm{d}y\right| &\leq \int_{\mathbb{R}^d\setminus Q^*}\big|K(x,y) - K(c_Q,y)\big|\,\mathrm{d}y\|f\|_\infty\\ &\leq \int_{y:|y-x|\geq d^{1/2}\ell(Q)}\frac{|x-c_Q|^\alpha}{|x-y|^{d+\alpha}}\,\mathrm{d}y\|f\|_\infty\\ &\leq |x-c_Q|^\alpha\int_{d^{1/2}\ell(Q)}^{+\infty}\frac{r^{d-1}}{r^{d+\alpha}}\,\mathrm{d}r\|f\|_\infty\\ &\leq \frac{1}{\alpha}|x-c_Q|^\alpha\frac{1}{\big(d^{1/2}\ell(Q)\big)^\alpha}\|f\|_\infty \leq \frac{1}{\alpha d^{\alpha/2}}\|f\|_\infty\end{aligned}$$

since $x \in Q$. Thus

$$\sup_Q\left|\int_{\mathbb{R}^d\setminus Q^*}\big(K(x,y) - K(c_Q,y)\big)f(y)\,\mathrm{d}y\right| \leq \frac{1}{\alpha}\|f\|_\infty.$$

Finally

$$\begin{aligned}\frac{1}{|Q|}|Tf(x)-\beta_Q|\,\mathrm{d}x &\le \frac{1}{|Q|}\int_Q |T[f\mathbf{1}_{Q^*}](x)|\,\mathrm{d}x+\sup_Q\left|\int_{\mathbb{R}^d\setminus Q^*}\big(K(x,y)-K(c_Q,y)\big)f(y)\,\mathrm{d}y\right|\\ &\le \left((1+2d^{1/2})^{d/2}\|T\|_{L^2\to L^2}+\frac{1}{\alpha d^{\alpha/2}}\right)\|f\|_\infty\end{aligned}$$

as desired. □

4.2. John-Nirenberg Inequality and interpolation. We have seen that $\log|x|$ belongs to $BMO(\mathbb{R}^d)$. This is, in a sense, the largest possible singularity. This is not a precise statement in the pointwise sense but can be shown for level sets. To start, if $f\in BMO(\mathbb{R}^d)$ with $\|f\|_{BMO}=1$ then, for every cube Q,

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,\mathrm{d}x\le 1.$$

Using Bienaymé-Chebicheff,

$$|\{x\in Q\,:|f(x)-f_Q|\ge\lambda\}|\le\frac{|Q|}{\lambda}.$$

This says that f can exceed its average f_Q by (say) 10 on Q on at most 1/10-th of Q.

It turns out that one can iterate the above fact to give far better estimates in the limit $\lambda\to+\infty$. This is because of a basic principle in harmonic analysis: bad behaviour on a proportionnally small exceptional set (such as 1/10 of any given ball) can often be iterated away if we know that the exceptional sets are small at every scale: f will then exceed its average by 20 on at most 1/10-th of 1/10-th of the cube...

The final statement is the following:

Theorem 7.26 (John-Nirenberg Inequality). *Let $f\in BMO(\mathbb{R}^d)$. Then for every cube Q and every $\lambda>0$,*

$$|\{x\in Q\,:|f(x)-f_Q|\ge\lambda\}|\le 20e^{-2^{-d}\lambda/\|f\|_{BMO}}|Q|.$$

Proof. Fix $f\in BMO(\mathbb{R}^d)$ with $\|f\|_{BMO}=1$. Note that this is not a restriction as we may replace f by $f/\|f\|_{BMO}$.

Let $\psi(\lambda)$ be the best possible constant that one can take in the inequality

$$\forall Q\qquad |\{x\in Q\,:|f(x)-f_Q|\ge\lambda\}|\le\psi(\lambda)|Q|.$$

First note that ψ is non-increasing since for $\lambda'>\lambda$

$$|\{x\in Q\,:|f(x)-f_Q|\ge\lambda'\}|\le|\{x\in Q\,:|f(x)-f_Q|\ge\lambda'\}|\le\psi(\lambda)|Q|$$

thus $\psi(\lambda')\le\psi(\lambda)$. Further, from Bienaymé-Chebicheff $\psi(\lambda)\le 1/\lambda$ and of course, we have the trivial bound $\psi(\lambda)\le 1$ *i.e.*

$$\psi(\lambda)\le\min\left(1,\frac{1}{\lambda}\right).$$

This is of course very bad. To improve this, we will use a variant of the Calderón-Zygmund decomposition. We start with a dyadic cube Q_0 and denote for $m\ge 1$ by $\mathcal{D}_m$ the set of all dyadic cubes Q of side length $\ell(Q)=2^{-m}\ell(Q_0)$. We will write $\mathcal{D}_0$ for the set of dyadic cubes $\subset Q_0$. We will also write $F(x)=|f(x)-f_{Q_0}|$.

Next we will consider $\Lambda>1$ (to be fixed later) and divide the cubes in $\mathcal{D}_0$ into good and bad cubes where a cube Q is bad if

$$\frac{1}{|Q|}\int_Q F(x)\,\mathrm{d}x>\Lambda.$$

As we assumed that $\|f\|_{BMO}=1$, $\frac{1}{|Q_0|}\int_{Q_0}F(x)\,\mathrm{d}x\le 1<\Lambda$ the original cube Q_0 is good (not bad). Further, each bad cube is contained in a maximal bad cube and we set $\mathcal{B}$ for the set of maximal bad cubes.

When $Q\in\mathcal{B}$ is maximal bad, then

$$\Lambda<\frac{1}{|Q|}\int_Q F(x)\,\mathrm{d}x\le 2^d\Lambda$$

Indeed, if $\tilde{Q}$ is the mother of Q, it is not bad so that

$$\frac{1}{2^d|Q|}\int_Q F(x)\,\mathrm{d}x \le \frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} F(x)\,\mathrm{d}x \le \Lambda.$$

But then, for $Q\in\mathcal{B}$,

$$|f_Q - f_{Q_0}| = \left|\frac{1}{|Q|}\int_Q f(x) - f_{Q_0}\,\mathrm{d}x\right| \le \frac{1}{|Q|}\int_Q |F(x)|\,\mathrm{d}x \le 2^d\Lambda.$$

Further, as maximal dyadic cubes are disjoint,

$$\begin{aligned}\sum_{Q\in\mathcal{B}}|Q| &\le \sum_{Q\in\mathcal{B}}\frac{1}{\Lambda}\int_Q F(x)\,\mathrm{d}x = \frac{1}{\Lambda}\int_{\bigcup_{Q\in\mathcal{B}}} F(x)\,\mathrm{d}x\\ &\le \frac{1}{\Lambda}\int_{Q_0} F(x)\,\mathrm{d}x \le \frac{|Q_0|}{\Lambda}\end{aligned}$$

since $\|f\|_{BMO}=1$.

On the other hand, from the Dyadic Maximal Theorem, or rather the Lebesgue differentiation theorem that results from it, for almost every $x\in Q_0\setminus\bigcup_{Q\in\mathcal{B}}Q$, $|f(x)-f_{Q_0}| := F(x)\le\Lambda$.

Now, consider $\lambda > 2^d\Lambda$, then

$$\begin{aligned}\big|\{x\in Q_0\,:\,|f(x)-f_{Q_0}|>\lambda\}\big| &\le \left|\left\{x\in\bigcup_{Q\in\mathcal{B}}Q\,:\,|f(x)-f_{Q_0}|>\lambda\right\}\right|\\ &\le \left|\left\{x\in\bigcup_{Q\in\mathcal{B}}Q\,:\,|f(x)-f_{Q}|>\lambda-|f_Q-f_{Q_0}\right\}\right|\\ &\le \sum_{Q\in\mathcal{B}}\big|\{x\in Q\,:\,|f(x)-f_Q|>\lambda-2^d\Lambda\}\big|\\ &\le \psi(\lambda-2^d\Lambda)\sum_{Q\in\mathcal{B}}|Q|\\ &\le \frac{\psi(\lambda-2^d\Lambda)}{\Lambda}|Q_0|.\end{aligned}$$

This means that $\psi(\lambda)\le\dfrac{\psi(\lambda-2^d\Lambda)}{\Lambda}$ as soon as $\lambda>2^d\Lambda$. We can now bootstrap the argument. Let $N\ge1$ be an integer such that $2^d\Lambda N\le\lambda\le 2^d\Lambda(N+1)$. Since ψ is non-increasing,

$$\psi(\lambda)\le\psi(2^d\Lambda N)\le\frac{\psi(2^d\Lambda N-2^d\Lambda)}{\Lambda}=\frac{\psi\big(2^d\Lambda(N-1)\big)}{\Lambda}=\frac{\psi(2^d\Lambda)}{\Lambda^{N-1}}$$

by a direct induction. As $\psi(x)\le 1/x$ we get

$$\psi(\lambda)\le\frac{1}{2^d\Lambda^N}=2^{-d}e^{-N\ln\Lambda}\le 2^{-d}e^{-(2^{-d}\lambda-1)\ln\Lambda}.$$

We can for instance chose $\Lambda=e$ then for $\lambda>2^de$,

$$\psi(\lambda)\le\frac{e}{2^d}e^{-2^{-d}\lambda}\le e^e e^{-2^{-d}\lambda}.$$

On the other hand, for $\lambda\le 2^de$, $\psi(\lambda)\le 1\le e^e e^{-2^{-d}\lambda}$. We get the result since $e^e\le 20$. □

4.3. *BMO* **and interpolation.** The aim of this section is to show that the space BMO can be used as a substitute of L^∞ in harmonic analysis.

LEMMA 7.27. *Let $1\le p<q<+\infty$ then there exists a constant $C=C(p,q,d)$ such that if $f\in L^p(\mathbb{R}^d)\cap BMO(\mathbb{R}^d)$, then $f\in L^q(\mathbb{R}^d)$ with*

$$\|f\|_{L^q(\mathbb{R}^d)}\le C\|f\|_{L^p(\mathbb{R}^d)}^{\frac{q}{p}}\|f\|_{BMO(\mathbb{R}^d)}^{1-\frac{q}{p}}.$$

PROOF. We may assume that $\|f\|_{BMO(\mathbb{R}^d)} \neq 0$ otherwise there is nothing to prove (f is constant and in L^p thus 0). Then, using homogeneity $f \to f/\|f\|_{BMO}$, we assume that $\|f\|_{BMO(\mathbb{R}^d)} = 1$. Finaly, using dilations $f \to f(\lambda x)$, we may assume that $\|f\|_{L^p} = 1$.

Throughout this proof, C_d is a constant depending on the dimension only and that changes from line to line.

We apply the Calderón-Zygmund decomposition to $|f|^p$ at level 1. This provides us with a family $\mathcal{B}$ of bad cubes. Then, for each $Q \in \mathcal{B}$,

$$|f_Q| = \left|\frac{1}{|Q|}\int_Q f(x)\,\mathrm{d}x\right| \leq \left(\frac{1}{|Q|}\int_Q |f(x)|^p\,\mathrm{d}x\right)^{1/p} \leq C_d. \tag{4.46}$$

with Hölder. Using the John-Nirenberg Inequality, we get, for each $Q \in \mathcal{B}$, and each $\lambda > C_d \geq |f_Q|$

$$\begin{aligned} |\{x \in Q \;\; |f(x)| > \lambda\}| &\leq |\{x \in Q \;\; |f(x) - f_Q| > \lambda - |f_Q|\}| \\ &\leq C_d e^{2^{-d}\lambda_Q} e^{-2^{-d}\lambda}|Q| \leq C_d e^{C_d \alpha^{1/p}} e^{-2^{-d}\lambda}|Q| \end{aligned}$$

with (4.46). Since $|f(x)| < C_d$ on $\mathbb{R}^d \setminus \bigcup_{Q \in \mathcal{B}} Q$, we get, for every $\lambda > C_d$,

$$\begin{aligned} |\{x \in \mathbb{R}^d \;\; |f(x)| > \lambda\}| &\leq \sum_{Q \in \mathcal{B}} |\{x \in Q \;\; |f(x)| > \lambda\}| \\ &\leq C_d e^{-C_d\lambda} \sum_{Q \in \mathcal{B}} |Q| \\ &\leq C_d e^{-C_d\lambda} \|f\|^p_{L^p(\mathbb{R}^d)} = C_d e^{-C_d\lambda} \end{aligned}$$

where we have used the fact that the bad cubes in the Calderón-Zygmund decomposition are disjoint and then that their total volume is controled by the L^1-norm of $|f|^p$.

On the other hand, we have

$$|\{x \in \mathbb{R}^d \;\; |f(x)| > \lambda\}| \leq \frac{\|f\|^p_{L^p(\mathbb{R}^d}}{\lambda^p} = \lambda^{-p}.$$

We conclude writing

$$\|f\|_q^q = \int_0^{+\infty} \lambda^{q-1} |\{x \in \mathbb{R}^d \;\; |f(x)| > \lambda\}|\,\mathrm{d}\lambda \leq \int_0^{C_d} \lambda^{q-p-1}\,\mathrm{d}\lambda + C_d \int_{C_d}^{+\infty} \lambda^{q-1} e^{-C_d\lambda}\,\mathrm{d}\lambda$$

which gives the result. □

REMARK 7.28. Note that if T is a Calderón-Zygmund operator, and $1 < p < q < +\infty$ then

$$\|T[f]\|_q \leq C\|T[f]\|_p^{q/p}\|T[f]\|_{BMO}^{1-q/p} \leq C'\|f\|_p^{q/p}\|f\|_\infty^{1-q/p}.$$

DEFINITION 7.29. The *sharp maximal function* is defined for $f \in L^1_{loc}(\mathbb{R}^d)$ by

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(x) - f_Q|\,\mathrm{d}x$$

where the supremum is taken over all cubes containing x. In particular, $f \in BMO(\mathbb{R}^d)$ if and only if $M^\sharp f \in L^\infty(\mathbb{R}^d)$ with $\|f\|_{BMO} = \|M^\sharp f\|_\infty$.

Note that, from the triangle inequality, if $x \in Q$

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,\mathrm{d}x \leq \frac{2}{|Q|}\int_Q |f(x)|\,\mathrm{d}x \leq 2\mathcal{M}^\square[f](x)$$

where $\mathcal{M}^\square$ is the uncentered Hardy-Littlewood maximal function associated to cubes (ℓ^∞-balls). It follows that $M^\sharp f \leq 2\mathcal{M}^\square[f]$ pointwise.

Recall that the dyadic maximal function was defined as

$$M^d f(x) = \sup_{x \in Q \in \mathcal{D}} \frac{1}{|Q|}\int_Q |f(x)|\,\mathrm{d}x$$

where the supremum is taken over all *dyadic* cubes containing x. The dyadic maximal function is bounded by the $\mathcal{M}^\square[f]$. A converse bound can not hold ($M^d[f]$ may be zero e.g. in dimension 1 when f is supported in $[0,+\infty)$ while $\mathcal{M}^{\square}[f]$ never is). As $\mathcal{M}^{\square}$ is of weak type $(1,1)$ and of strong type (p,p) for every $p>1$, so is M^d.

However, it is possible to reverse the inequality in the L^p-sense:

THEOREM 7.30 (Feffermann-Stein). *Let $1\leq p_0\leq p<+\infty$ then, for $f\in L^1_{loc}(\mathbb{R}^d)$ such that $M_d[f]\in L^{p_0}(\mathbb{R}^d)$,*

$$\|M^d[f]\|_p\leq C\|M^\sharp[f]\|_p$$

where C depends on d,p only.

The crux of the proof is the following *good-λ inequality.*

LEMMA 7.31. *Let $f\in L^1_{loc}(\mathbb{R}^d)$, $\lambda,\gamma>0$. Then*

$$|\{x\in\mathbb{R}^d\ :\ M^d[f](x)>2\lambda\,,\ M^\sharp[f](x)\leq\gamma\lambda\}|\leq 2^d\gamma|\{x\in\mathbb{R}^d\ :\ M^d[f](x)>\lambda\}| \tag{4.47}$$

PROOF. Again C_d will be a constant depending on d only.

