HOPF BIFURCATION IN AN AGE-STRUCTURED POPULATION MODEL WITH TWO DELAYS

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ABSTRACT. This paper is devoted to the study of an age-structured population system with Riker type birth function. Two time lag factors is considered for the model. One time lies in the birth process and the another is in the birth function. We investigate some dynamical properties of the equation by using integrated semigroup theory, through which we obtain some conditions of asymptotical stability and Hopf bifurcation occurring at positive steady state for the system. The obtained results show how the two delays affect these dynamical properties.

1. **Introduction.** In this work we study the problem of Hopf bifurcation in the following system with delayed birth process,

$$
\begin{cases}
\dfrac{\partial}{\partial t}u(t,a) = -\dfrac{\partial}{\partial a}u(t,a) - \mu u(t,a), & t > 0, \quad a > 0, \\
u(t,0) = \alpha h\left(\displaystyle\int_{-\tau_1}^{0} d\xi(\theta) \int_{0}^{+\infty} \beta(a)u(t+\theta,a)da\right), & t > 0, \\
u(s,a) = u_0(s,a), & s \in [-\tau_1, 0], \quad a > 0.
\end{cases}
\tag{1.1}
$$

where $u(t,a)$ represents the population density of certain species at time t with age a, $\mu > 0$, $\alpha > 0$, $\beta(\cdot) \in L_+^\infty(0,+\infty)$, $\xi(\cdot): [-\tau_1, 0] \to \mathbb{R}$ is a function of bounded variation, and the map $h: \mathbb{R} \to \mathbb{R}$ is defined by

$$
h(x) = xe^{-\gamma x}, \text{ for any } x \in \mathbb{R}.
\tag{1.2}
$$

The model (1.1) is viewed as an age(size)-structured population system (with delayed birth process), for example for the growth of trees or fish population,

where $a = 0$ is the minimal size. The growth of individuals is described by the term $\frac{\partial}{\partial a} u(t, a)$, which represents the average growth rate of individuals. The term $-\mu(t, a)$ describes the mortality process of individuals following an exponential law with mean $1/\mu$. The birth function given by $\alpha h(\int_0^{+\infty} \beta(a)u(t, a)da)$ is a Ricker type birth function ([30, 31]). This type of birth function has been commonly used in the literatures, to take into account some limitation of births when the population increases. In particular, the birth rate function is $\beta(a)$ when the total population is close to zero. We refer to Arino [4], Arino and Sanchez [5], Calsina and Saldana [9], Calsina and Sanchon [10], Webb [35], and Ackleh and Deng [1] (and references therein) for studies on age-structured models in the context of ecology and cell population dynamics.

As pointed out in [26], the existence of non-trivial periodic solutions in age structured models has been a very interesting and difficult problem. It is believed that such periodic solutions in age structured models are induced by Hopf bifurcation, but there is no general Hopf bifurcation theorem available for such models. Recently a center manifold theory has been developed for non-densely defined Cauchy problems in Magal and Ruan [26]. This center manifold theory allows us to obtain an abstract Hopf bifurcation theorem (see Liu, Magal and Ruan [21]). This Hopf bifurcation theorem has been successfully applied in [26] to the system (1.1) when

$$\beta(a) = (a - \tau)^n e^{\gamma(a-\tau)} 1_{[\tau, +\infty)}(a) \tag{1.3}$$

and there is no delay in the birth process. In Paper [11] Chu, Ducrot, Magal and Ruan added a diffusion term $\epsilon^2 \frac{\partial^2}{\partial a^2} u(t, a)$ in this model to describe the stochastic fluctuations around the tendency to growth. They have investigated how the diffusion rate ϵ^2 influences the stability and the Hopf bifurcation of the positive equilibrium of the system. In addition, as it can be seen there, the formula (1.3) indicates that there involves a time lag for the diffusion of the species. There are other works on Hopf bifurcation for related models by applying the center manifold theorem, see [27] and [32], for instance.

