Singular perturbation for an abstract non-densely defined Cauchy problem

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Abstract

This work is devoted to the study of a class of singularly perturbed non-densely defined abstract Cauchy problems. We extend the Tikhonov's theorem for ordinary differential equations to the case of abstract Cauchy problems. Roughly speaking we prove that the solutions rapidly evolve and stay in some neighbourhood of the slow manifold. As a consequence we conclude that the solutions of the problem converge on each compact time interval, as the singular parameter goes to zero, toward the solutions of the so-called reduced problem. These results are applied to an example of age-structured model as well as to a class of functional differential equations.

Keywords: Singular perturbation, Tikhonov theorem, Age-structured model, Functional differential equations.

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1 Introduction

In this work we study a class of singularly perturbed non-densely defined abstract Cauchy problems of the form

$$\begin{cases} \varepsilon \dfrac{du_\varepsilon(t)}{dt} = Au_\varepsilon(t) + F\left(u_\varepsilon(t), Lv_\varepsilon(t)\right), \\ \dfrac{dv_\varepsilon(t)}{dt} = Bv_\varepsilon(t) + G\left(u_\varepsilon(t), v_\varepsilon(t)\right), \end{cases} \quad , \ t \geq 0, \tag{1.1}$$

and supplemented with initial data

$$u_\varepsilon(0) = x_\varepsilon \in \overline{D(A)} \text{ and } v_\varepsilon(0) = y_\varepsilon \in \overline{D(B)}. \tag{1.2}$$

Here $\varepsilon > 0$ is a small parameter and the initial data satisfies

$$\lim_{\varepsilon \to 0^+} x_\varepsilon = x_0 \in \overline{D(A)} \text{ and } \lim_{\varepsilon \to 0^+} y_\varepsilon = y_0 \in \overline{D(B)}. \tag{1.3}$$

In (1.1), $A : D(A) \subset X \to X$ and $B : D(B) \subset Y \to Y$ are two given Hille-Yosida linear operators (see Definition 2.1 below) acting respectively on the Banach spaces $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ while $L \in \mathcal{L}(Y, Z)$ is a bounded linear operator from $(Y, \|.\|_Y)$ into the Banach space $(Z, \|.\|_Z)$. The maps $F : \overline{D(A)} \times Z \to X$ and $G : \overline{D(A)} \times \overline{D(B)} \to Y$ are Lipschitz continuous on bounded sets. The semilinear system of equations (1.1) is referred as a non-densely defined problem because the domain of the operator A (respectively B) is not supposed to be dense in X (respectively in Y).

The goal of this article is to study the convergence as $\varepsilon \to 0$ of the solution of system (1.1)-(1.2) toward the solution of the so-called reduced system, that is formally obtained by setting $\varepsilon = 0$ into (1.1)-(1.2) and that reads as

$$\text{(Reduced system)} \begin{cases} 0 = Au(t) + F(u(t), Lv(t)), \\ \dfrac{dv(t)}{dt} = Bv(t) + G(u(t), v(t)), \ v(0) = y_0. \end{cases} \tag{1.4}$$

Note that questions related to singular perturbation analysis have been extensively studied in the context of ordinary differential equations. As a special case of the so-called Tikhonov's theorem has been proved for ordinary differential equations and we refer for instance to [29, 30, 31, 32]. Here we aim at extending such a result in the case of abstract Cauchy problems of the form described in (1.1)-(1.2).

Our motivation to study such a class of singularly perturbed problems comes from age-structured models arising in population dynamics and also from delay differential equations.
As an illustration one may come back to the model proposed by Magal and McCluskey in [23] to describe a criss-cross transmission of bacteria between patients and health care workers within a hospital unit (see also [5] for more information about this topic). The model reads as the following age-structured system of equations

$$\varepsilon \frac{dH_C(t)}{dt} = -\nu_H H_C(t) + \beta_H \int_0^\infty \gamma(a) i(t,a) da (1 - H_C(t)), \tag{1.5}$$

and

$$\begin{cases} \dfrac{dS(t)}{dt} = \nu_P N_P - \nu_P S(t) - \beta_P H_C(t) S(t), \\ \dfrac{\partial i_P(t,a)}{\partial t} + \dfrac{\partial i_P(t,a)}{\partial a} = -\nu_P i_P(t,a), \\ i_P(t,0) = \beta_P H_C(t) S(t). \end{cases} \tag{1.6}$$

The above problem is supplemented with initial data

$$H_C(0) = H_{C0}, S(0) = S_0, \text{ and } i_P(0,.) = i_{P0} \in L^1_+(0, +\infty).$$

In the above system of equations $H_C(t)$ denotes the number of health care workers colonized by the bacteria at time t while the function $i_P(t,a)$ represents the density of the population of patients infected by the bacteria since a period of time a and at time t. All the parameters of the problem are positive while $0 < \varepsilon \ll 1$ is a small parameter describing the fast dynamics of health care workers contamination with respect to the length of stay of patients. A detailed scaling analysis of this singularly perturbed problem (1.5)-(1.6) has been given by Ducrot et al. in [7].
As an other motivating example, we consider the following class of functional differential equations

$$\begin{cases} \varepsilon \dfrac{dx(t)}{dt} = f(t, x_{t,\varepsilon}, y_t), \\ \dfrac{dy(t)}{dt} = g(t, x_{t,\varepsilon}, y_t), \end{cases} \quad t \geq 0, \tag{1.7}$$

and supplemented with the following initial data

$$(x_{0,\varepsilon}, y_0) = \left(\overline{\varphi}(\varepsilon \cdot), \overline{\psi}\right) \in C([-r,0], \mathbb{R}^n) \times C([-r,0], \mathbb{R}^m) \text{ with } n, m \in \mathbb{N} \setminus \{0\},$$

wherein $\overline{\varphi}, \overline{\psi} \in C([-r,0], \mathbb{R}^n)$ are two given functions. In (1.7) we have set

$$x_{t,\varepsilon}(\theta) = x(t + \varepsilon\theta) \text{ and } y_t(\theta) = y(t+\theta), \ \forall t \geq 0 \text{ and } \theta \in [-r, 0],$$

while the maps $g : [0,+\infty) \times C([-r,0], \mathbb{R}^n) \times C([-r,0], \mathbb{R}^m) \to \mathbb{R}^n$ and $f : [0,+\infty) \times C([-r,0], \mathbb{R}^n) \times C([-r,0], \mathbb{R}^m) \to \mathbb{R}^m$ are both Lipschitz continuous on bounded sets. Such a problem has already been considered by Artstein and Slemrod in [3] where the authors derived a general finite time convergence result. In this work we do not extend the general results proposed in the aforementioned work but we show that such a problem enters the general framework of system (1.1) and we obtain the finite time convergence to the reduced system by completely different methods.

Let us finally mention that the class of problems described by (1.1) may cover many other classes of finite and infinite dimensional dynamical systems. In addition to the two above examples one may observe that our framework can also be applied to study the following class of singularly perturbed functional differential equations in L^1 for $t \geq 0$,

$$\begin{cases} \varepsilon \dfrac{dx(t)}{dt} = f(\varepsilon, x_t, Ly(t)) \\ \dfrac{dy(t)}{dt} = B_1 y(t) + g(\varepsilon, x_t, y(t)) \end{cases} \tag{1.8}$$

with initial data

$$(x_0, y(0)) = (\varphi, y_0) \in \mathbb{R} \times L^1([-r,0], \mathbb{R}^n) \times Y_1 \text{ and } x(0) = \bar{x}.$$

One can first incorporate the parameters epsilon into the state variable as follows

$$\begin{cases} \varepsilon \dfrac{dx\,(t)}{dt} = f\,(z(t), x_t, Ly(t)) \\ \dfrac{dy\,(t)}{dt} = B_1 y(t) + g\,(z(t), x_t, y(t)) \\ \dfrac{dz(t)}{dt} = 0, \end{cases} \tag{1.9}$$

with initial data

$$(z(0), x_0, y(0)) = (\varepsilon, \varphi, y_0) \in \mathbb{R} \times L^1([-r,0], \mathbb{R}^n) \times Y_1 \text{ and } x(0) = \bar{x},$$

wherein $r > 0$, $L \in \mathcal{L}(Y_1, Z)$ and the maps $f : \mathbb{R} \times L^1([-r,0], \mathbb{R}) \times Y_1 \to \mathbb{R}^n$ and $g : \mathbb{R} \times L^1([-r,0], \mathbb{R}) \times Y_1 \to Y_1$ are both Lipschitz continuous on bounded sets while $B_1 : D(B) \subset Y_1 \to Y_1$ Hille-Yosida linear operators. In order to re-write the above problem in the framework of (1.1) we follow the methodology developed in [17] and we set

$$w(t,\theta) := x(t+\theta),\ \forall t \geq 0,\ \forall \theta \in [-r,0].$$

Using this notation (1.9) re-writes as

$$\begin{cases} \varepsilon \dfrac{dx\,(t)}{dt} = f\,(z(t), w(t,\cdot), y(t))\,, \\ \dfrac{\partial w(t,\theta)}{\partial t} - \dfrac{\partial w(t,\theta)}{\partial \theta} = 0, \text{ for } \theta \in (-r,0) \\ w(t,0) = x(t), \\ \dfrac{dy\,(t)}{dt} = By(t) + g\,(z(t), w(t,\cdot), y(t))\,, \\ \dfrac{dz(t)}{dt} = 0, \end{cases}$$

together with the set of initial conditions

$$(z(0), x(0), w(0,\cdot), y(0)) = (\varepsilon, \bar{x}, \varphi, y_0) \in \mathbb{R} \times \mathbb{R}^n \times L^1([-r,0], \mathbb{R}^n) \times Y_1.$$

Hence we can rewrite the system as

$$\begin{cases} \varepsilon \dfrac{dx\,(t)}{dt} = f\,(z(t), w(t,.), y(t))\,, \\ \dfrac{dy\,(t)}{dt} = B_1 y(t) + g\,(z(t), w(t,\cdot), y(t))\,, \\ \dfrac{d}{dt}\begin{pmatrix} 0_{\mathbb{R}^n} \\ w(t,.) \end{pmatrix} = B_2 \begin{pmatrix} 0_{\mathbb{R}^n} \\ w(t,.) \end{pmatrix} + \begin{pmatrix} x(t) \\ 0_{L^1} \end{pmatrix} \\ \dfrac{dz(t)}{dt} = 0, \end{cases} \tag{1.10}$$

where $B_2 : D(B_2) \subset Y_2 \to Y_2$ is a linear operator on $Y_2 := \mathbb{R}^n \times L^1([-r,0], \mathbb{R}^n)$ and

$$B_2 \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} := \begin{pmatrix} -\varphi(0) \\ \varphi' \end{pmatrix} \text{ with } D\,(B_2) := \{0_{\mathbb{R}^n}\} \times W^{1,1}([-r,0], \mathbb{R}^n).$$

Hence by using similar techniques as the ones developed in Section 5 one can reformulate (1.10) into the suitable form of system (1.1) and then apply the results derived in this work.

Despite their importance for applications, singular perturbations for structured population models have been scarcely studied. We refer to [2, 11, 8] for the study of few examples going in that direction. To our best knowledge the results presented in this article are not known in the context of age-structured models.

However perturbed functional differential equations and more specifically singularly perturbed delay differential equations have attracted a lot of interest and we refer to Hale in [12] for a nice survey on this topic. Such questions have also been studied by Magalhaes [24, 25, 26] in the case of linear perturbation. We also refer to [3, 7, 9, 10, 13, 16, 27] for more results on singularly perturbed delay differential equations.

Let us also mention that general results have been obtained for some specific classes of singularly perturbed abstract Cauchy problems. In particular Buckdahn and Guatteri consider in [4] similar questions in the context of stochastic differential equations in infinite dimensional Hilbert spaces. This work is devoted to stochastic densely defined Cauchy problems and their results do not cover the case of non-densely defined abstract Cauchy problems.
Moreover Henry in [14, Chapter 9 p. 275] considered a special case of system (1.1) where A is a sectorial operator and B is a bounded linear operator. Note that his analysis of such a problem strongly relies on the boundedness of the linear operator B. Indeed such an assumption allows to transform the singular perturbation problem into a regular one (by using a suitable change of time scale). Here we would like to emphasize that it is of particular interest to consider (1.1) with non-densely defined operators. Indeed, coming back to the above example of age-structured models, namely (1.5)-(1.6), one may observe that it reformulates as system (1.1) with a bounded operator A and a non-densely defined operator B (see Section 5). Moreover problem (1.7) also reformulates as system (1.1) and the corresponding operators A and B are both unbounded and non-densely defined linear operator (see also Section 5). The paper is organized as follows, in Section 2 we state and discuss our main assumptions and present the results of this paper. Sections 3 and 4 are concerned with the proof of our main results while Section 5 is devoted to the application of our results to the age-structured model and to the system of functional differential equations presented in this introduction.

2 Assumptions, main results and corollaries

Before going to our assumptions and results, let us recall the definition of a Hille-Yosida operator.

Definition 2.1 *Let $A : D(A) \subset E \to E$ be a linear operator on a Banach space $(E, \|.\|_E)$. Let $\omega_A \in \mathbb{R}$ and $M_A \geq 1$ be given. We say that $(A, D(A))$ is a Hille-Yosida operator on E if* *(i)* $(\omega_A, \infty) \subset \rho(A)$ *the resolvent set of* A.

(ii) *For each* $\lambda \in \rho(A)$, $R_\lambda(A) := (\lambda I - A)^{-1}$ *the resolvent operator of* A *satisfies the following estimate:*

$$\left\| R_\lambda(A)^k \right\|_{\mathcal{L}(E)} \leq M_A \left(\lambda - \omega_A\right)^{-k}, \ \forall \lambda > \omega_A, \ k \geq 1.$$

Assumption 2.2 *Let* $(X, \|.\|_X), (Y, \|.\|_Y)$ *and* $(Z, \|.\|_Z)$ *be three Banach spaces. Let* $A : D(A) \subset X \to X$ *and* $B : D(B) \subset Y \to Y$ *be two Hille-Yosida operators satisfying the Hille-Yosida condition respectively with the constants* (ω_A, M_A) *and* (ω_B, M_B). *Since these operators are possibly not densely defined (i.e.* $X \neq \overline{D(A)}$ *or* $Y \neq \overline{D(B)}$*) we define*

$$X_0 := \overline{D(A)} \text{ and } Y_0 := \overline{D(B)}.$$

Let $F : X_0 \times Z \to X$ *and* $G : X_0 \times Y_0 \to Y$ *be Lipschitz continuous on bounded sets. We assume in addition that* $(u, z) \mapsto F(u, z)$ *is continuously differentiable on* $X_0 \times Z$ *and the map* $(u, z) \mapsto \partial_u F(u, z)$ *is Lipschitz continuous on bounded sets from* $X_0 \times Z$ *into* $\mathcal{L}(X_0, X)$.

Remark 2.3 *In what follows we will use the notation* $\|.\|$ *instead of* $\|.\|_X$, $\|.\|_Y$ *and* $\|.\|_Z$ *to denote the norm in the Banach spaces* X, Y *and* Z *whenever no confusion is possible.*

By using Assumption 2.2, the results in Magal [19] or in Magal and Ruan [21] ensures that for each $\varepsilon > 0$ system (1.1) generates a maximal semiflow $\mathcal{S}_\varepsilon$ on $X_0 \times Y_0$. In other word, for each $\varepsilon > 0$ and for each initial data $(x_\varepsilon, y_\varepsilon) \in X_0 \times Y_0$, problem (1.1) has a maximal mild solution $t \mapsto \mathcal{S}_\varepsilon(t)(x_\varepsilon, y_\varepsilon) = (u_\varepsilon(t), v_\varepsilon(t))$ defined on some time interval $[0, \tau_\varepsilon(x_\varepsilon, y_\varepsilon))$ with the maximal time of existence $\tau_\varepsilon(x_\varepsilon, y_\varepsilon) \in (0, +\infty]$. Moreover if $\tau_\varepsilon(x_\varepsilon, y_\varepsilon)$ is finite we have

$$\lim_{t \nearrow \tau_\varepsilon(x_\varepsilon, y_\varepsilon)} \|\mathcal{S}_\varepsilon(t)(x_\varepsilon, y_\varepsilon)\| = +\infty.$$

Next we will assume the following set of conditions for the maximal semiflow $\mathcal{S}_\varepsilon$.

