MAXIMUM REGULARITY OF HUA HARMONIC FUNCTIONS

P. JAMING

1. The Siegel upper half space

1.1. **Setting.** On $\mathbb{C}^2$ consider $r(z) = Im(z_2) - |z_1|^2$. The Siegel upper half space and its boundary are given by

$$\mathcal{U} = \{z = (z_1, z_2) \,:\, r(z) > 0\}$$
$$\partial\mathcal{U} = \{z = (z_1, z_2) \,:\, r(z) = 0\}$$

The Heisenberg group is given by $\mathbb{H} = \mathbb{C} \times \mathbb{R}$ with the action given by

$$[\zeta, t][\eta, s] = [\zeta + \eta, t + s + 2Im(\zeta\bar{\eta})]$$

acts on $\mathcal{U}$ and $\partial\mathcal{U}$ bu

$$[\zeta, t]z = (z_1 + \zeta, z_2 + t + 2iz_2\bar{\zeta} + i|\zeta|^2).$$

This action is simply transitive on $\partial\mathcal{U}$ but not on $\mathcal{U}$. For this, one needs to add dilations: if $a > 0$ then $a \cdot z = (a^{1/2}z_1, az_2)$. One can then identify $S = \mathbb{H} \succeq \mathbb{R}_+^*$ with $\mathcal{U}$ by identifying $[\zeta, t, a] \in S$ with $[\zeta, t, a] \cdot i$ where the action of S on $\mathcal{U}$ is given by

$$[\zeta, t, a] \cdot z = [\zeta, t] \cdot a \cdot z.$$

The group law is then

$$[\zeta, t, a][\eta, s, b] = \big[[\zeta, t][a^{1/2}\eta, as], ab\big] = ...$$

One can then define left invariant vector field :
— on $\mathbb{H}$ by

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, Y = \frac{\partial}{\partial y} + 2x\frac{\partial}{\partial t}, Z_1 = \frac{1}{2}(X - iY), T = \frac{\partial}{\partial t};$$

— on S by

$$a^{1/2}X, a^{1/2}Y, a^{1/2}Z_1, aT, a\partial_a, aZ_2 = \frac{a}{2}(T - i\partial_a).$$

One may then identify functions on $\mathcal{U}$ with functions on S and extend notions such as holomorphy, pluriharmonicity... from functions on $\mathcal{U}$ to functions on S. For instance

	on $\mathcal{U}$	on S
f holomorphic	$\frac{\partial f}{\partial z_1} = \frac{\partial f}{\partial z_2} = 0$	$\overline{Z}_1 f = \overline{Z}_2 f = 0$
f pluri-harmonic	$\frac{\partial^2 f}{\partial z_k \partial \bar{z}_j} = 0$	$Z_2\overline{Z}_1 f = Z_1\overline{Z}_2 f = Z_2\overline{Z}_2 f = 0$ and $(Z_1\overline{Z}_1 + 2i\overline{Z_2})f = 0.$

Joint work with A. Bonami (Orléans), D. Buraczewski, E. Damek, A. Hulanicki (Wrocław).

Finally we will consider the following family of second order differential operators ($\alpha \in \mathbb{R}$)

$$\mathcal{L}_\alpha = -\frac{1}{2}(Z_1\overline{Z}_1 + \overline{Z}_1 Z_1) + i\alpha T, \quad \textit{on } \mathbb{H}$$

(we simply write $\mathcal{L} = \mathcal{L}_0$) and

$$L = -\frac{1}{2}a(\mathcal{L} + \partial_a) + a^2(\partial_a^2 + T^2), \quad \textit{on } S.$$

This operator has a Poisson kernel :
recall that the Haar measure on $\mathbb{H} \sim \mathbb{R}^3$ is the Lebesgue measure of $\mathbb{R}^3$ and that convolution is given by

$$f * g[\zeta, t] = \int_{\mathbb{H}} f[\eta, s] g([\eta, s]^{-1}[\zeta, t]) d\eta ds.$$

For exists $a > 0$, there exists a unique function P_a on $\mathbb{H}$ with $\int_{\mathbb{H}} P_a = 1$, called the Poisson kernel, such that F is L harmonic and bounded if and only if $F = f * P_a$.

