Kinetic modelling and control of epidemic dynamics with social heterogeneity

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Infectious Disease Outbreaks (IDO)

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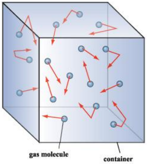
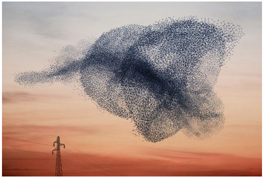
Collective behavior and self-organization

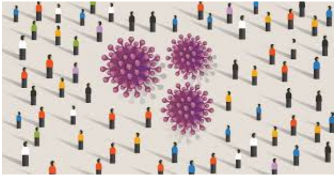
- The mathematical description of emerging collective phenomena and self–organization in systems composed of a large number of individuals has gained an increasing interest in heterogeneous research communities in biology, robotics and social sciences.





- In order reduce the computational cost of microscopic models ruling the dynamics of individual agents, it is of utmost importance to derive the corresponding kinetic and macroscopic dynamics.

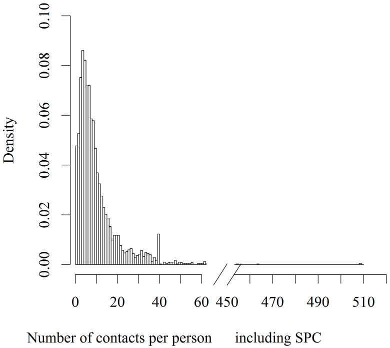
Collective behavior and self-organization

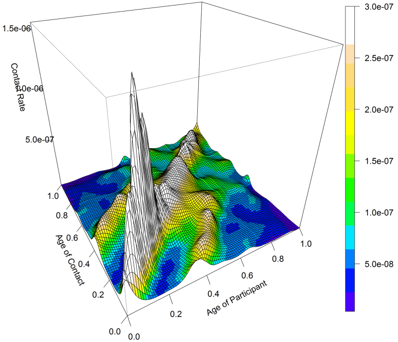
 

- The usual approach of mathematical epidemiology is based on compartmental population dynamics where the whole population is subdivided in several classes whose size evolves in time (see e.g. SIDARTHE, SEPIAR). [1]
- Several epidemic dynamics can be thought as a multiscale process involving interactions between a large number of individuals that may transmit the infection. [2]
- The collective compliance with the so-called non-pharmaceutical interventions has been essential to guarantee public health in absence of effective treatments. [3]

[1] F. Brauer '08; P. Magal, S. Ruan '08; G. Giordano et al. '20; M. Gatto et al. '20
[2] N. Loy, A. Tosin '20; G. Dimarco, B. Perthame, G. Toscani, M. Z. '20-'21
[3] A. Bertozzi et al. '21; G. Dimarco, G. Toscani, M. Z. '22

Social mixing

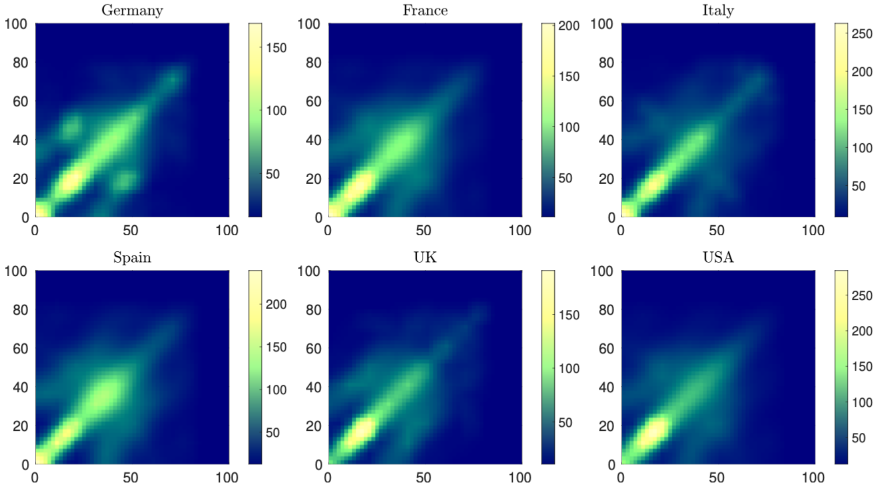




- Recent studies estimate the probability of contacts between individuals, and consequently of potential pathogen transmission. [4]
- Effects of non pharmaceutical interventions can only be measured at the macroscopic scale.