The aim of this paper is to consider the Hopf bifurcation problem for System (1.1) under two delays. one delay is for the birth process denoting that the birth process depends on the past population. In fact, there is often a time lag between conception and birth like in the models for host-parasite interactions (see [8]). The another delay appears in the birth function $\beta(\cdot)$ like in (6.1). This, as shown in [11], reflects the time lag in the population of the species. It will be shown in our results how the two delays affect the stability and Hopf bifurcations phenomenon for System (1.1).

This paper is organized as: In Section 2, we collect some notations and results on theory of C_0−semigroup and integrated semigroup which will be used in the later sections. In Section 3, we reformulate (1.1) as a non-densely defined Cauchy problem, and show the existence and uniqueness of solutions to this non-densely defined Cauchy problem. The positive equilibrium of the system is studied in Section 4. Then we linearize here the system at the positive equilibrium and discuss the spectral properties of the linearized equation. The characteristic equation is also given in this section. Based on the work of Section 4, in Section 5, we present the results of stability and Hopf bifurcation of the system (1.1). In Section 6, considering α as a parameter, we study the existence of Hopf bifurcation for System (1.1) when $\beta(\cdot)$ is defined by (6.1). Finally, in Section 7 we present some numerical simulations of the model to illustrate the obtained results.

2. **Preliminary.** Firstly we state in this section some notations and basic facts from the theory of semigroups and their asymptotic properties. For a closed linear operator $(A, D(A))$ acting on a Banach space X, we denote by $\sigma(A)$ the spectrum of A, i.e.

$$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda - A : D(A) \to X \text{ is not bijective}\}.$$

The spectral bound of A, $s(A)$, is defined as

$$s(A) = \sup\{\text{Re}\lambda : \lambda \in \sigma(A)\}.$$

The set

$$\rho(A) := \mathbb{C} \setminus \sigma(A)$$

i.e. the complement of $\sigma(A)$ in $\mathbb{C}$ is called the resolvent of operator A. Hence, for every $\lambda \in \rho(A)$, $R(\lambda, A) = (\lambda - A)^{-1}$ exists and is continuous operator on X.

If, moreover, A is the generator of a strongly continuous semigroup e^{tA}, a crucial quantity associated to it is the growth bound type of e^{tA}, $\omega_0(A)$, which is defined as

$$\omega_0(A) := \lim_{t \to +\infty} \frac{\ln\|e^{tA}\|}{t}.$$

If $\omega_0(A) < 0$, then there exist $\gamma > 0$ and $M \geq 1$ such that $\|e^{tA}x_0\| \leq Me^{-\gamma t}$ for each $t \geq 0$ and any $x_0 \in X$.

The essential growth bound $\omega_{ess}(A)$ of A is defined by

$$\omega_{0,ess}(A) := \lim_{t \to +\infty} \frac{\ln(\|e^{tA}\|_{ess})}{t},$$

where $\|e^{tA}\|_{ess}$ is the essential norm of e^{tA} given by

$$\|e^{tA}\|_{ess} = \kappa\left(e^{tA}B(0,1)\right),$$

here $B(0,1) = \{x \in X : \|x\| \leq 1\}$, and $\kappa(\cdot)$ denotes the Kuratovsky measure of non-compactness. It is known that (cf. [16, 35])

$$\omega_0(A) = \max\{\omega_{ess}(A), \ s(A)\}.$$

When an operator A is not densely defined but, nevertheless, satisfies the necessary condition on the resolvent in order to be a generator, it is called a Hille-Yosida operator. More precisely, we have the following definition:

Definition 2.1. A linear operator $(A, D(A))$ acting on a Banach space X is called to be a Hille-Yosida operator if there exists $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$\sup\left\{\|(\lambda - \omega)^n(\lambda - A)^{-n}\| : \lambda > \omega, \ n \in \mathbb{N}\right\} < +\infty.$$

If the constant ω can be chosen smaller than 0, then A is said of negative type.