Without loss of generality, we can assume that $\Omega_\lambda:=\{x\in\mathbb{R}^d\ :\ M^d[f](x)>\lambda\}$ has finite measure. Then, for each $x\in\Omega_\lambda$, there is a *maximal* dyadic cube Q_x such that

$$\frac{1}{|Q_x|}\int_{Q_x}|f(y)|\,\mathrm{d}y>\lambda$$

(otherwise Ω_λ would have infinite measure). Write $\{Q_j\}_{j\in J}$ to be the collection of maximal dyadic cubes obtained from Ω_λ *i.e.* $\{Q_j\}_{j\in J}=\{Q_x\ :x\in\Omega_\lambda\}$. Further, each x belongs to one Q_j and maximal dyadic cubes are disjoint so that $\bigcup_{j\in J}Q_j=\Omega_x$ is a partition. It is thus sufficient to prove that

$$|\{x\in Q_j\ :\ M^d[f](x)>2\lambda\,,\ M^\sharp[f](x)\leq\gamma\lambda\}|\leq 2^d\gamma|Q_j|$$

and to sum over $j\in J$ to obtain (4.47).

From now on, j is fixed and we can drop the index. Let $x\in Q$ be such that $M^d[f](x)>2\lambda$. Note that

$$M^d[f](x)=\sup_{R\in\mathcal{D},x\in R}\frac{1}{|R|}\int_R|f(x)|\,\mathrm{d}x$$

so that the supremum is taken over dyadic cubes that intersect Q (since both contain x) that is dyadic cubes that are either included in Q or contain Q. In the case $Q\subset R$, $Q\neq R$, the maximality of Q implies that

$$\frac{1}{|R|}\int_R|f(y)|\,\mathrm{d}y\leq\lambda$$

and such a cube can be discarded from the maximum since $M^d[f](x)>2\lambda$. That is, if $x\in Q$ ie such that $M^d[f](x)>2\lambda$, then

$$M^d[f](x)=\sup_{R\in\mathcal{D},x\in R\subset Q}\frac{1}{|R|}\int_R|f(x)|\,\mathrm{d}x.$$

In particular, we may replace f by $f\mathbf{1}_Q$ and assume that $M^d[f\mathbf{1}_Q](x)>2\lambda$.

Now let Q' be the mother of Q (the unique dyadic cube containing Q of twice the size) and note that the maximality of Q implies that

$$|f_{Q'}|=\left|\frac{1}{|Q'|}\int_{Q'}f(x)\,\mathrm{d}x\right|\leq\frac{1}{|Q'|}\int_{Q'}|f(x)|\,\mathrm{d}x\leq\lambda.$$

Therefore, for $x\in Q$,

$$M_d[(f-f_{Q'})\mathbf{1}_Q]\geq M_d[f\mathbf{1}_Q]-f_{Q'}>2\lambda-\lambda=\lambda.$$

We conclude that

$$|\{x\in Q\ :\ M^d[f](x)>2\lambda\}|\leq|\{x\in Q\ :\ M_d[(f-f_{Q'})\mathbf{1}_Q]>2\lambda\}|.$$

As the dyadic maximal function is of weak-type $(1,1)$ (with constant 1) we get

$$\begin{aligned}
|\{x\in Q\ :\ M^d[f](x)>2\lambda\}| &\leq \frac{1}{\lambda}\int_Q |(f(x)-f_{Q'}|\,\mathrm{d}x\\
&\leq \frac{2^d|Q|}{\lambda}\frac{1}{|Q'|}\int_{Q'}|(f(x)-f_{Q'}|\,\mathrm{d}x\\
&\leq \frac{2^d|Q|}{\lambda}M^\sharp[f](\xi) \qquad (4.48)
\end{aligned}$$

for any $\xi\in Q'$ thus also for any $\xi\in Q$.

Now observe that, either there is no $\xi\in Q$ such that $M^\sharp[f](\xi)>\gamma\lambda$, in which case

$$|\{x\in Q_j\ :\ M^d[f](x)>2\lambda\,,\ M^\sharp[f](x)\leq\gamma\lambda\}|=0$$

or such a ξ exists and then, putting it into (4.48), we get

$$|\{x\in Q_j\ :\ M^d[f](x)>2\lambda\,,\ M^\sharp[f](x)\leq\gamma\lambda\}|\leq|\{x\in Q\ :\ M^d[f](x)>2\lambda\}|\leq\frac{2^d|Q|}{\lambda}\gamma\lambda=2^d\gamma|Q|$$

as claimed. □

We can now conclude:

Proof of Theorem 7.30. We fix $p_0\leq p<+\infty$ and, for $r>0$, we write

$$I(r)=\int_0^r p\lambda^{p-1}|\{x\in\mathbb{R}^d\ :\ M_d[f]>\lambda\}\,\mathrm{d}\lambda.$$

First note that, as $p>p_0$

$$I(r)=\frac{p}{p_0}r^{p-p_0}\int_0^r p_0\lambda^{p_0-1}|\{x\in\mathbb{R}^d\ :\ M_d[f]>\lambda\}\,\mathrm{d}\lambda\leq\frac{p}{p_0}r^{p-p_0}\|M_d[f]\|^{p_0}_{L^{p_0}(\mathbb{R}^d)}<+\infty.$$

Next, changing variable $\lambda\to2\lambda$ and using the previous Lemma, we get

$$\begin{aligned}
I(r) &= 2^p\int_0^{r/2}p\lambda^{p-1}|\{x\in\mathbb{R}^d\ :\ M_d[f]>2\lambda\}\,\mathrm{d}\lambda\\
&\leq 2^p\int_0^{r/2}p\lambda^{p-1}|\{x\in\mathbb{R}^d\ :\ M_d[f]>2\lambda,\ M^\sharp[f]\leq\lambda\gamma\}\,\mathrm{d}\lambda\\
&\quad+2^p\int_0^{r/2}p\lambda^{p-1}|\{x\in\mathbb{R}^d\ :\ M^\sharp[f]>\lambda\gamma\}\,\mathrm{d}\lambda\\
&\leq 2^p2^d\gamma\int_0^{r/2}p\lambda^{p-1}|\{x\in\mathbb{R}^d\ :\ M_d[f]>\lambda\}\,\mathrm{d}\lambda\\
&\quad+2^p\int_0^{r/2}p\lambda^{p-1}|\{x\in\mathbb{R}^d\ :\ M^\sharp[f]>\lambda\gamma\}\,\mathrm{d}\lambda\\
&\leq 2^p2^d\gamma I(r)+2^p\int_0^{r/2}p\lambda^{p-1}|\{x\in\mathbb{R}^d\ :\ M^\sharp[f]>\lambda\gamma\}\,\mathrm{d}\lambda.
\end{aligned}$$

We now chose $\gamma=2^{-(p+d+1)}$ so that the factor in front of $I(r)$ on the right-hand side is $1/2$ and it can be put to the left-hand side. We get

$$\begin{aligned}
I(r) &\leq 2^{p+1}\int_0^{r/2}p\lambda^{p-1}|\{x\in\mathbb{R}^d\ :\ M^\sharp[f]>2^{-(p+d+1)}\lambda\}|\,\mathrm{d}\lambda\\
&= 2^{p+1+p(p+d+1)}\int_0^{2^{-(p+d+2)r}}p\lambda^{p-1}|\{x\in\mathbb{R}^d\ :\ M^\sharp[f]>\lambda\}|\,\mathrm{d}\lambda\\
&\leq 2^{2p+d+2+(p-1)(p+d+1)}\|M^\sharp[f]\|^p_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

Letting $r\to+\infty$ in the left-hand side, we get the result. □

From the Lebesgue differentiation theorem associated to M^d, we get $|f|\leq M^d[f]$ a.e., thus $\|f\|_p\leq\|M_d[f]\|_p$. We conclude that

COROLLARY 7.32 (Feffermann-Stein). *Let $1 \leq p_0 \leq p < +\infty$ then, for $f \in L^1_{loc}(\mathbb{R}^d)$ such that $M_d[f] \in L^{p_0}(\mathbb{R}^d)$,*

$$\|f\|_p \leq C\|M^\sharp[f]\|_p$$

where C depends on d, p only.

THEOREM 7.33. *Let $1 \leq p_0 < p < +\infty$. Let T be a linear operator, bounded from $L^{p_0}(\mathbb{R}^d)$ to $L^{p_0}(\mathbb{R}^d)$ with bound A_{p_0} and from $L^\infty(\mathbb{R}^d)$ to $BMO(\mathbb{R}^d)$ with bound A_∞. Then T extends to a bounded linear operator from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ with*

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \leq CA_{p_0}^{\frac{p_0}{p}} A_\infty^{1-\frac{p_0}{p}} \|f\|_{L^p(\mathbb{R}^d)}. \tag{4.49}$$

Here C depends on p, p_0 and d only.

PROOF. First, $T(f)$ is *a priori* only defined on $L^p(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ but (4.49) then allows to extend T to all of $L^p(\mathbb{R}^d)$.

We consider the sub-linear operator $S = M^\sharp[Tf]$ which is bounded from L^{p_0} to itself with bound $C_{p_0,d}A_0p_0$ (if p_0=1 it sends L^1 to the weak L^1 space) and from L^∞ to itself with bound C_dA_∞ since $M^\sharp$ sends BMO into L^∞ with a bound depending on the dimension only.

It remains to apply Marcinkiewicz interpolation to conclude. □

CHAPTER 8

Litllewood-Paley and multipliers

1. Fourier-multiplies

Among the most common operators met in mathematics, one finds operators that take the form of convolutions $Tf = K * f$. For instance, Calderón-Zygmund operators with standard kernel in the form $K(x, y) = k(x - y)$ can be considered as being of this form. The previous theory then applies to give a meaning to the associated operator and shows that is also of strong type (p, p) for $1 < p < +\infty$, of weak type $(1, 1)$... once it is of strong type $(2, 2)$.

For L^2-boundedness, we have a strong tool at hand, which is the Fourier transform

$$\widehat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}^d} f(x)e^{-2i\pi\langle x,\xi\rangle}\,\mathrm{d}x$$

(first defined on $\mathcal{S}(\mathbb{R}^d)$ and then extended to $L^2(\mathbb{R}^d)$ thanks to Parseval's relation $\|\mathcal{F}[f]\|_2 = \|f\|_2$. If k is a nice function then, using the convolution theorem $\widehat{k * f} = \widehat{k}\widehat{f}$ we can write

$$k * f = \mathcal{F}^{-1}[\widehat{k}\widehat{f}]$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform, given on $\mathcal{S}(\mathbb{R}^d)$ by

$$\mathcal{F}^{-1}[\varphi](x) = \int_{\mathbb{R}^d} \varphi(\xi)e^{2i\pi\langle x,\xi\rangle}\,\mathrm{d}\xi.$$

This leads us to the following definition:

Definition 8.1. Let $m \in L^\infty(\mathbb{R}^d)$ be a bounded function and $1 \leq p < +\infty$. The *Fourier multiplier* associated to m is the operator defined on $L^2(\mathbb{R}^d)$ given by

$$T_m[f] = \mathcal{F}^{-1}[m\widehat{f}].$$

If T_m extends to a bounded linear operator on L^p, we say that m is an *L^p-Fourier-multiplier* and write $m \in \mathcal{M}^p(\mathbb{R}^d)$ and

$$\|m\|_{\mathcal{M}^p} = \|T_m\|_{L^p \to L^p}.$$

Example 8.2. The Hilbert transform is the Fourier multiplier associated to $m(\xi) = -i\,\mathrm{sign}(\xi)$.

The first observation is that T_m is well-defined since $f \in L^2$ imply $\widehat{f} \in L^2$ and, as $m \in L^\infty$, $m\widehat{f} \in L^2$ thus $\mathcal{F}^{-1}[m\widehat{f}]$ is well defined. Further

$$\|T_m[f]\|_2 = \|\mathcal{F}^{-1}[m\widehat{f}]\|_2 = \|m\widehat{f}\|_2 \leq \|m\|_\infty\|\widehat{f}\|_2 = \|m\|_\infty\|f\|_2$$

thus T_m is bounded with $\|m\|_{\mathcal{M}^2} = \|T_m\|_{L^2 \to L^2} \leq |m\|_\infty$.

On the other hand, for $0 < \varepsilon < \|m\|_\infty$ let E_ε be a set of finite measure on which $|m| \geq \|m\|_\infty - \varepsilon$ and $f = \mathcal{F}^{-1}[\mathbf{1}_{E_\varepsilon}]$. Then

$$\|T_m[f]\|_2 = \|\mathcal{F}^{-1}[m\mathbf{1}_{E_\varepsilon}]\|_2 = \|m\mathbf{1}_{E_\varepsilon}\|_2 \geq (\|m\|_\infty - \varepsilon)\|\mathbf{1}_{E_\varepsilon}\|_2 = (\|m\|_\infty - \varepsilon)\|f\|_2$$

which shows that $\|T_m\|_{L^2 \to L^2} \geq \|m\|_\infty - \varepsilon$. Letting $\varepsilon \to 0$ we finally get

Proposition 8.3. *If $m \in L^\infty$, then the Fourier-multiplier T_m associated to f is bounded $L^2 \to L^2$ and $\|m\|_{\mathcal{M}^2} = \|m\|_\infty$.*

We are now interested in its extension to L^p. The first observation is the following:

Proposition 8.4. *Let $m \in L^\infty(\mathbb{R}^d)$, $1 < p < +\infty$ and $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.*

(1) *The adjoint of T_m is $T_m^* = T_{\bar{m}}$.*

(2) *$m\in\mathcal{M}^p(\mathbb{R}^d)$ if and only if $\bar m\in L^p(\mathbb{R}^d)$ if and only if $m\in\mathcal{M}^{p'}(\mathbb{R}^d)$ with*

$$\|m\|_{\mathcal{M}^p}=\|\bar m\|_{\mathcal{M}^p}=\|m\|_{\mathcal{M}^{p'}}$$

(3) *if $m\in\mathcal{M}^p(\mathbb{R}^d)$ then $m\in\mathcal{M}^q(\mathbb{R}^d)$ for every $q\in[p,p']$ and*

$$\|m\|_\infty\le\|m\|_{\mathcal{M}^q}\le\|m\|_{\mathcal{M}^p}.$$

PROOF. For $f,g\in\mathcal{S}(\mathbb{R}^d)$, we have from Parseval

$$\begin{aligned}\langle T_mf,g\rangle=\int_{\mathbb{R}^d}T_mf(x)\overline{g(x)}\,\mathrm{d}x&=\int_{\mathbb{R}^d}m(\xi)\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,\mathrm{d}\xi\\&=\int_{\mathbb{R}^d}\widehat{f}(\xi)\overline{\overline{m(\xi)}\widehat{g}(\xi)}\,\mathrm{d}\xi=\int_{\mathbb{R}^d}f(x)\overline{T_{\bar m}g(x)}\,\mathrm{d}x=\langle f,T_{\bar m}g\rangle.\end{aligned}$$

This shows that $T_m^*=T_{\bar m}$. Then, using $L^p-L^{p'}$-duality

$$\begin{aligned}\|m\|_{\mathcal{M}^p}=\|T_m\|_{L^p\to L^p}&=\sup_{f,g\in\mathcal{S}(\mathbb{R}^d),\|f\|_p=\|g\|_{p'}=1}|\langle T_mf,g\rangle|\\&=\sup_{f,g\in\mathcal{S}(\mathbb{R}^d),\|f\|_p=\|g\|_{p'}=1}|\langle f,T_{\bar m}g\rangle|=\|T_{\bar m}\|_{L^{p'}\to L^{p'}}=\|\bar m\|_{\mathcal{M}^{p'}}.\end{aligned}$$

It remains to notice that if we denote by $f^*(x)=\overline{f(-x)}$ then $\widehat{f^*}=\overline{\widehat{f}}$ so that

$$\langle T_mf^*,g^*\rangle=\int_{\mathbb{R}^d}\overline{\bar m(\xi)\widehat{f}(\xi)}\widehat{g}(\xi)\,\mathrm{d}\xi=\overline{T_{\bar m}f,g}$$

thus

$$\begin{aligned}\|m\|_{\mathcal{M}^p}=\|T_m\|_{L^p\to L^p}&=\sup_{f,g\in\mathcal{S}(\mathbb{R}^d),\|f\|_p=\|g\|_{p'}=1}|\langle T_mf,g\rangle|=\sup_{f,g\in\mathcal{S}(\mathbb{R}^d),\|f\|_p=\|g\|_{p'}=1}|\langle T_mf^*,g^*\rangle|\\&=\sup_{f,g\in\mathcal{S}(\mathbb{R}^d),\|f\|_p=\|g\|_{p'}=1}|\langle T_{\bar m}f,g\rangle|=\|T_{\bar m}\|_{L^p\to L^p}=\|\bar m\|_{\mathcal{M}^p}.\end{aligned}$$

For the last part, Let $q\in[p,p']$ and let $\theta\in(0,1)$ be such that $\dfrac{1}{q}=\dfrac{\theta}{p}+\dfrac{1-\theta}{p'}$. Using Riesz-Thorin, we get

$$\|m\|_{\mathcal{M}^q}=\|T_m\|_{L^q\to L^q}\le\|T_m\|_{L^p\to L^p}^\theta\|T_m\|_{L^{p'}\to L^{p'}}^{1-\theta}=\|m\|_{\mathcal{M}^p}^\theta\|m\|_{\mathcal{M}^{p'}}^{1-\theta}=\|m\|_{\mathcal{M}^p}.$$

In particular, when $q=2$, we get $\|m\|_\infty=\|T_m\|_{L^2\to L^2}\le\|m\|_{\mathcal{M}^p}$. □

REMARK 8.5. One could defined Fourier multipliers for $m\in L^1_{loc}$ via $T_mf=\mathcal{F}^{-1}[m\widehat{f}]$ where f is such that (say) $\widehat{f}$ is bounded with compact support (this set of functions is dense in L^p, $1\le p<+\infty$). One can then show along the lines above that T_m is bounded on L^2 if and only if $m\in L^\infty$.