[4] L. Fumanelli et al. '12; G. Beraud et al. '17

Age dependent social mixing: different countries



The cases of Wuhan and Shanghai

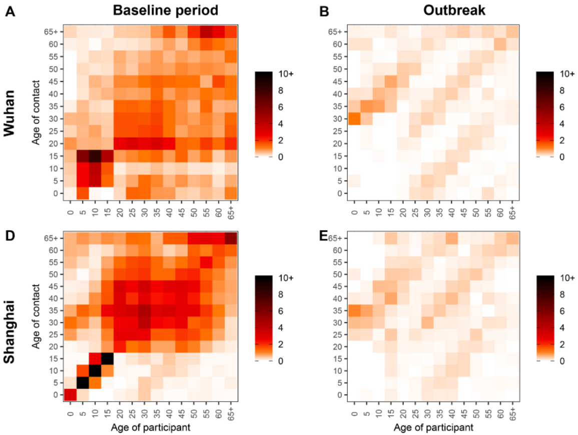


Figura: Daily contacts were reduced 7-8-fold during the COVID-19 social distancing period. [5]

[5] J. Zhang, A. Vespignani et al. '20; Z. McCarthy, J. Wu et al. '20

(Prototype) kinetic models for epidemic dynamics

- We denote by $f_J = f_J(x,t)$, $J \in \{S, I, R\}$, the distribution at time $t \geq 0$ of the number of social contacts of the population belonging to the class J such that

$$\sum_{J \in \{S,I,R\}} f_J(x,t) = f(x,t), \qquad \text{and} \qquad \int_{\mathbb{R}^+} f(x,t)dx = 1.$$

- Therefore the quantities

$$J(t) = \int_{\mathbb{R}^+} f_J(x,t)dx, \qquad J \in \{S, I, R\}$$

denote the fractions of population belonging to the class $J \in \{S, I, R\}$. We also define the local moments of order $\alpha > 0$ as follows

$$J(t)x_{J,\alpha}(t) = \int_{\mathbb{R}^+} x^\alpha f_J(x,t)dx,$$

Notation

We indicate with x_J the local mean of f_J.

(Prototype) kinetic models for epidemic dynamics

- We combine the epidemic process with the contact dynamics as follows

$$\partial_t f_S(x,t) = -K(f_S, f_I)(x,t) + \frac{1}{\epsilon} Q_S(f_S)(x,t)$$
$$\partial_t f_I(x,t) = K(f_S, f_I)(x,t) - \gamma f_I(x,t) + \frac{1}{\epsilon} Q_I(f_I)(x,t)$$
$$\partial_t f_R(x,t) = \gamma f_I(x,t) + \frac{1}{\epsilon} Q_R(f_R)(x,t)$$

where $\gamma > 0$ is the recovery rate while the transmission of the infection is governed by the local incidence rate

$$K(f_S, f_I)(x,t) = f_S(x,t) \int_{\mathbb{R}^+} \kappa(x,y) f_I(y,t)\, dy.$$

where

$$\kappa(x,y) = \beta x^\alpha y^\alpha,$$

is the contact function and $\alpha, \beta > 0$.

(Prototype) kinetic models for epidemic dynamics

- The operators $Q_J(f_J)$ characterize the thermalization of the distribution of social contacts in terms of repeated interactions and is such that

$$\int_{\mathbb{R}^+} Q_J(f_J)(x,t)dx = 0$$

for all $t \geq 0$.