Any Hille-Yosida operator gives rise to a strongly continuous semigroup on the closure of the domain, that is, (cf. [29])

Proposition 1. *Let $(A, D(A))$ be a Hille-Yosida operator on Banach space X and set $X_0 := (\overline{D(A)}, \|\cdot\|)$. then the part of A in X_0, A_0, which is defined as*

$$A_0x := Ax, \ \ for \ x \in D(A_0),$$

with the domain

$$D(A_0) = \{x \in D(A) : AX \in X_0)\},$$

generates a strongly continuous semigroup on X_0. Moreover, $\rho(A) \subseteq \rho(A_0)$ and $(\lambda - A_0)^{-n}$ is the restriction of $(\lambda - A)^{-n}$ to X_0

Meanwhile, a Hille-Yosida operator generates an integrated semigroup on space X. We end this section by stating some notations and basic results on integrated semigroup. For the theory of integrated semigroup we refer to Arendt [2], Thieme [34], Kellermann and Hieber [19], and the book of Arendt et al. [3] for details on this subject

Definition 2.2. Let X be a Banach space. An integrated semigroup is a family $(S(t))_{t\geq 0}$ of bounded linear operators $S(t)$ on X with the following properties:

(i) $S(0) = 0$;

(ii) $t \to S(t)$ is strongly continuos;

(ii) $S(t)S(s) = \int_0^t [S(s+\tau) - S(\tau)]d\tau$, for all $t, s \geq 0$.

Definition 2.3. An operator A is called to be the generator of an integrated semigroup if there exits $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and there exists a strongly continuous exponently bounded family $(S(t))_{t\geq 0}$ of bounded linear operators such that $S(0) = 0$ and $(\lambda I - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t)dt$ exists for all λ with $\lambda > \omega$.

Proposition 2. *Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then for all $x \in X$ and $t \geq 0$,*

$$\int_0^t S(s)xds \in D(A), \quad and \quad S(t)x = A\int_0^t S(s)xds + tx.$$

Definition 2.4. (i) An integrated semigroup $(S(t))_{t\geq 0}$ is called Locally Lipschitz continuous, if, for all $\tau > 0$, there exists a constant $L > 0$ such that

$$\|S(t) - S(s)\| \leq L|t - s|, \quad t, s \in [0, \tau].$$

(ii) An integrated semigroup $(S(t))_{t\geq 0}$ is called nondegenerate if $S(t)x = 0$ for all $t \geq 0$ implies that $x = 0$.

Proposition 3. *The following assertions are equivalent:*

(i) A is the generator of non-degenerate, locally Lipschitz continuous integrated semigroup;

(ii) A satisfies the Hille-Yosida condition.

For the existence of solutions of the following Cauchy problem:

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + f(t), & t > 0, \\ x(0) = x_0 \in X, \end{cases}$$

where A satisfies the Hille-Yosida condition without being densely defined, one has

Proposition 4. *Let $f : [0, a] \to X$ is a continuous function. Then for $y_0 \in \overline{D(A)}$, there is a unique continuous function $y : [0, a] \to X$ such that*

(i) $\int_0^t y(s)ds \in D(A)$, $t \in [0.a]$;

(ii) $y(t) = y_0 + A\int_0^t y(s)ds + \int_0^t f(s)ds$, $t \in [0.a]$;

(ii) $\|y(t)\| \leq Me^{\omega t}[\|y_0\| + \int_0^t e^{-\omega s}\|f(s)\|ds]$, $t \in [0.a]$.

3. **Abstract equations.** We will use the integrated semigroup theory to study such a PDE (1.1), so in the sequel, we rewrite (1.1) into an abstract non-densely defined Cauchy problem by several steps.

Consider the space

$$Y := \mathbb{R} \times L^1(0, +\infty)$$

endowed with the usual product norm

$$\left(\begin{array}{c} \alpha \\ \varphi \end{array} \right) = |\alpha| + \|\varphi\|_{L^1}.$$

Define the linear operator $B : D(B) \subset Y \to Y$ by

$$B\left(\begin{array}{c} 0 \\ \varphi \end{array} \right) = \left(\begin{array}{c} -\varphi(0) \\ -\varphi' - \mu\varphi \end{array} \right)$$

with the domain

$$D(B) = \{0\} \times W^{1,1}(0, +\infty).$$

Then

$$Y_0 := \overline{D(B)} = \{0\} \times L^1(0, +\infty).$$

Let

$$C_B = \left\{ \left(\begin{array}{c} \alpha(\cdot) \\ \phi(\cdot) \end{array} \right) \in C([-\tau_1, 0], Y) : \ \alpha(0) = 0 \right\}.$$