The proposition also holds in this case so that if $m\in\mathcal{M}_p$ then $m\in L^\infty$. There is therefore no gain in weakening the assumption $m\in L^\infty$.

We have just seen that L^p multipliers are always bounded, but the opposite is not true. The difficulty in the theory of multipliers is precisely to get away from the case $p=2$.

Let us start with a simple case. One may write $T_mf=(\mathcal{F}^{-1}m)*f$ and $\mathcal{F}^{-1}m\in\mathcal{S}'(\mathbb{R}^d)$ (at least when $f\in\mathcal{S}$). If it happens that $K=\mathcal{F}^{-1}m$ is an L^1-function (or equivalently $\widehat{m}\in L^1$), then we can use Young's Inequality and get

$$\|T_mf\|_p=\|K*f\|_p\le\|K\|_1\|f\|_p$$

which shows that $m\in\mathcal{M}^p$ with $\|m\|_{\mathcal{M}^p}\le\|\widehat{m}\|_1$. A simple condition on m that ensures this is given in the scale of Sobolev spaces:

DEFINITION 8.6. For $s\ge0$, the *Sobolev space* $W^{s,2}(\mathbb{R}^d)$ is the space of L^2-functions such that

$$\|f\|_{W^{2,s}}^2=\int_{\mathbb{R}^d}|\widehat{f}(\xi)|^2(1+|\xi|^2)^s\,\mathrm{d}\xi<+\infty.$$

PROPOSITION 8.7. *If $s>d/2$ and $m\in W^{s,2}(\mathbb{R}^d)$, then $\widehat{m}\in L^1(\mathbb{R}^d)$ with $\|\widehat{m}\|_1\le C_{s,d}\|m\|_{W^{2,s}}$. In particular, for every $1<p<+\infty$, $m\in\mathcal{M}^p$ with $\|m\|_{\mathcal{M}^p}\le C_{s,d}\|m\|_{W^{2,s}}$.*

PROOF. This follows from Cauchy-Schwarz writing

$$\int_{\mathbb{R}^d} |\widehat{m}(\xi)|\,\mathrm{d}\xi = \int_{\mathbb{R}^d} |\widehat{m}(\xi)|(1+|\xi|^2)^{s/2}(1+|\xi|^2)^{-s/2}\,\mathrm{d}\xi \leq C_{s,d}\int_{\mathbb{R}^d} |\widehat{m}(\xi)|^2(1+|\xi|^2)^s\,\mathrm{d}\xi$$

with $C_{s,d} = \displaystyle\int_{\mathbb{R}^d}(1+|\xi|^2)^{-s}\,\mathrm{d}\xi < +\infty$ when $s > d/2$. □

2. Littlewood-Paley theory

2.1. Littlewood-Paley decomposition. We here build a smooth Littlewood-Paley decomposition, that will be fixed in the remaining of this chapter. This is done as follows:

– we fix a function $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$, radial, $0 \leq \phi \leq 1$, $\phi(\xi) = 1$ when $|\xi| \leq 1$ and $\phi(\xi) = 0$ when $|\xi| \geq 2$;

– we define

$$\psi(\xi) = \phi(\xi) - \phi(2\xi)$$

and notice that $\psi \in \mathcal{C}^\infty$, is radial, supported in $\{\xi \in \mathbb{R}^d : 1/2 \leq |\xi| \leq 2\}$ and $0 \leq \psi \leq 1$. This is because $\phi(\xi) = \phi(2\xi) = 1$ when $0 \leq |\xi| \leq 1/2$, $0 \leq \phi(2\xi) \leq 1 = \phi(\xi)$ when $1/2 \leq |\xi| \leq 1$, $0 = \phi(2\xi) \leq \phi(\xi) \leq 1$ when $1 \leq |\xi| \leq 2$ and $\phi(\xi) = \phi(2\xi) = 0$ when $|\xi| \geq 2$. Note that $\psi(u) = 1$ if $|u| = 1$.

– for $j \in \mathbb{Z}$, we then define $\psi_j(\xi) = \psi(2^{-j}\xi)$ and notice that ψ_j is supported in $\{\xi \in \mathbb{R}^d : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ and forms a *partition of unity*

$$\sum_{j\in\mathbb{Z}} \psi_j(\xi) = 1 \qquad \xi \in \mathbb{R}^d \setminus \{0\}.$$

Indeed, note that $\psi_j(\xi) = 0$ unless $2^{j-1} < |\xi| < 2^{j+1}$ and $\psi_j(2^j u) = 1$ when $|u| = 1$. In particular, for $|u| = 1$, $\psi_j(2^\ell u) = \delta_{j,\ell}$ thus, if $|\xi| = 2^\ell$ (there is at most one such ℓ)

$$\sum_{j\in\mathbb{Z}} \psi_j(\xi) = \sum_{j\in\mathbb{Z}} \delta_{j,\ell} = 1.$$

On the other hand, for $\xi \neq 0$ with $|\xi| \neq 2^k$, $k \in \mathbb{Z}$, there is a unique ℓ such that $2^\ell < |\xi| < 2^{\ell+1}$ and then $\psi_j(\xi) = 0$ unless $j = \ell$ or $j = \ell + 1$. Then

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \psi_j(\xi) &= \psi_\ell(\xi) + \psi_{\ell+1}(\xi) = \phi(2^{-\ell}\xi) - \phi(2^{-\ell+1}\xi) + \phi(2^{-\ell-1}\xi) - \phi(2^{-\ell}\xi)\\ &= \phi(2^{-\ell-1}\xi) - \phi(2^{-\ell+1}\xi) = 1 - 0\end{aligned}$$

since $|2^{-\ell-1}\xi| \leq 1$ and $|2^{-\ell+1}\xi| \geq 2$.

Further, through the same computation

$$\sum_{j=-\infty}^{k} \psi_j(\xi) = \phi(\xi/2^k)$$

when $\xi \neq 0$. This is extended by continuity for $\xi = 0$.

Next we associate some multiplicators to this partition: for $f \in \mathcal{S}(\mathbb{R}^d)$, let

– $\Delta_j f = \mathcal{F}^{-1}[\psi_j \widehat{f}]$ and

– $S_k f = \displaystyle\sum_{j=-\infty}^{k} \Delta_j f$.

Those operators are better defined on the Fourier side as

$$\widehat{\Delta_j f}(\xi) = \psi_j(\xi)\widehat{f}(\xi) \quad \text{and} \quad \widehat{S_k f}(\xi) = \phi(\xi/2^k)\widehat{f}(\xi).$$

In particular, both operators consist in multiplying $\widehat{f}$ by a smooth compactly supported function (thus preserving $\mathcal{S}$) and then taking inverse Fourier transform (which also preserves $\mathcal{S}$). Thus $\Delta_j, S_k : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. Using Parseval, it is trivial to see that $\|\Delta_j f\|_2, \|S_k f\|_2 \leq \|f\|_2$ so that they both extend into bounded linear operators $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

We now list their simplest properties:

PROPOSITION 8.8. *For $f \in L^2(\mathbb{R}^d)$,*

(1) $\Delta_j f = S_j f - S_{j-1} f$;

(2) $\lim_{k\to-\infty} S_k f = 0$ *and* $\lim_{k\to+\infty} S_k f = f$ *(with limits in the L^2 sense);*
(3) *For every $f \in L^2(\mathbb{R}^d)$, in the L^2-sense,*

$$\sum_{j\in\mathbb{Z}} \Delta_j f = f.$$

We leave this simple proposition as an exercice, all statements follow from Parseval and dominated convergence.

The third statement is called the Littlewood-Paley decomposition of $f \in L^2(\mathbb{R}^d)$ and consists in breaking up f in pieces which are localized on the Fourier side around $|\xi| \approx 2^j$.

One may further notice that $\psi_j(x) = \delta_{2^{-j}}\psi(x)$ where $\delta_\lambda f(x) = f(\lambda x)$ has inverse Fourier transform given by $2^{jd}\delta_{2^j}\widehat{\psi}$ (since the inverse Fourier transform of ψ is also its Fourier transform). Using the fact that the Fourier transform of a convolution is the product of Fourier transfoms, we may identify

$$\Delta_j f = 2^{jd}(\delta_{2^j}\widehat{\psi}) * f$$

and, in the same way $S_k f = 2^{kd}(\delta_{2^k}\widehat{\phi}) * f$. As $\widehat{\phi}, \widehat{\psi} \in \mathcal{S}(\mathbb{R}^d)$, these operators are thus well defined on every L^p-space thanks of Young's Inequality. One may also show that, taking Fourier transform in the sense of distributions,

$$\widehat{\Delta_j f} = \delta_{2^{-j}}\psi\widehat{f}$$

which is thus supported in $\{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$.

Our aim is to extend the decomposition of f to $L^p(\mathbb{R}^d)$.

2.2. Littlewood-Paley decomposition and differentiation. Recall that $\widehat{\partial^\alpha f}(\xi) = (2i\pi\xi)^\alpha \widehat{f}(\xi)$ so that

$$|\widehat{\nabla f}(\xi)|^2 = 4\pi^2|\xi|^2|\widehat{f}(\xi)|^2.$$

In particular, if Δ_j is as in the previous section, then $\widehat{\Delta_j f}$ is supported in an annulus $\{\xi : 2^{-j-1} \leq |\xi| \leq 2^{j+1}\}$ so that

$$\pi^2 2^{2j}|\widehat{\Delta_j f}(\xi)|^2 \leq |\widehat{\nabla\Delta_j f}(\xi)|^2 \leq 16\pi^2 2^{2j}|\widehat{\Delta_j f}(\xi)|^2.$$

Parseval then implies that

$$\pi 2^j\|\Delta_j f\|_2 \leq \|\nabla\Delta_j f\|_2 \leq 4\pi 2^j\|\Delta_j f\|_2.$$

This fact is valid in any L^p-space:

PROPOSITION 8.9. *Let $1 < p < +\infty$. There exists a constant C depending on d, p only such that, if $f \in L^p(\mathbb{R}^d)$ and $j \in \mathbb{Z}$, then*

$$frac1C 2^j\|\Delta_j f\|_p \leq \|\nabla\Delta_j f\|_p \leq C2^j\|\Delta_j f\|_p.$$

PROOF. Recall that

$$\Delta_j f(x) = 2^{jd}(\delta_{2^j}\widehat{\psi}) * f(x) = 2^{jd}\int_{\mathbb{R}^d}\widehat{\psi}\big(2^j(x-y)\big)f(y)\,\mathrm{d}y$$

As ψ is a $\mathcal{C}^\infty$ with compact support, $\psi \in \mathcal{S}(\mathbb{R}^d)$ and thus $\widehat{\psi} \in \mathcal{S}(\mathbb{R}^d)$. One easily checks that one may differentiate this integral

$$\nabla\Delta_j f(x) = 2^j\int_{\mathbb{R}^d} 2^{jd}\nabla\widehat{\psi}\big(2^j(x-y)\big)f(y)\,\mathrm{d}y = 2^j\int_{\mathbb{R}^d} 2^{jd}\nabla\widehat{\psi}(2^j y)f(x-y)\,\mathrm{d}y = 2^j\big(2^{jd}\delta_{2^j}\nabla\widehat{\psi}\big) * f$$

It remains to apply Young's Inequality to get

$$\|\nabla\Delta_j f\|_p \leq 2^j\left\|2^{jd}\delta_{2^j}\nabla\widehat{\psi}\right\|_1\|f\|_p = 2^j\|\nabla\psi\|_1\|f\|_p$$

which gives the upper bound we are looking for (that depends on ψ). Note that this implies that if $f \in L^p$ then $\nabla\Delta_j f \in L^p$.

We now turn to the lower bound. This is done essentially by inverting ∇. To do so, we introduce a second Littlewood-Paley function $\rho \in \mathcal{C}^\infty$ that is radial, $0 \leq \rho \leq 1$, supported in $\{\xi : 1/4 \leq |\xi| \leq 4\}$ and such that $\rho = 1$ on $\{\xi : 1/2 \leq |\xi| \leq 2\}$.