- The classical SIR model can be therefore obtained for the evolution of mass fractions by choosing $\alpha = 0$ and $\beta > 0$. Indeed we would obtain

$$\partial_t \int_{\mathbb{R}^+} f_S(x,t)dx = -\beta \int_{\mathbb{R}^+} f_S(x,t)dx \int_{\mathbb{R}^+} f_I(x,t)dx$$
$$\partial_t \int_{\mathbb{R}^+} f_I(x,t)dx = \beta \int_{\mathbb{R}^+} f_S(x,t)dx \int_{\mathbb{R}^+} f_I(x,t)dx - \gamma \int_{\mathbb{R}^+} f_I(x,t)dx$$
$$\partial_t \int_{\mathbb{R}^+} f_R(x,t)dx = \gamma \int_{\mathbb{R}^+} f_I(x,t)dx$$

Interaction operators

- The microscopic updates of social contacts of individuals can be considered of the form

$$x'_J = x - \Phi^\delta(x/x_J) + \eta x,$$

with η a r.v. such that $\langle \eta \rangle = 0$ and $\langle \eta^2 \rangle = \lambda > 0$ (finite moments up to order three). Furthermore, we consider the transition function [6] in terms of $s = x/x_J$

$$\Phi(s) = \mu \frac{e^{(s^\delta - 1)/\delta} - 1}{e^{(s^\delta - 1)/\delta} + 1}, \qquad 0 < \delta \leq 1, \mu \in (0,1).$$

- For a given density $f_J(x,t)$, $J \in \{S, I, R\}$, the action of the transition operator $Q_J(x,t)$ is given in weak form by

$$\frac{d}{dt}\int_{\mathbb{R}^+} \varphi(x) f_J(x,t)dx = \left\langle \int_{\mathbb{R}^+} B(x)(\varphi(x^*) - \varphi(x)) f_J(x,t)dx \right\rangle$$

where $B(x)$ is the interaction kernel (exponential convergence to equilibrium)[7]

[6] L. Preziosi, G. Toscani, M. Z. '20

[7] G. Furioli, A. Pulvirenti, E. Terraneo, G. Toscani '20

Interaction operators

- Several examples are highlighting the structure of the operators $Q_J(f_J)(x,t)$:

  $i)$ Fokker-Planck-type (coherent with Boltzmann-type description in the quasi-invariant limit)

  $$Q_J^{\mathrm{FP}}(f_J) = \frac{\mu}{2\delta}\frac{\partial}{\partial x}\left\{x^{1-\delta}\left[\left(\frac{x}{x_J}\right)^{\delta}-1\right]f_J(x,t)\right\} + \frac{\lambda}{2}\frac{\partial^2}{\partial x^2}(x^{2-\delta}f_J(x,t))$$

  whose equilibrium is given by Gamma densities

  $$f_{J,\infty}(x;\theta,\chi,\delta) = \frac{\delta}{\theta^\chi}\frac{1}{\Gamma(\chi/\delta)}x^{\chi-1}\exp\{-(x/\theta)^\delta\}.$$

  with $\chi = \nu/\delta + \delta - 1$, $\theta = \bar{x}_J(\delta^2/\nu)^{1/\delta}$, $\nu = \mu/\lambda$ and

  $$\int_{\mathbb{R}+} x f_{J,\infty}(x,\theta,\nu,\delta)dx = x_J, \qquad \int_{\mathbb{R}+} x^2 f_{J,\infty}(x,\theta,\nu,\delta)dx = \frac{\nu+1}{\nu}x_J^2.$$

  $ii)$ BGK-type

  $$Q_J^{\mathrm{BGK}}(f_J) = -(f_J(x,t) - f_{J,\infty}(x)).$$

Analytical aspects

Theorem

Let $Q_J(f_J)$ be the defined Fokker-Planck-type operator and f_J be a solution of the Cauchy problem

$$\begin{cases} \partial_t f_J(x,t) = Q_J(f_J)(x,t), \qquad J \in \{S, I, R\} \\ f_J(x,0) = f_J^0(x). \end{cases} \tag{1}$$

If $f_J^0 \in L^1(\mathbb{R}_+)$ then the L^1 norm of f_J is non-increasing for $t \geq 0$.

