# Nazarov's uncertainty principle in higher dimension

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# Outline of talk

# Definitions

## Definition

*Let $S, \Sigma$ subsets of $\mathbb{R}^d$.*

- $(S, \Sigma)$ *is an annihilating pair if*

$$\operatorname{supp} f \subset S \quad \& \quad \operatorname{supp} \widehat{f} \subset \Sigma \quad \Rightarrow \quad f = 0;$$

- $(S, \Sigma)$ *is a strong annihilating pair if* $\exists C = C(S, \Sigma)$ *s.t.* $\forall f \in L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |f(x)|^2 \, dx \leq C \left( \int_{\mathbb{R}^d \setminus S} |f(x)|^2 \, dx + \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \, d\xi \right).$$

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$$\int_{\Sigma} |\widehat{f}(\xi)|^2 \, d\xi \leq D \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \, d\xi$$

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— PDE's...

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# Main Theorem

**Theorem**

*Let $S, \Sigma \subset \mathbb{R}^d$ have finite measure. Then*

- *(Benedicks 1974-1985) $(S, \Sigma)$ is weakly annihilating.*
- *(Amrein-Berthier 1977) $(S, \Sigma)$ is strongly annihilating.*
- *(Nazarov $d = 1$ 1993) $C(S, \Sigma) \leq ce^{c|S||\Sigma|}$*
- *(J. $d \geq 2$ 2007) $C(S, \Sigma) \leq ce^{c\min\left(|S||\Sigma|, |S|^{1/d}\omega(\Sigma), \omega(S)|\Sigma|^{1/d}\right)}$*
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- $f = e^{-\pi|x|^2} = \widehat{f}$, $S = \Sigma = B(0,R)$

$$\int_{\mathbb{R}^d} |f(x)|^2\,dx \le c e^{(2\pi+\varepsilon)R^2} \left( \int_{\mathbb{R}^d \setminus B(0,R)} e^{-2\pi|x|^2}\,dx + \int_{\mathbb{R}^d \setminus B(0,R)} e^{-2\pi|\xi|^2}\,d\xi \right).$$

- Optimal: $C(S,\Sigma) \le c e^{(2\pi+\varepsilon)(|S||\Sigma|)^{1/d}}$.
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## Benedick's proof

$|S|, |\Sigma| < +\infty$, $f \in L^2(\mathbb{R})$, $\operatorname{supp} f \subset S$ & $\operatorname{supp} \widehat{f} \subset \Sigma$.

1. WLOG $|S| = 1/2$
2. $\displaystyle\int_{[0,1]} \sum_k \chi_\Sigma(\xi + k)\, d\xi = |\Sigma| < +\infty \Rightarrow$
   for a.a. $\xi \in \mathbb{R}$, $\mathrm{Card}\,\{k \in \mathbb{Z} : \xi + k \in \Sigma\}$ *finite*
3. $\displaystyle\int_{[0,1]} \underbrace{\sum_k \chi_S(x + k)}_{=0 \text{ or } \geq 1}\, dx = |S| = 1/2 \Rightarrow \exists F \subset [0,1], |F| > 0$
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4. by Poisson Summation

$$\sum_{k\in\mathbb{Z}} f(x+k)e^{2i\pi\xi(x+k)} = \sum_{k\in\mathbb{Z}} \widehat{f}(\xi+k)e^{2i\pi kx}.$$

   By 2, the RHS is a trigonometric polynomial $Z(f)(x)$ in $x$ (for a.a. $\xi$)
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## Random Periodization

**Lemma (Nazarov, $d = 1$)**

$\varphi \in L^1(\mathbb{R})$, $\varphi \geq 0$,

$$\int_1^2 \sum_{k\in\mathbb{Z} \quad \setminus\{0\}} \varphi(v \quad k \ ) \, \mathrm{d}v \qquad \simeq \int_{\|x\|\geq 1} \varphi(x) \, \mathrm{d}x$$

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**Lemma (Nazarov, $d = 1$)**

$\varphi \in L^1(\mathbb{R}^d)$, $\varphi \geq 0$,

$$\int_{SO(d)} \int_1^2 \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \varphi\big(v\,\rho(k)\big)\,\mathrm{d}v\,\mathrm{d}\nu_d(\rho) \simeq \int_{\|x\| \geq 1} \varphi(x)\,\mathrm{d}x$$

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## Second proof: scaling

$|S|, |\Sigma| < +\infty$, $f \in L^2(\mathbb{R})$, $\operatorname{supp} f \subset S$ & $\operatorname{supp} \widehat{f} \subset \Sigma$.