Assumption 2.4 *We assume that the following properties are satisfied*

(a) *We assume that the maximal time of existence of solutions* $\tau_\varepsilon(x_\varepsilon, y_\varepsilon)$ *is bounded from below uniformly with respect to* ε. *That is to say that there exists* $\widehat{\tau} \in (0, +\infty]$ *(*$\widehat{\tau}$ *can be* $+\infty$*) such that*

$$\widehat{\tau} \leq \tau_\varepsilon(x_\varepsilon, y_\varepsilon), \forall \varepsilon \in (0, 1].$$

(b) *We assume that all the trajectories belong to a bounded set which is a Cartesian product. That is to say that there exist $\tau \in (0, \widehat{\tau}]$ (τ can be $+\infty$) and a closed bounded set*

$$\mathcal{M} := \mathcal{M}_X \times \mathcal{M}_Y \subset X_0 \times Y_0$$

such that

$$\mathcal{S}_\varepsilon(t)(x_\varepsilon, y_\varepsilon) \in \mathcal{M}, \ \ \forall t \in [0, \tau), \ \forall \varepsilon \in (0, 1].$$

Remark 2.5 *Note that Assumption 2.4-(b) in particular implies that*

$$\mathcal{M}_0 := \{(x_\varepsilon, y_\varepsilon) : \varepsilon \in [0, 1]\} \subset \mathcal{M}.$$

Hence $\mathcal{M}_0$ is a bounded set.

Now to simplify the notations we define

$$(u_\varepsilon(t), v_\varepsilon(t)) := \mathcal{S}_\varepsilon(t)(x_\varepsilon, y_\varepsilon) \in \mathcal{M}_X \times \mathcal{M}_Y, \quad \forall t \in [0, \tau), \ \forall \varepsilon \in (0, 1].$$

Using this notation and since L is a bounded linear operator, Assumption 2.4-(b) also implies that

$$\mathcal{M}_Z := \overline{L(\mathcal{M}_Y)} \supset \overline{\{Lv_\varepsilon(t) : t \in [0, \tau) \text{ and } \varepsilon \in (0, 1]\}} \tag{2.1}$$

is a bounded set in Z.

In our application to system (1.5)-(1.6), the set of initial data $\mathcal{M}_0$ will be chosen in a positively invariant bounded set so that, in that case, the corresponding set $\mathcal{M}$ will be a bounded set and $\tau = +\infty$.
In order to deal with the application to the non-autonomous problem (1.7), we shall extend the system to include the time variable into the state space. As a consequence the corresponding trajectory are not uniformly bounded for all time. In that case we will work on a finite time interval, namely $\tau < \infty$.

Besides the above technical assumptions and notations, our main assumption is related to the dynamics of the fast component of system (1.1). Let $z \in \mathcal{M}_Z$ be given, the fast dynamics corresponds to the following z-parametrized evolution equation

$$\frac{du(t)}{dt} = Au(t) + F(u(t), z), \ t \geq 0, \ u(0) = x \in \overline{D(A)}. \tag{2.2}$$

We are now in position to state our first main assumption in order to derive a Tikhonov like theorem in the general context of (1.1).

Assumption 2.6 (Tikhonov like conditions) *For each $z \in \mathcal{M}_Z$ we assume that system (2.2) generates a unique globally defined nonlinear semiflow $\{U_z(t)\}_{t \geq 0}$ on X_0. Moreover we assume that this semiflow satisfies the following properties:*

(a) ***(Equilibrium points)*** *For each $z \in \mathcal{M}_Z$ there exists an equilibrium solution $H(z) \in X_0$ of $\{U_z(t)\}_{t\geq 0}$ in X_0. That is to say that for each $z \in \mathcal{M}_Z$ there exists $H(z) \in D(A)$ satisfying*

$$AH(z) + F(H(z), z) = 0. \tag{2.3}$$

Moreover we assume that the map $z \mapsto H(z)$ is bounded from $\mathcal{M}_Z$ into X.

(b) ***(Global exponential stability)*** *There exist two constants $\kappa \geq 1$ and $\alpha > 0$ such that for all $x \in X_0$ and $z \in \mathcal{M}_Z$ one has*

$$\|U_z(t)x - H(z)\| \leq \kappa e^{-\alpha t} \|x - H(z)\|, \ \forall t \geq 0. \tag{2.4}$$

Remark 2.7 *Combining (a) and (b) we deduce that $\{U_z(t)\}_{t\geq 0}$ has a unique equilibrium in X_0.*

The above assumption contains the usual ingredients of the Tikhonov theorem for ordinary differential equations. We refer to [29, 32] for more details.

In infinite dimensional spaces, we shall need an extra assumption to deal with the compactness of the family of trajectories. This assumption is stated in term of uniform regularity. More precisely, we will need to combine the properties of the bounded linear operator $L \in \mathcal{L}(Y, Z)$ together with the solutions of the following non-homogeneous abstract Cauchy problem

$$\frac{dv(t)}{dt} = Bv(t) + g(t), \ t > 0, \ v(0) = y \in Y_0, \tag{2.5}$$

where $g \in L^1((0,\tau); Y)$. Since B is a Hille-Yosida operator, the mild solution of (2.5) can be expressed by using the constant variation formula involving the integrated semigroup $\{S_B(t)\}_{t\geq 0}$ generated by B, and the C_0-semigroup $\{T_{B_0}(t)\}_{t\geq 0}$ generated by the linear operator $B_0 : D(B_0) \subset Y_0 \to Y_0$, the part of B in Y_0. We refer to [21] for more details on integrated semigroup associated to Hille-Yosida operators (and for some other classes of linear operators). In that context, the solution of (2.6) can be written as follows

$$v(t) = T_{B_0}(t)y + (S_B \diamond g)(t), \ t \in [0, \tau),$$

with

$$(S_B \diamond g)(t) = \frac{d}{dt}\int_0^t S_B(t-s)g(s)ds = \lim_{\lambda \to +\infty} \int_0^t T_{B_0}(t-s)\lambda R_\lambda(B)g(s)\,ds.$$

Moreover the following estimate holds true

$$\|v(t)\| \leq M_B e^{\omega_B t}\|x\| + M_B \int_0^t e^{\omega_B(t-s)}\|g(s)\|\,ds, \forall t \in [0, \tau).$$

Using the above notation and recalling that $\tau \in (0, \infty]$ is given and fixed, our regularity assumptions reads as:

Assumption 2.8 (Regularity conditions) *We assume that the following properties are satisfied*

(a) *The family of maps* $\{L \circ T_{B_0}(.)y_\varepsilon : \varepsilon \in (0,1]\}$ *is uniformly equicontinuous from* $[0,\tau)$ *into* Z.

(b) *There exists a closed subspace* $Y_G \subset Y$ *such that*

$$G(X_0 \times Y_0) \subset Y_G$$

and the family of maps

$$\left\{ L \circ (S_B \diamond g)(.) : g \in L^\infty(0,\tau; Y_G) \text{ with } \sup_{t\in[0,\tau)} \|g(t)\|_Y \leq 1 \right\},$$

is uniformly equicontinuous from $[0,\tau)$ *into* Z.

We are now able to present the main result of this work that roughly asserts that the solution of (1.1) approaches the graph of H in the short time and then stays close to the graph up to the time τ, that is possible for all time if we have chosen $\tau = \infty$. Our precise statement reads as follows.

Theorem 2.9 (Slow manifold approximation) *Let Assumptions 2.2, 2.4, 2.6 and 2.8 be satisfied. Let* $t_0 \in (0,\tau)$ *be given. Then for each* $\delta > 0$ *there exists* $\widehat{\varepsilon} := \widehat{\varepsilon}(\delta) > 0$ *such that the solutions* $(u_\varepsilon(t), v_\varepsilon(t)) = \mathcal{S}_\varepsilon(t)(x_\varepsilon, y_\varepsilon)$ *for* $\varepsilon \in (0,\widehat{\varepsilon})$ *satisfy:*

$$\sup_{t\in[t_0,\tau)} \|u_\varepsilon(t) - H(Lv_\varepsilon(t))\| \leq \delta, \ \forall \varepsilon \in (0,\widehat{\varepsilon}).$$

Remark 2.10 *It is important to note that if* τ *is finite then the supremum in* $[t_0,\tau)$ *in Theorem 2.9 can be replaced by the supremum in* $[t_0,\tau]$.

As a consequence of the above theorem we will derive the following corollary, that roughly asserts the convergence of the solution of system (1.1)-(1.2) to the solution of system (1.4). The result reads as follows.

Corollary 2.11 (Finite time convergence) *Let Assumptions 2.2, 2.4, 2.6 and 2.8 be satisfied. Assume in addition that*

(i) τ *is finite;*

and

(ii) there exists a unique solution $t \to v(t) \in \mathcal{M}_Y$ *on* $[0,\tau]$ *of*

$$\frac{dv(t)}{dt} = Bv(t) + G(H(Lv(t)), v(t)), \ t \in [0,\tau] \text{ and } v(0) = y_0,$$

where y_0 *is the limit defined in (1.3).*

Then the following convergence results hold true

$$\lim_{\varepsilon \to 0^+} \sup_{t \in [t_0, \tau]} \| u_\varepsilon (t) - H (Lv (t)) \| = 0, \quad \forall t_0 \in (0, \tau), \tag{2.6}$$

and

$$\lim_{\varepsilon \to 0^+} \sup_{t \in [0, \tau]} \| v_\varepsilon (t) - v (t) \| = 0. \tag{2.7}$$

The above corollary only ensures the local uniform convergence of v_ε to v. Without specific assumption on the dynamical behaviour of the reduced problem, one cannot expect to get a more refined convergence property. However a uniform convergence property on the unbounded time interval $[0, \infty)$ as well as convergence to an heteroclinic orbit have been obtained by Ducrot et al. [7] for some specific singularly perturbed delay differential equation by coupling a slow manifold approach estimate reminiscent from Theorem 2.9 together with the dynamical properties of the reduced system.

Remark 2.12 *We would like to note that the above results also hold true in the case where the maps F and G smoothly depend on the small parameter ε, namely $F \equiv F(\varepsilon, x, Ly)$ and $G \equiv G(\varepsilon, x, y)$. Indeed, as for the reformulation of (1.8), such a parametrized case can be re-written as a special case of* (1.1) *by extending the problem as follows*

$$\begin{cases} \varepsilon \dfrac{du_\varepsilon (t)}{dt} = Au_\varepsilon (t) + F (z_\varepsilon (t), u_\varepsilon (t), Lv_\varepsilon (t)), \\ \dfrac{dv_\varepsilon (t)}{dt} = Bv_\varepsilon (t) + G (z_\varepsilon (t), u_\varepsilon (t), v_\varepsilon (t)), \\ \dfrac{dz_\varepsilon (t)}{dt} = 0, \end{cases}$$

together with the initial data

$$u_\varepsilon (0) = x_\varepsilon \in \overline{D (A)}, v_\varepsilon (0) = y_\varepsilon \in \overline{D (B)} \text{ and } z_\varepsilon(0) = \varepsilon.$$

3 Proof of Theorem 2.9

This section is devoted to the proof of Theorem 2.9. We firstly state a preliminary result on abstract Cauchy problems. Then we derive some basic regularity properties of the graph H before investigating the short time behaviour of the solution of (1.1). Then a suitable fixed point reformulation for the solution of (1.1) is obtained and used together with a contraction property around the graph to complete the proof of the theorem.

3.1 Preliminary

Let $\widehat{z} \in \mathcal{M}_Z$ be given. Recalling the definition of the semiflow U_z in Assumption 2.6, the map $t \to U_{\widehat{z}}(t)x$ is a mild solution of

$$\frac{du(t)}{dt} = Au(t) + F(u(t), \widehat{z}), \; t \geq 0, \; u (0) = x \in \overline{D (A)}.$$

Since $x \to F(x, \widehat{z})$ is continuously differentiable, by using Proposition 5.1 in [21] we deduce that $x \to U_{\widehat{z}}(t)x$ is continuously differentiable, and if we set

$$V(t)y := \partial_x U_{\widehat{z}}(t)\left(H\left(\widehat{z}\right)\right)y, \ \ \forall y \in \overline{D(A)},$$

then $\{V(t)\}_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators on $\overline{D(A)}$ and $t \to V(t)y$ is a mild solution of

$$\frac{dV(t)y}{dt} = AV(t)y + CV(t)y, \ t \geq 0 \text{ with } V\left(0\right)y = y \in \overline{D\left(A\right)}.$$

Here we have set

$$Cy := \partial_u F\left(H\left(\widehat{z}\right), \widehat{z}\right)y, \forall y \in \overline{D(A)}.$$

Now since for all $h \in \overline{D(A)}$ one has

$$\begin{aligned} \|V(t)h\| &= \lim_{\delta \searrow 0} \frac{\left\|U_{\widehat{z}}(t)\left(\delta h + H\left(\widehat{z}\right)\right) - U_{\widehat{z}}(t)\left(H\left(\widehat{z}\right)\right)\right\|}{\delta} \\ &= \lim_{\delta \searrow 0} \frac{\left\|U_{\widehat{z}}(t)\left(\delta h + H\left(\widehat{z}\right)\right) - H\left(\widehat{z}\right)\right\|}{\delta}, \end{aligned}$$

we deduce by using Assumption 2.6-(b) that

$$\|V(t)h\| \leq \kappa e^{-\alpha t}\left\|h\right\|, \forall t \geq 0.$$

Since A is a Hille-Yosida operator and C is bounded linear, the linear operator $A + C : D(A) \to X$ is also a Hille-Yosida operator (see Arendt et al. [1]) with respect to the constants (ω_{A+C}, M_{A+C}). Moreover one has

$$\omega_{A+C} \leq -\alpha < 0.$$

Indeed, $\{V(t)\}_{t\geq 0}$ is the strongly continuous semigroup generated by $(A+C)_0$ (the part of $A + C$ in $\overline{D\left(A\right)}$). We deduce that the growth rate of $\{V(t)\}_{t\geq 0}$ satisfies

$$\omega\left((A+C)_0\right) := \lim_{t \to +\infty} \frac{\ln\left(\|V(t)\|\right)}{t} \leq -\alpha.$$

Let $\widehat{\alpha} \in (0, \alpha)$ be given and fixed. We obtained that $(-\widehat{\alpha}, +\infty) \subset \rho\left((A+C)_0\right)$ the resolvent set of $(A+B)_0$, and we can find $\widehat{M} \geq 0$ such that

$$\left\|\left(\lambda I - (A+C)_0\right)^{-n}\right\|_{\mathcal{L}\left(\overline{D(A)}\right)} \leq \frac{\widehat{M}}{\left(\lambda + \widehat{\alpha}\right)^n}, \forall \lambda > -\widehat{\alpha}, \forall n \geq 1. \qquad (3.1)$$

Moreover since $A + C$ is a Hille-Yosida operator, the resolvent set $\rho\left(A + C\right)$ of $(A + C)$ is non empty. Let $\mu > \omega_{A+C}$ be given. By using Lemma 2.1 in [22] ensures that

$$\rho(A + C) = \rho\left((A + C)_0\right),$$

and for each $\lambda > -\widehat{\alpha}$ one has

$$\left(\lambda I - (A+C)\right)^{-1} = (\mu - \lambda)\left(\lambda I - (A+C)_0\right)^{-1}\left(\mu I - (A+C)\right)^{-1} + \left(\mu I - (A+C)\right)^{-1}.$$

Therefore we get

$$\left\|(\lambda+\widehat{\alpha})\left(\lambda I-(A+C)\right)^{-1}\right\| \leq \left[|(\mu-\lambda)|\,\widehat{M}+(\lambda+\widehat{\alpha})\right]\left\|(\mu I-(A+C))^{-1}\right\|,$$

whenever $\lambda \in (-\widehat{\alpha}, \omega_{A+C}+1)$, and

$$\left\|(\lambda+\widehat{\alpha})\left(\lambda I-(A+C)\right)^{-1}\right\| \leq \frac{(\lambda+\widehat{\alpha})}{(\lambda-\omega_{A+C})},$$

whenever $\lambda \geq \omega_{A+C}+1$. Hence this yields

$$\sup_{\lambda>-\widehat{\alpha}}\left\|(\lambda+\widehat{\alpha})\left(\lambda I-(A+C)\right)^{-1}\right\| < +\infty. \tag{3.2}$$

Combining (3.1) and (3.2) we obtain the following lemma.

Lemma 3.1 *Let Assumptions 2.2, 2.4 and 2.6 be satisfied. Then $A+C$ is a Hille-Yosida operator and we can choose the constants (ω_{A+C}, M_{A+C}) such that*

$$M_{A+C} \geq 1 \text{ and } \omega_{A+C} \in (-\alpha, 0).$$

By using the above lemma it follows that replacing A by $A+C$ and $F(x,z)$ by $F(x,z)-C$, we can assume (without loss of generality) that A is a Hille-Yosida operator with the constants (ω_A, M_A) satisfying

$$M_A \geq 1 \text{ and } \omega_A \in (-\alpha, 0).$$

3.2 Short time behaviour

In this part, we investigate the short time behaviour of the solutions. Roughly speaking we will show that the solution $u_\varepsilon(t)$ quickly approaches the graph of $H(Lv_\varepsilon(t))$. In order to prove this result, let us first derive some regularity properties of the graph of H.