1.2. **Harmonicity and pluri-harmonicity 1.**

Theorem 1.1. *Assume that $F = f * P_a$ with f bounded. Then*

(1) *F is holomorphic if and only if $\mathcal{L}_1 f = 0$,*
(2) *F is anti-holomorphic if and only if $\mathcal{L}_{-1} f = 0$,*
(3) *F is pluri-harmonic if and only if $(\mathcal{L}^2 + T^2)f = \mathcal{L}_{-1}\mathcal{L}_1 f = 0$.*

Sketch of proof. Denote by C the Cauchy-Szegö projection. There is a well-known function Φ such that, if f is smooth and fastly decreasing, then $f - C(f) = \mathcal{L}_1(f * \Phi)$.

Apply this to $f\varphi_R$ with φ_R a dilation of a cutoff function, compose with $\bar{Z}_1$ (this removes $C(f\varphi_R)$) estimate the convolution, noting that it is an integration over a shell and let $R \to 0$, this gives $\bar{Z}_1 f = 0$.

The second point is obtained by conjugation. For the 3rd point, note that $\mathcal{L}_1 F$ is anti-holomorphic, one obtains pluri-harmonicity by computing commutators. □

1.3. **Automatic regularity of harmonic functions.**

We will say that a function G on S has a *boundary distribution* if, for every $\psi \in \mathcal{C}_c^\infty(\mathbb{H})$,

$$\lim_{a \to 0} \int_{\mathbb{H}} G(\omega, a)\psi(\omega) d\omega$$

exists. It is obvious that if G has a boundary distribution, so do $\mathcal{L}G$ and TG.

Theorem 1.2. *If F is harmonic with a boundary distribution, then $\partial_a F$ has also a boundary distribution.*

Sketch of proof. Let $\psi \in \mathcal{C}_c^\infty(\mathbb{H})$ and set

$$\varphi(a) = \int_{\mathbb{H}} \partial_a F(\omega, a)\psi(\omega) dw.$$

We want to show that ffi has a limit as $a \to 0$. We have

$$a\partial_a^2 F(a) - \frac{1}{2}\partial_a F = (\frac{1}{2}\mathcal{L} + aT^2)F$$

and the right hand side has a boundary distribution. Multiplying by ψ and integrating over $\mathbb{H}$ gives

$$\partial_a\varphi - \frac{1}{2}\varphi = g(a)$$

for some function g that has a limit as $a \to 0$. Solving this differential equation, we get

$$\varphi(a) = \lambda a^{1/2} + a^{1/2}\int_1^a \frac{g(t)}{t^{1+1/2}}dt$$

and, as $1/2 > 0$ it is easy to show that this has a limit as $a \to 0$. □

1.4. **Maximal regularity of t-invariant harmonic functions.** If F is L-harmonic and t-invariant (i.e $TF = 0$), then $\Lambda F = 0$ with

$$\Lambda = \frac{1}{2}a(\Delta - \partial_a) + a^2\partial_a^2$$

where Δ is the Euclidean Laplacian.

Proposition 1.3. *Assume that F is bounded and Λ-harmonic such that, for every $\psi \in \mathcal{S}(\mathbb{C}^n)$ and $k = 0, 1$ or 2,*

$$\text{(BR)} \qquad \sup_{a\leq 1}\left|\int_{\mathbb{C}^n}\partial_a^k F(\zeta, a)\psi(\zeta)d\zeta\right| < +\infty.$$

Then F is constant.