Recall that, if $f \in L^p$ then $\widehat{\Delta_j f}$ (Fourier transform taken in the sense of distributions) is supported in $\{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ then

$$\rho(\xi/2^j)\widehat{\partial_k\Delta_j f}(\xi) = 2i\pi\xi_k\rho(\xi/2^j)\widehat{\Delta_j f}(\xi) = 2i\pi\xi_k\widehat{\Delta_j f}(\xi)$$

since $\rho(\xi/2^k)=1$ on the support of $\widehat{\Delta_j f}$. Multiplying by ξ_k and summing, we thus get

$$\sum_{k=1}^{d}\rho(\xi/2^j)\xi_k\widehat{\partial_k\Delta_j f}(\xi)=2i\pi|\xi|^2\widehat{g}(\xi)$$

that we re-write

$$\widehat{\Delta_j f}(\xi)=\sum_{k=1}^{d}\frac{\xi_k\rho(\xi/2^j)}{2i\pi|\xi|^2}\widehat{\partial_k\Delta_j f}(\xi).$$

Inverting the Fourier transform, we have

$$\Delta_j f=2^{-j}\sum_{k=1}^{d}K_{j,k}*(\partial_k\Delta_j f)$$

with

$$K_{j,k}(x)=2^j\int_{\mathbb{R}^d}\rho(\xi/2^j)\frac{\xi_k}{2i\pi|\xi|^2}e^{2i\pi\langle\xi,x\rangle}\,\mathrm{d}\xi=2^{jd}\int_{|\eta|\geq 1/4}\rho(\eta)\frac{\eta_k}{2i\pi|\eta|^2}e^{2i\pi 2^j\langle\eta,x\rangle}\,\mathrm{d}\eta$$

with the change of variable $\eta=\xi/2^j$ and the fact that $\rho(\eta)=0$ for $|\eta|\leq 1/4$. As

$$\left|\frac{\eta_k}{2i\pi|\eta|^2}\right|\leq\frac{2}{\pi}$$

for $|\eta|\geq 1/4$, we get

$$|K_{j,k}(x)|\leq\frac{2}{\pi}2^{jd}\|\rho\|_1.$$

Further,

$$e^{2i\pi 2^j\langle\eta,x\rangle}=2^{-j}\frac{1}{2i\pi 2^j x_\ell}\partial_{\eta_\ell}e^{2i\pi 2^j\langle\eta,x\rangle}$$

thus, integrating by parts, (using that ρ is compactly supported) we get

$$\begin{aligned}K_{j,k}(x)&=\;=2^{jd}\frac{1}{2i\pi 2^j x_\ell}\int_{|\eta|\geq 1/4}\rho(\eta)\frac{\eta_k}{2i\pi|\eta|^2}\partial_{\eta_\ell}e^{2i\pi 2^j\langle\eta,x\rangle}\,\mathrm{d}\eta\\&=-2^{jd}\frac{1}{2i\pi 2^j x_\ell}\int_{|\eta|\geq 1/4}\partial_{\eta_\ell}\left(\rho(\eta)\frac{\eta_k}{2i\pi|\eta|^2}\right)e^{2i\pi 2^j\langle\eta,x\rangle}\,\mathrm{d}\eta.\end{aligned}$$

It is not hard to see that $\partial_{\eta_\ell}\frac{\eta_k}{2i\pi|\eta|^2}$ is bounded over $\{|\eta|\geq 1/4\}$ so that

$$|K_{j,k}(x)|\leq C2^{jd}\frac{\|\rho\|_1+\|\partial_\ell\rho\|_1}{2^j|x_\ell|}.$$

Now, if $2^j\|x\|_\infty\geq 1$ then for at least one ℓ, $2^j|x_\ell|\geq 1$ so that $2^j|x_\ell|\geq\dfrac{1}{2}(1+2^j|x_\ell|)$ thus

$$|K_{j,k}(x)|\leq 2^{jd}\frac{2C(\|\rho\|_1+\sum_\ell\|\partial_\ell\rho\|_1)}{1+2^j|x_\ell|}.$$

On the other hand, if $2^j\|x\|_\infty\leq 1$, $(1+2^j\|x\|_\infty)^{-1}\geq 1/2$ and we will use the bound

$$|K_{j,k}(x)|\leq 2^{jd}\|\rho\|_1\leq 2^{jd}\|\rho\|_1\frac{2\|\rho\|_1}{1+2^j|x_\ell|}\leq 2^{jd}\|\rho\|_1\frac{2\max(1,C)(\|\rho\|_1+\sum_\ell\|\partial_\ell\rho\|_1}{1+2^j|x_\ell|}.$$

In all cases, we have a bound of the form

$$|K_{j,k}(x)|\leq C2^{jd}(1+2^j\|x\|_\infty)^{-1}$$

(with C depending on ρ). Now iterating the argument and using multiple integration by parts, we get a bound of the form

$$|K_{j,k}(x)|\leq C_N2^{jd}(1+2^j\|x\|_\infty)^{-N}$$

for every N, where C depends on N but *not* on j. In particular, taking $N = d+1$ we obtain that $K_{j,k} \in L^1(\mathbb{R}^d)$ and $\|K_{j,k}\|_1 \leq C_{d+1}\|(1+|x|_\infty)^{-d-1}\|_1$ a bound that does *not* depend on j. But then

$$\begin{aligned}\|\Delta_j f\|_p &= 2^{-j}\sum_{k=1}^d \|K_{j,k} * (\partial_k \Delta_j f)\|_p \leq 2^{-j}\sum_{k=1}^d \|K_{j,k}\|_1 \|\partial_k \Delta_j f\|_p \\ &\leq 2^{-j} d C_{d+1} \|(1+\|x\|_\infty)^{-d-1}\|_1 \|\nabla \Delta_j f\|_p\end{aligned}$$

as claimed. □

2.3. The main theorem. The general heuristic of the Littlewood-Paley decomposition is that, the $\Delta_j f$ having (almost) separated support in the Fourier domain, they behave almost independently. In particular, one should have

$$\left|\sum_j \Delta_j f\right| \simeq \left(\sum_j |\Delta_j f|^2\right)^{1/2}.$$

This is of course a too strong statement, however, it may hold in the L^p-sense. First, it holds in the L^2-sense: with Parseval

$$\left\|\sum_j \Delta_j f\right\|_2^2 = \left\|\mathcal{F}[\sum_j \Delta_j f]\right\|^2 = \left\|\sum_j \mathcal{F}[\Delta_j f]\right\|^2 = \int_{\mathbb{R}^d}\left|\sum_j \psi(\xi/2^j)\right|^2 |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi = \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi.$$

On the other hand

$$\begin{aligned}\left\|\left(\sum_j |\Delta_j f|^2\right)^{1/2}\right\|_2^2 &= \int_{\mathbb{R}^d}\sum_j |\Delta_j f(x)|^2 \,\mathrm{d}x \\ &= \int_{\mathbb{R}^d}\sum_j |\widehat{\Delta_j f}(\xi)|^2 \,\mathrm{d}\xi = \int_{\mathbb{R}^d}\sum_j \psi(\xi/2^j)^2 |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi.\end{aligned}$$

Now, for fixed j, there are at most 2 consecutive j's for which $\psi(\xi/2^j)^2 \neq 0$. Call them $k, k+1$, then we have

$$\begin{aligned}1 = \big(\psi(\xi/2^k)+\psi(\xi/2^{k+1})\big)^2 &= \psi(\xi/2^k)^2 + \psi(\xi/2^{k+1})^2 + 2\psi(\xi/2^k)\psi(\xi/2^{k+1}) \\ &\leq 2\big(\psi(\xi/2^k)^2 + \psi(\xi/2^{k+1})^2\big) \leq 2\end{aligned}$$

since $0 \leq \psi \leq 1$. It follows that

$$1 \leq \sum_j \psi(\xi/2^j)^2 \leq 2$$

thus

$$\int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi \leq \int_{\mathbb{R}^d}\sum_j \psi(\xi/2^j)^2 |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi \leq 2\int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi.$$

The following result is the central result of this chapter:

THEOREM 8.10. *Defined the* Littlewood-Paley square function *as*

$$S[f] = \left(\sum_j |\Delta_j f|^2\right)^{1/2}.$$

Then, for every $1 < p < +\infty$, there exists a constant C depend on p, d only, such that for every $f \in L^p(\mathbb{R}^d)$,

$$\frac{1}{C}\|f\|_p \leq \|S(f)\|_p \leq C\|f\|_p.$$

PROOF. We define the vector-valued operator $\vec{S}[f] = \big(\Delta_j f\big)_{j\in\mathbb{Z}}$ so that $S[f] = \|\vec{S}[f]\|_{\ell^2(\mathbb{Z})}$. We want to show that $\|\vec{S}[f]\|_{L^p(\mathbb{R}^d,\ell^2(\mathbb{Z}))}$ is comparable to $\|f\|_{L^p(\mathbb{R}^d)}$. We have just shown that $\vec{S}$

is a bounded linear operator from $L^2(\mathbb{R}^d)$ to $L^2\big(\mathbb{R}^d,\ell^2(\mathbb{Z})\big)$. Further, the operator $\vec{S}$ is associated to the kernel

$$\vec{K}(x,y)=\big\{2^{jd}\widehat{\psi}\big(2^j(x-y)\big)\big\}_{j\in\mathbb{Z}}.$$

The L^p-boundedness will follow immediately from vector-valued Calderón-Zygmund theory once we show the following:

LEMMA 8.11. *The kernel $\vec{K}$ is a vector-valued Calderón-Zygmund kernel.*

We postpone the proof of the lemma and first show that the converse bound also holds. The first observation is that, if $\vec{g}=\{g_j\}_{\ell^2(\mathbb{Z})}:\mathbb{R}^d\to\ell^2(\mathbb{Z})$ then

$$\begin{aligned}\int_{\mathbb{R}^d}\Big\langle\vec{S}[f](x),\vec{g}(x)\Big\rangle_{\ell^2(\mathbb{Z})}\,\mathrm{d}x &= \int_{\mathbb{R}^d}\sum_{j\in\mathbb{Z}}\Delta_j f(x)\overline{g_j(x)}\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d}\sum_{j\in\mathbb{Z}}\psi(\xi/2^j)\widehat{f}(\xi)\overline{\widehat{g_j}(\xi)}\,\mathrm{d}\xi\\ &= \int_{\mathbb{R}^d}\sum_{j\in\mathbb{Z}}\widehat{f}(\xi)\overline{\psi(\xi/2^j)\widehat{g_j}(\xi)}\,\mathrm{d}\xi\\ &:= \int_{\mathbb{R}^d}f(x)\overline{S^*\vec{g}(x)}\,\mathrm{d}x.\end{aligned}$$

By duality, ℓ^2-Cauchy-Schwarz and Hölder,

$$\begin{aligned}\|S^*\vec{g}(x)\|_{L^{p'}} &= \sup_{\|f\|_p=1}\left|\int_{\mathbb{R}^d}f(x)\overline{S^*\vec{g}(x)}\,\mathrm{d}x\right| = \sup_{\|f\|_p=1}\left|\int_{\mathbb{R}^d}\Big\langle\vec{S}[f](x),\vec{g}(x)\Big\rangle_{\ell^2(\mathbb{Z})}\mathrm{d}x\right|\\ &\le \sup_{\|f\|_p=1}\int_{\mathbb{R}^d}\left|\Big\langle\vec{S}[f](x),\vec{g}(x)\Big\rangle_{\ell^2(\mathbb{Z})}\right|\mathrm{d}x \le \sup_{\|f\|_p=1}\int_{\mathbb{R}^d}\|\vec{S}[f](x)\|_{\ell^2}\|\vec{g}(x)\|_{\ell^2}\,\mathrm{d}x\\ &\le \sup_{\|f\|_p=1}\left(\int_{\mathbb{R}^d}\|\vec{S}[f](x)\|_{\ell^2}^p\,\mathrm{d}x\right)^{1/p}\left(\int_{\mathbb{R}^d}\|\vec{g}(x)\|_{\ell^2}^{p'}\,\mathrm{d}x\right)^{1/p'}\\ &= \left\|\vec{S}\right\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d,\ell^2)}\|g\|_{L^{p'}(\mathbb{R}^d,\ell^2)}.\end{aligned}$$

Now we repeat the Littlewood-Paley decomposition but starting with

$$\tilde{\psi}(\xi)=\varphi(\xi/4)-\varphi(4\xi)$$

that is, we set

$$\widehat{\tilde{\Delta}_j f}(\xi)=\tilde{\psi}(\xi/2^j)\widehat{f}(\xi).$$

We still have that $\sum\tilde{\psi}(\xi/2^k)$ is bounded from above and below while the previous arguments still show that there is a constant C such that

$$\left\|\left(\sum_{j\in\mathbb{Z}}|\tilde{\Delta}_j f|^2\right)^{1/2}\right\|_p\le C\|f\|_p$$

and

$$\left\|\sum_{j\in\mathbb{Z}}\tilde{\Delta}_j g_j\right\|_p\le C\|\vec{g}\|_{L^p(\mathbb{R}^d,\ell^2)}$$

thus

$$\left\|\sum_{j\in\mathbb{Z}}\tilde{\Delta}_j\Delta_j f\right\|_p\le C\left\|\vec{S}[f]\right\|_{L^p(\mathbb{R}^d,\ell^2)}\le C'\|f\|_p.$$

The support properties of $\tilde{\psi}$ and the fact that this function is 1 on $\{1/2\le|\xi|\le 2\}$ imply that $\tilde{\Delta}_j\Delta_j f=\Delta_j f$. we thus get

$$\|f\|_p=\left\|\sum_{j\in\mathbb{Z}}\Delta_j f\right\|_p=\left\|\sum_{j\in\mathbb{Z}}\tilde{\Delta}_j\Delta_j f\right\|_p\le C\left\|\vec{S}[f]\right\|_{L^p(\mathbb{R}^d,\ell^2)}\le C'\|f\|_p$$

which is what we want to prove.

It remains to prove the lemma.

Proof of Lemma 8.11. As ψ thus $\widehat{\psi}\in\mathcal{S}(\mathbb{R}^d)$, there is a $C>0$ such that

$$|\widehat{\psi}(\xi)|\leq C\min(1,|\xi|^{-d-1}),$$

the first bound being better when $|\xi|\leq 1$ while the second one is better for $|\xi|\geq 1$. Now fix $x\neq y$ and let $J\in\mathbb{Z}$ be such that $\frac{1}{2}\leq 2^J|x-y|\leq 2$ (such a J exists) and note that, if $j\leq J$, $2^j|x-y|\leq 2$ while for $j>J$, $2^j|x-y|\geq 2$. We write

$$\left|2^{jd}\widehat{\psi}\big(2^j(x-y)\big)\right|\leq\begin{cases}C2^{jd} & \text{if } j\leq J\\ \dfrac{C2^{jd}}{(2^j|x-y|)^{d+1}} & \text{if } j\geq J+1\end{cases}.$$

Then

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\left|2^{jd}\widehat{\psi}\big(2^j(x-y)\big)\right|^2 &= \sum_{-\infty<j\leq J}\left|2^{jd}\widehat{\psi}\big(2^j(x-y)\big)\right|^2+\sum_{j\geq J+1}\left|2^{jd}\widehat{\psi}\big(2^j(x-y)\big)\right|^2\\ &\leq C\sum_{-\infty\leq j\leq J}2^{2jd}+\frac{C}{|x-y|^{2d+4}}\sum_{j\geq J+1}\frac{1}{2^{4j}}\\ &= C\frac{2^{2Jd}}{1-2^{-2d}}+\frac{C}{|x-y|^{2d+4}}\frac{2^{-4(J+1)}}{1-2^{-4}}\\ &\leq \frac{C}{1-2^{-2d}}\left(\frac{2}{|x-y|}\right)^d+\frac{C}{2^4-1}\frac{1}{|x-y|^{2d+4}}(2|x-y|)^4\end{aligned}$$

which is the desired bound

$$\|\vec{K}(x,y)\|_{\ell^2(\mathbb{Z})}\leq\frac{C'}{|x-y|^d}.$$

For the smoothness bound, we note that

$$\nabla_y\vec{K}(x,y)=-\nabla_x\vec{K}(x,y)=\{2^{j(d+1)}\nabla\widehat{\psi}\big(2^j(x-y)\big)\}_{j\in\mathbb{Z}}.$$

We leave as an exercice to adapt the previous proof to obtain the estimate

$$\|\nabla_x\vec{K}(x,y)\|_{\ell^2(\mathbb{Z})}\leq\frac{C''}{|x-y|^{d+1}}.$$

From there, the proof of the smoothness estimate follows as in the scalar case. □

This concludes the proof of the theorem. □

3. The Hörmander-Mikhlin Theorem

We have already seen that a function $m\in W^{2,s}(\mathbb{R}^d)$ with $s>d/2$ defines a multiplier for every $1<p<+\infty$. The flavour of this result is that, some smoothness together with controllable local singularities and some global decay will give a multiplier.