Lemma 3.2 *Let Assumptions 2.2, 2.4 and 2.6 be satisfied. Then H is Lipschitz continuous from $\mathcal{M}_Z$ into $D(A)$ endowed with the graph norm, that is there exists a constant $C_H > 0$ such that for all $z, \bar{z} \in \mathcal{M}_Z$*

$$\|AH(z)-AH(\bar{z})\|_X + \|H(z)-H(\bar{z})\|_X \leq C_H \|z-\bar{z}\|_Z,$$

and

$$\|H(z)\| \leq C_H.$$

Proof. Let $z, \bar{z} \in \mathcal{M}_Z$ be given. Recall κ and α are defined in (2.4), and set

$$\nu := \frac{\ln(2\kappa)}{\alpha} \Leftrightarrow \kappa e^{-\alpha\nu} = \frac{1}{2}. \tag{3.3}$$

First note that due to (2.4) one has

$$\begin{aligned}\|H(z)-H(\bar{z})\| &\leq \|H(z)-U_z(\nu)H(\bar{z})\|+\|U_z(\nu)H(\bar{z})-H(\bar{z})\| \\ &\leq \kappa e^{-\alpha\nu}\|H(z)-H(\bar{z})\|+\|U_z(\nu)H(\bar{z})-H(\bar{z})\|.\end{aligned}$$

Hence it follows from (3.3) that

$$\|H(z)-H(\bar{z})\| \leq 2\|U_z(\nu)H(\bar{z})-H(\bar{z})\|. \tag{3.4}$$

Let us now derive an estimate for $\|U_z(\nu)H(\bar{z})-H(\bar{z})\|$. Note that as a consequence the variation of the constants formula one obtains that for all $t \geq 0$,

$$U_z(t)H(\bar{z}) = T_{A_0}(t)H(\bar{z})+\lim_{\lambda\to+\infty}\int_0^t T_{A_0}(t-s)\lambda R_\lambda(A)F(U_z(t)H(\bar{z}),z)\,ds, \tag{3.5}$$

and

$$U_{\bar{z}}(t)H(\bar{z}) = T_{A_0}(t)H(\bar{z})+\lim_{\lambda\to+\infty}\int_0^t T_{A_0}(t-s)\lambda R_\lambda(A)F(U_{\bar{z}}(t)H(\bar{z}),\bar{z})\,ds. \tag{3.6}$$

Next observe that

$$U_{\bar{z}}(t)H(\bar{z}) = H(\bar{z}),\ \forall t \geq 0.$$

Then replacing $U_{\bar{z}}(t)H(\bar{z})$ by $H(\bar{z})$ into (3.6), subtracting (3.5) to (3.6) and recalling that A is a Hille-Yosida operator with type (ω_A, M_A) yield for each $t \geq 0$,

$$\|U_z(t)H(\bar{z})-H(\bar{z})\| \leq M_A\int_0^t e^{\omega_A(t-s)}\|F(U_z(s)H(\bar{z}),z)-F(H(\bar{z}),\bar{z})\|\,ds.$$

Since by Assumptions 2.6 the map $z \to H(z)$ is bounded, by using (2.4) we deduce that

$$\sup_{t\geq 0, z,\bar{z}\in\mathcal{M}_Z}\|U_z(t)H(\bar{z})\| \leq \sup_{z,\bar{z}\in\mathcal{M}_Z}[\kappa\|H(\bar{z})-H(z)\|+\|H(z)\|] < +\infty.$$

Since F is Lipschitz continuous on bounded sets, we can find a constant $C_F > 0$ such that for each $t \in [0,\nu]$

$$\|U_z(t)H(\bar{z})-H(\bar{z})\| \leq M_A C_F\int_0^t e^{\omega_A(t-s)}[\|U_z(s)H(\bar{z})-H(\bar{z})\|+\|z-\bar{z}\|]\,ds,$$

so that

$$\begin{aligned}e^{-\omega_A t}\|U_z(t)H(\bar{z})-H(\bar{z})\| &\leq M_A C_F\nu e^{|\omega_A|\nu}\|z-\bar{z}\| \\ &\quad +M_A C_F\textstyle\int_0^t e^{-\omega_A s}\|U_z(s)H(\bar{z})-H(\bar{z})\|\,ds.\end{aligned}$$

Hence applying Gronwall's inequality provides

$$\|U_z(\nu)H(\bar{z})-H(\bar{z})\| \leq M_A C_F\nu e^{2|\omega_A|\nu}e^{M_A C_F\nu}\|z-\bar{z}\|,$$

and by plugging this last inequality into (3.4) we obtain

$$\|H(z) - H(\bar{z})\| \leq 2M_A C_F \nu e^{2|\omega_A|\nu} e^{M_A C_F \nu} \|z - \bar{z}\|.$$

The proof is completed by using again the fact that F is Lipschitz on bounded sets and equation (2.3). ■

Using this regularity property we will prove a first lemma which express the fact that the solutions are approaching rapidly the graph of H.

Lemma 3.3 (Fast approach to the graph of H) *Let Assumptions 2.2, 2.4 and 2.6 be satisfied. We can find a constant $C_0 > 0$ such that $(u_\varepsilon(t), v_\varepsilon(t))$ a mild solution of system (1.1) satisfies*

$$\begin{aligned}\|u_\varepsilon(\varepsilon t) - H(Lv_\varepsilon(\varepsilon t))\| \leq \kappa e^{-\alpha t} \|x_\varepsilon - H(Ly_\varepsilon)\| \\ + C_0(1+t)e^{C_0 t} \sup_{s\in[0,\varepsilon t]} \|Lv_\varepsilon(s) - Lv_\varepsilon(0)\|.\end{aligned}$$

whenever $\varepsilon \in (0, 1]$ and $t > 0$.

Proof. By Assumption 2.6 and since $Lv_\varepsilon(0) = Ly_\varepsilon \in \mathcal{M}_Z$, for each $\varepsilon \in (0, 1]$, we can find $\bar{u}_\varepsilon \in C\left([0, +\infty), \overline{D(A)}\right)$ a mild solution of

$$\frac{d\bar{u}_\varepsilon(t)}{dt} = A\bar{u}_\varepsilon(t) + F(\bar{u}_\varepsilon(t), Lv_\varepsilon(0)),\ t \geq 0 \text{ and } \bar{u}_\varepsilon(0) = x_\varepsilon.$$

Define $\widehat{u}_\varepsilon \in C\left([0, +\infty), \overline{D(A)}\right)$ by

$$\widehat{u}_\varepsilon(t) := \bar{u}_\varepsilon\left(\frac{t}{\varepsilon}\right) = U_{Ly_\varepsilon}\left(\frac{t}{\varepsilon}\right) x_\varepsilon,\ \forall t \geq 0,$$

the mild solution of

$$\varepsilon \frac{d\widehat{u}_\varepsilon(t)}{dt} = A\widehat{u}_\varepsilon(t) + F(\widehat{u}_\varepsilon(t), Lv_\varepsilon(0)),\ t \geq 0 \text{ and } \widehat{u}_\varepsilon(0) = x_\varepsilon.$$

Therefore recalling that $u_\varepsilon \in C\left([0, \tau), \overline{D(A)}\right)$ is the mild solution of

$$\varepsilon \frac{du_\varepsilon(t)}{dt} = Au_\varepsilon(t) + F(u_\varepsilon(t), Lv_\varepsilon(t)),\ t \geq 0 \text{ and } u_\varepsilon(0) = x_\varepsilon,$$

and setting

$$\Delta_\varepsilon(t) := u_\varepsilon(t) - \widehat{u}_\varepsilon(t),\ \forall t \in [0, \tau),$$

one has for all $t \in [0, \tau)$

$$\Delta_\varepsilon(t) = \varepsilon^{-1} \lim_{\lambda\to+\infty} \int_0^t T_{A_0}\left(\frac{t-s}{\varepsilon}\right) \lambda R_\lambda(A) \left[F(\Delta_\varepsilon(s) + \widehat{u}_\varepsilon(s), Lv_\varepsilon(s)) - F(\widehat{u}_\varepsilon(s), Lv_\varepsilon(0))\right] ds.$$

Since F is Lipschitz on bounded sets, by using the boundedness Assumption 2.4-(b), and (2.1) we deduce that we can find a constant $C_F > 0$ such that for all $t \geq 0$ and all $\varepsilon \in (0, 1]$

$$\|\Delta_\varepsilon(t)\| \leq \frac{M_A C_F}{\varepsilon} \int_0^t e^{\frac{\omega_A}{\varepsilon}(t-s)} \left[\|Lv_\varepsilon(s) - Lv_\varepsilon(0)\| + \|\Delta_\varepsilon(s)\|\right] ds.$$

By making a change of variable $l = s/\varepsilon$ in this integral, we obtain for all $t \geq 0$ and all $\varepsilon \in (0, 1]$,

$$\|\Delta_\varepsilon(\varepsilon t)\| \leq C_1 \int_0^t e^{\omega_A(t-l)} \left[\|Lv_\varepsilon(\varepsilon l) - Lv_\varepsilon(0)\| + \|\Delta_\varepsilon(\varepsilon l)\|\right] dl,$$

where $C_1 := M_A C_F$.

It follows that for all $t \in [0, \nu]$ (since we can assume that $\omega_A < 0$) we have

$$e^{-\omega_A t} \|\Delta_\varepsilon(\varepsilon t)\| \leq C_1 \nu \sup_{l \in [0,\nu]} \|Lv_\varepsilon(\varepsilon l) - Lv_\varepsilon(0)\| + C_1 \int_0^t e^{-\omega_A l} \|\Delta_\varepsilon(\varepsilon l)\| dl.$$

By applying the Gronwall's lemma we obtain for each $t \in [0, \nu]$

$$e^{-\omega_A t} \|\Delta_\varepsilon(\varepsilon t)\| \leq C_1 \nu \sup_{l \in [0,\nu]} \|Lv_\varepsilon(\varepsilon l) - Lv_\varepsilon(0)\| \exp(C_1 t),$$

that provides that

$$\|\Delta_\varepsilon(\varepsilon \nu)\| \leq C_1 \nu \sup_{s \in [0,\varepsilon\nu]} \|Lv_\varepsilon(s) - Lv_\varepsilon(0)\| \exp([\omega_A + C_1]\nu). \tag{3.7}$$

Finally we observe that

$$\begin{aligned} \|u_\varepsilon(\varepsilon\nu) - H(Lv_\varepsilon(\varepsilon\nu))\| &\leq \|u_\varepsilon(\varepsilon\nu) - \widehat{u}_\varepsilon(\varepsilon\nu)\| \\ &\quad + \|\widehat{u}_\varepsilon(\varepsilon\nu) - H(Lv_\varepsilon(0))\| \\ &\quad + \|H(Lv_\varepsilon(0)) - H(Lv_\varepsilon(\varepsilon\nu))\|, \end{aligned} \tag{3.8}$$

and due to Assumption 2.6-(b) we also have

$$\|\widehat{u}_\varepsilon(\varepsilon\nu) - H(Lv_\varepsilon(0))\| \leq \kappa e^{-\alpha\nu} \|\bar{u}_\varepsilon(0) - H(Lv_\varepsilon(0))\|. \tag{3.9}$$

Hence using the fact that H is Lipschitz continuous on $\mathcal{M}_Z$ the result follows from (3.7)-(3.9). ∎

3.3 Fixed point reformulation of the u_ε-equation

The proof of Theorem 2.9 will be performed by a fixed point argument. To do so, we will reformulate the u_ε-equation of (1.1) into a more convenient form and derive a fixed point problem. Define

$$C(z) := \partial_u F(H(z), z), \forall z \in \mathcal{M}_Z,$$

and

$$K\left(u,z\right):=F\left(u,z\right)-F\left(H\left(z\right),z\right)-C\left(z\right)\left[u-H\left(z\right)\right],\forall z\in\mathcal{M}_{Z}. \tag{3.10}$$

First note that one has

$$F\left(u,z\right)-F\left(H\left(z\right),z\right)=\int_{0}^{1}\partial_{u}F(\theta u+\left(1-\theta\right)H\left(z\right),z)\left[u-H\left(z\right)\right]d\theta,$$

so that

$$K\left(u,z\right)=\int_{0}^{1}\left[\partial_{u}F(\theta u+\left(1-\theta\right)H\left(z\right),z)-\partial_{u}F(H(z),z)\right]\left[u-H\left(z\right)\right]d\theta. \tag{3.11}$$

By using Assumption 2.6-(a) we have $AH(z)+F(H(z),z)=0$ therefore

$$\begin{aligned}Au+F(u,Lv)&=A(u-H(Lv))+AH(Lv)+F(u,Lv)\\&=A(u-H(Lv))-F(H(Lv),Lv)+F(u,Lv)\\&=A(u-H(Lv))+K(u,Lv)+C(Lv)(u-H(Lv)).\end{aligned}$$

Therefore the u_ε-equation of (1.1) re-writes as

$$\begin{cases}\varepsilon\dfrac{du_{\varepsilon}(t)}{dt}=\left[A+C\left(Lv_{\varepsilon}(t)\right)\right]\left[u_{\varepsilon}(t)\ -H\left(Lv_{\varepsilon}(t)\right)\right]+K\left(u_{\varepsilon}(t),Lv_{\varepsilon}(t)\right),\ \text{for } t\geq 0,\\ u_{\varepsilon}\left(0\right)=x_{\varepsilon}\in\overline{D\left(A\right)}.\end{cases}$$

We also have for any $v,w\in Y$ and $u\in X$

$$\begin{aligned}(A+C(Lv))(u-H(Lv))&=(A+C(Lw))(u-H(Lv))-(C(Lw)-C(Lv))(u-H(Lv))\\&=(A+C(Lw))u-(A+C(Lw))H(Lv)\\&\quad+(C(Lv)-C(Lw))(u-H(Lv)),\end{aligned}$$

hence for each $l\geq 0$, u_ε is a mild solution of the following abstract Cauchy problem

$$\begin{aligned}\varepsilon\frac{du_{\varepsilon}\left(t\right)}{dt}&=\left[A+C\left(Lv_{\varepsilon}\left(l\right)\right)\right]u_{\varepsilon}\left(t\right)\\&\quad-\left[A+C\left(Lv_{\varepsilon}\left(l\right)\right)\right]H\left(Lv_{\varepsilon}\left(t\right)\right)\\&\quad+K\left(u_{\varepsilon}\left(t\right),Lv_{\varepsilon}\left(t\right)\right)\\&\quad+\left[C\left(Lv_{\varepsilon}\left(t\right)\right)-C\left(Lv_{\varepsilon}\left(l\right)\right)\right]\left[u_{\varepsilon}\left(t\right)-H\left(Lv_{\varepsilon}\left(t\right)\right)\right],\end{aligned} \tag{3.12}$$

with

$$u_{\varepsilon}\left(0\right)=x_{\varepsilon}\in\overline{D\left(A\right)}.$$

In the sequel we will concentrate our study on system (3.12) parametrized by $l\in[0,\tau)$. For more simplicity in the notations we define for each $l\in[0,\tau)$

$$T_{l}\left(t\right)x:=T_{\left(A+C\left(Lv_{\varepsilon}\left(l\right)\right)\right)_{0}}\left(t\right)x,\ \forall t\geq 0 \text{ and } \forall x\in\overline{D\left(A+C\left(Lv_{\varepsilon}\left(l\right)\right)\right)}=\overline{D\left(A\right)},$$

where $\left\{T_{(A+C(Lv_{\varepsilon}(l)))_0}(t)\right\}_{t\geq 0} \subset \mathcal{L}\left(\overline{D(A)}\right)$ is the strongly continuous semigroup generated by $(A+C(Lv_{\varepsilon}(l)))_0$ the part of $A+C(Lv_{\varepsilon}(l))$ in $\overline{D(A)}$.

By Assumption 2.4-(b) and Assumption 2.6-(a) the subset

$$\widehat{\mathcal{M}}_1 = \overline{\mathcal{M}_X \cup \{H(z) : z \in \mathcal{M}_Z\}},$$

is closed and bounded and the solutions of system (1.1) satisfies

$$u_{\varepsilon}(t) \in \widehat{\mathcal{M}}_1 \text{ and } H(Lv_{\varepsilon}(t)) \in \widehat{\mathcal{M}}_1 \ \forall t \in [0,\tau),\ \forall \varepsilon \in (0,1].$$

We also note that since F and $\partial_u F$ are Lipschitz continuous on bounded sets it follows that there exists a constant $C_F > 0$ such that for all $(u,z),\ (\bar{u},\bar{z}) \in \widehat{\mathcal{M}}_1 \times \mathcal{M}_Z$

$$\begin{cases} \|F(u,z) - F(\bar{u},\bar{z})\| \leq C_F \|u-\bar{u}\| + C_F \|z-\bar{z}\|, \\ \|\partial_u F(u,z) - \partial_u F(\bar{u},\bar{z})\|_{\mathcal{L}\left(\overline{D(A)},X\right)} \leq C_F \|u-\bar{u}\| + C_F \|z-\bar{z}\|, \end{cases} \tag{3.13}$$

and for all $(u,z) \in \widehat{\mathcal{M}}_1 \times \mathcal{M}_Z$

$$\|F(u,z)\| \leq C_F \text{ and } \|\partial_u F(u,z)\|_{\mathcal{L}\left(\overline{D(A)},X\right)} \leq C_F. \tag{3.14}$$

Note that by using (3.11) and (3.13)-(3.14) we obtain for all $(u,z),(\bar{u},\bar{z}) \in \widehat{\mathcal{M}}_1 \times \mathcal{M}_Z$

$$\|K(u,z)\| \leq C_F \|u - H(z)\|^2, \tag{3.15}$$

while using (3.10) combined with (3.13)-(3.14) yields

$$\begin{aligned} K(u,z) - K(\bar{u},z) &= F(u,z) - F(\bar{u},z) - \partial_u F(H(z),z)[\bar{u}-u] \\ &= \int_0^1 [\partial_u F((1-\theta)(\bar{u}-u)+u,z) - \partial_u F(H(z),z)][\bar{u}-u]d\theta, \end{aligned}$$

so that

$$\|K(u,z) - K(\bar{u},z)\| \leq C_F [\|u-\bar{u}\| + \|u-H(z)\|] \|u-\bar{u}\|. \tag{3.16}$$

Let us now prove the following two lemmas that are crucial in obtaining the fixed point formulation.