1. WLOG $|S| = C_0$ small enough (see bellow)
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3. $\displaystyle\int_0^1 \int_1^2 \underbrace{\sum_k \chi_S((x+k)/v)}_{=0 \text{ or } \geq 1}\,\mathrm{d}v\,\mathrm{d}x \leq (C_1+2)|S| := 1/2$ if $C_0$
   small enough $\Rightarrow \exists F \subset [0,1], \exists V \subset [1,2]$, $|F| > 0$ $|V| > 0$
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## Turan type Lemma

### Lemma (Nazarov, $d = 1$, Fontes-Merz $d \geq 2$)

- $p(\theta_1, \ldots, \theta_d) = \sum_{k_1=0}^{m_1} \cdots \sum_{k_d=0}^{m_d} c_{k_1,\ldots,k_d} e^{2i\pi(r_{1,k_1}\theta_1 + \cdots r_{r_d,k_d}\theta_d)}$ *a trigonometric polynomial in d variables.*
- $E \subset \mathbb{T}^d$,
- *Then*

$$\sup_{(\theta_1,\ldots,\theta_d)\in\mathbb{T}^d} |p(\theta_1, \ldots, \theta_d)| \leq \left(\frac{14d}{|E|}\right)^{m_1+\cdots+m_d} \sup_{(\theta_1,\ldots,\theta_d)\in E} |p(\theta_1, \ldots, \theta_d)|.$$

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## Turan type Lemma

### Lemma (Nazarov, $d = 1$, Fontes-Merz $d \geq 2$)

- $p(\theta_1, \ldots, \theta_d) = \sum_{k_1=0}^{m_1} \cdots \sum_{k_d=0}^{m_d} c_{k_1,\ldots,k_d} e^{2i\pi(r_{1,k_1}\theta_1 + \cdots r_{r_d,k_d}\theta_d)}$ *a trigonometric polynomial in d variables.*
- $E \subset \mathbb{T}^d$,
- *Then*
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## Average order

### Lemma

- $\Sigma$ *be a relatively compact open set with* $0 \in \Sigma$
- $\Lambda = \Lambda(\rho, v) := \{v\,{}^t\rho(j) \;:\; j \in \mathbb{Z}^d\}$ *a random lattice*
- $\mathcal{M}_{\rho,v} = \{k \in \mathbb{Z}^d \;:\; v\,{}^t\rho(k) \in \Sigma\} = \Lambda \cap \Sigma$
- *then*

$$\mathbb{E}_{\rho,v}\left(\mathrm{ord}\,\mathcal{M}_{\rho,v} - d\right) \leq C\omega(\Sigma).$$

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If order $\to$ size of support, $\mathbb{E}_{\rho,v}\left(\mathrm{Card}\,\mathcal{M}_{\rho,v} - d\right) \leq C|\Sigma|$.

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## End of Proof 1/4

Scale to have $|S| = 2^{-d-1}$ and take $f \in L^2$ with supp $f \subset S$

Set $\Gamma_{\rho,v}(t) = \dfrac{1}{v^{d/2}} \displaystyle\sum_{k \in \mathbb{Z}^d} f\left(\frac{\rho(k+t)}{v}\right)$

Set $E_{\rho,v} = \{t \in [0,1] : \Gamma_{\rho,v}(t) = 0\}$

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## End of Proof 2/4

From the Lattice averaging lemma, one can choose $\rho, v$ s.t.

— $\|R_{\rho,v}\|_2^2 \leq C \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \, d\xi$ (w.h.p)

— ord $P_{\rho,v} \leq C(\omega(\Sigma) + d)$ (w.h.p)

— $|E_{\rho,v}| \geq 1/2$ (certain)

— $\widehat{f}(0) \leq |P_{\rho,v}(0)|$ (certain).

$\rho, v$ s.t. all 4 properties hold.

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On $E_{\rho,v}$, we have $\Gamma_{\rho,v} = 0$, thus $P_{\rho,v} = -R_{\rho,v}$ so

$$\int_{E_{\rho,v}} |P_{\rho,v}(t)|^2 \,\mathrm{d}t = \int_{E_{\rho,v}} |R_{\rho,v}(t)|^2 \,\mathrm{d}t \leq C \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi$$

So $E := \{t \in E_{\rho,v} \,:\, |P_{\rho,v}(t)|^2 \leq 4C \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi\}$ has $|E| \geq 1/4$.