Define the operators $H : Y_0 \to Y$ by

$$H\left(\left(\begin{array}{c} 0 \\ \varphi \end{array} \right) \right) = \left(\begin{array}{c} \alpha h \left(\int_0^{+\infty} \beta(a)\varphi(a)da \right) \\ 0 \end{array} \right),$$

and $\widehat{H} : C_B \to Y$ by

$$\widehat{H}\left(\left(\begin{array}{c} \alpha(\cdot) \\ \phi(\cdot) \end{array} \right) \right) = H\left(\left(\begin{array}{c} 0 \\ \int_{-\tau_1}^0 d\xi(\theta)\phi(\theta)(a) \end{array} \right) \right)$$

$$= \left(\begin{array}{c} \alpha h \left(\int_{-\tau_1}^0 d\xi(\theta) \int_0^{+\infty} \beta(a)\phi(\theta)(a)da \right) \\ 0 \end{array} \right).$$

Then by identifying $u(t)$ with $u(t, a)$ and $y(t) = \left(\begin{array}{c} 0 \\ u(t) \end{array} \right)$ the equation (1.1) can be rewritten as the following Cauchy problem

$$\left\{ \begin{array}{ll} \dfrac{d}{dt}y(t) = By(t) + \widehat{H}(y_t), & t > 0, \\ y(\theta) = y_0(\theta) \in C_B, & \end{array} \right. \tag{3.1}$$

here $y_t = y_t(\theta) = y(t + \theta) \in C_B$ and $y_0(\theta) = \left(\begin{array}{c} 0 \\ u_0(\theta, \cdot) \end{array} \right).$

This is a abstract (non-densely defined) functional differential equation. To apply the integrated semigroup theory we need further to rewrite (3.1) to an abstract Cauchy problem of ODE.

Define $v \in C([0, +\infty) \times [-\tau_1, 0]; Y)$ by

$$v(t, \theta) = y(t + \theta), \text{ for any } t \geq 0 \text{ and } \theta \in [-\tau_1, 0].$$

Note that

$$\frac{\partial v(t, \theta)}{\partial t} = y'(t + \theta) = \frac{\partial v(t, \theta)}{\partial \theta},$$

we must have

$$\frac{\partial v(t, \theta)}{\partial t} - \frac{\partial v(t, \theta)}{\partial \theta} = 0, \text{ for any } t \geq 0 \text{ and } \theta \in [-\tau_1, 0].$$

Moreover, for $\theta = 0$, we obtain

$$\frac{\partial v(t, 0)}{\partial \theta} = y'(t) = By(t) + \widehat{H}(y_t) = Bv(t, 0) + \widehat{H}(v(t, \cdot)), \qquad t \geq 0.$$

Therefore, we deduce formally that v must satisfy a PDE

$$\begin{cases} \dfrac{\partial v(t,\theta)}{\partial t} - \dfrac{\partial v(t,\theta)}{\partial \theta} = 0, \\ \dfrac{\partial v(t,0)}{\partial \theta} = Bv(t,0) + \widehat{H}\left(v(t,\cdot)\right), \qquad t \geq 0, \\ v(0,\cdot) = y_0(\cdot) \in C_B, \end{cases} \tag{3.2}$$

In order to rewrite the PDE (3.2) as an abstract non-densely defined Cauchy problem, we extend the state space to take into account the boundary condition. This can be accomplished by adopting the following state space

$$X = Y \times C([-\tau_1, 0]; Y)$$

taken with the usual product norm

$$\left(\begin{array}{c} f \\ \phi \end{array} \right) = \|f\|_Y + \|\phi\|_C.$$

Define the linear operator $A : D(A) \subset X \to X$ by

$$A\left(\begin{array}{c} 0_Y \\ \phi \end{array} \right) = \left(\begin{array}{c} -\phi'(0) + B\phi(0) \\ \phi' \end{array} \right), \quad \text{for any } \left(\begin{array}{c} 0_Y \\ \phi \end{array} \right) \in D(A),$$

where the domain

$$D(A) = \{0\} \times \left\{ \phi \in C^1([-\tau_1, 0]; Y), \ \phi(0) \in D(B) \right\}.$$