Lemma 3.4 *Let Assumptions 2.2, 2.4 and 2.6 be satisfied. Then the following two properties hold true*

(i) *For each $l \in [0,\tau)$ and $\varepsilon \in (0,1]$ the semigroup $\{T_l(t)\}_{t\geq 0} \subset \mathcal{L}\left(\overline{D(A)}\right)$ satisfies*

$$\|T_l(t)x\| \leq \kappa e^{-\alpha t}\|x\|,\ \forall t \geq 0,\ \forall x \in \overline{D(A)}, \tag{3.17}$$

and we have

$$(-\alpha, +\infty) \subset \rho(A + C(Lv_{\varepsilon}(l))). \tag{3.18}$$

(ii) *The resolvent* $R_\lambda(l) := [\lambda - A - C(Lv_\varepsilon(l))]^{-1}$ *with* $\lambda \in \rho(A + C(Lv_\varepsilon(l)))$ *satisfies*

$$\limsup_{\lambda \to +\infty} \|\lambda R_\lambda(l)\|_{\mathcal{L}(X)} \leq M_A.$$

Proof. Assertion (i) is proved in section 3.1 and it remains to prove (ii). To that aim note that due to (3.14) one has

$$\sup_{\varepsilon \in (0,1], l \in [0,\tau)} \|C(Lv_\varepsilon(l))\|_{\mathcal{L}\left(\overline{D(A)} \times \overline{D(B)}, X \times Y\right)} \leq C_F,$$

so that for all $l \in [0,\tau)$ and $\lambda > \omega_A + M_A C_F$, the resolvent $R_\lambda(l)$ exists and it is the unique fixed point of

$$R_\lambda(l) = R_\lambda(A) + R_\lambda(A) C(Lv_\varepsilon(l)) R_\lambda(l).$$

For $l \in [0,\tau)$ and $\lambda \in \rho(A + C(Lv_\varepsilon(l)))$ large enough one has

$$\|R_\lambda(l)\| \leq \frac{M_A}{\lambda - \omega_A} + \frac{M_A}{\lambda - \omega_A} C_F \|R_\lambda(l)\| \Longrightarrow \|R_\lambda(l)\|_{\mathcal{L}(X)} \leq \frac{M_A}{\lambda - \omega_A - M_A C_F},$$

and the result follows. ■

The second lemma is the following

Lemma 3.5 *Let* $\widehat{A} : D\left(\widehat{A}\right) \to X$ *be a Hille-Yosida operator such that* $0 \in \rho\left(\widehat{A}\right)$. *Then for each* $x \in X$ *and each* $t \geq 0$ *one has*

$$\widehat{A}^{-1}x = T_{\widehat{A}_0}(t)\widehat{A}^{-1}x - \lim_{\lambda \to +\infty} \int_0^t T_{\widehat{A}_0}(t-l)\lambda\left(\lambda I - \widehat{A}\right)^{-1} x dl,$$

where $\widehat{A}_0$ *denotes the part of* $\widehat{A}$ *in* $\overline{D(\widehat{A})}$ *and* $\left\{T_{\widehat{A}_0}(t)\right\}_{t \geq 0} \subseteq \mathcal{L}\left(\overline{D\left(\widehat{A}\right)}, X\right)$ *is the strongly continuous semigroup generated by* $\widehat{A}_0$.

Remark 3.6 *By using the above formula we deduce that for each* $t, s \in \mathbb{R}$ *with* $t \geq s$

$$\widehat{A}^{-1}x = T_{\widehat{A}_0}(t-s)\widehat{A}^{-1}x - \lim_{\lambda \to +\infty} \int_s^t T_{\widehat{A}_0}(t-l)\lambda\left(\lambda I - \widehat{A}\right)^{-1} x dl.$$

Proof. This lemma follows by observing that $u(t) = \widehat{A}^{-1}x$, $t \geq 0$ is a mild stationary solution of the problem

$$\frac{du(t)}{dt} = \widehat{A}u(t) - x, \ t \geq 0 \text{ and } u(0) = \widehat{A}^{-1}x,$$

and since the mild solution of the above equation is also given by

$$u(t) = T_{\widehat{A}_0}(t)\widehat{A}^{-1}x + \lim_{\lambda \to +\infty} \int_0^t T_{\widehat{A}_0}(t-l)\lambda\left(\lambda I - \widehat{A}\right)^{-1} x dl,$$

the result follows. ■

To define our fixed point problem we will use a variation of constants formula. In fact applying the results in [21] to equation (3.12) yields for each $t \geq t_0$

$$\begin{aligned} u_\varepsilon(t) &= T_l\left(\frac{t-t_0}{\varepsilon}\right)u_\varepsilon(t_0) + \frac{1}{\varepsilon}\lim_{\lambda\to+\infty}\int_{t_0}^{t} T_l\left(\frac{t-r}{\varepsilon}\right)\lambda R_\lambda(l)\, K\left(u_\varepsilon(r), Lv_\varepsilon(r)\right)dr \\ &\quad -\frac{1}{\varepsilon}\lim_{\lambda\to+\infty}\int_{t_0}^{t} T_l\left(\frac{t-r}{\varepsilon}\right)\lambda R_\lambda(l)\left[A + C\left(Lv_\varepsilon(l)\right)\right]H\left(Lv_\varepsilon(r)\right)dr \\ &\quad +\frac{1}{\varepsilon}\lim_{\lambda\to+\infty}\int_{t_0}^{t} T_l\left(\frac{t-r}{\varepsilon}\right)\lambda R_\lambda(l)\left[C\left(Lv_\varepsilon(\varepsilon r)\right) - C\left(Lv_\varepsilon(l)\right)\right]\left[u_\varepsilon(r) - H\left(Lv_\varepsilon(r)\right)\right]dr. \end{aligned} \tag{3.19}$$

Due to (3.18) one obtains from Lemma 3.5 that for all $t \geq t_0$ and $x \in X$

$$\left[A + C\left(Lv_\varepsilon(l)\right)\right]^{-1}x = T_l\left(\frac{t-t_0}{\varepsilon}\right)\left[A + C\left(Lv_\varepsilon(l)\right)\right]^{-1}x - \lim_{\lambda\to+\infty}\int_{\frac{t_0}{\varepsilon}}^{\frac{t}{\varepsilon}} T_l\left(\frac{t}{\varepsilon} - s\right)\lambda R_\lambda(l)\, x\, ds,$$

hence that for $x = \left[A + C\left(Lv_\varepsilon(l)\right)\right]H\left(Lv_\varepsilon(t)\right)$ one has

$$\begin{aligned} H\left(Lv_\varepsilon(t)\right) =& T_l\left(\frac{t-t_0}{\varepsilon}\right)H\left(Lv_\varepsilon(t)\right) \\ & - \lim_{\lambda\to+\infty}\int_{\frac{t_0}{\varepsilon}}^{\frac{t}{\varepsilon}} T_l\left(\frac{t}{\varepsilon} - s\right)\lambda R_\lambda(l)\left[A + C\left(Lv_\varepsilon(l)\right)\right]H\left(Lv_\varepsilon(t)\right)ds. \end{aligned}$$

Next the change of variables $r = \varepsilon s$ in the foregoing integral provides that

$$\begin{aligned} H\left(Lv_\varepsilon(t)\right) =& T_l\left(\frac{t-t_0}{\varepsilon}\right)H\left(Lv_\varepsilon(t)\right) \\ & - \frac{1}{\varepsilon}\lim_{\lambda\to+\infty}\int_{t_0}^{t} T_l\left(\frac{t-r}{\varepsilon}\right)\lambda R_\lambda(l)\left[A + C\left(Lv_\varepsilon(l)\right)\right]H\left(Lv_\varepsilon(t)\right)dr. \end{aligned} \tag{3.20}$$

Finally plugging (3.19) into (3.20) and setting

$$w_\varepsilon(t) := u_\varepsilon(t) - H\left(Lv_\varepsilon(t)\right),\ \forall t \in [0, \tau),$$

yields for each $t \geq t_0$

$$\begin{aligned} & w_\varepsilon(t) = T_l\left(\tfrac{t-t_0}{\varepsilon}\right)\left(u_\varepsilon(t_0) - H\left(Lv_\varepsilon(t)\right)\right) \\ & +\tfrac{1}{\varepsilon}\lim_{\lambda\to+\infty}\int_{t_0}^{t} T_l\left(\tfrac{t-r}{\varepsilon}\right)\lambda R_\lambda(l)\, K\left(w_\varepsilon(r) + H\left(Lv_\varepsilon(r)\right), Lv_\varepsilon(r)\right)dr \\ & -\tfrac{1}{\varepsilon}\lim_{\lambda\to+\infty}\int_{t_0}^{t} T_l\left(\tfrac{t-r}{\varepsilon}\right)\lambda R_\lambda(l)\left[A + C\left(Lv_\varepsilon(l)\right)\right]\left[H\left(Lv_\varepsilon(r)\right) - H\left(Lv_\varepsilon(t)\right)\right]dr \\ & +\tfrac{1}{\varepsilon}\lim_{\lambda\to+\infty}\int_{t_0}^{t} T_l\left(\tfrac{t-r}{\varepsilon}\right)\lambda R_\lambda(l)\left[C\left(Lv_\varepsilon(r)\right) - C\left(Lv_\varepsilon(l)\right)\right]w_\varepsilon(r)\, dr. \end{aligned} \tag{3.21}$$

3.4 Proof of Theorem 2.9

This section is devoted to the proof of Theorem 2.9. To do so note that this result will follow from the following claim.

Claim 3.7 *Let $t_0 \in (0,\tau)$ be given. For each $\delta > 0$ we can find $\varepsilon_1 := \varepsilon_1(\delta) > 0$ satisfying*

$$\sup_{t\in[t_0,\tau)} \|w_\varepsilon(t)\| \leq \delta,\ \forall \varepsilon \in (0,\varepsilon_1).$$

Proof. We will prove the claim for each $\delta > 0$ small enough. Let $\{t_\varepsilon\}_{\varepsilon\in(0,1]} \subset \mathbb{R}_+$ be a given family such that

$$t_\varepsilon \to 0 \text{ as } \varepsilon \to 0^+.$$

At this stage of the proof the family $\{t_\varepsilon\}_{\varepsilon\in(0,1]} \subset \mathbb{R}_+$ does not need to be specified. This will be done at the final step of the proof after fixing some parameters. Consider the following Banach spaces

$$\mathcal{BC}_\varepsilon := \left\{ w \in C\left([t_\varepsilon,\tau), \overline{D(A)}\right) : \|w\|_\infty := \sup_{t\in[t_\varepsilon,\tau)} \|w(t)\| < +\infty \right\}.$$

We now re-write equation (3.21) as

$$w_\varepsilon = \Gamma^\varepsilon(w_\varepsilon) \text{ with } w_\varepsilon \in \mathcal{BC}_\varepsilon,$$

where the map Γ^ε is defined by

$$\Gamma^\varepsilon(w)(t) := \mathcal{T}_1^\varepsilon(t) + \mathcal{S}_1^\varepsilon(w)(t) + \mathcal{T}_2^\varepsilon(t) + \mathcal{S}_2^\varepsilon(w)(t),\ \forall t \in [t_\varepsilon,\tau), \tag{3.22}$$

with

$$\mathcal{T}_1^\varepsilon(t) := T_l\left(\frac{t-t_\varepsilon}{\varepsilon}\right)(u_\varepsilon(t_\varepsilon) - H(Lv_\varepsilon(t))), \tag{3.23}$$

$$\mathcal{S}_1^\varepsilon(w)(t) := \frac{1}{\varepsilon}\lim_{\lambda\to+\infty}\int_{t_\varepsilon}^t T_l\left(\frac{t-r}{\varepsilon}\right)\lambda R_\lambda(l)\, K(w(r) + H(Lv_\varepsilon(r)), Lv_\varepsilon(r))\, dr, \tag{3.24}$$

$$\mathcal{T}_2^\varepsilon(t) := -\frac{1}{\varepsilon}\lim_{\lambda\to+\infty}\int_{t_\varepsilon}^t T_l\left(\frac{t-r}{\varepsilon}\right)\lambda R_\lambda(l)\,[A + C(Lv_\varepsilon(l))]\,[H(Lv_\varepsilon(r)) - H(Lv_\varepsilon(t))]\, dr, \tag{3.25}$$

$$\mathcal{S}_2^\varepsilon(w)(t) := \frac{1}{\varepsilon}\lim_{\lambda\to+\infty}\int_{t_\varepsilon}^t T_l\left(\frac{t-r}{\varepsilon}\right)\lambda R_\lambda(l)\,[C(Lv_\varepsilon(r)) - C(Lv_\varepsilon(l))]\, w(r)\, dr. \tag{3.26}$$

Next we derive some estimates of $\mathcal{T}_1^\varepsilon$, $\mathcal{T}_2^\varepsilon$, $\mathcal{S}_1^\varepsilon$ and $\mathcal{S}_2^\varepsilon$.

By using the formula $C(z) = \partial_u F(H(z), z)$, Lemma 3.2 and equation (3.13) we obtain for all $t, s \in [t_\varepsilon, \tau)$

$$\|C(Lv_\varepsilon(t)) - C(Lv_\varepsilon(s))\|_{\mathcal{L}\left(\overline{D(A)}\times\overline{D(B)}, X\times Y\right)} \leq C_F(C_H + 1)\,\|Lv_\varepsilon(t) - Lv_\varepsilon(s)\|, \tag{3.27}$$

and for all $t \in [t_\varepsilon, \tau]$

$$\|C(Lv_\varepsilon(t))\|_{\mathcal{L}\left(\overline{D(A)}\times\overline{D(B)}, X\times Y\right)} \leq C_F. \tag{3.28}$$

Furthermore for each $\varepsilon \in (0,1]$ we denote by $\omega(.; Lv_\varepsilon) : [0, \tau - t_\varepsilon) \to [0, +\infty)$ the modulus of continuity of Lv_ε that is

$$\omega(\delta; Lv_\varepsilon) := \sup_{s,t\in[t_\varepsilon,\tau):|t-s|=\delta} \|Lv_\varepsilon(s) - Lv_\varepsilon(t)\|, \forall \delta < \tau - t_\varepsilon,$$

so that

$$\|Lv_\varepsilon(s) - Lv_\varepsilon(t)\| \leq \omega(|t-s|; Lv_\varepsilon), \forall s, t \in [t_\varepsilon, \tau).$$

Due to Assumption 2.8 one has

$$\lim_{h\to 0^+} \omega(h; Lv_\varepsilon) = 0.$$

Estimate for $\mathcal{T}_1^\varepsilon$: For each $t \in [t_\varepsilon, \tau)$ by writing

$$\begin{aligned}\mathcal{T}_1^\varepsilon(t) &= T_l\left(\frac{t-t_\varepsilon}{\varepsilon}\right)(u_\varepsilon(t_\varepsilon) - H(Lv_\varepsilon(t_\varepsilon))) \\ &\quad + T_l\left(\frac{t-t_\varepsilon}{\varepsilon}\right)(H(Lv_\varepsilon(t_\varepsilon)) - H(Lv_\varepsilon(t))),\end{aligned}$$

and using (3.17) combined with the fact that H is Lipschitz continuous on $\mathcal{M}_Z$ it easy to obtain that for all $t \in [t_\varepsilon, t_\varepsilon + \varepsilon|\ln\varepsilon|]$ (up to reduce ε such that $[t_\varepsilon, t_\varepsilon + \varepsilon|\ln\varepsilon|] \subset [t_\varepsilon, \tau)$)

$$\|\mathcal{T}_1^\varepsilon(t)\| \leq \kappa\|u_\varepsilon(t_\varepsilon) - H(Lv_\varepsilon(t_\varepsilon))\| + \kappa C_H \omega(\varepsilon|\ln\varepsilon|; Lv_\varepsilon), \tag{3.29}$$

and for all $t \in [t_\varepsilon + \varepsilon|\ln\varepsilon|, \tau)$ by using (3.17) combined with the uniform boundedness of H on $\mathcal{M}_Z$ one gets

$$\|\mathcal{T}_1^\varepsilon(t)\| \leq \kappa e^{-\alpha|\ln\varepsilon|}\|(u_\varepsilon(t_\varepsilon) - H(Lv_\varepsilon(t_\varepsilon)))\| + 2\kappa C_H e^{-\alpha|\ln\varepsilon|}. \tag{3.30}$$