Corollary

Let f_J be a solution of the Cauchy problem (1) with initial condition $f_{J,0} \in L^1(\mathbb{R}_+)$. If $f_{J,0} \geq 0$ in $\mathbb{R}_+$ then $f_J \geq 0$ for all $t \geq 0$.

If $f_{J,0} \in L^1(\mathbb{R}_+)$ for all J and the contact function $\kappa(x,t)$ is bounded then the solution to the kinetic model is unique. The result holds for both FP and BGK operators.

(Prototype) Social-SIR model

The choice $\epsilon \ll 1$ identifies a faster adaption of individuals' social contacts with respect to the epidemic dynamics.

We recall that a simple example of connection-dependent model can be obtained by considering the case of symmetric interactions

$$\kappa(x,y) = \beta\, x^\alpha\, y^\alpha.$$

If $\alpha = 1$ we then obtain the evolution of mass fractions

$$\begin{aligned} \frac{d}{dt}S(t) &= -\beta x_S(t) x_I(t) S(t) I(t), \\ \frac{d}{dt}I(t) &= \beta x_S(t) x_I(t) S(t) I(t) - \gamma I(t) \\ \frac{d}{dt}R(t) &= \gamma I(t). \end{aligned} \tag{2}$$

System (2) is not closed since it depends on the local mean number of contacts $x_J(t)$.

(Prototype) Social-SIR

Since the considered operators $Q_J(x,t)$ are momentum preserving we have

$$\frac{d}{dt}(x_S(t)S(t)) = -\beta x_{S,2}(t)x_I(t)S(t)I(t).$$

If $\epsilon \ll 1$ we have exponential convergence to the local Gamma equilibrium and we can rewrite $x_{S,2}$ as follows

$$x_{S,2}(t) = \int_{\mathbb{R}^+} x^2 f_{S,\infty}(x)dx = \frac{\nu+1}{\nu}x_S^2(t), \qquad \nu = \mu/\lambda.$$

Therefore, we should add to the previous system the system for the mean number of contacts

$$\begin{aligned}
&\frac{d}{dt}x_S(t) = -\frac{\beta}{\nu}x_S^2(t)x_I(t)I(t),\\
&\frac{d}{dt}x_I(t) = \beta x_S(t)x_I(t)\left(\frac{\nu+1}{\nu}x_S(t) - x_I(t)\right)S(t)\\
&\frac{d}{dt}x_R(t) = \gamma\frac{I(t)}{R(t)}(x_I(t) - x_R(t)).
\end{aligned} \tag{3}$$

The case of saturated incidence rate

Assuming $x_I(t) = \tilde{x}_I$, for any $\alpha > 0$ the first equation of (3) gives

$$\frac{d}{dt}x_S(t) = -\frac{\beta}{\nu}x_S^{1+\alpha}(t)\tilde{x}_I I(t) \Rightarrow x_S(t) = \frac{x_S(0)}{\left(1 + \frac{\beta\alpha x_S^\alpha(0)\tilde{x}_I^\alpha}{\nu}\int_0^t I(s)ds\right)^{1/\alpha}},$$

Therefore we obtain the following closed system for mass fractions with saturated incidence rate