Note that A is also non-densely defined because

$$X_0 := \overline{D(A)} = \{0_Y\} \times C_B \neq X.$$

We define $\widetilde{H} : X_0 \to X$ by

$$\widetilde{H}\left(\begin{array}{c} 0_Y \\ \phi \end{array} \right) = \left(\begin{array}{c} \widehat{H}(\phi) \\ 0_C \end{array} \right).$$

Finally, set

$$x(t) := \left(\begin{array}{c} 0 \\ v(t) \end{array} \right).$$

Then we can consider the PDE (3.2) as the following non-densely defined Cauchy problem

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + \widetilde{H}\left(x(t)\right), \qquad t > 0, \\ x(0) = \left(\begin{array}{c} 0_Y \\ y_0 \end{array} \right) \in X_0, \end{cases} \tag{3.3}$$

Putting

$$\Omega = \left\{ \lambda \in \mathbb{C} : \ \mathrm{Re}\lambda > -\mu \right\}.$$

Then we have the following

Theorem 3.1. *For the operators A and B defined above, there hold that*

(i) For each $\lambda \in \Omega$ with $\mathrm{Re}\lambda > -\mu$, one has $\lambda \in \rho(B)$ and

$$(\lambda - B)^{-1}\left(\begin{array}{c} \alpha \\ \psi \end{array} \right) = \left(\begin{array}{c} 0 \\ \varphi \end{array} \right) \Leftrightarrow \varphi(a) = \alpha e^{-(\lambda+\mu)a} + \int_0^a e^{-(\lambda+\mu)(a-l)}\psi(l)dl;$$

(ii) $\rho(A) = \rho(B)$. Moreover, for each $\lambda \in \rho(A)$, we also have the following explicit formula for the resolvent of A:

$$(\lambda - A)^{-1}\left(\begin{array}{c} f \\ \psi \end{array} \right) = \left(\begin{array}{c} 0_Y \\ \phi \end{array} \right) \Leftrightarrow \phi(a) = e^{\lambda\theta}(\lambda - B)^{-1}[\psi(0) + f] + \int_\theta^0 e^{\lambda(\theta-s)}\psi(s)ds;$$

(iii) The operators B and A are Hille-Yosida operators on Y and X, respectively.

Proof. Assertion (ii) and (iii) are proved in Theorem 3.5 and Lemma 3.6 of [15], and Assertion (i) is proved in Section 6 of [15]. $\qquad\square$

By using the results in Thieme [33], Magal [23], and Magal and Ruan [25], we have the following theorem.

Theorem 3.2 (Existence). *There exists a unique continuous semiflow $X(t)_{t\geq 0}$ on X_{0+} such that for any $x \in X_{0+}$, $t \to X(t)x$ is the unique integrated solution of*

$$\begin{cases} \dfrac{dX(t)x}{dt} = AX(t)x + \widetilde{H}(X(t)x) \\ X(0)x = x \end{cases}$$

or equivalently,

$$X(t)x = x + A \int_0^t X(l)x\,dl + \int_0^t \widetilde{H}(X(l)x)\,dl, \ t \geq 0.$$

Furthermore, since B and A are Hille-Yosida operators, they generate non-degenerated integrated semigroups $(S_B(t))_{t\geq 0}$ and $(S_A(t))_{t\geq 0}$ on Y and X respectively. Introduce their parts B_0 and A_0 on $\overline{Y_0}$ and X_0 respectively, namely,

$$B_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = B \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$

with $D(B_0) = \left\{ \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in D(B) : \ B \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in Y_0 \right\}$, and

$$A_0 \begin{pmatrix} 0_Y \\ \phi \end{pmatrix} = A \begin{pmatrix} 0_Y \\ \phi \end{pmatrix}$$

with

$$D(A_0) = \left\{ \begin{pmatrix} 0_Y \\ \phi \end{pmatrix} \in D(A) : \ A \begin{pmatrix} 0_Y \\ \phi \end{pmatrix} \in X_0 \right\}.$$

Then the operators $(B_0, D(B_0))$ and $(A_0, D(A_0))$ generate, respectively, C_0- semi-groups $(T_{B_0}(t))_{t\geq 0}$ and $(T_{A_0}(t))_{t\geq 0}$ on Y_0 and X_0.