Therefore one obtains for all $t \in [t_\varepsilon, \tau)$

$$\begin{aligned}\|\mathcal{T}_1^\varepsilon(t)\| &\leq \kappa\|u_\varepsilon(t_\varepsilon) - H(Lv_\varepsilon(t_\varepsilon))\| + \kappa C_H \omega(\varepsilon|\ln\varepsilon|; Lv_\varepsilon) \\ &\quad + \kappa e^{-\alpha|\ln\varepsilon|}[2C_H + \|(u_\varepsilon(t_\varepsilon) - H(Lv_\varepsilon(t_\varepsilon)))\|].\end{aligned} \tag{3.31}$$

Estimate for $\mathcal{S}_1^\varepsilon$: By using (3.24) we obtain for all $t \geq t_\varepsilon$

$$\|\mathcal{S}_1^\varepsilon(w)(t)\| \leq \frac{\kappa^2}{\varepsilon}\int_{t_\varepsilon}^t e^{-\frac{\alpha}{\varepsilon}(t-r)}\|K(w(r) + H(Lv_\varepsilon(r)), Lv_\varepsilon(r))\| dr.$$

By using (3.15) we obtain for all $r \in [t_\varepsilon, t]$

$$\|K(w(r) + H(Lv_\varepsilon(r)), Lv_\varepsilon(r))\| \leq C_F\|w(r)\|^2,$$

providing that

$$\left\|\mathcal{S}_1^{\varepsilon}(w)(t)\right\| \leq \frac{\kappa^2}{\alpha} C_F \left\|w\right\|_{\infty}^2, \ \forall t \in [t_{\varepsilon}, \tau). \tag{3.32}$$

Estimate for $\mathcal{T}_2^{\varepsilon}$: By using the fact that H is Lipschitz continuous on $\mathcal{M}_Z$ with respect to the graph norm of A combined together with (3.28) we obtain

$$\left\|\mathcal{T}_2^{\varepsilon}(t)\right\| \leq \frac{\kappa^2}{\varepsilon}(C_H + C_F)\int_{t_{\varepsilon}}^{t} e^{-\frac{\alpha}{\varepsilon}(t-r)} \left\|Lv_{\varepsilon}(r) - Lv_{\varepsilon}(t)\right\| dr, \ \forall t \geq t_{\varepsilon}.$$

Hence if $t \in [t_{\varepsilon}, t_{\varepsilon} + \varepsilon\left|\ln\varepsilon\right|]$ we have

$$\begin{aligned} \left\|\mathcal{T}_2^{\varepsilon}(t)\right\| &\leq \frac{\kappa^2}{\varepsilon}(C_H + C_F)\int_{t_{\varepsilon}}^{t} e^{-\frac{\alpha}{\varepsilon}(t-r)} dr \omega\left(\varepsilon\left|\ln\varepsilon\right|; Lv_{\varepsilon}\right), \qquad (3.33) \\ &\leq \frac{\kappa^2}{\alpha}(C_H + C_F)\,\omega\left(\varepsilon\left|\ln\varepsilon\right|; Lv_{\varepsilon}\right), \end{aligned}$$

and if $t \in [t_{\varepsilon} + \varepsilon\left|\ln\varepsilon\right|, \tau)$ we have

$$\begin{aligned} \left\|\mathcal{T}_2^{\varepsilon}(t)\right\| &\leq \frac{\kappa^2}{\varepsilon}(C_H + C_F)\int_{t_{\varepsilon}}^{t-\varepsilon\left|\ln\varepsilon\right|} e^{-\frac{\alpha}{\varepsilon}(t-r)} \left\|Lv_{\varepsilon}(r) - Lv_{\varepsilon}(t)\right\| dr \qquad (3.34) \\ &\quad + \frac{\kappa^2}{\varepsilon}(C_H + C_F)\int_{t-\varepsilon\left|\ln\varepsilon\right|}^{t} e^{-\frac{\alpha}{\varepsilon}(t-r)} \left\|Lv_{\varepsilon}(r) - Lv_{\varepsilon}(t)\right\| dr \\ &\leq 2\left\|Lv_{\varepsilon}\right\|_{\infty} \frac{\kappa^2}{\varepsilon}(C_H + C_F)\int_{t_{\varepsilon}}^{t-\varepsilon\left|\ln\varepsilon\right|} e^{-\frac{\alpha}{\varepsilon}(t-r)} dr \\ &\quad + \frac{\kappa^2}{\varepsilon}(C_H + C_F)\int_{t-\varepsilon\left|\ln\varepsilon\right|}^{t} e^{-\frac{\alpha}{\varepsilon}(t-r)} dr \omega\left(\varepsilon\left|\ln\varepsilon\right|; Lv_{\varepsilon}\right) \\ &\leq 2\left\|Lv_{\varepsilon}\right\|_{\infty} \frac{\kappa^2}{\alpha}(C_H + C_F)\, e^{-\alpha\left|\ln\varepsilon\right|} + \frac{\kappa^2}{\alpha}(C_H + C_F)\,\omega\left(\varepsilon\left|\ln\varepsilon\right|; Lv_{\varepsilon}\right). \end{aligned}$$

Thus combining (3.33) with (3.34) yields for all $t \in [t_{\varepsilon}, \tau)$

$$\left\|\mathcal{T}_2^{\varepsilon}(t)\right\| \leq 2\left\|Lv_{\varepsilon}\right\|_{\infty} \frac{\kappa^2}{\alpha}(C_H + C_F)\, e^{-\alpha\left|\ln\varepsilon\right|} + \frac{\kappa^2}{\alpha}(C_H + C_F)\,\omega\left(\varepsilon\left|\ln\varepsilon\right|; Lv_{\varepsilon}\right). \tag{3.35}$$

Estimate for $\mathcal{S}_2^{\varepsilon}$: By using (3.26) combined with (3.27) one obtains

$$\left\|\mathcal{S}_2^{\varepsilon}(w)(t)\right\| \leq \frac{\kappa^2}{\varepsilon} C_F (C_H + 1)\int_{t_{\varepsilon}}^{t} e^{-\frac{\alpha}{\varepsilon}(t-r)} \left\|Lv_{\varepsilon}(r) - Lv_{\varepsilon}(l)\right\| \left\|w(r)\right\| dr, \ \forall t \in [t_{\varepsilon}, \tau).$$

Since $l \in [t_{\varepsilon}, \tau)$ is arbitrary by setting $l = t$ and proceeding similarly as for the estimate of $\mathcal{T}_2^{\varepsilon}$ we obtain for all $t \in [t_{\varepsilon}, \tau)$

$$\begin{aligned} \left\|\mathcal{S}_2^{\varepsilon}(w)(t)\right\| &\leq 2\left\|Lv_{\varepsilon}\right\|_{\infty} \frac{\kappa^2}{\alpha} C_F (C_H + 1)\, e^{-\alpha\left|\ln\varepsilon\right|} \left\|w\right\|_{\infty} \qquad (3.36) \\ &\quad + \frac{\kappa^2}{\alpha} C_F (C_H + 1)\,\omega\left(\varepsilon\left|\ln\varepsilon\right|; Lv_{\varepsilon}\right) \left\|w\right\|_{\infty}. \end{aligned}$$

Final step (estimate of Γ^ε and fixed point argument): We first fix some constants that will appear in the sequel. This allows us to make clear the choice of $\delta \in (0,1)$ and $\varepsilon \in (0,1]$. To do so define

$$C_1 := (1 + 2\sup(\mathcal{M}_Z))\,\frac{\kappa^2}{\alpha}\max\{C_F + C_H; C_F(C_H+1); 3C_F; C_H\}. \tag{3.37}$$

From now on let $\delta \in (0,1)$ be given and fixed such that

$$C_1\delta < \frac{1}{4}. \tag{3.38}$$

To prove our claim we will use a fixed point argument on

$$B_{\mathcal{BC}_\varepsilon}(0,\delta) := \{w \in \mathcal{BC}_\varepsilon : \|w\|_\infty \le \delta\},$$

with a careful choice of $\varepsilon \in (0,1]$ and t_ε. To do this note that by using (3.37) we obtain respectively from (3.31)

$$\|\mathcal{T}_1^\varepsilon(t)\| \le \kappa\|u_\varepsilon(t_\varepsilon) - H(Lv_\varepsilon(t_\varepsilon))\| + C_1\left[\omega(\varepsilon|\ln\varepsilon|; Lv_\varepsilon) + e^{-\alpha|\ln\varepsilon|}\right],\ \forall t \in [t_\varepsilon,\tau),$$

from (3.32)

$$\|\mathcal{S}_1^\varepsilon(w)(t)\| \le C_1\|w\|_\infty^2,\ \forall t \in [t_\varepsilon,\tau),$$

from (3.35)

$$\|\mathcal{T}_2^\varepsilon(t)\| \le C_1\left[\omega(\varepsilon|\ln\varepsilon|; Lv_\varepsilon) + e^{-\alpha|\ln\varepsilon|}\right],\ \forall t \in [t_\varepsilon,\tau),$$

and from (3.36)

$$\|\mathcal{S}_2^\varepsilon(w)(t)\| \le C_1\left[\omega(\varepsilon|\ln\varepsilon|; Lv_\varepsilon) + e^{-\alpha|\ln\varepsilon|}\right]\|w\|_\infty, \forall t \in [t_\varepsilon,\tau). \tag{3.39}$$

We now show that Γ^ε maps $B_{\mathcal{BC}_\varepsilon}(0,\delta)$ into itself for $\varepsilon \in (0,1]$ small enough and $t_\varepsilon = O(\varepsilon)$. Recalling that

$$\Gamma^\varepsilon(w)(t) = \mathcal{T}_1^\varepsilon(t) + \mathcal{S}_1^\varepsilon(w)(t) + \mathcal{T}_2^\varepsilon(t) + \mathcal{S}_2^\varepsilon(w)(t),\ \forall t \in [t_\varepsilon,\tau),$$

it follows that for all $w \in B_{\mathcal{BC}_\varepsilon}(0,\delta)$ and $t \in [t_\varepsilon,\tau)$

$$\|\Gamma^\varepsilon(w)(t)\| \le \kappa\|u_\varepsilon(t_\varepsilon) - H(Lv_\varepsilon(t_\varepsilon))\| + (2C_1 + C_1)\left[\omega(\varepsilon|\ln\varepsilon|; Lv_\varepsilon) + e^{-\alpha|\ln\varepsilon|}\right] + C_1\delta^2.$$

Observe that from Lemma 3.3 we have for all $\nu > 0$

$$\|u_\varepsilon(\varepsilon\nu) - H(Lv_\varepsilon(\varepsilon\nu))\| \le \kappa e^{-\alpha\nu}\|x_\varepsilon - H(Ly_\varepsilon)\| + C_0(1+\nu)e^{C_0\nu}\omega(\varepsilon\nu; Lv_\varepsilon),$$

with $C_0 > 0$. Then since $\|x_\varepsilon - H(Ly_\varepsilon)\|$ is uniformly bounded with respect to $\varepsilon \in (0,1]$, there exists $\nu_0 > 0$ large enough such that

$$\|u_\varepsilon(\varepsilon\nu_0) - H(Lv_\varepsilon(\varepsilon\nu_0))\| \le \frac{\delta}{2} + C_0(1+\nu_0)e^{C_0\nu_0}\omega(\varepsilon\nu_0; Lv_\varepsilon),$$

so that for

$$t_\varepsilon = \varepsilon\nu_0,\ \forall\varepsilon \in (0,1],$$

we obtain for all $t \in [\varepsilon\nu_0, \tau)$

$$\|\Gamma^\varepsilon(w)(t)\| \leq \frac{\delta}{2} + C_0(1+\nu_0)e^{C_0\nu_0}\omega(\varepsilon\nu_0; Lv_\varepsilon) + (2C_1+C_1)\left[\omega(\varepsilon|\ln\varepsilon|; Lv_\varepsilon) + e^{-\alpha|\ln\varepsilon|}\right] + C_1\delta^2,$$

Now we infer from (3.38) combined with $\lim_{\varepsilon\to 0}\omega(\varepsilon\nu_0; Lv_\varepsilon) = 0$ and $\lim_{\varepsilon\to 0}\omega(\varepsilon|\ln\varepsilon|; Lv_\varepsilon) = 0$ that there exists $\varepsilon_1 := \varepsilon_1(\delta) > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$

$$\begin{cases} (2C_1 + C_1)\left[\omega(\varepsilon|\ln\varepsilon|; Lv_\varepsilon)e^{-\alpha|\ln\varepsilon|}\right] < \dfrac{\delta}{8}, \\ C_0(1+\nu_0)e^{C_0\nu_0}\omega(\varepsilon\nu_0; Lv_\varepsilon) < \dfrac{\delta}{8}, \\ C_1\delta^2 < \dfrac{\delta}{4}. \end{cases} \tag{3.40}$$

That yields

$$\|\Gamma^\varepsilon(w)(t)\| \leq \delta,\ \forall t \in [\varepsilon\nu_0, \tau) \text{ and } w \in B_{\mathcal{BC}_\varepsilon}(0,\delta).$$

Next we will show that Γ^ε is Lipschitz continuous on $B_{\mathcal{BC}_\varepsilon}(0,\delta)$ wherein $t_\varepsilon = \varepsilon\nu_0$, $\varepsilon \in (0, \varepsilon_1)$ is now given and fixed.
Since $\mathcal{T}_1^\varepsilon$ and $\mathcal{T}_2^\varepsilon$ do not depend on w, the Lipschitz estimate of $\mathcal{T}^\varepsilon$ is given by those of $\mathcal{S}_2^\varepsilon$ and $\mathcal{S}_1^\varepsilon$. Thus using the fact that $\mathcal{S}_2^\varepsilon$ is linear with respect to $w \in \mathcal{BC}_\varepsilon$ we obtain from (3.39) that for all $w, \bar{w} \in B_{\mathcal{BC}_\varepsilon}(0,\delta)$

$$\|\mathcal{S}_2^\varepsilon(w) - \mathcal{S}_2^\varepsilon(\bar{w})\|_\infty \leq C_1\left[\omega(\varepsilon|\ln\varepsilon|; Lv_\varepsilon) + e^{-\alpha|\ln\varepsilon|}\right]\|w - \bar{w}\|_\infty, \tag{3.41}$$

and by using (3.16) and (3.24) we obtain

$$\begin{aligned} \|\mathcal{S}_1^\varepsilon(w)(t) - \mathcal{S}_1^\varepsilon(\bar{w})(t)\| &\leq \frac{\kappa^2}{\varepsilon}\int_{\varepsilon\nu_0}^t e^{-\frac{\alpha}{\varepsilon}(t-r)}C_F[\|w-\bar{w}\|_\infty + \|w\|_\infty]\|w-\bar{w}\|_\infty dr \\ &\leq \frac{\kappa^2}{\varepsilon}3\delta C_F\int_{t_\varepsilon}^t e^{-\frac{\alpha}{\varepsilon}(t-r)}dr\|w-\bar{w}\|_\infty \\ &\leq \frac{\kappa^2}{\alpha}3\delta C_F\|w-\bar{w}\|_\infty,\ \forall t \in [\varepsilon\nu_0, \tau), \end{aligned}$$

that is

$$\|\mathcal{S}_1^\varepsilon(w) - \mathcal{S}_1^\varepsilon(\bar{w})\|_\infty \leq \frac{\kappa^2}{\alpha}3\delta C_F\|w-\bar{w}\|_\infty. \tag{3.42}$$

Therefore we deduce from (3.41), (3.42) and (3.37) that for all $w, \bar{w} \in B_{\mathcal{BC}_\varepsilon}(0,\delta)$

$$\|\Gamma^\varepsilon(w) - \Gamma^\varepsilon(\bar{w})\| \leq C_1\left[\omega(\varepsilon|\ln\varepsilon|; Lv_\varepsilon) + e^{-\alpha|\ln\varepsilon|} + \delta\right]\|w-\bar{w}\|_\infty.$$

Finally it is easy to see from (3.37) and (3.40) that Γ^ε define a contraction on $B_{\mathcal{BC}_\varepsilon}(0,\delta)$ for all $\varepsilon \in (0, \varepsilon_1)$. The proof of the claim is completed. ■

4 Proof of Corollary 2.11

In this section we will give the proof of Corollary 2.11. To do so we will first show that $\|v_\varepsilon(t) - v(t)\|$ converges uniformly to 0 in $[0,\tau]$ as ε goes to zero. Then we end the proof by using a simple argument combined together with the fact that H is Lipschitz continuous on M_Z.