We present now two refinements, usually called Hörmander multiplier theorems. The first one starts with a function $m\in L^\infty$ which will garantee that the associated multiplier is of strong type $(2,2)$. The function will further be assumed to be smooth away from the origin and with derivatives that decay at least as fast as their order:

Theorem 8.12 (First Hörmander-Mikhlin Theorem). *Let m be a bounded function, that is $\mathcal{C}^\infty$ on $\mathbb{R}^d\setminus\{0\}$ and such that, for every $\alpha\in\mathbb{N}^d$, there exists a constant $C_{m,\alpha}$ with*

$$|\partial^\alpha m(\xi)|\leq\frac{C_{m,\alpha}}{|\xi|^{|\alpha|}}\quad\textit{for all } \xi\in\mathbb{R}^d\setminus\{0\}.$$

Then $k=\mathcal{F}^{-1}[m]$ agrees with a $\mathcal{C}^\infty$ function on $\mathbb{R}^d\setminus\{0\}$ and, for every $\alpha\in\mathbb{N}^d$, there exists a constant $D_{d,\alpha}$ with

$$|\partial^\alpha k(x)|\leq\frac{D_{d,\alpha}}{|x|^{d+|\alpha|}}\quad\textit{for all } x\in\mathbb{R}^d\setminus\{0\}.$$

In particular $K(x,y)=k(x-y)$ *is a Calderón-Zygmund kernel and* T_m *is of strong type* (p,p) *for every* $1<p<+\infty$.

Here $k=\mathcal{F}^{-1}[m]$ has to be taken in the sense of distributions.

EXAMPLE 8.13. The Hilbert transform is a particular case of this theorem with $m(\xi)=i\,\mathrm{sign}(\xi)$.

PROOF. We will use the Littlewood-Paley decomposition to write, for $\xi\neq 0$,

$$m(\xi)=\sum_{j\in\mathbb{Z}}\psi(\xi/2^j)m(\xi):=\sum_{j\in\mathbb{Z}}m_j(\xi).$$

Each m_j is $\mathcal{C}^\infty$ and supported in the annulus $A_j=\{\xi\,:\,2^{j-1}\leq|\xi|\leq 2^{j+1}\}$. We may thus define

$$k_j(x)=\int_{\mathbb{R}^d}m_j(\xi)e^{2i\pi\langle x,\xi\rangle}\,\mathrm{d}\xi$$

the inverse Fourier transform of m_j. Let us now see how the hypothesis on m transfers into estimates on k_j. First, we notice that

$$\partial^\alpha k_j(x)=\int_{\mathbb{R}^d}(2i\pi\xi)^\alpha m_j(\xi)e^{2i\pi\langle x,\xi\rangle}\,\mathrm{d}\xi$$

from which we immediately deduce that

$$|\partial^\alpha k_j(x)|\leq\int_{A_j}(2\pi)^\alpha|\xi^\alpha||m_j(\xi)|\,\mathrm{d}\xi\leq(2\pi)^\alpha\|m\|_\infty|A_j|(2^{j+1})^{|\alpha|}\leq B_{d,\alpha}2^{j(d+|\alpha|)}$$

where we use that $|\xi^\alpha|\leq|\xi|^{|\alpha|}\leq(2^{j+1})^{|\alpha|}$ when $\xi\in A_j$ and that $|A_j|\leq|B(0,2^{j+1})|\leq 2^d|B(0,1)|2^{jd}$.

Further,

$$\langle x,\nabla_\xi\rangle e^{2i\pi\langle x,\xi\rangle}:=\sum_{\ell=1}^d x_\ell\frac{\partial}{\partial\xi_\ell}e^{2i\pi\langle x,\xi\rangle}=2i\pi|x|^2e^{2i\pi\langle x,\xi\rangle}$$

that is

$$\frac{\langle x,\nabla_\xi\rangle}{2i\pi|x|^2}e^{2i\pi\langle x,\xi\rangle}=e^{2i\pi\langle x,\xi\rangle}$$

when $x\neq 0$. Thus, for every $M>0$, and $x\neq 0$,

$$\begin{aligned}\partial^\alpha k_j(x)&=\int_{\mathbb{R}^d}(2i\pi\xi)^\alpha m_j(\xi)\left(\frac{\langle x,\nabla_\xi\rangle}{2i\pi|x|^2}\right)^M e^{2i\pi\langle x,\xi\rangle}\,\mathrm{d}\xi\\&=\sum_{\ell=1}^d\frac{x_\ell}{2i\pi|x|^2}\int_{A_j}(2i\pi\xi)^\alpha m_j(\xi)\frac{\partial}{\partial\xi_\ell}\left(\frac{\langle x,\nabla_\xi\rangle}{2i\pi|x|^2}\right)^{M-1}e^{2i\pi\langle x,\xi\rangle}\,\mathrm{d}\xi\\&=-\sum_{\ell=1}^d\frac{x_\ell}{2i\pi|x|^2}\int_{A_j}\frac{\partial}{\partial\xi_\ell}[(2i\pi\xi)^\alpha m_j(\xi)]\left(\frac{\langle x,\nabla_\xi\rangle}{2i\pi|x|^2}\right)^{M-1}e^{2i\pi\langle x,\xi\rangle}\,\mathrm{d}\xi\qquad(3.50)\end{aligned}$$

after an integration by parts. Now note that, if we denote $(e_\ell)_{\ell=1,\ldots,d}$ is the standard basis of $\mathbb{R}^d$ then

$$\begin{aligned}\frac{\partial}{\partial\xi_\ell}[(2i\pi\xi)^\alpha\psi(\xi/2^j)m(\xi)]&=\alpha_\ell(2i\pi\xi)^{\alpha-e_\ell}\psi(\xi/2^j)m(\xi)\\&\quad+(2i\pi\xi)^\alpha 2^{-j}\frac{\partial}{\partial\xi_\ell}\psi(\xi/2^j)m(\xi)+(2i\pi\xi)^\alpha\psi(\xi/2^j)\frac{\partial}{\partial\xi_\ell}m(\xi).\end{aligned}$$

Now, this is either 0 or $\xi\in A_j$ thus $2^{j-1}|\xi|\leq 2^{j+1}$ in which case

(i) $|(2i\pi\xi)^{\alpha-e_\ell}\psi(\xi/2^j)m(\xi)|\leq(4\pi)^{j(|\alpha|-1}\|m\|_\infty 2^{j(|\alpha|-1)}$,

(ii) $\left|(2i\pi\xi)^\alpha 2^{-j}\frac{\partial}{\partial\xi_\ell}\psi(\xi/2^j)m(\xi)\right|\leq(4\pi)^{j(|\alpha|}\|\nabla\psi\|_\infty\|m\|_\infty 2^{j(|\alpha|-1)}$,

(iii) $\left|(2i\pi\xi)^\alpha\psi(\xi/2^j)\frac{\partial}{\partial\xi_\ell}m(\xi)\right|\leq(4\pi)^{j(|\alpha|+1}C_{m,\alpha}2^{j(|\alpha|-1)}$.

All in one, we get

$$\left|\frac{\partial}{\partial\xi_\ell}[(2i\pi\xi)^\alpha\psi(\xi/2^j)m(\xi)]\right| \leq \kappa_1 2^{j(|\alpha|-1)}$$

for some constant κ_1. It then follows from (3.50) with $M=1$ that

$$|\partial^\alpha k_j(x)| \leq \sum_{\ell=1}^d \frac{|x_\ell|}{2\pi|x|^2}\kappa_{\alpha,1}2^{j(|\alpha|-1)}|A_j| \leq \frac{\tilde\kappa_{\alpha,1}}{|x|}2^{j(d+|\alpha|-1)}.$$

One may then
– pursue integration by parts in (3.50)
– use Leibnitz formula to estimate $\partial^\beta[(2i\pi\xi)^\alpha\psi(\xi/2^j)m(\xi)]$
and we finally get that, for every α, M, there is a constant $\tilde\kappa_{\alpha,M}$ such that

$$|\partial^\alpha k_j(x)| \leq \frac{\tilde\kappa_{\alpha,M}}{|x|^M}2^{j(d+|\alpha|-M)}.$$

Summarising all estimates, for each M,α, we get a constant $\sigma_{M,\alpha}$ such that

$$|\partial^\alpha k_j(x)| \leq \sigma_{M,\alpha}\min(2^{j(d+|\alpha|)}, |x|^{-M}2^{j(d+|\alpha|-M)}).$$

Next we estimate

$$\sum_{j\in\mathbb{Z}:2^j\leq 1/|x|}|\partial^\alpha k_j(x)| \leq \sigma_{0,\alpha}\sum_{j\in\mathbb{Z}:2^j\leq 1/|x|}2^{j(d+|\alpha|)} \leq \tilde\sigma_{0,\alpha}|x|^{-(d+|\alpha|)}$$

while, for $M>d+|\alpha|$,

$$\begin{aligned}\sum_{j\in\mathbb{Z}:2^j\geq 1/|x|}|\partial^\alpha k_j(x)| &\leq \sigma_{M,\alpha}|x|^{-M}\sum_{j\in\mathbb{Z}:2^j\leq 1/|x|}2^{j(d+|\alpha|-M)}\\ &\leq \tilde\sigma_{M,\alpha}|x|^{-M}|x|^{-(d+|\alpha|-M)} \leq \tilde\sigma_\alpha|x|^{-(d+|\alpha|)}.\end{aligned}$$

In particular, for each α, the series $\sum_{j\in\mathbb{Z}}\partial^\alpha k_j$ is normally convergent over every compact set $E\subset\mathbb{R}^d\setminus\{0\}$. It follows that $\sum_{j\in\mathbb{Z}}k_j$ converges over $\mathbb{R}^d\setminus\{0\}$ to some function $\tilde k\in\mathcal{C}^\infty(\mathbb{R}^d\setminus\{0\})$ and we have also proven the claimed bound

$$|\partial^\alpha\tilde k(x)| \leq \frac{B_\alpha}{|x|^{d+|\alpha|}}.$$

It remains to show that $k=\tilde k$ on $\mathbb{R}^d\setminus\{0\}$, that is

$$\langle k,\varphi\rangle = \langle\tilde k,\varphi\rangle \quad\text{for every } \varphi\in\mathcal{C}_c^\infty(\mathbb{R}^d) \text{ with } \operatorname{supp}\varphi\subset\mathbb{R}^d\setminus\{0\}.$$

But, by definition of the Fourier transform,

$$\langle k,\varphi\rangle = \langle\widehat{k},\widehat{\varphi}\rangle = \langle m,\widehat{\varphi}\rangle = \int_{\mathbb{R}^d}m(\xi)\widehat{\varphi}(\xi)\,\mathrm{d}\xi$$

since $\widehat\varphi\in\mathcal{S}(\mathbb{R}^d)$ and $m\in L^\infty\subset L^1_{loc}$.

On the other hand, as φ is compactly supported away from 0, $\sum k_j$ converges uniformly to $\tilde k$ over the support of φ thus

$$\begin{aligned}\langle\tilde k,\varphi\rangle = \int_{\mathbb{R}^d}\left(\sum_{j\in\mathbb{Z}}k_j(x)\right)\varphi(x)\,\mathrm{d}x &= \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}k_j(x)\varphi(x)\,\mathrm{d}x\\ &= \sum_{j\in\mathbb{Z}}\langle k_j,\varphi\rangle = \sum_{j\in\mathbb{Z}}\langle\widehat{k_j},\widehat\varphi\rangle = \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}\widehat{k_j}(\xi)\widehat\varphi(\xi)\,\mathrm{d}\xi\\ &= \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}\psi(\xi/2^j)m(\xi)\widehat\varphi(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^d}\left(\sum_{j\in\mathbb{Z}}\psi(\xi/2^j)\right)m(\xi)\widehat\varphi(\xi)\,\mathrm{d}\xi\end{aligned}$$

since $\sum_{j\in\mathbb{Z}}\psi(\xi/2^j)$ converges a.e. (to 1) and partial sums are bounded by 1 so that Lebesgue's Dominated Convergence shows that one may invert summation over j and integration. Finally we get

$$\langle\tilde{k},\varphi\rangle=\int_{\mathbb{R}^d}m(\xi)\widehat{\varphi}(\xi)\,\mathrm{d}\xi=\langle k,\varphi\rangle$$

as expected.

We now conclude by noting that T_m is a Calderón-Zygmund operator associated to $K(x,y)=\tilde{k}(x-y)$:

– K is a standard kernel since, when $x\neq y$

$$|K(x,y)|=|\tilde{k}(x-y)|\leq\frac{B_0}{|x-y|^d}\quad|\nabla_xK(x,y)|=|\nabla_yK(x,y)|=|\nabla\tilde{k}(x-y)|\leq\frac{B_1}{|x-y|^{d+1}}.$$

– T_m is bounded on L^2 since $m\in L^\infty$;

– if $f\in L^2(\mathbb{R}^d)$ has compact support and $x\notin\operatorname{supp}f$ then, for every $y\in\operatorname{supp}f$, $x-y\notin f$ so that

$$T_mf(x)=k*f(x)=\int_{\mathbb{R}^d}\tilde{k}(x-y)f(y)\,\mathrm{d}y$$

and T_m is the operator with kernel K.

It follows from general Calderón-Zygmund theory that T_m is of strong-type (p,p). □

The result is not optimal and one does not need as many derivatives.

EXAMPLE 8.14. A typical example of an m that satisfies the hypothesis is $m_\beta(\xi)=\dfrac{\xi^\beta}{|\xi|^2}$ when β is a multi-index of length 2. This kernel is usefull for the following reason: for $f\in\mathcal{S}(\mathbb{R}^d)$, we have

$$\widehat{\partial^\beta f}(\xi)=(2i\pi\xi)^\beta\widehat{f}(\xi)=\frac{(2i\pi\xi)^\beta}{(2i\pi|\xi|)^2}(2i\pi|\xi|)^2\widehat{f}(\xi)=m_\beta(\xi)\widehat{\Delta f}(\xi)$$

that is $\partial^\beta f=T_{m_\beta}[\Delta f]$. As m_β satisfies the conditions of the theorem, T_{m_β} is of strong type (p,p) for any $1<p<\infty$. As a consequence

$$\left\|\partial^\beta f\right\|_p\leq C_p\|\Delta f\|_p$$

i.e. all derivatives of order 2 are controlled by the Laplace operator.

We now give a sharper result.

THEOREM 8.15 (Hörmander-Mikhlin Multiplier Theorem). *Let n be the smallest integer $>d/2$ (i.e. $d=2(n-1)$ when d is even and $d=2n-1$ when d is odd). Let m be a bounded function on $\mathbb{R}^d$, of class $\mathcal{C}^n$ on $\mathbb{R}^d\setminus\{0\}$ and such that, for every $\alpha\in\mathbb{N}^d$ with $|\alpha|\leq n$, there is a constant C_α such that*

$$|\partial^\alpha m(\xi)|\leq\frac{C_\alpha}{|\xi|^{|\alpha|}}.$$

Let $K=\mathcal{F}^{-1}[m]$. Then K agrees with a locally integrable function $\tilde{K}$ on $\mathbb{R}^d\setminus\{0\}$. Further, there exists C such that, for every $y\in\mathbb{R}^d\setminus\{0\}$,

$$\int_{|x|>2|y|}|K(x-y)-K(x)|\,\mathrm{d}x\leq C.$$

Further, $m\in\mathcal{M}^p$ for every $1<p<+\infty$.

PROOF. We will not take care of constants in this proof and write C for a constant that depends on the dimension d only and that may change from one occurence to the following.

As in the previous proof, we only have to control the pieces K_j. For this, let β be a multiindex. Then

$$\int_{\mathbb{R}^d}|(-2i\pi x)^\beta K_j(x)|^2\,\mathrm{d}x=\int_{\mathbb{R}^d}|\partial^\beta m_j(\xi)|^2\,\mathrm{d}\xi.$$

Thus, as long as $\ell \le n$, there is a constant C (that depends on ℓ only but not on j and in the end will depend on d only) such that

$$\int_{\mathbb{R}^d} (|x|^\ell)^2 |K_j(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^d} (x_1^2 + \cdots + x_d^2)^\ell |K_j(x)|^2 \,\mathrm{d}x \le C_\ell \int_{\mathbb{R}^d} \sum_{|\beta|=2\ell} |\partial^\beta m_j(\xi)|^2 \,\mathrm{d}\xi \le C_\ell 2^{jd} 2^{-2j\ell}$$

where the factor 2^{jd} comes from the fact that we integrate over an annulus $2^{j-1} \le |\xi| \le 2^{j+1}$ which has measure $\le C2^{jd}$ while the second factor comes from the fact that annulus, the integrant is $\le (2^{-j\ell})^2$.