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$$\begin{aligned}
|\widehat{f}(0)|^2 &\leq |\widehat{P_{\rho,v}}(0)|^2 \leq \left(\sum_{k\in\mathbb{Z}^d} |\widehat{P_{\rho,v}}(k)|\right)^2 \leq \left(\sup_{x\in\mathbb{T}^d} |P_{\rho,v}(x)|\right)^2 \\
&\leq \left[\left(\frac{14d}{|E|}\right)^{\operatorname{ord} P_{\rho,v}-1} \sup_{x\in E} |P_{\rho,v}(x)|\right]^2 \\
&\leq \left[\left(\frac{14d}{1/4}\right)^{\operatorname{ord} P_{\rho,v}-1} \left(4C\int_{\mathbb{R}^d\setminus\Sigma} |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi\right)^{1/2}\right]^2 \\
&\leq Ce^{C\omega(\Sigma)} \int_{\mathbb{R}^d\setminus\Sigma} |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi.
\end{aligned}$$

Apply to $f \to f_y(x) = f(x)e^{-2i\pi xy}$, $\Sigma \to \Sigma_y = \Sigma - y$ and integrate over $y \in \Sigma$ QED

## Almost time & band-limited functions: Landau-Slepian-Pollak

$$\mathcal{S}_{\varepsilon,T,\Omega} := \left\{ f \in L^2 \ : \ \int_{|t|>T} |f(t)|^2 \mathrm{d}t < \varepsilon \|f\|^2 \right.$$
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### Theorem (Landau-Slepian-Pollak)

$\exists$ an orthonormal system $\{\gamma_k\}_{k=0,1,\ldots}$ s.t., $\forall f \in \mathcal{S}_{\varepsilon,T,\Omega}$,

$$\|f - P_{4T\Omega} f\| \leq 7\|f\|.$$

$f \in L^2(\mathbb{R})$ supp $\widehat{f} \subset [-\Omega, \Omega]$, $h < \pi/\Omega$

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# The Umbrella Theorem

**The Umbrella Theorem (J.-Powell)**

$\varphi, \psi \in L^2(\mathbb{R})$. $\exists N = N(\varphi, \psi)$ s.t.

- $(e_k)_{k \in I} \subset L^2(\mathbb{R})$ orthonormal
- $\forall k \in I$, $|e_k| \leq \varphi$ and $|\widehat{e}_k| \leq \psi$,

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- $\varepsilon > 0$ et $T, \Omega > 0$ s.t.

$$\int_{|x|>T} |\varphi(x)|^2 \, \mathrm{d}x < \varepsilon \quad , \quad \int_{|x|>\Omega} |\psi(x)|^2 \, \mathrm{d}x < \varepsilon$$

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# Estimates on spherical codes

$(e_1, \ldots, e_N) \in \mathbb{S}^{d-1}$ $[-\alpha, \alpha]$-spherical code if $\langle e_i, e_j \rangle \in [-\alpha, \alpha]$.

- (linear independence) $\alpha < \frac{1}{d} \Rightarrow N \le d$;
- (volume counting) $N \le \left(\frac{2-\alpha}{1-\alpha}\right)^d$;
- (Delsarte-Goethals-Siedel) $\alpha < \frac{1}{\sqrt{d}}$ $N \le \frac{1-\alpha^2}{1-\alpha^2 d} d$ and equality only possible for $\{-\alpha, \alpha\}$-codes.
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- (volume counting) $N \leq \left(\frac{2-\alpha}{1-\alpha}\right)^d$;
- (Delsarte-Goethals-Siedel) $\alpha < \frac{1}{\sqrt{d}}$ $N \leq \frac{1-\alpha^2}{1-\alpha^2 d} d$ and equality only possible for $\{-\alpha, \alpha\}$-codes.
- (Siedel) $\{-\alpha, \alpha\}$-codes have cardinality $\leq \frac{d(d-1)}{2}$ (best $\geq \frac{2}{9} d^2$)
- DSG bound met for some $\alpha$'s.
- (J - Guedon 20??) $\alpha = d^{(-1+\gamma)/2}/6$, $\gamma > 0$ then $N_{max} \geq e^{d^\gamma / C}$.

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