Proof of Corollary 2.11. The mild solutions $v(t)$ and $v_\varepsilon(t)$ are defined for all $t \in [0,\tau]$ by

$$v(t) = T_{B_0}(t)y_0 + \lim_{\lambda\to+\infty} \int_0^t T_{B_0}(t-s)\,\lambda R_\lambda(B) G\left(H\left(Lv(t)\right), v(t)\right) ds, \quad (4.1)$$

and

$$v_\varepsilon(t) = T_{B_0}(t)y_\varepsilon + \lim_{\lambda\to+\infty} \int_0^t T_{B_0}(t-s)\,\lambda R_\lambda(B) G\left(u_\varepsilon(t), v_\varepsilon(t)\right) ds. \quad (4.2)$$

By subtracting (4.1) to (4.2) yields

$$\begin{aligned}
&\|v_\varepsilon(t) - v(t)\| \le \|T_{B_0}(t)(y_\varepsilon - y_0)\| \\
&+ \lim_{\lambda\to+\infty} \int_0^t \|T_{B_0}(t-s)\,\lambda R_\lambda(B)[G(u_\varepsilon(s), v_\varepsilon(s)) - G(H(Lv(s)), v(s))]\|\, ds \\
&\le \|T_{B_0}(t)(y_\varepsilon - y_0)\| \\
&+ \lim_{\lambda\to+\infty} \int_0^t \|T_{B_0}(t-s)\,\lambda R_\lambda(B)[G(u_\varepsilon(s), v_\varepsilon(s)) - G(H(Lv_\varepsilon(s)), v(s))]\|\, ds \\
&+ \lim_{\lambda\to+\infty} \int_0^t \|T_{B_0}(t-s)\,\lambda R_\lambda(B)[G(H(Lv_\varepsilon(s)), v(s)) - G(H(Lv(s)), v(s))]\|\, ds.
\end{aligned} \quad (4.3)$$

Recall that $Lv(t)$, $Lv_\varepsilon(t) \in \mathcal{M}_Z = \overline{L(\mathcal{M}_Y)}$ and $u_\varepsilon(t) \in \mathcal{M}_X$. Recall also that H is Lipschitz continuous on $\mathcal{M}_Z$ and G is Lipschitz continuous on bounded sets, therefore there exists $C_G > 0$ such that for all $t \in [0,\tau]$

$$\|G(H(Lv_\varepsilon(t)), v(t)) - G(H(Lv(t)), v(t))\| \le C_G C_H \|L\|_{\mathcal{L}(Y,Z)} \|v_\varepsilon(t) - v(t)\|, \quad (4.4)$$

and

$$\|G(u_\varepsilon(t), v_\varepsilon(t)) - G(H(Lv_\varepsilon(t)), v(t))\| \le C_G\left[\|u_\varepsilon(t) - H(Lv_\varepsilon(t))\| + \|v_\varepsilon(t) - v(t)\|\right], \quad (4.5)$$

Then since B is a Hille-Yosida operator with constants (ω_B, M_B) by using (4.3)-(4.5) we obtain for all $t \in [0,\tau]$

$$\begin{aligned}
\|v_\varepsilon(t) - v(t)\| \le{}& M_B e^{\omega_B t}\|y_\varepsilon - y_0\| \\
&+ M_B^2[C_G + C_G C_H \|L\|_{\mathcal{L}(Y,Z)}] \int_0^t e^{\omega_B(t-s)}\|v_\varepsilon(s) - v(s)\| ds \\
&+ M_B^2 C_G \int_0^t e^{\omega_B(t-s)}\|u_\varepsilon(s) - H(Lv_\varepsilon(s)\| ds.
\end{aligned}$$

Thus by setting $N_0 := \max\{M_B^2, M_B^2[C_G + C_G C_H \|L\|_{\mathcal{L}(Y,Z)}]\}$ we obtain for each $\eta > 0$ (small enough) and $t \in [0, \tau]$

$$\begin{aligned} e^{-\omega_B t}\|v_\varepsilon(t) - v(t)\| &\leq N_0\|y_\varepsilon - y_0\| + N_0 \int_0^t e^{-\omega_B s}\|v_\varepsilon(s) - v(s)\|ds \\ &+ N_0 \int_0^t e^{-\omega_B s}\|u_\varepsilon(s) - H(Lv_\varepsilon(s)\|ds \\ &\leq N_0\|y_\varepsilon - y_0\| + N_0 \int_0^t e^{-\omega_B s}\|v_\varepsilon(s) - v(s)\|ds \\ &+ N_0 \int_0^\eta \|u_\varepsilon(s) - H(Lv_\varepsilon(s)\|ds + N_0 \int_\eta^t \|u_\varepsilon(s) - H(Lv_\varepsilon(s)\|ds. \end{aligned}$$

Since $\|u_\varepsilon(t) - H(Lv_\varepsilon(t)\|$ is uniformly bounded with respect to $t \in [0, \tau]$ and $\varepsilon \in (0, 1]$, by denoting N_1 its upper bound we get for all $t \in [0, \tau]$

$$\begin{aligned} e^{-\omega_B t}\|v_\varepsilon(t) - v(t)\| &\leq N_0\|y_\varepsilon - y_0\| + N_0 \int_0^t e^{-\omega_B s}\|v_\varepsilon(s) - v(s)\|ds \\ &+ N_1\eta + N_0\tau \sup_{t\in[\eta,\tau]} \|u_\varepsilon(t) - H(Lv_\varepsilon(t)\|. \end{aligned}$$

Next Gronwall's lemma applies and ensures

$$\|v_\varepsilon(t) - v(t)\| \quad \leq N_0 e^{(N_0+\omega_B)t}[\|y_\varepsilon - y_0\| + \frac{N_1}{N_0}\eta + \tau \sup_{t\in[\eta,\tau]} \|u_\varepsilon(t) - H(Lv_\varepsilon(t)\|], \quad t \in [0, \tau].$$

Now using Theorem 2.9 combined with the fact that η is arbitrarily small it follows that

$$\lim_{\varepsilon\to 0^+} \sup_{t\in[0,\tau]} \|v_\varepsilon(t) - v(t)\| \leq \frac{N_1}{N_0}\eta \Longrightarrow \lim_{\varepsilon\to 0^+} \sup_{t\in[0,\tau]} \|v_\varepsilon(t) - v(t)\| = 0,$$

Finally the proof is completed by observing that for each $t_0 \in [0, \tau]$

$$\sup_{t\in[t_0,\tau]} \|u_\varepsilon(t) - H(Lv_\varepsilon(t))\| \leq \sup_{t\in[t_0,\tau]} \|u_\varepsilon(t) - H(Lv_\varepsilon(t))\| + \sup_{t\in[t_0,\tau]} \|H(Lv_\varepsilon(t)) - H(Lv(t))\|.$$

and using the fact that H is Lipschitz continuous on $\mathcal{M}_Z$. ■

5 Applications

This section is devoted to the application of our general results to the system (1.5)-(1.6) and to the problem (1.7).

5.1 Application to the age-structured model

In this section we will apply our main result to system (1.5)-(1.6). We recall it here for more convenience

$$\varepsilon \frac{dH_C(t)}{dt} = -\nu_H H_C(t) + \beta_H \int_0^\infty \gamma(a) i(t, a) da (1 - H_C(t)), \tag{5.1}$$

and

$$\begin{cases} \dfrac{dS(t)}{dt} = \nu_P N_P - \nu_P S(t) - \beta_P H_C(t) S(t), \\ \dfrac{\partial i_P(t,a)}{\partial t} + \dfrac{\partial i_P(t,a)}{\partial a} = -\nu_P i_P(t,a), \\ i_P(t,0) = \beta_P H_C(t) S(t), \end{cases} \tag{5.2}$$

supplemented with the initial conditions

$$H_C(0,.) = H_{C0}, S(0) = S_0, \text{ and } i_P(0,.) = i_{P0} \in L^1_+(0,+\infty).$$

Then we make the following assumptions.

Assumption 5.1 *We assume that $\nu_H, \nu_P, \beta_H, \beta_P, N_P > 0$ and $\gamma \in L^\infty_+(0,+\infty)$.*

Age-structured models have been considered first by using Volterra's integral equations (see Webb [34] and Iannelli [15]). Here we will use an abstract Cauchy problem reformulation introduced by Thieme [28] (see also Magal [19]).
Abstract reformulation : We first reformulate the linear part of (5.2). To that aim define the Banach space

$$Y := \mathbb{R} \times \mathbb{R} \times L^1((0,+\infty), \mathbb{R}),$$

endowed with the usual product norm and set

$$Y_0 := \mathbb{R} \times \{0_\mathbb{R}\} \times L^1((0,+\infty), \mathbb{R}).$$

Consider the linear operator $B : D(B) \subset Y \to Y$ given by

$$B\left(\begin{array}{c} \zeta \\ \left(\begin{array}{c} 0_\mathbb{R} \\ \psi \end{array}\right) \end{array}\right) = \left(\begin{array}{c} 0_\mathbb{R} \\ \left(\begin{array}{c} -\psi(0) \\ -\psi' - \nu_P \psi \end{array}\right) \end{array}\right), \tag{5.3}$$

with

$$D(B) = \mathbb{R} \times \{0_\mathbb{R}\} \times W^{1,1}((0,+\infty), \mathbb{R}), \tag{5.4}$$

and observe that

$$\overline{D(B)} = Y_0.$$

Since the linear part of the fast component (5.1) is defined on $\mathbb{R}$ we obviously have

$$D(A) = \mathbb{R} = X = X_0 \text{ and } Ax = -\nu_H x, \ \forall x \in \mathbb{R}.$$

In order to reformulate the non linear part of (5.2) as well as the non linear part of (5.1) we introduce respectively the maps $G : \mathbb{R} \times Y_0 \to Y$

$$G\left(x, \left(\begin{array}{c} \zeta \\ \left(\begin{array}{c} 0_\mathbb{R} \\ \psi \end{array}\right) \end{array}\right)\right) = \left(\begin{array}{c} \nu_P N_P - \nu_P \zeta - \beta_P x \zeta \\ \left(\begin{array}{c} \beta_P x \zeta \\ 0_{L^1} \end{array}, \right) \end{array}\right).$$

and $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$F(x,z) = \beta_H z(1-x),$$

while $L : Y_0 \to \mathbb{R}$ is defined by

$$L\left(\begin{array}{c}\zeta\\ \left(\begin{array}{c}0_{\mathbb{R}}\\ \psi\end{array}\right)\end{array}\right) := \int_0^{+\infty} \gamma(a)\psi(a)\,da. \tag{5.5}$$

Hence the Banach space Z is equal to $\mathbb{R}$. Therefore by setting

$$u(t) := H_C(t) \text{ and } v(t) := \left(\begin{array}{c}S(t)\\ \left(\begin{array}{c}0_{\mathbb{R}}\\ i_P(t,.)\end{array}\right)\end{array}\right),\ t>0, \tag{5.6}$$

and

$$x_0 := H_{C0} \in \mathbb{R} \text{ and } y_0 := \left(\begin{array}{c}S_0\\ \left(\begin{array}{c}0_{\mathbb{R}}\\ i_{P0}\end{array}\right)\end{array}\right) \in Y_0, \tag{5.7}$$

system (5.1)-(5.2) re-writes as the following non-densely defined Cauchy problem

$$\begin{cases} \varepsilon \dfrac{du(t)}{dt} = Au(t) + F(u(t), Lv(t)), \\ \dfrac{dv(t)}{dt} = Bv(t) + G(u(t), v(t)), \end{cases} \tag{5.8}$$

with the initial data

$$u(0) = x_0 \in \mathbb{R} \text{ and } v(0) = y_0 \in \overline{D(B)}.$$

Note that here we use the subscribe 0 for the initial data since in this example they do not depend on ε.

Checking the assumptions : In order to deal with the existence and the positivity of the solutions, we denote by

$$Y_+ := \mathbb{R}_+ \times \mathbb{R}_+ \times L^1_+((0,+\infty), \mathbb{R}),$$

the positive cone of Y and we set

$$Y_{+0} := Y_+ \cap Y_0 = \mathbb{R}_+ \times \{0_{\mathbb{R}}\}_+ \times L^1_+((0,+\infty), \mathbb{R}).$$

Then the following lemma holds true.

Lemma 5.2 *For each $\lambda > 0$ the following estimates for the resolvent of A and B are satisfied*

$$\left|(\lambda I - A)^{-1}\right| \le \frac{1}{\lambda} \text{ and } \left\|(\lambda I - B)^{-1}\right\|_{\mathcal{L}(Y)} \le \frac{1}{\lambda}.$$

Furthermore one has

$$(\lambda I - A)^{-1}\mathbb{R}_+ \subset \mathbb{R}_+ \text{ and } (\lambda I - B)^{-1} Y_+ \subset Y_+,\ \forall \lambda > 0.$$

Proof. The proof of this lemma is classical. We refer for instance to [23]. ■

Next define the following closed bounded set

$$\mathcal{M} := \mathcal{M}_X \times \mathcal{M}_Y$$

with

$$\mathcal{M}_X := [0,1]$$

and

$$\mathcal{M}_Y := \left\{ y := \left(\begin{array}{c} \zeta \\ \left(\begin{array}{c} 0_{\mathbb{R}} \\ \psi \end{array} \right) \end{array} \right) \in Y_{+0} : \zeta + \int_0^\infty \psi(a)da \leq N_P \right\}.$$

By using the results in [19, 21, 28] we obtain the following lemma.

Lemma 5.3 *There exists a unique semiflow* $\{\mathcal{S}_\varepsilon(t)\}_{t\geq 0}$ *on* $\mathcal{M}$ *such that for each* $(x,y) \in \mathcal{M}$

$$(u_\varepsilon(t), v_\varepsilon(t)) := \mathcal{S}_\varepsilon(t)(x,y), \ t \geq 0,$$

is the unique mild solution of system (5.8). Moreover $\mathcal{M}$ *is positively invariant with respect to the semiflow* $\{\mathcal{S}_\varepsilon(t)\}_{t\geq 0}$ *that is to say that*

$$\mathcal{S}_\varepsilon(t)\mathcal{M} \subset \mathcal{M}, \ \forall t \geq 0 \text{ and } \varepsilon \in (0,1].$$

In this example with $Z = \mathbb{R}$ and for each $(x,y) \in \mathcal{M}$ the linear operator $L : Y_0 \to \mathbb{R}$ defined in (5.5) satisfies

$$Lv_\varepsilon(t) \in [0, \|\gamma\|_\infty N_P], \ \forall t \geq 0 \text{ and } \varepsilon \in (0,1].$$

Therefore $\mathcal{M}_{\mathbb{R}} := \overline{L(\mathcal{M}_Y)}$ is included into $[0, \|\gamma\|_\infty N_P]$.
In order to define our graph we need to solve the following equation

$$Au + F(u,z) = 0, \ \forall z \in \mathcal{M}_{\mathbb{R}} \subset [0, \|\gamma\|_\infty N_P],$$

that is equivalent to

$$-\nu_H u + \beta_H z(1-u) = 0, \ \forall z \in \mathcal{M}_{\mathbb{R}} \subset [0, \|\gamma\|_\infty N_P].$$

Thus the map $H : \mathcal{M}_{\mathbb{R}} \to \mathbb{R} = D(A)$ is defined by

$$H(z) = \frac{\beta_H z}{\nu_H + \beta_H z}, \ \forall z \in \mathcal{M}_{\mathbb{R}}.$$

Let us now focus on the z-parametrized system

$$\frac{du(t)}{dt} = Au(t) + F(u(t), z), \ t \geq 0, \tag{5.9}$$

with

$$u(0) = x \in \overline{D(A)} = \mathbb{R} \text{ and } z \in \mathcal{M}_{\mathbb{R}}.$$

This equation can simply be re-written as a scalar ordinary differential equation

$$\frac{du\left(t\right)}{dt}=-\nu_{H}u\left(t\right)+\beta_{H}z(1-u(t)).$$

The following lemma is devoted to the exponential stability condition of system (5.9) and the existence of a globally defined semiflow. The proof is classical and thus omitted.