In particular, for $R > 0$, Cauchy-Schwarz gives

$$\int_{|x|\le R} |K_j(x)| \,\mathrm{d}x \lesssim R^{d/2} \left(\int_{\mathbb{R}^d} |K_j(x)|^2 \,\mathrm{d}x \right)^{1/2} \le C 2^{jd/2} R^{d/2}$$

and

$$\begin{aligned} \int_{|x|\ge R} |K_j(x)| \,\mathrm{d}x &= \int_{|x|\ge R} |x|^{-n}|x|^n |K_j(x)| \,\mathrm{d}x \le \left(\int_{|x|\ge R} |x|^{-2n} \,\mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^d} |x|^{2n} |K_j(x)|^2 \,\mathrm{d}x \right)^{1/2} \\ &\le C R^{d/2-k} 2^{jd/2 - jn} \end{aligned} \tag{3.51}$$

since $d/2 - n < 0$. In particular, chosing $R = 2^{-j}$ gives

$$\int_{\mathbb{R}^d} |K_j(x)| \,\mathrm{d}x \lesssim 2^{jd/2} (2^{-j})^{d/2} + (2^{-j})^{d/2-n} 2^{jd/2-jn} \le C.$$

Using the Leibnitz rule, one can extend this computation to the derivatives of K_j. First one proves

$$\int_{\mathbb{R}^d} |(-2i\pi x)^\beta \partial^\alpha K_j(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^d} |\partial^\beta [2i\pi\xi)^\alpha m_j](\xi)|^2 \,\mathrm{d}\xi \lesssim 2^{jd} 2^{-2j\ell} 2^{2j|\alpha|}.$$

Then, the same cut-off and Cauchy-Schwarz shows that

$$\int_{\mathbb{R}^d} |\partial^\alpha K_j(x)| \,\mathrm{d}x \lesssim 2^{j|\alpha|}.$$

Next, let $h \in \mathbb{R}^d \setminus \{0\}$ and write $h = |h|h'$, then

$$\begin{aligned} \int_{\mathbb{R}^d} |K_j(x+h) - K_j(x)| \,\mathrm{d}x &= \int_{\mathbb{R}^d} \left| \int_0^{|h|} \langle h', \nabla K_j(x + th') \rangle \,\mathrm{d}t \right| \mathrm{d}x \\ &\le \int_0^{|h|} \int_{\mathbb{R}^d} |\nabla K_j(x+th')| \,\mathrm{d}x \,\mathrm{d}t \le C 2^j |h| \end{aligned}$$

by the previous estimate for $|\alpha| = 1$.

As a first consequence

$$\sum_{2^j \lesssim |y|^{-1}} \int_{|x|\ge 2|y|} |K_j(x+y) - K_j(x)| \,\mathrm{d}x \lesssim \sum_{2^j \lesssim |y|^{-1}} 2^j |y| \lesssim 1.$$

On the other hand, writing $|K_j(x+y) - K_j(x)| \le |K_j(x+y)| + |K_j(x)|$ and noting that $|x+y| \ge |y|$ when $|x| \ge 2|y|$, we get

$$\begin{aligned} \sum_{2^j \gtrsim |y|^{-1}} \int_{|x|\ge 2|y|} |K_j(x+y) - K_j(x)| \,\mathrm{d}x &\le 2 \sum_{2^j \ge C^{-1}|y|^{-1}} \int_{|x|\ge |y|} |K_j(x)| \,\mathrm{d}x \\ &\le C \sum_{2^j \ge C^{-1}|y|^{-1}} 2^{dj/2} 2^{-kj} |y|^{\frac{d}{2}-k} \le C \end{aligned}$$

with (3.51).

As in the previous proof, we conclude that K coincides on $\mathbb{R}^d \setminus \{0\}$ with the locally integrable function $\sum K_j$. and that

$$\int_{|x|\ge 2|y|} |K(x+y) - K(x)| \,\mathrm{d}x \le C$$

for $y\neq 0$ and that, if $f\in L^2(\mathbb{R}^d)$ has compact support and $0\notin\operatorname{supp} f$ then

$$T_mf(x)=\int_{\mathbb{R}^d}K(x-y)f(y)\,\mathrm{d}y.$$

Further, as $m\in L^\infty$, T_m is bounded on L^2. We are going to show that T_m is also of weak-type $(1,1)$ thus, by interpolation, it is of strong type (p,p) for $1<p\leq 2$ and, by duality, it is of strong type (p,p) for $2\leq p<+\infty$.

We now take a general $f\in L^1$ and $\lambda>0$. Take its Calderón-Zygmund decomposition at level λ, $f=g+\sum_Q b_Q$ with g a good piece and b_Q bad pieces. We have

$$|\{|Tf|>\lambda\}|\leq|\{|Tg|>\lambda/2\}|+|\{|\sum_Q Tb_Q|>\lambda/2\}|.$$

The good part is easy to deal with since T_m is of strong type $(2,2)$ so that $\|T_mg\|_2\leq C\|g\|_2\leq C\lambda^{1/2}\|f\|_1^{1/2}$. From Bienaymé-Tchebichev, we get

$$|\{|Tg|>\lambda/2\}|\leq C\frac{\|g\|_2^2}{\lambda^2}\leq C\frac{\lambda\|f\|_1}{\lambda^2}=C\frac{\|f\|_1}{\lambda}$$

which has the desired form.

The bad pieces are delt with as follows: b_Q is supported in a cube Q with center c_Q and we denote $Q^*=(1+2d^{1/2})Q$. We then write

$$|\{|\sum Tb_Q|>\lambda/2\}|\leq\left|\bigcup Q^*\right|+|\{x\notin\bigcup Q^*\ :\ |\sum_Q Tb_Q(x)|>\lambda/2\}|.$$

But

$$\left|\bigcup Q^*\right|\leq\sum|Q^*|\leq C\sum|Q|\leq C\frac{\|f\|_1}{\lambda}.$$

As $\displaystyle\int b_Q(y)\,\mathrm{d}y=0$ we have

$$\begin{aligned}\int_{\mathbb{R}^d\setminus Q^*}\left|\int_Q K(x-y)b_Q(y)\,\mathrm{d}y\right|\mathrm{d}x &= \int_{\mathbb{R}^d\setminus Q^*}\left|\int_Q\big(K(x-y)-K(x-c_Q)\big)b_Q(y)\,\mathrm{d}y\right|\mathrm{d}x\\ &\leq \int_{\mathbb{R}^d\setminus Q^*}\int_Q\big|K(x-y)-K(x-c_Q)\big|\,|b_Q(y)|\,\mathrm{d}y\,\mathrm{d}x\\ &\leq \int_Q\left(\int_{\mathbb{R}^d\setminus Q^*}\big|K(x-y)-K(x-c_Q)\big|\,\mathrm{d}x\right)|b_Q(y)|\,\mathrm{d}y.\end{aligned}$$

Now if $y\in Q$ and $x\notin Q^*$ then $|x-c_Q|\geq 2|y-c_Q|$ thus

$$\int_{\mathbb{R}^d\setminus Q^*}\big|K(x-y)-K(x-c_Q)\big|\,\mathrm{d}x=\int_{\mathbb{R}^d\setminus Q^*}\big|K\big(x-c_q-(y-c_Q)\big)-K(x-c_Q)\big|\,\mathrm{d}x\leq C$$

so that

$$\int_{\mathbb{R}^d\setminus Q^*}\left|\int_Q K(x-y)b_Q(y)\,\mathrm{d}y\right|\mathrm{d}x\leq C\|b_Q\|_1\leq C\|Q\|.$$

It follows that

$$\begin{aligned}\int_{\mathbb{R}^d\setminus\bigcup Q^*}|\sum_Q Tb_Q(x)|\,\mathrm{d}x &= \sum_Q\int_{\mathbb{R}^d\setminus Q^*}\left|\int_Q K(x-y)b_Q(y)\,\mathrm{d}y\right|\mathrm{d}x\\ &\leq C\lambda\sum_Q\|Q\|\leq C\lambda\frac{\|f\|_1}{\lambda}=C\|f\|_1\end{aligned}$$

and Bienaymé-Tchebichev shows that the last term is bounded by

$$|\{x\notin\bigcup Q^*\ :\ |\sum Tb_Q(x)|>\lambda/2\}|\leq 2\frac{\|\sum Tb_Q(x)\|_{L^1(\mathbb{R}^d\setminus\bigcup Q^*)}}{\lambda}\leq C\frac{\|f\|_1}{\lambda}$$

which has also the requested form. □

APPENDIX A

Integrating over the sphere and the Bessel function

1. The Γ and β functions

Recall that the Γ function is defined for $x>0$

$$\Gamma(x)=\int_0^{+\infty} t^x e^{-t}\,\frac{\mathrm{d}t}{t}.$$

Obviously, Γ is well defined and holomorphic over $\{z\in\mathbb{C}\,:\,\Re(z)>0\}$ and its derivatives are given by

$$\Gamma^{(k)}(x)=\int_0^{+\infty}\ln(t)^k t^x e^{-t}\,\frac{\mathrm{d}t}{t}.$$

This shows that Γ is log-convex over $\mathbb{R}^+$.

A further result is that, first for $0<\Re(z)<1$, Γ satisfies the functional equation

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z} \tag{1.52}$$

which allows to define Γ as a meromorphic function over $\mathbb{C}$ with poles at the non-positive integers with resudie $\mathrm{Res}\,(\Gamma,-n)=\dfrac{(-1)^n}{n!}$ with no zeroes.

Γ also satisfies the duplication formula

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)=2^{1-2z}\sqrt{\pi}\Gamma(2z). \tag{1.53}$$

A direct computation shows that $\Gamma(1)=\Gamma(2)=1$, any of the functional equation shows that $\Gamma(1/2)=\sqrt{\pi}$ while integration by parts shows that $\Gamma(x+1)=x\Gamma(x)$ so that $\Gamma(n+1)=n!$ and $\Gamma\left(n+\dfrac{1}{2}\right)=\dfrac{(2n)!}{2^{2n}n!}\sqrt{\pi}$.

Γ satisfies the asymptotic (Stirling) formula

$$\Gamma(x)=\sqrt{2\pi}x^{x-\frac{1}{2}}\left(1+\frac{1}{12x}+O(x^{-2})\right).$$

The β function is closely related to the Γ function. Recall that it is defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t$$

for $x,y>0$. Using Fubini we can write

$$\Gamma(x)\Gamma(y)=\int_0^{+\infty}\int_0^{+\infty} t^{x-1}e^{-t}s^{y-1}e^{-s}\,\mathrm{d}t\,\mathrm{d}s$$

and changing variable $u=s+t$, $v=\dfrac{s}{s+t}$ (that is $s=uv$, $t=u(1-v)$) we conclude that

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

which allows to extend it analytically.

Further, simplie changes of variables give various expressions

- $s=1-t$ shows that $B(y,x)=B(x,y)$;
- $t=\sin^2\theta$ shows that $B(x,y)=\displaystyle\int_0^{\frac{\pi}{2}}\sin^{2x-1}\theta\cos^{2y-1}\theta\,\mathrm{d}\theta$;
- $t=\dfrac{1}{1+s}$ shows that $B(x,y)=\displaystyle\int_0^{+\infty}\frac{s^{y-1}}{(1+s)^{x+y}}\,\mathrm{d}s$.

The link with the Γ function also shows that
- $B(x+1,y)=\dfrac{x}{x+y}B(x,y)$ and $B(x,y+1)=\dfrac{y}{x+y}B(x,y)$;
- $B(x,y)B(x+y,1-y)=\dfrac{\pi}{x\sin\pi y}$
- $B(x,x)=2^{1-2x}B\left(\frac{1}{2},2x\right)$.

2. Spherical coordinates

Spherical coodinates extend the 2-dimensional polar coordinates to higher dimensions. First note that if $x\in\mathbb{R}^d$, $x\neq 0$, then $x=r(\zeta_1,\dots,\zeta_d)$ with $r=|x|=(x_1^2+\cdots+x_d^2)^{1/2}$ and $(\zeta_1^2+\cdots+\zeta_{d-1}^2)+\zeta_d^2=1$

We can then find $\theta_{d-1}\in[0,\pi]$ such that $\zeta_d=\cos\theta_{d-1}$ while $\zeta_1^2+\cdots+\zeta_{d-1}^2=\sin^2\theta_{d-1}$. Either this last quantity is 0 or we divide it by $\sin^2\theta_{d-1}$ so that

$$\big((\zeta_1/\sin\theta_{d-1})^2+\cdots+(\zeta_{d-2}/\sin\theta_{d-1})^2\big)+(\zeta_{d-1}/\sin\theta_{d-1})^2=1.$$

It follows that there is a $\theta_{d-2}\in[0,\pi]$ such that $\zeta_{d-1}/\sin\theta_{d-1}=\cos\theta_{d-2}$ and $\big((\zeta_1/\sin\theta_{d-1})^2+\cdots+(\zeta_{d-2}/\sin\theta_{d-1})^2\big)=\sin^2\theta_{d-2}$ *i.e.* $\zeta_{d-1}=\cos\theta_{d-2}\sin\theta_{d-1}$ and $\zeta_1^2+\cdots+\zeta_{d-2}^2=(\sin\theta_{d-2}\sin\theta_{d-1})^2$. We keep on like this till we obtain $\zeta_1^2+\zeta_2^2=(\sin\theta_2\cdots\sin\theta_{d-1})^2$ so that there is $\theta_1\in(-\pi,\pi)$ for which $\zeta_1=\sin\theta_1\sin\theta_2\cdots\sin\theta_{d-1}$ and $\zeta_2=\cos\theta_1\sin\theta_2\cdots\sin\theta_{d-1}$. In summary

$$\begin{cases} x_1 &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{d-1}\\ x_2 &= r\cos\theta_1\sin\theta_2\cdots\sin\theta_{d-1}\\ \vdots &= \vdots\\ x_{d-1} &= r\cos\theta_{d-2}\sin\theta_{d-1}\\ x_d &= r\cos\theta_{d-1}\end{cases}$$

with $r\geq 0$, $\theta_1\in(-\pi,\pi]$, and $\theta_j\in[0,\pi]$ for $j=2,\dots,d-1$. This leads to a $\mathcal{C}^1$ bijection $\Pi:(r,\theta_1,\dots,\theta_{d-1})\to(x_1,\dots,x_d)$ from $]0,+\infty)\times(-\pi,\pi]\times[0,\pi]^2$ onto $\mathbb{R}^d\setminus\{0\}$. If we fix $r>0$, the image of $\{r\}\times(-\pi,\pi]\times[0,\pi]^2$ under Π is the sphere $r\mathbb{S}^{d-1}=\{x\in\mathbb{R}^d\ :\ |x|=r\}$.