Proof. Let Q_a be the Poisson kernel for Λ, so that $F = Q_a * f$ with f bounded. Let $\varphi \in \mathcal{S}(\mathbb{C}^n)$ such that $0 \notin supp\hat{\varphi}$, we will prove that $\left\langle \hat{f}, \hat{\varphi}\hat{Q}_a\right\rangle = \int F_a(\zeta)\varphi(\zeta)d\zeta = 0$.

Once this is done, $\hat{f}$ is a distribution supported in $\{0\}$ so f is bounded polynomial, thus a constant.

To prove the claim, we first identify Q_a : $\hat{Q}_a(\xi) = z(a|\xi|^2/2)$ where z is the unique bounded solution of

$$(a\partial_a^2 - \frac{1}{2}\partial_a - 1)z(a) = 0$$

with $z(0) = 1$ (z is the Legendre function).

Then we study the regularity of z :

$$\partial^2 z(a|\xi|^2/2) = \gamma(a|\xi|^2) + a^{-1/2}|\xi|^{-1}\tilde{\gamma}(a|\xi|^2)$$

with $\gamma, \tilde{\gamma}$ smooth functions, $\tilde{\gamma}(0) \neq 0$.

The condition (BR) implies the claim. □

1.5. **Fourier analysis on $\mathbb{H}$.** The represntations f $\mathbb{H}$ on $L^2(\mathbb{R})$ are given by ($\lambda \in \mathbb{R}^*$) :

$$R^\lambda[u, v, t]\Phi(x) = e^{2i\pi\lambda(ux+uv/2+t/4)}\Phi(x + v).$$

The fourier transform of a function f on $\mathbb{H}$ is the operator defined by

$$\left\langle \hat{f}(\lambda)\phi, \psi\right\rangle = \int_{\mathbb{H}}\left\langle R^\lambda(w)^*\phi, \psi\right\rangle f(w)dw.$$

It is an Hilbert-Schmidt operator with kernel $K_f^\lambda(x,v)\mathcal{F}_{1,3}f(\lambda(x+v)/2, x-v, \lambda/4)$. The inversion formula is given by

$$f(u,v,t)=\int_{\mathbb{R}^*} tr\big(\hat{f}(\lambda)R^\lambda(u,v,t)\big)|\lambda|^n d\lambda.$$

The trace in this formula may be written using h_k^λ, a properly scaled Hermite basis, and

$$e_k^\lambda(\omega)=\left\langle R^\lambda(\omega)h_k^\lambda, h_k^\lambda\right\rangle.$$

1.6. **Identification of P_a.** Notons ${}_\omega f(w)=f(\omega w)$. Alors

Lemma 1.4. *For $\lambda\neq 0$ and $k\in\mathbb{N}$, set*

$$g_k^\lambda(\omega,a)=\left\langle \widehat{{}_\omega P_a}(\lambda)h_k^\lambda, h_k^\lambda\right\rangle = e_k^\lambda * \check{P}_a(\omega^{-1}).$$

Then $g_k^\lambda(\omega,a)=e_k^\lambda(\omega^{-1})g(|\lambda|a)$ where g is the unique bounded solution of

$$\left(\partial_a^2-\left(\frac{k+1}{a}+1\right)\right)g(a)=0$$

with $g(0)=1$.

Such a g is a confluent hypergeometric function.
The proof uses the fact that e_k^λ are eigenfunctions of $\mathcal{L}$ and T.

1.7. **An orthogonality property.** For $\psi\in\mathcal{S}(\mathbb{R})$ and $k\in\mathbb{N}$, write

$$e_k^\psi(\omega)=\int_{\mathbb{R}} e_k^\lambda(\omega)\psi(\lambda)d\lambda.$$

Proposition 1.5. *Let f be a bounded function on $\mathbb{H}$ and $F=f*P_a$. Assume that F satisfies*

$$\text{(BR)}\qquad \sup_{a\leq 1}\left|\int_{\mathbb{H}}\partial_a^k F(\omega,a)\psi(\omega)d\omega\right|<+\infty$$