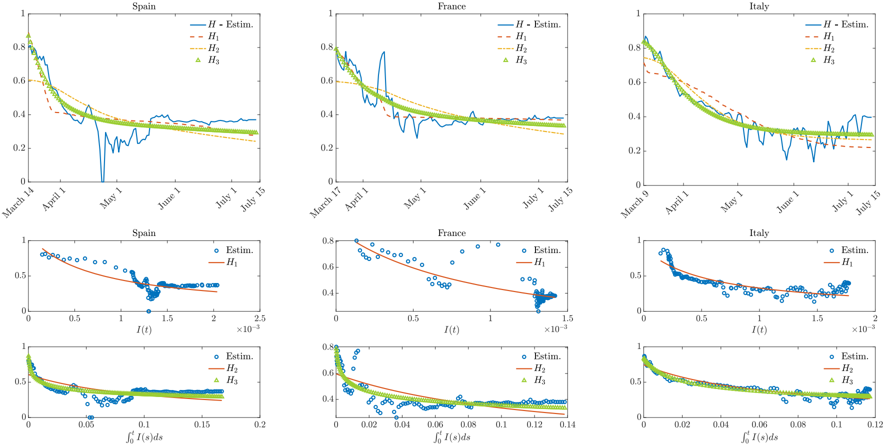
$$\begin{aligned}
\frac{d}{dt}S(t) &= -\beta H(I(t),t)S(t)I(t), \\
\frac{d}{dt}I(t) &= \beta H(I(t),t)S(t)I(t) - \gamma I(t), \\
\frac{d}{dt}R(t) &= \gamma I(t).
\end{aligned}$$

where [8]

$$H(I(t),t) = \frac{1}{\left(1 + \bar{\beta}/\nu \int_0^t I(s)ds\right)^{1/\alpha}}, \qquad \bar{\beta} = \alpha\beta x_S^\alpha(0)\tilde{x}_I^\alpha.$$

[8] V. Capasso, G. Serio ’78; P. K. Maini ’05; A. Medaglia, M. Z. ’21

The case of saturated incidence rate



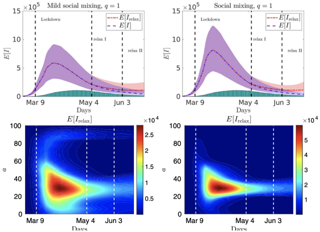
Control policies and epidemic dynamics

The action of a policy maker can be modelled by introducing a control over the compartmental system. [9]

- Typical examples are based on NPI takin into account social structure

$$\min_{u\in\mathcal{U}} J(u)$$

sbj $\dot{s}(a,t)$ where $\beta(a,a_*) \to \beta(a,a_*) - u(a,a_*)$.



- Standard solution of control problems applied to kinetic equations can be obtained via Pontryagin's maximum principle. For large systems the main drawback is the high computational effort. [10]
- We will follow a different path by introducing a Boltzmann-type kinetic control that is upscaled at the mesoscopic level. [11]

[9] S. Lee, G. Chowell, C. Castillo-Chàvez '10; L. Bolzoni, M. Groppi, R. Della Marca '18-'22
[10] M. Fornasier, B. Piccoli, F. Rossi '14; A. Bensoussan, J. Frehse, P. Yam '13
[11] G. Albi, M. Herty, L. Pareschi, M. Z. '14-'20; G. Albi, Y.-P. Choi, D. Kalise, M. Fornasier '17

Observable effect of selective social restrictions

We can model lockdown measures on the compartmentalization $\mathcal{C}$ through the introduction of an additive control at the level of microscopic interactions [12]

$$x' = x - \Phi\left(\frac{x}{x_J}\right) x + \sqrt{\epsilon} W(x) u_J + \eta x, \qquad J \in \mathcal{C}$$

where

$$u_J^* = \arg\min_{u \in \mathcal{U}} \frac{1}{2} \mathbb{E}\left[(x' - x_T)^2 + \nu_J u_J^2\right] \Rightarrow u_J^* = -\frac{\sqrt{\epsilon} W(x)}{\nu + \epsilon W^2(x)} (x - x_T - \Phi(x/x_J) x)$$

with $x_T > 0$ the target number of social contacts and $\nu_J > 0$ a penalization. At the kinetic level we get

$$\frac{\partial \mathbf{f}(x,t)}{\partial t} = \mathbf{P}(x, \mathbf{f}(x,t)) + \mathbf{C}(\mathbf{f}(x,t)) + \frac{1}{\epsilon} \mathbf{Q}(\mathbf{f}(x,t)),$$

being $\mathbf{f} = (f_J)_{J \in \mathcal{C}}$, $\mathbf{P}$ the vector of the transition rates between compartments and $\mathbf{Q} = (Q_J)_{J \in \mathcal{C}}$ Boltzmann-type interaction operators.