For $d=2$ we have just constructed polar coordinates while for $d=3$

$$\begin{cases} x &= r\sin\theta\sin\varphi\\ y &= r\sin\theta\cos\varphi\\ z &= r\cos\theta\end{cases}\qquad r\geq 0,\ 0\leq\theta\leq\pi,\ 0\leq\varphi\leq 2\pi.$$

For instance, for $d=3$, the Jacobian matrix of this change of variable is

$$\begin{pmatrix}\sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi\\ \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi\\ \cos\theta & -r\sin\theta & 0\end{pmatrix}$$

and its determinant is

$$\begin{aligned} J &= \cos\theta\begin{bmatrix} r\cos\theta\sin\varphi & r\sin\theta\cos\varphi\\ r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi\end{bmatrix}+r\sin\theta\begin{bmatrix}\sin\theta\sin\varphi & r\sin\theta\cos\varphi\\ \sin\theta\cos\varphi & -r\sin\theta\sin\varphi\end{bmatrix}\\ &= r^2\cos^2\theta\sin\theta\big(-\sin^2\varphi-\cos^2\varphi\big)+r\sin^3\theta\big(-\sin^2\varphi-\cos^2\varphi\big)\\ &= -r\sin\theta(\cos^2\theta+\sin^2\theta)=-r\sin\theta.\end{aligned}$$

In particular, if we write f in spherical coordinates $\tilde f(r,\theta,\varphi)=f(r\sin\theta\sin\varphi,r\sin\theta\cos\varphi,r\cos\theta)$ then

$$\int_{\mathbb{R}^d}f(x)\,\mathrm{d}x=\int_0^{+\infty}\int_0^{2\pi}\int_0^{\pi}f(r\sin\theta\sin\varphi,r\sin\theta\cos\varphi,r\cos\theta)r\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}r.$$

We can then define for $\Psi:\mathbb{S}^2\to\mathbb{C}$,

$$\int_{\mathbb{S}^2}\Psi(\zeta)\,\mathrm{d}\sigma_2(\zeta)=\int_0^{2\pi}\int_0^{\pi}\Psi(\sin\theta\sin\varphi,\sin\theta\cos\varphi,\cos\theta)\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta.$$

For arbitrary dimension d, a similar computation shows that

$$\int_{\mathbb{S}^{d-1}}\Psi(\zeta)\,\mathrm{d}\sigma_{d-1}(\zeta)=\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\Psi(\Theta)\prod_{j=2}^{d-1}\sin^{j-1}\theta_j\,\mathrm{d}\theta_{d-1}\cdots\mathrm{d}\theta_2\,\mathrm{d}\theta_1\tag{2.54}$$

with

$$\Theta = (\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{d-1}, \cos\theta_1 \sin\theta_2 \cdots \sin\theta_{d-1}, \ldots, \cos\theta_{d-1})$$

and then

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \int_0^{+\infty} \int_{S^{d-1}} f(r\zeta)\,\mathrm{d}\sigma_{d-1}(\zeta)\, r^{d-1}\,\mathrm{d}r.$$

In particular, if f is radial, $f(x) = f_0(|x|)$ then

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \sigma_{d-1}(\mathbb{S}^{d-1}) \int_0^{+\infty} f_0(r) r^{d-1}\,\mathrm{d}r.$$

We can now compute the volume of the euclidean ball $|B_d(\rho)|$ and the surface area of the sphere in $\mathbb{R}^d$. Indeed, if $f = \mathbf{1}_{B_d(\rho)}$ so that $f(x) = f_0(|x|)$ with $f_0 = \mathbf{1}_{[0,\rho]}$ we have

$$\begin{aligned} |B_d(\rho)| &= \int_{\mathbb{R}^d} \mathbf{1}_{B_d(\rho)}(x)\,\mathrm{d}x = \sigma_{d-1}(\mathbb{S}^{d-1}) \int_0^{+\infty} \mathbf{1}_{[0,\rho]}(r) r^{d-1}\,\mathrm{d}r \\ &= \sigma_{d-1}(\mathbb{S}^{d-1}) \int_0^{\rho} r^{d-1}\,\mathrm{d}r = \frac{\sigma_{d-1}(\mathbb{S}^{d-1})}{d}\rho^d. \end{aligned}$$

This already shows that $|B_d(\rho)| = |B_d(1)|\rho^d$ and that $\sigma_{d-1}(\mathbb{S}^{d-1}) = d|B(0,1)|$.

On the other hand, if $x \in B_d(\rho)$ then each coordinate of x is $\leq \rho$. Thus

$$\begin{aligned} B_d(\rho) &= \{(x_1, \ldots, x_{d-1}, x_d) : (x_1^2 + \cdots + x_{d-1}^2) + x_d^2 = \rho^2\} \\ &= \{(x_1, \ldots, x_{d-1}, x_d) : (x_1^2 + \cdots + x_{d-1}^2) = \sqrt{\rho^2 - x_d^2}^{\,2}\} \\ &= \{(\bar{x}, x_d) : x_d \in [-\rho, \rho],\ \bar{x} \in B_{d-1}(\sqrt{\rho^2 - x_d^2})\}. \end{aligned}$$

Using Fubini (we only integrate non-negative quantities), we find

$$\begin{aligned} |B_d(\rho)| &= \int_{\mathbb{R}^d} \mathbf{1}_{B(\rho)}(x)\,\mathrm{d}x = \int_{-\rho}^{\rho} \int_{\mathbb{R}^{d-1}} \mathbf{1}_{B_{d-1}(\sqrt{\rho^2 - x_d^2})}(\bar{x})\,\mathrm{d}\bar{x}\,\mathrm{d}x_d \\ &= \int_{-\rho}^{\rho} |B_{d-1}(\sqrt{\rho^2 - x_d^2})|\,\mathrm{d}x_d \\ &= |B_{d-1}(1)| \int_{-\rho}^{\rho} (\rho^2 - x_d^2)^{(d-1)/2}\,\mathrm{d}x_d. \end{aligned}$$

This gives an induction formula

$$\begin{aligned} |B_d(1)| &= |B_{d-1}(1)| \int_{-1}^{1} (1 - t^2)^{(d-1)/2}\,\mathrm{d}t = 2|B_{d-1}(1)| \int_0^1 (1 - t^2)^{(d-1)/2}\,\mathrm{d}t \\ &= |B_{d-1}(1)| \int_0^1 (1 - x)^{(d-1)/2} x^{-1/2}\,\mathrm{d}s = \mathbf{B}\left(\frac{d+1}{2}, \frac{1}{2}\right) |B_{d-1}(1)| \end{aligned}$$

which the change of variable $x = t^2$ and B the β function.

It follows that

$$|B_d(1)| = B(d/2 + 1, 1/2)|B_{d-1}(1)| = \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{d}{2} + 1\right)}|B_{d-1}(1)| = \sqrt{\pi}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2} + 1\right)}|B_{d-1}(1)|.$$

Iterating this formula, we get

$$|B_d(1)| = \pi \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2} + 1\right)} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)} |B_{d-2}(1)| = \frac{2\pi}{d}|B_{d-2}(1)|.$$

But, when $d = 1$ of course $|B_1(1)| = |[-1, 1]| = 2$ while for $d = 2$, the area of a disc af radius 1 $|B_2(1)| = \pi$ (by definition of π). An immediate induction then gives

$$|B_d(1)| = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right)} = \begin{cases} \frac{\pi^p}{p!} & \text{si } d = 2p \\ \frac{2(4\pi)^p p!}{(2p+1)!} & \text{si } d = 2p + 1 \end{cases}$$

(the first fomula follows from the second one) while

$$\sigma_{d-1}(\mathbb{S}^{d-1}) = d|B_d(1)| = \frac{d\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}.$$

This formula could also be obtained by an induction, starting with $\sigma_1(\mathbb{S}^1) = 2\pi$ and the fact that (2.54) shows that

$$\sigma_{d-1}(\mathbb{S}^{d-1}) = \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\prod_{j=2}^{d-1}\sin^{j-1}\theta_j\,\mathrm{d}\theta_{d-1}\cdots\mathrm{d}\theta_2\,\mathrm{d}\theta_1 = \sigma_{d-2}(\mathbb{S}^{d-2})\int_0^{\pi}\sin^{d-2}\theta_{d-1}\,\mathrm{d}\theta_{d-1}.$$

A key property of the surface measure of $\mathbb{S}^{d-1}$ is that it is rotation invariant, that is, if $R \in SO(d)$ (a $d\times d$ unitary matrix with determinant 1) then, for any continuous function φ on $\mathbb{S}^{d-1}$,

$$\int_{\mathbb{S}^{d-1}}\varphi(R\zeta)\,\mathrm{d}\sigma_{d-1}(\zeta) = \int_{\mathbb{S}^{d-1}}\varphi(\zeta)\,\mathrm{d}\sigma_{d-1}(\zeta).$$

This property follows immediately from the same property for the Lebesgue measure on $\mathbb{R}^d$.

As a consequence, we can derive a formula for the integral of a function that depends only on the scalar product with a fixed direction $\varphi(\zeta) = \varphi_0(\langle\zeta, x\rangle)$. To do so, write $x = |x|Re_d$ where $e_d = (0,\ldots,0,1)$ and $R \in SO(d)$, then

$$\begin{aligned}\int_{\mathbb{S}^{d-1}}\varphi(\zeta)\,\mathrm{d}\sigma_{d-1}(\zeta) &= \int_{\mathbb{S}^{d-1}}\varphi_0(|x|\langle\zeta, Re_d\rangle)\,\mathrm{d}\sigma_{d-1}(\zeta)\\ &= \int_{\mathbb{S}^{d-1}}\varphi_0(|x|\langle R^*\zeta, e_d\rangle)\,\mathrm{d}\sigma_{d-1}(\zeta)\\ &= \int_{\mathbb{S}^{d-1}}\varphi_0(|x|\langle\zeta, e_d\rangle)\,\mathrm{d}\sigma_{d-1}(\zeta)\\ &= \sigma_{d-2}(\mathbb{S}^{d-2})\int_0^{\pi}\varphi_0(|x|\cos\theta_{d-1})\sin^{d-2}\theta_{d-1}\,\mathrm{d}\theta_{d-1}\end{aligned}$$

with (2.54). Changing variable $t = \cos\theta_{d-1}$ gives

$$\begin{aligned}\int_{\mathbb{S}^{d-1}}\varphi_0(\langle\zeta, x\rangle)\,\mathrm{d}\sigma_{d-1}(\zeta) &= \sigma_{d-2}(\mathbb{S}^{d-2})\int_{-1}^{1}\varphi_0(|x|t)(1-t^2)^{\frac{d}{2}-1}\frac{\mathrm{d}t}{\sqrt{t^2-1}}\\ &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\int_{-1}^{1}\varphi_0(|x|t)(1-t^2)^{\frac{d}{2}-1}\frac{\mathrm{d}t}{\sqrt{t^2-1}}.\end{aligned}\tag{2.55}$$

3. The Bessel function

The *Bessel function* is defined for $\mathrm{Re}\,\nu > -\frac{1}{2}$ via the Poisson representation formula as

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \frac{1}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1}e^{ist}(1-s^2)^\nu\frac{\mathrm{d}t}{\sqrt{1-s^2}}.$$

Alternatively, J_ν can be defined via the power series

$$J_\nu(t) = \sum_{n=0}^{+\infty}\frac{1}{\Gamma(\nu+n+1)}\frac{(-1)^n}{n!}\left(\frac{t}{2}\right)^{\nu+2n}.$$

This function is of class $\mathcal{C}^\infty$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}[t^{-\nu}J_\nu] = -t^{-\nu}J_{\nu+1}(t) \quad\text{and}\quad \frac{\mathrm{d}}{\mathrm{d}t}[t^\nu J_\nu] = t^\nu J_{\nu-1}(t)$$

the first one being valid for $\mathrm{Re}\,\nu > -\frac{1}{2}$ and the second only for $\mathrm{Re}\,\nu > \frac{1}{2}$. One may further verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}J_\nu(t) = \frac{1}{2}\big(J_{\nu-1}(t) - J_{\nu+1}(t)\big).$$

and that J_ν satisfies a Sturm-Liouville differential equation

$$t^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}J_\nu + t\frac{\mathrm{d}}{\mathrm{d}t}J_\nu + (t^2-\nu^2)J_\nu = 0.$$

A further property is that J_ν has a “transmutation” property

$$\int_0^1 J_\mu(st)s^{\mu+1}(1-s^2)^\nu \,\mathrm{d}s = \frac{2^\nu\Gamma(\nu+1)}{t^{\nu+1}} J_{\mu+\nu+1}(t). \tag{3.56}$$

We leave this formula as an exercice based on the series representation of the Bessel function and the properties of the Γ function.

The most important facts for us are that J_ν have the asymptotic behavior

$$J_\nu(t) \sim \frac{t^\nu}{2^\nu\Gamma(\nu+1)} \qquad \text{, when } t\to 0$$

and

$$J_\nu(t) \sim \sqrt{\frac{2}{\pi}}\frac{\cos\left(t-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}{t^{1/2}} \qquad \text{, when } t\to +\infty.$$

4. The Fourier transform of the surface measure of the sphere

The importance of the Bessel function in Fourier analysis comes from the fact that it appears in the expression of the Fourier transform of the surface measure of the sphere.

LEMMA A.1. *For $d\geq 2$, the Fourier transform of σ_{d-1} is given by*

$$\widehat{\sigma_{d-1}}(\xi) = 2\pi\frac{J_{\frac{d-2}{2}}(2\pi|\xi|)}{|\xi|^{\frac{d-2}{2}}}.$$

PROOF. By definition

$$\widehat{\sigma_{d-1}}(\xi) = \int_{\mathbb{S}^{d-1}} e^{-2i\pi\langle x,\xi\rangle}\,\mathrm{d}\sigma_{d-1}(x) = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\int_{-1}^1 e^{-2i\pi t|\xi|}(1-t^2)^{\frac{d}{2}-1}\frac{\mathrm{d}t}{\sqrt{t^2-1}}$$

with (2.55). Using the definition of the Bessel function, we then get

$$\widehat{\sigma_{d-1}}(\xi) = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\frac{2^{\frac{d-2}{2}}\Gamma\left(\frac{d-2}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{(2\pi|\xi|)^{\frac{d-2}{2}}}J_{\frac{d-2}{2}}(2\pi|\xi|).$$

As $\Gamma(1/2)=\pi^{1/2}$, the result follows. □

5. Fourier transform of radial functions

LEMMA A.2. *Let $f\in L^1(\mathbb{R}^d)$ be a radial function and define f_0 by $f(x)=f_0(|x|)$. Then the Fourier transform of f is given by*

$$\widehat{f}(\xi) = \frac{(2\pi)^{\frac{d}{2}}}{(|\xi|)^{\frac{d-2}{2}}}\int_0^{+\infty} f_0(r)J_{\frac{d-2}{2}}(r|\xi|)\,r^{\frac{d}{2}-1}\,\mathrm{d}r.$$

PROOF. Observe that $f_0\in L^1(\mathbb{R}^+, r^{d-1}\,\mathrm{d}r)$. It is enough to compute

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{\mathbb{R}^d} f_0(|x|)e^{-2i\pi\langle x,\xi\rangle}\,\mathrm{d}x = \int_0^{+\infty} f_0(r)\int_{\mathbb{S}^{d-1}} e^{-2i\pi r\langle\zeta,\xi\rangle}\,\mathrm{d}\sigma(\zeta)\,r^{d-1}\,\mathrm{d}r\\
&= \int_0^{+\infty} f_0(r)\int_{\mathbb{S}^{d-1}} e^{-2i\pi\langle\zeta,r\xi\rangle}\,\mathrm{d}\sigma(\zeta)\,r^{d-1}\,\mathrm{d}r = \int_0^{+\infty} f_0(r)2\pi\frac{J_{\frac{d-2}{2}}(2\pi r|\xi|)}{(r|\xi|)^{\frac{d-2}{2}}}\,\mathrm{d}r\\
&= \frac{2\pi}{|\xi|^{\frac{d-2}{2}}}\int_0^{+\infty} f_0(r)J_{\frac{d-2}{2}}(r|\xi|)\,r^{\frac{d}{2}-1}\,\mathrm{d}r
\end{aligned}$$

and the asymptotics of the Bessel function in 0 and ∞ show that all integrals are well defined. □

EXAMPLE A.3. If we take $f_0=\mathbf{1}_{[0,1]}$ we get

$$\widehat{\mathbf{1}_{B(0,1)}}(\xi) = \frac{2\pi}{|\xi|^{\frac{d-2}{2}}}\int_0^1 J_{\frac{d}{2}-1}(2\pi r|\xi|)\,r^{\frac{d}{2}-1}\,\mathrm{d}r = \frac{1}{|\xi|^{\frac{d}{2}}}J_{\frac{d}{2}+1}(2\pi|\xi|)$$

with the transmutation property (3.56) with $\mu=\dfrac{d}{2}-1$ and $\nu=0$.