for $k=0,1,2$ and $\psi\in\mathcal{S}(\mathbb{H})$. Then for $k\neq 0$, $w\in\mathbb{H}$, ψ with $\hat{\psi}\in\mathcal{C}_c^\infty(\mathbb{R}\setminus\{0\})$,

$$\int_{\mathbb{H}} {}_w f(\omega)e_k^\psi(\omega)d\omega=0.$$

1.8. **Final result.**

Theorem 1.6. *Let f be a bounded function on $\mathbb{H}$ and $F=f*P_a$. Assume that F satisfies*

$$\text{(BR)}\qquad \sup_{a\leq 1}\left|\int_{\mathbb{H}}\partial_a^k F(\omega,a)\psi(\omega)d\omega\right|<+\infty$$

for $k=0,1,2$ and $\psi\in\mathcal{S}(\mathbb{H})$. Then F is pluri-harmonic.

Sketch of proof. By replacing f by $\psi * f$ with ψ smooth, we may assume that f is smooth and has 4 bounded derivatives.

Write $g = (\mathcal{L}^2 + T^2)f$ and $G = g * P_a = (\mathcal{L}^2 + T^2)F$. Then G still satisfies the hypothesis of the theorem.

According to Theorem 1.1, it is enough to prove that $g = 0$ and, with Harnack, that g is a constant. With Proposition 1.3, it is enough to prove that G does not depend on t.

But Proposition 1.5 states that

$$\int_{\mathbb{H}} g(\omega) e_k^{\psi}(\omega) d\omega = 0$$

for $k = 0$ and ψ with $0 \notin \operatorname{supp} \hat{\psi}$. As $g = (\mathcal{L}^2 + T^2)f$, the same is true for $k = 0$, then, by Fourier inversion on the Heisenberg group, g is a polynomial in the t variable, and by boundedness, does not depend on t. So G does not depend on the t variable. $\square$

2. Hua operators on tube type domains

2.1. **Jordan algebras.** Let $V = Sym(2, \mathbb{R})$ the symetric 2×2 matrices with product $A \cdot B = \frac{1}{2}(AB + BA)$ and scalar product $\langle A, B \rangle = tr(AB^t)$. This is an Euclidean Jordan algebra.

Let $c_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $c_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$c_i^2 = c_i, \; c_i c_j = 0, \; c_1 + c_2 = I$$

and is maximal for this property. This is a Jordan frame of V.

Let $\Omega = Int\{A \cdot A, A \in V\} = Sym_+(2, \mathbb{R})$ the set of positive definite symmetric 2×2 matrices. This is an irreducible cone in V.

Write $L(A)B = AB$ an endomorphism of V and let $V(c_i, \lambda)$ be the eigenspaces of $L(c_i)$. The only eigenvalues are $0, 1/2$ and 1. Here $V_{i,i} := V(c_i, 1) = \mathbb{R}c_i$ $i = 1, 2$ and $V_{i,j} := V(c_i, 1/2) \cap V(c_j, 1/2) = \mathbb{R}c_{1,2}$ with $c_{1,2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so that $V = \bigoplus_{1 \le i \le j \le r} V_{1,1} \oplus V_{1,2} \oplus V_{2,2}$. This is the Peirce decomposition of V.

2.2. **Automorphisms of Ω.** Let G be the component of I of the group of $g \in GL(V)$ s.t $g \cdot \Omega \subset \Omega$. Let $\mathcal{G}$ be its Lie algebra.