Lemma 5.4 *For each $z\in\mathcal{M}_{\mathbb{R}}$ the ordinary differential equation (5.9) generates a unique semiflow $\left\{U_{z}\left(t\right)\right\}_{t\geq0}$ on $\mathbb{R}$ that satisfies*

$$\left|U_{z}\left(t\right)x-H\left(z\right)\right|\leq e^{-\nu_{H}t}\left|x-H\left(z\right)\right|,\ \forall t\geq0 \text{ and } x\in\mathbb{R}.$$

Next we will investigate the regularity conditions of Assumptions 2.8. Denote by $\left\{T_{B_{0}}\left(t\right)\right\}_{t\geq0}\subset\mathcal{L}\left(Y_{0}\right)$ the strongly continuous semigroup generated by B_{0} the part of B in Y_{0}. Let $\left\{S_{B}\left(t\right)\right\}_{t\geq0}\subset\mathcal{L}\left(Y\right)$ be the integrated semigroup generated by B. Moreover

$$G\left(\mathbb{R}\times Y_{0}\right)\subset Y_{G}$$

where

$$Y_{G}:=\mathbb{R}\times\mathbb{R}\times\left\{0_{L^{1}}\right\}.$$

Lemma 5.5 *Let $y_{0}\in Y_{0}$ be fixed. The map $t\rightarrow L\circ T_{B_{0}}\left(t\right)y_{0}$ is uniformly continuous from $[0,+\infty)$ into $\mathbb{R}$.*

Proof. Let $y_{0}=\left(\begin{array}{c}S_{0}\\ \left(\begin{array}{c}0_{\mathbb{R}}\\ i_{P0}\end{array}\right)\end{array}\right)\in\overline{D(B)}$ be given. Recall that for each $t\geq0$ one has

$$T_{B_{0}}(t)\left(\begin{array}{c}S_{0}\\ \left(\begin{array}{c}0_{\mathbb{R}}\\ i_{P0}\end{array}\right)\end{array}\right)=\left(\begin{array}{c}S_{0}\\ \left(\begin{array}{c}0_{\mathbb{R}}\\ \hat{T}_{B_{0}}(t)(i_{P0})\end{array}\right)\end{array}\right),\tag{5.10}$$

with

$$\hat{T}_{B_{0}}(t)(i_{P0})(a):=\left\{\begin{array}{ll}e^{-\nu_{P}t}i_{P0}(a-t) & \text{if } a\geq t,\\ \\ 0 & \text{if } a<t.\end{array}\right.\tag{5.11}$$

The map $t\rightarrow\hat{T}_{B_{0}}(t)(i_{P0})(a)$ is uniformly continuous from $[0,+\infty)$ into $L^{1}(0,+\infty)$, since this map is continuous on $[0,+\infty)$ and converges to 0 as t goes $+\infty$. Hence the result follows by using the boundedness of L. ∎

Lemma 5.6 *The family of maps $\left\{L\circ\left(S_{B}\diamond g\right)\left(t\right):g\in Y_{G} \text{ with } \sup_{t\geq0}\left\|g\left(t\right)\right\|\leq1\right\}$ is uniformly equicontinuous from $[0,+\infty)$ into $\mathbb{R}$. More precisely for each $h\geq0$ and $t\geq0$ we have*

$$\left|L\circ\left(S_{B_{0}}\diamond g\right)\left(t+h\right)-L\circ\left(S_{B_{0}}\diamond g\right)\left(t\right)\right|\leq\left|h\right|\left\|\gamma\right\|_{L^{\infty}}+\left\|e^{-\nu_{P}(h+.)}\gamma(h+.)-e^{-\nu_{P}}\gamma(.)\right\|_{L^{1}}.$$

Proof. Let $g \in L^{\infty}([0,+\infty), Y_G)$ be given. Recall that $t \to (S_{B_0} \diamond g)(t)$ is the mild solution of

$$\frac{dv(t)}{dt} = Bv(t) + g(t), \ t \geq 0 \text{ and } v(0) = 0. \tag{5.12}$$

Set $v(t) = \begin{pmatrix} y_1(t) \\ \begin{pmatrix} 0_{\mathbb{R}} \\ y_2(t,.) \end{pmatrix} \end{pmatrix}$ and $g(t) = \begin{pmatrix} g_1(t) \\ \begin{pmatrix} g_2(t) \\ 0 \end{pmatrix} \end{pmatrix} \in Y_G$ then system (5.12) becomes equivalent to the PDE

$$\begin{cases} y_1'(t) = g_1(t), \\ \dfrac{\partial y_2(t,a)}{\partial t} + \dfrac{\partial y_2(t,a)}{\partial a} = -\nu_P y_2(t,a), \\ y_2(t,0) = g_2(t). \end{cases} \tag{5.13}$$

with initial conditions

$$y_1(0) = 0_{\mathbb{R}} \text{ and } y_2(0,.) = 0_{L^1}.$$

For the sake of simplicity recall the definition of L in (5.5) and set

$$\widehat{L}(t) := L\begin{pmatrix} y_1(t) \\ \begin{pmatrix} 0_{\mathbb{R}} \\ y_2(t,.) \end{pmatrix} \end{pmatrix} = \int_0^{\infty} \gamma(a) y_2(t,a)\, da.$$

By integrating (5.13) along the characteristics curves one gets

$$y_2(t,a) = \begin{cases} e^{-\nu_P a} g_2(t-a), & if \ t > a, \\ 0, & if \ t \leq a. \end{cases}$$

Then for each $h \geq 0$ and $t \geq 0$ one has

$$\begin{aligned} \widehat{L}(t+h) - \widehat{L}(t) &= \int_0^{t+h} e^{-\nu_P a}\gamma(a) g_2(t+h-a)\, da - \int_0^t e^{-\nu_P a}\gamma(a) g_2(t-a)\, da \\ &= \int_{-h}^{t} e^{-\nu_P (a+h)}\gamma(a+h) g_2(t-a)\, da - \int_0^t e^{-\nu_P a}\gamma(a) g_2(t-a)\, da \\ &= \int_{-h}^{0} e^{-\nu_P (a+h)}\gamma(a+h) g_2(t-a)\, da \\ &\quad + \int_0^t \left[e^{-\nu_P (a+h)}\gamma(a+h) - e^{-\nu_P a}\gamma(a)\right] g_2(t-a)\, da, \end{aligned}$$

and the result follows. ■

Convergence result: With the above verifications, we are now able to obtain a convergence result. Here the initial distribution does not depending on ε, therefore the subset $\mathcal{M}_0$ is reduced to the single point

$$\mathcal{M}_0 := \left\{ \left(H_{C0}, \begin{pmatrix} S_0 \\ \begin{pmatrix} 0_{\mathbb{R}} \\ i_{P0} \end{pmatrix} \end{pmatrix} \right) \right\} \subset \mathcal{M},$$

and by applying Theorem 2.9 and Corrolary 2.11 provide the following result.

Theorem 5.7 *Let $t \in [0,+\infty) \to (H_C^\varepsilon(t,\cdot), S^\varepsilon(t), i^\varepsilon(t,\cdot))$ be the unique mild solution of (5.1)-(5.2) associated with the initial condition $(H_{C0}, S_0, i_{P0}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times L^1_+((0,+\infty),\mathbb{R})$ such that*

$$H_{C0} \leq 1 \text{ and } S_0 + \int_0^\infty i_{P0}(a)da \leq N_P.$$

Let $t \in [0,+\infty) \to (S(t), i(t,\cdot))$ be the unique mild solution of the reduced system

$$\begin{cases} \dfrac{dS(t)}{dt} = \nu_P N_P - \nu_P S(t) - \beta_P H\left(\int_0^{+\infty} \gamma(a) i(t,a)da\right) S(t), \\ \dfrac{\partial i_P(t,a)}{\partial t} + \dfrac{\partial i_P(t,a)}{\partial a} = -\nu_P i_P(t,a), \\ i_P(t,0) = \beta_P H\left(\int_0^{+\infty} \gamma(a) i(t,a)da\right) S(t), \end{cases} \tag{5.14}$$

with initial conditions

$$S(0) = S_0 \text{ and } i_P(0,.) = i_{P0} \in L^1_+(0,+\infty),$$

and

$$H\left(\int_0^{+\infty} \gamma(a) i(t,a)da\right) = \frac{\beta_H \int_0^{+\infty} \gamma(a) i(t,a)da}{\nu_H + \beta_H \int_0^{+\infty} \gamma(a) i(t,a)da}.$$

Then for each $\tau > 0$ we have

$$\lim_{\varepsilon \to 0^+} \sup_{t \in [0,\tau]} \|(S^\varepsilon(t), i^\varepsilon(t,\cdot)) - (S(t), i(t,\cdot))\| = 0.$$

Moreover, for each $t_0 \in (0,\tau)$ we have

$$\lim_{\varepsilon \to 0^+} \sup_{t \in [t_0,\tau]} \left\| H_C^\varepsilon(t) - H\left(\int_0^{+\infty} \gamma(a) i^\varepsilon(t,a)da\right)\right\| = 0.$$

5.2 Application to the delay differential equations

In this section we will apply our result to the system of delay differential equation

$$\begin{cases} \varepsilon \dfrac{dx(t)}{dt} = f(t, x_{t,\varepsilon}, y_t), \\ \dfrac{dy(t)}{dt} = g(t, x_{t,\varepsilon}, y_t), \end{cases} \tag{5.15}$$

with initial condition

$$(x_{0,\varepsilon}, y_0) = \left(\overline{\varphi}(\varepsilon\cdot), \overline{\psi}\right) \in C([-r,0], \mathbb{R}^n) \times C([-r,0], \mathbb{R}^m).$$

In order to state our assumptions, let us formally set $\varepsilon = 0$ in system (5.15) in order to define the reduced problem

$$\begin{cases} 0 = f\left(t, x_{t,0}, y_t\right), \\ \dfrac{dy\left(t\right)}{dt} = g\left(t, x_{t,0}, y_t\right), \end{cases} \tag{5.16}$$

with the initial condition

$$y_0 = \overline{\psi} \in C\left(\left[-r, 0\right], \mathbb{R}^m\right).$$

In the rest of this section the parameter $\tau > 0$ is given and fixed. Then our assumptions concerning system (5.15) and (5.16) are the following.

Assumption 5.8 *Assume that*

(i) There exists a unique continuous solution $t \in [0, \tau] \to \left(x^{\varepsilon}_{t,\varepsilon}, y^{\varepsilon}_t\right) \in C\left(\left[-r, 0\right], \mathbb{R}^n\right) \times C\left(\left[-r, 0\right], \mathbb{R}^m\right)$ *of system (5.15) with*

$$x^{\varepsilon}_{t,\varepsilon}\left(\theta\right) = x^{\varepsilon}\left(t + \varepsilon\theta\right) \text{ and } y^{\varepsilon}_t\left(\theta\right) = y^{\varepsilon}\left(t + \theta\right), \ \forall t \in [0, \tau] \text{ and } \theta \in [-r, 0].$$

(ii) There exists a unique continuous solution $t \in [0, \tau] \to \left(x_{t,0}, y_t\right) \in C\left(\left[-r, 0\right], \mathbb{R}^n\right) \times C\left(\left[-r, 0\right], \mathbb{R}^m\right)$ *of system (5.16) with*

$$x_{t,0}\left(\theta\right) = x\left(t\right) \text{ and } y_t\left(\theta\right) = y\left(t + \theta\right), \ \forall t \in [0, \tau] \text{ and } \theta \in [-r, 0].$$

(iii) There exists a closed bounded subset $\widehat{\mathcal{M}}_1 \times \widehat{\mathcal{M}}_2 \subset C\left(\left[-r, 0\right], \mathbb{R}^n\right) \times C\left(\left[-r, 0\right], \mathbb{R}^m\right)$ *such that*

$$\left(x^{\varepsilon}_{t,\varepsilon}, y^{\varepsilon}_t\right) \in \widehat{\mathcal{M}}_1 \times \widehat{\mathcal{M}}_2 \text{ and } \left(x_{t,0}, y_t\right) \in \widehat{\mathcal{M}}_1 \times \widehat{\mathcal{M}}_2, \ \forall t \in [0, \tau] \text{ and } \varepsilon \in (0, 1].$$

*(iv) (**Equilibrium points**) There exists a uniformly bounded map* $\widehat{H} : [0, \tau] \times \widehat{\mathcal{M}}_2 \to \mathbb{R}^n$ *such that for each* $(t, \psi) \in [0, \tau] \times \widehat{\mathcal{M}}_2$ *there exists a unique* $\widehat{H}\left(t, \psi\right) \in \mathbb{R}^n$ *with*

$$0 = f\left(t, \widehat{H}\left(t, \psi\right) \mathbb{1}_{[-r,0]}(\cdot), \psi\right).$$

*(v) (**Global exponential stability**) For each* $\psi \in \widehat{\mathcal{M}}_2$ *and* $t \in [0, \tau]$ *the Cauchy problem*

$$\frac{dx\left(s\right)}{ds} = f\left(t, x_s, \psi\right), \ x_0 = \varphi \in C\left(\left[-r, 0\right], \mathbb{R}^n\right), \tag{5.17}$$

generates a unique semiflow $\left\{U_{\psi,t}\left(s\right)\right\}_{s \geq 0}$ *on* $C\left(\left[-r, 0\right], \mathbb{R}^n\right)$. *Moreover there exists two constants* $\alpha > 0$ *and* $\kappa \geq 1$ *such that*

$$\left\| U_{\psi,t}\left(s\right)\varphi - \widehat{H}\left(t, \psi\right) \right\|_C \leq \kappa e^{-\alpha s} \left\| \varphi - \widehat{H}\left(t, \psi\right) \right\|_C, \ \forall s \geq 0.$$

In order to apply our result to system (5.15)-(5.16) we will re-write it as an autonomous non-densely defined Cauchy problem by adding the time t as a state variable.

Abstract reformulation: Define for each $s \in [0, \tau]$, $\theta \in [-r, 0]$ and $\varepsilon \in (0,1]$

$$i_\varepsilon (s, \theta) := x^\varepsilon (s + \varepsilon\theta), \ \ l_\varepsilon (s, \theta) := y^\varepsilon (s + \theta) \text{ and } t(s) := s.$$

Then $s \to (i_\varepsilon (s, .), l_\varepsilon (s, .), t(s))$ satisfies formally for each $s \in [0, \tau]$, $\theta \in [-r, 0]$ and $\varepsilon \in (0, 1]$ the following partial differential equation (see for instance [6])

$$\begin{cases} \varepsilon\dfrac{\partial i_\varepsilon (s,\theta)}{\partial s} - \dfrac{\partial i_\varepsilon (s,\theta)}{\partial \theta} = 0, \text{ for } t \geq 0 \text{ and } \theta \in [-r, 0], \\ \dfrac{\partial i_\varepsilon (s,0)}{\partial \theta} = f(t(s), i_\varepsilon (s,.), l_\varepsilon (s,.)), \\ \dfrac{dt(s)}{ds} = 1, \\ \dfrac{\partial l_\varepsilon (s,\theta)}{\partial s} - \dfrac{\partial l_\varepsilon (s,\theta)}{\partial \theta} = 0, \\ \dfrac{\partial l_\varepsilon (s,0)}{\partial \theta} = g(t(s), i_\varepsilon (s,.), l_\varepsilon (s,.)), \end{cases} \tag{5.18}$$

with the initial conditions

$$i_\varepsilon (0, \theta) = \overline{\varphi}(\varepsilon\theta), \ t(0) = 0, \ l_\varepsilon (0,\theta) = \overline{\psi}(\theta), \ \forall \theta \in [-r, 0].$$

Next we use the approach introduced in Liu, Magal and Ruan [18] to reformulate the above partial differential equations as an abstract Cauchy problem by defining the Banach spaces

$$X := \mathbb{R}^n \times C([-\tau, 0], \mathbb{R}^n) \text{ and } Y := \mathbb{R} \times \mathbb{R}^m \times C([-\tau, 0], \mathbb{R}^m),$$

endowed with the usual product norm. We also set

$$X_0 := \{0_{\mathbb{R}^n}\} \times C([-\tau, 0], \mathbb{R}^n) \text{ and } Y_0 := \mathbb{R} \times \{0_{\mathbb{R}^m}\} \times C([-\tau, 0], \mathbb{R}^m).$$

To re-write (5.18) as an autonomous non-densely defined Cauchy problem we introduce the linear operators $A : D(A) \to X$ and $B : D(B) \to Y$ given respectively by

$$A \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) \\ \varphi' \end{pmatrix} \text{ with } D(A) = \{0_{\mathbb{R}^n}\} \times C^1([-\tau, 0], \mathbb{R}^n), \tag{5.19}$$

and

$$B \begin{pmatrix} t \\ 0_{\mathbb{R}^m} \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ -\psi'(0) \\ \psi' \end{pmatrix} \text{ with } D(B) = \mathbb{R} \times \{0_{\mathbb{R}^m}\} \times C^1([-\tau, 0], \mathbb{R}^m). \tag{5.20}$$

Finally we introduce the non linear maps $F : X_0 \times Y_0 \to X$ and $G : X_0 \times Y_0 \to Y$ defined respectively by

$$F\left(\left(\begin{array}{c} 0_{\mathbb{R}^n} \\ \varphi \end{array}\right), \left(\begin{array}{c} t \\ 0_{\mathbb{R}^m} \\ \psi \end{array}\right)\right) = \left(\begin{array}{c} f(t, \varphi, \psi) \\ 0_C \end{array}\right),$$

and

$$G\left(\left(\begin{array}{c} 0_{\mathbb{R}^n} \\ \varphi \end{array}\right), \left(\begin{array}{c} t \\ 0_{\mathbb{R}^m} \\ \psi \end{array}\right)\right) = \left(\begin{array}{c} 1 \\ g(t, \varphi, \psi) \\ 0_C \end{array}\right).$$

Hence by setting

$$u_\varepsilon(s) := \left(\begin{array}{c} 0_{\mathbb{R}^n} \\ i_\varepsilon(s, .) \end{array}\right) \text{ and } v_\varepsilon(s) := \left(\begin{array}{c} t(s) \\ 0_{\mathbb{R}^m} \\ l(s, .) \end{array}\right), \ t \geq 0$$

and

$$x_\varepsilon = \left(\begin{array}{c} 0_{\mathbb{R}^n} \\ \overline{\varphi}(\varepsilon \cdot) \end{array}\right) \text{ and } y_0 = \left(\begin{array}{c} 0 \\ 0_{\mathbb{R}^m} \\ \overline{\psi} \end{array}\right), \ \forall \varepsilon \in (0, 1],$$

we formally obtain the following autonomous non-densely defined Cauchy system

$$\begin{cases} \varepsilon \dfrac{du_\varepsilon(s)}{ds} = Au_\varepsilon(s) + F(u_\varepsilon(s), v_\varepsilon(s)), \ s \in [0, \tau], \\ \dfrac{dv_\varepsilon(s)}{ds} = Bv_\varepsilon(s) + G(u_\varepsilon(s), v_\varepsilon(s)), \ s \in [0, \tau], \end{cases} \tag{5.21}$$

with

$$u_\varepsilon(0) = x_\varepsilon \text{ and } v_\varepsilon(0) = y_0, \ \forall \varepsilon \in (0, 1].$$

The relationship between system (5.15) and (5.21) is given by the following lemma.