[12] G. Dimarco, G. Toscani, M. Z. '21

Observable effect of selective social restrictions

We can derive the operator $\mathbf{C}(\mathbf{f}(x,t))$ defined as follows

$$C_J(f_J(x,t)) = \frac{1}{\nu_J}\frac{\partial}{\partial x}\left[\frac{x - x_{T,J}}{x} W^2(x) f_J(x,t)\right], \qquad J \in \mathcal{C},$$

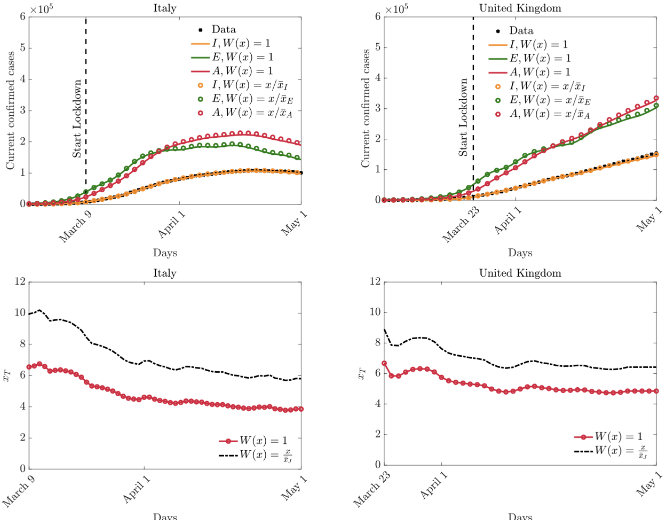
and quantifies the effects of the control on the epidemic spreading. In the macroscopic limit $\epsilon \ll 1$ we get

$$C_J(f_J^\infty) = \begin{cases} \dfrac{1}{\nu_J}\left(\dfrac{\lambda}{\lambda - 1}\dfrac{x_{T,J}}{x_J} - 1\right) & W(x) = 1 \\ \dfrac{1}{\nu_J}\left(\dfrac{\lambda + 1}{\lambda} x_J - x_{T,J}\right) & W(x) = \dfrac{x}{x_J}. \end{cases}$$

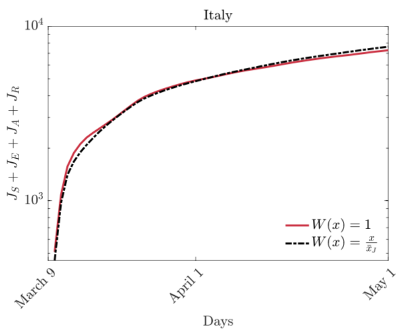
It is worth to remark that for small penalizations $\nu_J > 0$ the effect of the selective control on the number of connections is to steer the mean number of connections towards the values

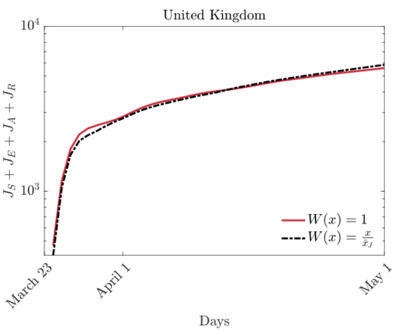
$$x_J^\infty = \begin{cases} x_{T,J}\dfrac{\lambda}{\lambda - 1} > x_{T,J}, & W(x) = 1 \\ x_{T,J}\dfrac{\lambda}{\lambda + 1} < x_{T,J}, & W(x) = \dfrac{x}{x_J} \end{cases}$$