Let $\mathcal{A}$ be the abelian part of $\mathcal{G}$, then $\mathcal{A} = Vect\{L(c_1), L(c_2)\}$ and note that this are given by

$$L(c_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad L(c_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\mathcal{N}$ be the nilpotent part of $\mathcal{G}$, $\mathcal{N} = \bigoplus_{i<j} \mathcal{N}_{i,j}$ where $\mathcal{N}_{i,j} = Vect\{c_{i,j} \square c_i\}$ and the $\square$ operation is defined by

$$A \square B = L(A \cdot B) + [L(A), L(B)].$$

Here $\mathcal{N} = \mathcal{N}_{1,2} = \mathbb{R}c_{1,2} \square c_1 = \mathbb{R} \begin{pmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}.$

Let $\mathcal{S}_0 = \mathcal{A} \oplus \mathcal{N}$ and $S_0 = \exp \mathcal{S}_0 = \left\{ \begin{pmatrix} a^2 & 0 & 0 \\ \gamma a & ab & 0 \\ \gamma^2/2 & \gamma b & b^2 \end{pmatrix}, a, b > 0 \gamma \in \mathbb{R} \right\}$.

This acts simply transitively on Ω.

2.3. **Tube domains.** We now consider the tube domain $T_\Omega = V + i\Omega \subset V^{\mathbb{C}}$. Then VS_0 acts on T_Ω by $(x, I) \cdot (u + iv) = x + u + uv$ and $(0, s) \cdot (u + iv) = s \cdot u + is \cdot v$. The group operation is therefore $(v, s)(v', s') = (v + s \cdot v', s \cdot s')$.

The action is simply transitive, so we may identify VS_0 with T_Ω by $(u + iv) \sim (v, s) \cdot iI$.

The Lie algebra of VS_0 is $V \oplus S_0 = V \oplus \mathcal{A} \oplus \bigoplus_{i,j} \mathcal{N}_{i,j}$.

A basis of order 1 invariant differential operators is given by $X_1, X_2, X_{1,2}$ that correspond to $c_1, c_2, c_{1,2} \in V$, $Y_{1,2}$ that corresponds to $2e_{1,2} \in \mathcal{N}$ and H_1, H_2 that correspond to $L(c_1), L(c_2)$ in $\mathcal{A}$.

2.4. **Hua operators and main theorem.** We that set

$$\Delta_1 = X_1^2 + H_1^2 - H_1, \quad \Delta_2 = X_2^2 + H_2^2 - H_2, \quad \Delta_{1,2} = X_{1,2}^2 + Y_{1,2}^2 - H_2.$$

At the point iI, these correspond to $\frac{\partial^2}{\partial z_1 \partial \bar{z}_1}$, $\frac{\partial^2}{\partial z_2 \partial \bar{z}_2}$ and $\frac{\partial^2}{\partial z_{1,2} \partial \bar{z}_{1,2}}$ and one can show that a function is pluriharmonic if and only if $\Delta_1 f = \Delta_2 f = \Delta_{1,2} f = 0$.

The (strongly diagonal) Hua operators are then given by

$$\mathbb{H}_j = \Delta_j + \frac{1}{2} \sum_{k<j} \Delta_{k,j} + \frac{1}{2} \sum_{k>j} \Delta_{j,k}$$

so that here they are $\mathbb{H}_1 = \Delta_1 + \frac{1}{2}\Delta_{1,2}$ and $\mathbb{H}_2 = \Delta_2 + \frac{1}{2}\Delta_{1,2}$. It is known that the bounded Hua-harmonic functions are the Poisson integrals.

Theorem 2.1. *Every bounded Hua-harmonic function f has a finite number of derivatives with a boundary distribution, unless it is pluri-harmonic.*

The proof is an induction on the "rank" of the cone (=the number of elements in the Jordan frame).

One first isolates everything that contains the index 2 by averaging f over the other variables (against a test function ψ). This way, one obtains a function over a subgroup of S which turns out to be the extension of the Heisenberg group presented earlier. Moreover, the function f_ψ one obtains is L-harmonic, (the operator L has been built for that purpose). Further if f has a given number of boundary derivatives, so does the function f_ψ one obtains by averaging. From the previous section, we know that if one asks for too many derivatives, this only occurs if f_ψ is pluri-harmonic. This then implies that $\Delta_2 f = \Delta_{1,2} f = 0$ so that $\mathbb{H}_1 f = 0$ reduces to $\Delta_1 f = 0$ and finally f is pluri-harmonic.