More generally, if we write $x_+ = \max(0, x)$ and $m_\nu(x) = (1 - |x|)_+^\nu$. Then, for $\nu > -d$,

$$\widehat{m_\nu}(\xi) = \frac{1}{(|\xi|)^{\frac{d-2}{2}}} \int_0^1 J_{\frac{d}{2}-1}(2\pi r|\xi|)\, r^{\frac{d}{2}-1}(1-r^2)^\nu \,\mathrm{d}r = \frac{2^\nu \Gamma(\nu+1)}{(2\pi)^\nu |\xi|^{\frac{d}{2}+\nu}} J_{\frac{d}{2}+\nu}.$$

APPENDIX B

The Schur test

In this chapter, we will focus on operators $L^p(\Omega,\mu)\to L^q(\tilde\Omega,\nu)$ defined through a kernel K:

$$T_Kf(x)=\int_\Omega K(x,y)f(y)\,\mathrm{d}\mu(y). \tag{0.57}$$

More precisely, we assume that T_K is defined this way on a dense subset of $L^p(\Omega,\mu)$ and we are interrested in conditions on K that ensure that T_K extends to a bounded operator $L^p(\Omega,\mu)\to L^q(\tilde\Omega,\nu)$.

This is rather easy when $p=1$. Indeed, from Minkovski's Inequality, we get

$$\begin{aligned}\left\|\int_\Omega K(x,y)f(y)\,\mathrm{d}\mu(y)\right\|_{L^q(\tilde\Omega,\nu)} &\leq \int_\Omega \|K(x,y)f(y)\|_{L^q_x(\tilde\Omega,\nu)}\,\mathrm{d}\mu(y)\\ &= \int_\Omega \|K(x,y)\|_{L^q_x(\tilde\Omega,\nu)}|f(y)|\,\mathrm{d}\mu(y)\\ &\leq \sup_{y\in\Omega}\|K(x,y)\|_{L^q_x(\tilde\Omega,\nu)}\int_\Omega |f(y)|\,\mathrm{d}\mu(y)\end{aligned}$$

This shows the following:

Proposition B.1. *Let (Ω,μ), $(\tilde\Omega,\nu)$ be two σ-finite measure spaces and let $1\leq q\leq\infty$. Let $K:\tilde\Omega\times\Omega\to\mathbb{C}$. Assume that*

$$M:=\begin{cases}\sup\limits_{y\in\Omega}\left(\int_{\tilde\Omega}|K(x,y)|^q\,\mathrm{d}\nu(x)\right)^{1/q} & \text{when } q<+\infty\\ \sup_{(x,y)\in\Omega\times\tilde\Omega}|K(x,y)| & \text{otherwise}\end{cases}$$

is finite. Then the linear operator defined by

$$T_Kf(x)=\int_\Omega K(x,y)f(y)\,\mathrm{d}\mu(y)$$

is bounded $L^1(\Omega,\mu)\to L^q(\tilde\Omega,\nu)$ with $\|T_K\|_{L^1(\Omega,\mu)\to L^q(\tilde\Omega,\nu)}\leq M$.

The dual case, *i.e.* with final space L^∞, is a simple: take $1\leq p\leq+\infty$ and p' be the dual exponent, $\dfrac{1}{p}+\dfrac{1}{p'}=1$. Then Hölder shows that

$$\sup_{x\in\tilde\Omega}\left|\int_\Omega K(x,y)f(y)\,\mathrm{d}\mu(y)\right|\leq\sup_{x\in\tilde\Omega}\|K(x,y)\|_{L^{p'}_y(\Omega,\mu)}\|f\|_{L^p(\Omega,\mu)}.$$

This shows the following:

Proposition B.2. *Let (Ω,μ), $(\tilde\Omega,\nu)$ be two σ-finite measure spaces and let $1\leq p\leq\infty$ and p' such that $\dfrac{1}{p}+\dfrac{1}{p'}=1$. Let $K:\tilde\Omega\times\Omega\to\mathbb{C}$. Assume that*

$$M:=\begin{cases}\sup\limits_{x\in\tilde\Omega}\left(\int_{\Omega}|K(x,y)|^{p'}\,\mathrm{d}\mu(y)\right)^{1/p'} & \text{when } p>1\\ \sup_{(x,y)\in\Omega\times\tilde\Omega}|K(x,y)| & \text{when } p=1\end{cases}$$

is finite. Then the linear operator defined by

$$T_Kf(x)=\int_\Omega K(x,y)f(y)\,\mathrm{d}\mu(y)$$

is bounded $L^p(\Omega,\mu)\to L^\infty(\tilde\Omega,\nu)$ with $\|T_K\|_{L^p(\Omega,\mu)\to L^\infty(\tilde\Omega,\nu)}\leq M$.

For the case $L^p \to L^p$, we will now prove the following powerful tool:

THEOREM B.3 (Schur Test). *Let* (Ω, μ), $(\tilde{\Omega}, \nu)$ *be two σ-finite measure spaces and let* $1 \le p \le \infty$. *Let* $K : \tilde{\Omega} \times \Omega \to \mathbb{C}$. *Assume that*

$$A := \sup_{x \in \tilde{\Omega}} \int_\Omega |K(x,y)|^p \, \mathrm{d}\mu(y) < +\infty$$

and

$$B := \sup_{y \in \Omega} \int_{\tilde{\Omega}} |K(x,y)|^p \, \mathrm{d}\nu(x) < +\infty$$

Then the linear operator defined by

$$T_K f(x) = \int_\Omega K(x,y) f(y) \, \mathrm{d}\mu(y)$$

is bounded $L^p(\Omega, \mu) \to L^p(\tilde{\Omega}, \nu)$ *with* $\|T_K\|_{L^p(\Omega,\mu) \to L^p(\tilde{\Omega},\nu)} \le A^{1/p'} B^{1/p}$.

We take the convention $X^{1/\infty} = 1$.

PROOF. Note that when $A = 0$ or $B = 0$ then $K = 0$ a.e. and $T_K = 0$. We thus exclude this case.

We have already considered the cases $p = 1$ and $p = \infty$ so we assume that $1 < p < +\infty$ and take for p' the dual exponent $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. We want to show that, for every $f \in L^p(\Omega, \mu)$ and every $g \in L^{p'}(\tilde{\Omega}, \nu)$

$$\left| \int_{\tilde{\Omega}} T_K f(x) g(x) \, \mathrm{d}\nu(y) \right| \le C \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\nu)}$$

where C is a constant to be determined. This would then imply

$$\begin{aligned} \|T_K f\|_{L^p(\nu)} &= \sup_{\|g\|_{L^{p'}(\tilde{\Omega},\nu)}=1} \left| \int_{\tilde{\Omega}} T_K f(x) g(x) \, \mathrm{d}\nu(y) \right| \\ &\le \sup_{\|g\|_{L^{p'}(\tilde{\Omega},\nu)}=1} C \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\nu)} = C \|f\|_{L^p(\mu)}. \end{aligned}$$

Next notice that

$$\left| \int_{\tilde{\Omega}} T_K f(x) g(x) \, \mathrm{d}\nu(y) \right| \le \int_{\tilde{\Omega}} |T_K f(x)| \, |g(x)| \, \mathrm{d}\nu(y)$$

while

$$|T_K f(x)| = \left| \int_\Omega K(x,y) f(y) \, \mathrm{d}\mu(y) \right| \le \int_\Omega |K(x,y)| \, |f(y)| \, \mathrm{d}\mu(y).$$

But then, from Fubini,

$$\int_{\tilde{\Omega}} |T_K f(x)| \, |g(x)| \, \mathrm{d}\nu(y) = \int_\Omega \int_{\tilde{\Omega}} |K(x,y)| \, |f(x)| \, |g(y)| \, \mathrm{d}\mu(y) \, \mathrm{d}\nu(x)$$

(order of integration is not relevant as we integrate non-negative quantities). We thus have to show

$$\int_\Omega \int_{\tilde{\Omega}} |K(x,y)| \, |f(x)| \, |g(y)| \, \mathrm{d}\mu(y) \, \mathrm{d}\nu(x) \le C \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\nu)}. \tag{0.58}$$

The next step is a reduction to the case $A = B = 1$. To do so, define a new measure $\mathrm{d}\tilde{\mu} = \dfrac{1}{A} \mathrm{d}\mu$ so that

$$\sup_{x \in \tilde{\Omega}} \int_\Omega |K(x,y)|^p \, \mathrm{d}\tilde{\mu}(y) = 1$$

while

$$\|f\|_{L^p(\tilde{\mu})} = \left(\int_\Omega |f(x)|^p \, \frac{1}{A} \mathrm{d}\mu(x) \right)^{1/p} = A^{-1/p} \|f\|_{L^p(\mu)}.$$

If we set $\mathrm{d}\tilde{\nu} = \dfrac{1}{B} \mathrm{d}\nu$ then

$$\sup_{y \in \Omega} \int_{\tilde{\Omega}} |K(x,y)|^p \, \mathrm{d}\tilde{\nu}(x) = 1$$

while $\|g\|_{L^{p'}(\tilde\nu)} = B^{-1/p'}\|g\|_{L^{p'}(\nu)}$. Finally

$$\int_\Omega\int_{\tilde\Omega} |K(x,y)|\,|f(x)|\,|g(y)|\,\mathrm{d}\tilde\mu(y)\,\mathrm{d}\tilde\nu(x) = A^{-1}B^{-1}\int_\Omega\int_{\tilde\Omega} |K(x,y)|\,|f(x)|\,|g(y)|\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x).$$

Thus, if we show the case $A = B = 1$, that is,

$$\int_\Omega\int_{\tilde\Omega} |K(x,y)|\,|f(x)|\,|g(y)|\,\mathrm{d}\tilde\mu(y)\,\mathrm{d}\tilde\nu(x) \le \|f\|_{L^p(\tilde\mu)}\|g\|_{L^{p'}(\tilde\nu)} \tag{0.59}$$

then

$$A^{-1}B^{-1}\int_\Omega\int_{\tilde\Omega} |K(x,y)|\,|f(x)|\,|g(y)|\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x) \le A^{-1/p}\|f\|_{L^p(\mu)}B^{-1/p'}\|g\|_{L^{p'}(\nu)}$$

which is (0.58) with $C = A^{1-1/p}B^{1-1/p'} = A^{1/p'}B^{1/p}$.

From now on, we thus assume $A = B = 1$ and will show (0.59). As a last reduction, notice that (0.59) is homogeneous in f and g, so may replace f and g by $f/\|f\|_{L^p(\tilde\mu)}$ and $g/\|g\|_{L^{p'}(\tilde\nu)}$. We thus want to show that

$$\int_\Omega\int_{\tilde\Omega} |K(x,y)|\,|f(x)|\,|g(y)|\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x) \le 1$$

when $\|f\|_{L^p(\tilde\mu)} = \|g\|_{L^{p'}(\tilde\nu)} = 1$ and

$$\sup_{x\in\tilde\Omega}\int_\Omega |K(x,y)|^p\,\mathrm{d}\mu(y) = \sup_{y\in\Omega}\int_{\tilde\Omega} |K(x,y)|^p\,\mathrm{d}\nu(x) = 1.$$

For this, we will use convexity of the exponential function, for $a, b > 0$,

$$e^{\frac{1}{p}\ln a + \frac{1}{p'}\ln b} \le \frac{1}{p}e^{\ln a} + \frac{1}{p'}e^{\ln b}$$

i.e. $a^{1/p}b^{1/p'} \le \frac{1}{p}a + \frac{1}{p'}b$ that is, for $u, v > 0$

$$uv \le \frac{1}{p}u^p + \frac{1}{p'}v^{p'}.$$

Note that this is trivial when $u = 0$ and when $v = 0$ so that

$$|f(x)g(x)| \le \frac{1}{p}|f(x)|^p + \frac{1}{p'}|g(x)|^{p'}.$$

It follows that

$$\begin{aligned}
\int_\Omega\int_{\tilde\Omega} &|K(x,y)|\,|f(x)|\,|g(y)|\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x)\\
&\le \frac{1}{p}\int_\Omega\int_{\tilde\Omega} |K(x,y)|\,|f(x)|^p\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x)\\
&\qquad + \frac{1}{p'}\int_\Omega\int_{\tilde\Omega} |K(x,y)|\,|g(y)|^{p'}\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x)\\
&= \frac{1}{p}\int_{\tilde\Omega}\left(\int_\Omega |K(x,y)|\,\mathrm{d}\mu(y)\right)|f(x)|^p\,\mathrm{d}\nu(x)\\
&\qquad + \frac{1}{p'}\int_\Omega\left(\int_{\tilde\Omega} |K(x,y)|\,\mathrm{d}\nu(x)\right)|g(y)|^{p'}\,\mathrm{d}\mu(y)
\end{aligned}$$

with Fubini (everything is non-negative). The first integral is bounded by

$$\begin{aligned}
\int_{\tilde\Omega}&\left(\int_\Omega |K(x,y)|\,\mathrm{d}\mu(y)\right)|f(x)|^p\,\mathrm{d}\nu(x)\\
&\le \left(\sup_{x\in\tilde\Omega}\int_\Omega |K(x,y)|\,\mathrm{d}\mu(y)\right)\int_{\tilde\Omega}|f(x)|^p\,\mathrm{d}\nu(x)\\
&= \int_{\tilde\Omega}|f(x)|^p\,\mathrm{d}\nu(x) = \|f\|^p_{L^p(\nu)} = 1
\end{aligned}$$

while for the second one

$$\int_\Omega\left(\int_{\tilde\Omega}|K(x,y)|\,\mathrm{d}\nu(x)\right)|g(y)|^{p'}\,\mathrm{d}\mu(y)$$
$$\leq\left(\sup_{y\in\Omega}\int_\Omega\int_{\tilde\Omega}|K(x,y)|\,\mathrm{d}\nu(x)\right)\int_\Omega|g(y)|^{p'}\,\mathrm{d}\mu(y)$$
$$=\int_\Omega|g(y)|^{p'}\,\mathrm{d}\mu(y)=\|g\|^p_{L^{p'}(\nu)}=1.$$

In conclusion

$$\int_\Omega\int_{\tilde\Omega}|K(x,y)|\,|f(x)|\,|g(y)|\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x)\leq\frac{1}{p}+\frac{1}{p'}=1$$

as claimed. □

Remark B.4. Schur test sometimes gives an optimal bound. For instance, if μ,ν are finite measures and $K\geq 0$ and if we know

$$\int_\Omega K(x,y)\,\mathrm{d}\mu(y)=A\quad\text{and}\quad\int_{\tilde\Omega}K(x,y)\,\mathrm{d}\nu(x)=B$$

then, integrating the first identity with respect to $\nu$and the second one with respect to μ, we get $A\nu(\tilde\Omega)=B\mu(\Omega)$. This reads $T_K\mathbf{1}_\Omega=A\mathbf{1}_{\tilde\Omega}$ while $\|\mathbf{1}_\Omega\|_p=\mu(\Omega)^{1/p}$ and $\|\mathbf{1}_{\tilde\Omega}\|_{p'}=\nu(\tilde\Omega)^{1/p'}$. In conclusion $\|T_K\mathbf{1}_\Omega\|_{p'}=A^{1/p'}B^{1/p}\|\mathbf{1}_\Omega\|_p$.

On the opposite, when K oscillates, one can't expect any optimality from Schur's test as no cancellation is taken into account. A striking example is $\Omega=\tilde\Omega=\mathbb{R}^d$ with $K(x,y)=e^{-2i\pi\langle x,y\rangle}$ so that $T_Kf=\widehat{f}$, the Fourier transform of f. From Plancherel we know that $\|T_Kf\|_2=\|f\|_2$ while $A=B=+\infty$ in Schur's test so that one can not even obtain boundedness.

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