Observable effect of selective social restrictions: Italy & UK

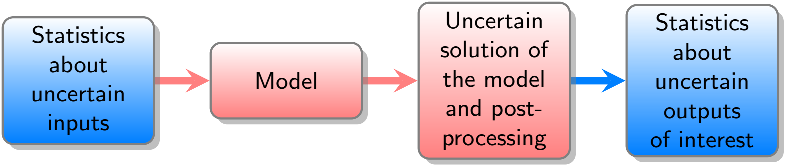


Observable effect of selective social restrictions: Italy & UK



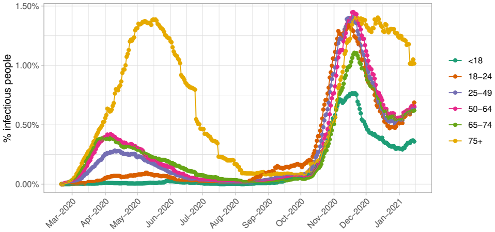


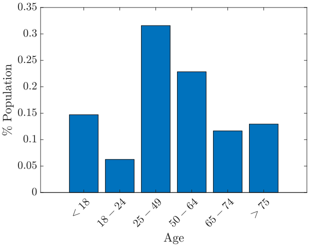
Data in epidemic modelling

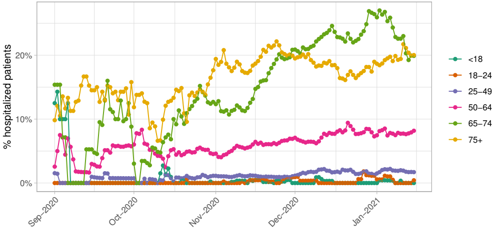


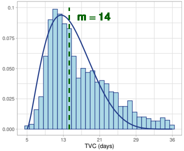
- Timely policies should take into account realistic previsions of the epidemic.
- The comprehension of epidemic dynamics is essentially based on available data and their heterogeneity.
- Main uncertainties:
    - Tracking of infected in official statistics is usually a lower bound on the real number of infected: presence of asymptomatic cases and weakly symptomatic cases + limited testing capacity (early phases)
    - Recovery time depending on the clinical history of each patient.
- Up-to-date forecasts and innovative interventions are essential to control the diffusion of the disease.

Province of Pavia: general trends

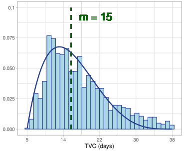




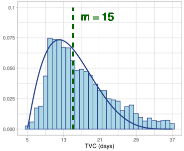




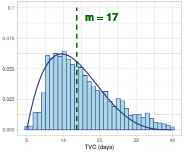
(a) Grup 0-17



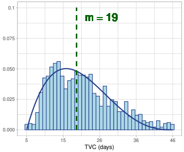
(b) Grup 18-24



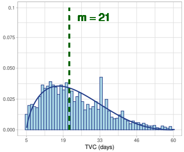
(c) Grup 25-49



(d) Grup 50-64



(e) Grup 65-74



(f) Grup 75+

Age-dependent model with asymptomatic cases

- We assume now that the contact distribution depends on the age variable $a \in \mathcal{A} = [0, 100]$ such that

$$f_S(x,a,t) + f_I(x,a,t) + f_A(x,a,t) + f_R(x,a,t) = f(x,a,t),$$

and

$$\int_{\mathcal{A}} \int_{\mathbb{R}^+} f(x,a,t) dx\, da = 1.$$

- This choice gives the following system of kinetic equations

$$\partial_t f_S = -K(f_S, f_I + f_A) + \frac{1}{\epsilon} Q_S(f_S),$$

$$\partial_t f_I = \xi(a) K(f_S, f_I + f_A) - \gamma_I(a) f_I + \frac{1}{\epsilon} Q_I(f_I)$$

$$\partial_t f_A = (1 - \xi(a)) K(f_S, f_I + f_A) - \gamma_A(a) f_A + \frac{1}{\epsilon} Q_A(f_A)$$

$$\partial_t f_R = \gamma_I(a) f_I + \gamma_A f_A + \frac{1}{\epsilon} Q_R(f_R)$$