It follows from Thieme [34] and Kellerman and Hieber [19] that the abstract Cauchy problem (3.3) has at most one integrated solution. The following theorem indicates the relationship of solutions $u(t, a)$, $y(t)$ and $x(t)$ to Systems (1.1), (3.1) and (3.3).

Theorem 3.3. *Let $x_0 = \begin{pmatrix} 0_Y \\ y_0(\cdot) \end{pmatrix}$ with $y_0 = \begin{pmatrix} 0 \\ u_0(\cdot)(a) \end{pmatrix} \in C([-\tau_1, 0]; Y)$, then there exists an unique integrated solution $t \to x(t)$ of the Cauchy problem (3.3) given by*

$$x(t) = \begin{cases} x_0(t), \ t \in [\tau_1, \ 0], \\ T_{A_0}(t)x_0(0) + \dfrac{d}{dt} \int_0^t S_A(t-s)\widetilde{H}(y(s))\,ds, \ t \geq 0. \end{cases}$$

It can be expressed explicitly by the following formula

$$x(t) = \begin{pmatrix} 0_Y \\ v(t) \end{pmatrix}$$

with

$$v(t)(\theta) = y(t+\theta), \ for \ t \geq 0, \ \theta \in [-\tau_1, 0],$$

where

$$y(t) = \begin{pmatrix} 0 \\ u(t,a) \end{pmatrix} = \begin{cases} y_0(t), \ t \in [\tau_1, \ 0], \\ T_{B_0}(t)u_0(0) + \dfrac{d}{dt}\displaystyle\int_0^t S_B(t-s)\widehat{H}(y(s))ds, \ t \geq 0. \end{cases}$$

4. **Equilibrium, linearized equation and spectral properties.** Now we consider the positive equilibrium solutions of Eq. (1.1).

Let $\begin{pmatrix} 0_Y \\ \phi \end{pmatrix} \in D(A) \times C([-\tau_1,0];Y)$ with $\phi = \begin{pmatrix} \alpha(\cdot) \\ u(\cdot)(a) \end{pmatrix}$, and set

$$A\begin{pmatrix} 0_Y \\ \phi \end{pmatrix} + \widetilde{H}\left(\begin{pmatrix} 0_Y \\ \phi \end{pmatrix}\right) = 0,$$

that is,

$$\begin{pmatrix} -\phi'(0) + B\phi(0) \\ \phi' \end{pmatrix} + \begin{pmatrix} \widehat{H}\left(\begin{pmatrix} \alpha(\cdot) \\ u(\cdot)(a) \end{pmatrix}\right) \\ 0_C \end{pmatrix}$$

$$= \begin{pmatrix} -\phi'(0) + B\phi(0) \\ \phi' \end{pmatrix} + \begin{pmatrix} H\left(\begin{pmatrix} 0 \\ \int_{-\tau_1}^0 d\xi(\theta)u(\theta)(a) \end{pmatrix}\right) \\ 0_C \end{pmatrix}$$

$$= \begin{pmatrix} -\phi'(0) + \begin{pmatrix} -u(0)(0) \\ -u'_a - \mu u \end{pmatrix} \\ \phi' \end{pmatrix} + \begin{pmatrix} \alpha h\left(\int_{-\tau_1}^0 d\xi(\theta)\int_0^{+\infty}\beta(a)u(\theta)(a)da\right) \\ 0 \\ 0_C \end{pmatrix}$$

$$= 0.$$

Then we get the equilibrium for (3.3)

$$\bar{x}(a) = \begin{pmatrix} 0_Y \\ 0 \\ ce^{-\mu a} \end{pmatrix}, \quad a > 0$$

and hence the (unique) positive equilibrium for (1.1) as

$$\bar{u}(t,a) = ce^{-\mu a}, \qquad t \geq -\tau_1, \ a > 0, \tag{4.1}$$

where c, from the boundary condition, is determined by the equation

$$c = \alpha h\left(c\int_{-\tau_1}^0 d\xi(\theta)\int_0^{+\infty}\beta(a)e^{-\mu a}da\right),$$

which, by expression (1.2), gives that

$$c = \frac{1}{\gamma K}\ln(\alpha K),$$

with

$$K = \int_{-\tau_1}^0 d\xi(\theta)\int_0^{+\infty}\beta(a)e^{-\mu a}da. \tag{4.2}$$