Lemma 5.9 *Let Assumption 5.8 be satisfied. Then system (5.21) admits a unique integrated solution $s \to (u_\varepsilon(s), v_\varepsilon(s))$ defined on $[0, \tau]$. Moreover for each $\varepsilon \in (0, 1]$, $s \in [0, \tau]$ and $\theta \in [-r, 0]$ we have the following relationship*

$$u_\varepsilon(s) = \left(\begin{array}{c} 0_{\mathbb{R}^n} \\ \widehat{u}_\varepsilon(s)(.) \end{array}\right) \textit{ and } v_\varepsilon(s) = \left(\begin{array}{c} s \\ 0_{\mathbb{R}^m} \\ \widehat{v}_\varepsilon(s)(.) \end{array}\right),$$

with

$$u_\varepsilon(s)(\theta) = \begin{cases} x^\varepsilon(s + \varepsilon\theta), \ \textit{if } s + \varepsilon\theta \geq 0, \\ \overline{\varphi}(s + \varepsilon\theta), \ \textit{if } s + \varepsilon\theta \leq 0, \end{cases} \textit{ and } \widehat{v}_\varepsilon(s)(\theta) = \begin{cases} y^\varepsilon(s + \theta), \ \textit{if } s + \theta \geq 0, \\ \overline{\psi}(s + \theta), \ \textit{if } s + \theta \leq 0, \end{cases}$$

while

$$x^{\varepsilon}(s) = \begin{cases} \overline{\varphi}(0) + \int_0^s \varepsilon^{-1} f\left(l, x_l^{\varepsilon}, y_l^{\varepsilon}\right) dl, & \text{if } s \geq 0, \\ \\ \overline{\varphi}(\varepsilon s), & \text{if } -r \leq s \leq 0. \end{cases}$$

and

$$y^{\varepsilon}(s) = \begin{cases} \overline{\psi}(0) + \int_0^s g\left(l, x_l^{\varepsilon}, y_l^{\varepsilon}\right) dl, & \text{if } s \geq 0, \\ \\ \overline{\psi}(s), & \text{if } -r \leq s \leq 0. \end{cases}$$

Checking the assumptions: It is important to observe that for system (5.21) we have

$$L = I_{\mathcal{L}(Y)} \text{ and } Z = Y.$$

Due to condition *(iv)* of Assumption 5.8 one can define the set $\mathcal{M}_Z$ as

$$\mathcal{M}_Z := [0, \tau] \times \{0_{\mathbb{R}^n}\} \times \widehat{\mathcal{M}}_2,$$

and the map $H : \mathcal{M}_Z \to D(A)$ by

$$H \begin{pmatrix} t \\ 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{H}(t, \psi) \mathbb{1}_{[-r,0]}(\cdot) \end{pmatrix},$$

where $\widehat{H}(t, \psi)$ is defined in *(iv)* of Assumption 5.8. Then for each $z := \begin{pmatrix} t \\ 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} \in \mathcal{M}_Z$ we have $H(z) \in D(A)$ and

$$AH(z) + F(H(z), z) = 0, \ \forall z \in \mathcal{M}_Z.$$

Furthermore the boundedness of H into $\mathcal{M}_Z$ follows directly from condition *(iv)* in Assumption 5.8. Next note that the unbounded linear operators A and B are Hille-Yosida operators. We refer for instance to [6] where such verification were already done. Now consider the following parametrized Cauchy problem

$$\frac{du(s)}{ds} = Au(s) + F(u(s), z), \ s \geq 0, \ u(0) = u_0 \in X_0 \text{ and } z \in \mathcal{M}_Z. \quad (5.22)$$

Recall that $z := \begin{pmatrix} t \\ 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} \in \mathcal{M}_Z$ and this last equation corresponds to the solution of the delay differential equation (5.17) where t and ϕ are regarded as parameters of this last equation. Then the following lemma holds true.

Lemma 5.10 *Let Assumption 5.8 be satisfied. Then system (5.22) generates a unique globally defined nonlinear flow $\{U_z(s)\}_{s \geq 0}$ on X_0. Furthermore for all $x \in X_0$ and $z \in \mathcal{M}_Z$ one has*

$$\|U_z(s)x - H(z)\| \leq \kappa e^{-\alpha s} \|x - H(z)\|, \ \forall s \geq 0.$$

Proof. The proof of this lemma follows directly from condition *(v)* stated in Assumption 5.8. ■

Next define the sets

$$\mathcal{M} := \widehat{\mathcal{M}}_1 \times \widehat{\mathcal{M}}_2 \text{ and } \mathcal{M}_0 := \left\{ \left(\left(\begin{array}{c} 0_{\mathbb{R}^n} \\ \overline{\varphi}(\varepsilon \cdot) \end{array} \right), \left(\begin{array}{c} 0_{\mathbb{R}} \\ 0_{\mathbb{R}^m} \\ \overline{\psi} \end{array} \right) \right) : \varepsilon \in [0,1] \right\}, \quad (5.23)$$

and observe that due to condition *(iii)* in Assumption 5.8 one has

$$(u_\varepsilon(s), v_\varepsilon(s)) \in \mathcal{M}, \ \forall s \in [0,\tau] \text{ and } \varepsilon \in (0,1].$$

Finally it remains to verify the regularity assumption. To do so let $\{T_{B_0}(s)\}_{s\geq 0} \subset \mathcal{L}(Y_0)$ be the strongly continuous semigroup generated by B_0 the part of B (defined in 5.20) in Y_0. Let $\{S_B(s)\}_{s\geq 0}$ be the integrated semigroup generated by the linear operator B. Then we have the following result

Lemma 5.11 *Let Assumption 5.8 be satisfied. Then*

i) The map $t \to T_{B_0}(t)y_0$ is uniformly continuous from $[0,\tau]$ into Y.

ii) Let $Y_G := \mathbb{R} \times \mathbb{R}^m \times \{0_C\}$ be the closed subspace containing $G(X_0 \times Y_0)$. Then the family of maps

$$\left\{ (S_B \diamond g)(.) : g \in L^\infty(0,\tau; Y_G) \text{ with } \sup_{s\in[0,\tau]} \|g(s)\|_Y \leq 1 \right\},$$

is uniformly equicontinuous from $[0,\tau]$ into Y.

Proof. The property *i)* is a direct consequence of the strong continuity of T_{B_0}. *Proof of ii).* Let $g \in L^\infty(0,\tau; Y_G)$ with $\sup_{s\in[0,\tau]} \|g(s)\|_Y \leq 1$. Since $g(s) \in Y_G$ for all $s \in [0,\tau]$ one can define

$$g(s) := \left(\begin{array}{c} \beta \\ \chi(s) \\ 0_C \end{array} \right) \in Y_G = \mathbb{R} \times \mathbb{R}^m \times \{0_C\},$$

with

$$|\beta| + \sup_{s\in[0,\tau]} \|\chi(s)\|_{\mathbb{R}^n} \leq 1.$$

Then using the results in [18], the map $s \to (S_B \diamond g)(s) \in C([-r,0],\mathbb{R}^m)$ is given by

$$(S_B \diamond g)(t) = \left(\begin{array}{c} \beta t \\ 0_{\mathbb{R}^m} \\ u(t,\cdot) \end{array} \right),$$

with

$$u(t,\theta) = \begin{cases} \int_0^{t+\theta} \chi(l)dl, & \text{if } t+\theta \geq 0, \\ 0, & \text{if } -r \leq t+\theta < 0. \end{cases}$$

Then for each $h \geq 0$ and each $t \in [0, \tau]$ with $t + h \in [0, \tau]$ we have

$$\|(S_B \diamond g)(t) - (S_B \diamond g)(t+h)\|_Y \leq h + \sup_{\theta \in [-r,0]} \|u(t,\theta) - u(t+h,\theta)\|_{\mathbb{R}^m}.$$

However one has

$$u(t+h,\theta) - u(t,\theta) = \begin{cases} \int_0^{t+h+\theta} \chi(l)dl - \int_0^{t+\theta} \chi(l)dl, \text{ if } t+\theta \geq 0, \\ \\ \int_0^{t+h+\theta} \chi(l)dl, \text{ if } t+\theta \leq 0 \text{ and } t+h+\theta \geq 0, \\ \\ 0, \text{ otherwise }, \end{cases}$$

therefore we conclude that

$$\|(S_B \diamond g)(t) - (S_B \diamond g)(t+h)\|_Y \leq 2h.$$

This completes the proof of the lemma. ■

Convergence result: With the above verifications we are ready to state our convergence result for the delay differential equation (5.15). By using the definition of $\mathcal{M}_0$ and $\mathcal{M}$ in (5.23), Corollary 2.11 applies to system (5.21) and provides the following result.

Theorem 5.12 *Let Assumption 5.8 be satisfied. Let $\eta \in (0, \tau)$ be given. Then we have*

$$\sup_{t \in [\eta,\tau]} \|x^{\varepsilon}_{t,\varepsilon} - \widehat{H}\,(t, y^{\varepsilon}_t)\, \mathbb{1}_{[-r,0]}(\cdot)\|_{\infty} \to 0 \text{ as } \varepsilon \to 0.$$

Assume in addition that for each $\psi \in \widehat{\mathcal{M}_2} \subset C\left([-r,0], \mathbb{R}^m\right)$, there exists $y \in C([-r,\tau], \mathbb{R}^m)$ a solution of the reduced equation

$$\frac{dy\,(t)}{dt} = g\left(t, \widehat{H}\,(t, y_t)\, \mathbb{1}_{[-r,0]}(\cdot), y_t\right), \forall t \in [0,\tau], \text{ and } y_0 = \psi.$$

Then we obtain

$$\sup_{t \in [0,\tau]} |y^{\varepsilon}(t) - y(t)| \to 0 \text{ as } \varepsilon \to 0.$$

References

[1] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems*, Birkhauser, Basel, 2001.

[2] O. Arino, E. Sanchez, R. Bravo de la Parra and P. Auger, A singular perturbation in an age-structured population model, *SIAM Journal on Applied Mathematics* **60** (1999), 408–436.

[3] Z. Artstein and M. Slemrod, On Singularly Perturbed Retarded Functional Differential Equations, *Journal of Differential Equations* **171** (2001), 88–109.

[4] R. Buckdahn and G. Guatteri, A stochastic Tikhonov theorem in infinite dimensions, *Applied Mathematics and Optimization* **53** (2006), 221–258.

[5] E.M.C. D'Agata, P. Magal, D. Olivier, S. Ruan and G.F. Webb, Modeling Antibiotic Resistance in Hospitals: The Impact of Minimizing Treatment Duration, *Journal of Theoretical Biology* **249** (2007), 487–499.

[6] A. Ducrot, Z. Liu and P. Magal, Projectors on the Generalized Eigenspaces for Neutral Functional Differential Equations in L^p Spaces, *Canadian Journal of Mathematics* **62** (2010), 74–93.

[7] A. Ducrot, P. Magal and O. Seydi, A singularly perturbed Delay Differential Equation modeling nosocomial infections, *Differential and Integral Equations,* to appear.

[8] A. Ducrot, P. Magal and O. Seydi, Nonlinear boundary conditions derived by singular pertubation in age structured population dynamics model, *Journal of Applied Analysis and Computation* **1** (2011), 373–395.

[9] E. Fridman, Decoupling Transformation of Singularly Perturbed Systems with Small Delays and Its Applications, *Zeitschrift für angewandte Mathematik und Mechanik: Ingenieurwissenschaftliche Forschungsarbeiten* **76** (1996), 201–204.

[10] V. Y. Glizer, Controllability of Nonstandard Singularly Perturbed Systems With Small State Delay, *IEEE Transaction on Automatic Control* **48** (2003), 1280–1285.

[11] G. Greiner, J.A.P. Heesterbeek and J.A.J. Metz, A singular perturbation theorem for evolution equations and time-scale arguments for structured population models, *Canadian Applied Mathematics Quarterly* **2** (1994), 435–459.

[12] J. K. Hale, Some problems in FDE. Functional Differential and Difference Equations, *Fields Institute Communications Series, American Mathematical Society, Providence, RI,* (2001), 195–222.

[13] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[14] D. Henry, *Geometric theory of semilinear parabolic equations*, Springer-Verlag, Berlin Heidelberg New York, 1981.

[15] M. Iannelli, *Mathematical Theory of Age-Structured Population Dynamics*, Giadini Editorie Stampatori in Pisa, 1994.

[16] M. Krupa and J. D. Touboul, Canard Explosion in Delay Differential Equations, *Journal of Dynamics and Differential Equations*, to appear.

[17] A. Ducrot, Z. Liu and P. Magal, Projectors on the Generalized Eigenspaces for Neutral Functional Differential Equations in L^p Spaces, *Canadian Journal of Mathematics* **62** (2010), 74–93.

[18] Z. Liu, P. Magal and S. Ruan, Projectors on the generalized eigenspaces for functional differential equations using integrated semigroups, *Journal of Differential Equations* **244** (2008), 1784–1809.

[19] P. Magal, Compact attractors for time periodic age-structured population models, *Electronic Journal of Differential Equations* **2001** (2001), 1–35.

[20] P. Magal and S. Ruan, On Integrated Semigroups and Age Structured Models in L^p Spaces, *Differential and Integral Equations* **20** (2007), 197–139.

[21] P. Magal and S. Ruan, On Semilinear Cauchy Problems with Non-dense Domain, *Advances in Differential Equations* **14** (2009), 1041–1084.

[22] P. Magal and S. Ruan, Center manifolds for semilinear equations with non-dense domain and applications to Hopf bifurcation in age structured models, Mem. Amer. Math. Soc., **202** (2009), no. 951.

[23] P. Magal and C.C. McCluskey, Two group infection age model: an application to nosocomial infection, *SIAM Journal on Applied Mathematics* **73** (2013), 1058–1095.

[24] L. T. Magalhães, Presistence and Smoothness of Hyperbolic Invariant Manifolds for Functional Differential Equations, *SIAM Journal on Mathematical Analysis* **18** (1987), 670–693.

[25] L. T. Magalhães, Convergence and boundary layers in singularly perturbed linear functional differential equations, *Journal of differential equations* **54** (1984), 295–309.

[26] L. T. Magalhães, Exponential estimates for singularly perturbed linear fuctional differential equations, *Journal of mathematical analysis and applications* **103** (1984), 443–460.

[27] S. Nagarajan, R. Narasimhan, J. J. H. Miller and V. Sigamani, A Parameter Uniform Method for an Initial Value Problem for a System of Singularly Perturbed Delay Differential Equations, *Advances in Applied Mathematics* **87** (2014), 127–138.

[28] H. R. Thieme, Semiflows generated by Lipschitz perturbations of non-densely defined operators, *Differential Integral Equations*, **3** (1990), 1035–1066.

[29] A. N. Tikhonov, A. B. Vasileva and A. G. Sveshnikov, *Differential Equations*, Springer, Berlin, 1985.

[30] A. N. Tikhonov. On system of differential equations containing parameters, *Mat. Sb.* **27** (1950), 147–156.

[31] A. N. Tikhonov. System of differential equations containing a small parameters at derivatives. *Mat.* Sbornik **31** (1952), 575–586.

[32] F. Verhulst, *Methods and Applications of Singular Perturbations*, Springer, New York, 2005.

[33] K. Wang, J. Peng and D. Jin, A new development for the Tikhonov Theorem in nonlinear singular perturbation systems, *Nonlinear Analysis: Theory, Methods and Applications* **74** (2011), 2869–2879.

[34] G. F. Webb, *Theory of Nonlinear Age-Dependent Population Dynamics*, Marcel Dekker, New York, 1985.