Age-dependent model with asymptomatic cases

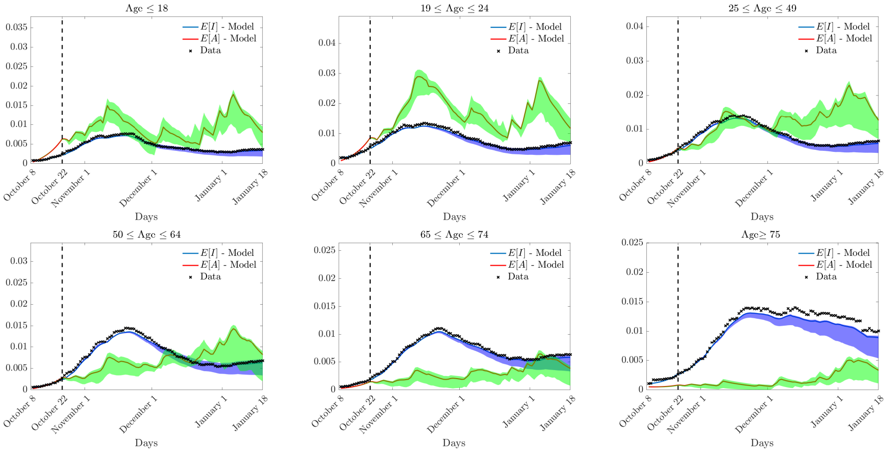
- Previous discussions on the operators $K(\cdot,\cdot)$ and $Q_J(\cdot)$ are still valid.
- At the macroscopic (observable) level we get the following system of equations

$$
\begin{aligned}
\frac{d}{dt}S(a,t) &= -\Lambda(a,t)\\
\frac{d}{dt}I(a,t) &= \xi(a)\Lambda(a,t) - \gamma_I(a)I(a,t)\\
\frac{d}{dt}A(a,t) &= (1-\xi(a))\Lambda(a,t) - \gamma_A A(a,t)\\
\frac{d}{dt}R(a,t) &= \gamma_I(a)I(a,t) + \gamma_A(a)A(a,t)
\end{aligned}
$$

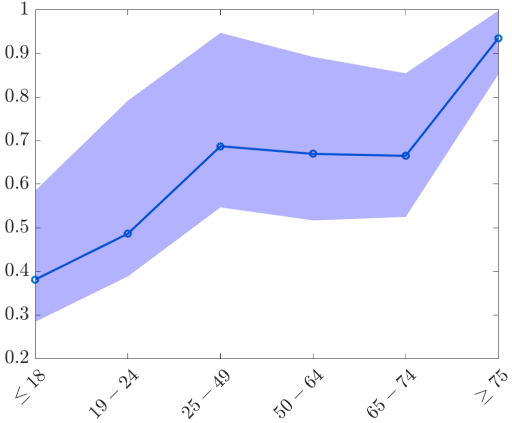
where

$$
\begin{aligned}
&\Lambda(a,t)\\
&\quad = \beta(x)H_S(a,I(t))S(a,t)\int_{\mathcal{A}(a)} H_I(y,I(t))I(y,t) + H_A(y,I(t))A(y,t)dy
\end{aligned}
$$

Age-dependent model with asymptomatic cases



Heterogeneity in contact reduction



Conclusion and perspectives

- We introduced a system of kinetic equations coupling the distribution of social contacts with the spreading of an infectious disease to quantify the evolution of the individuals' contacts.
- Optimal control policies can be considered in the introduced modelling setting and highlight the possible advantages in selectively control individuals with high number of contacts.
- The findings, based on an interplay of mathematical models and data, highlight the role of public awareness of the evolution of the disease. Structural uncertainties are often present and should be incorporated for modelling the dynamics.
- Follow-up questions
    - Emergence of variants
    - Fake news
    - Social related issues (emergence of inequalities)
    - ...

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