Next we deduce the linearized system for System (1.1) around the positive equilibrium $\bar{u}$. Actually, the linearized system for System (3.3) is given by

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + D\widetilde{H}\left(\bar{x}\right)x(t), \qquad t > 0, \\ x(t) = \bar{x}|_{[-\tau_1, \ 0]}, \end{cases} \tag{4.3}$$

where, for $x = \begin{pmatrix} 0_Y \\ \phi \end{pmatrix} = \begin{pmatrix} 0_Y \\ \alpha(\cdot) \\ u(\cdot)(a) \end{pmatrix} \in X_0,$

$$D\widetilde{H}(\bar{x})\, x = D\widetilde{H}(\bar{x}) \begin{pmatrix} 0_Y \\ \phi \end{pmatrix}$$

$$= \begin{pmatrix} \alpha h' \left(\int_{-\tau_1}^0 d\xi(\theta) \int_0^{+\infty} \beta(a)\bar{u}(a)da \right) \cdot \int_{-\tau_1}^0 d\xi(\theta) \int_0^{+\infty} \beta(a)u(\theta)(a)da \\ 0_{L^1} \\ 0_C \end{pmatrix}$$

$$= \begin{pmatrix} \eta(\alpha) \int_{-\tau_1}^0 d\xi(\theta) \int_0^{+\infty} \beta(a)u(\theta)(a)da \\ 0_{L^1} \\ 0_C \end{pmatrix}$$

with

$$\eta(\alpha) := \alpha h' \left(\int_{-\tau_1}^0 d\xi(\theta) \int_0^{+\infty} \beta(a)\bar{u}(a)da \right) = \frac{1}{K}\left[1 - \ln(\alpha K) \right]. \qquad (4.4)$$

The Cauchy problem (4.3) corresponds to the following linear partial differential equation

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,a) = -\dfrac{\partial}{\partial a}u(t,a) - \mu u(t,a), & t > 0, \quad a > 0, \\[2mm] u(t,0) = \eta(\alpha) \displaystyle\int_{-\tau_1}^0 d\xi(\theta) \int_0^{+\infty} \beta(a)u(t+\theta,a)da, & t > 0, \\[2mm] u(s,a) = \bar{u}(a), & s \in [-\tau_1,0], \quad a > 0. \end{cases} \qquad (4.5)$$

Now let $u(t,a) = e^{\lambda t}\phi(a)$ in (4.5), one can easily obtain the characteristic equation as

$$\Delta(\lambda,\ \alpha) = 1 - \eta(\alpha) \int_{-\tau_1}^0 e^{\lambda\theta} d\xi(\theta) \int_0^{+\infty} \beta(a)e^{-(\mu+\lambda)a}da. \qquad (4.6)$$

To simplify the notation, we define $B_\alpha : D(B_\alpha) \subset X \to X$ as

$$B_\alpha x = Ax + DH(\bar{x})x \text{ with } D(B_\alpha) = D(A),$$

and denote by $(B_\alpha)_0$ the part of B_α on X_0. Then we have

Theorem 4.1. *For each $\lambda \in \Omega$, there hold*

$$\lambda \in \rho(B_\alpha) \Leftrightarrow \Delta(\lambda,\ \alpha) \neq 0,$$

and the following explicit formula:

$$(\lambda - B_\alpha)^{-1} \begin{pmatrix} \beta \\ f \\ \beta_1(\cdot) \\ \varphi(\cdot) \end{pmatrix}$$

$$= \begin{pmatrix} 0_Y \\ e^{\lambda\theta}(\lambda - B)^{-1}\left[\begin{pmatrix} \beta_1(0) \\ \varphi(0)(a) \end{pmatrix} + \begin{pmatrix} \beta \\ f(a) \end{pmatrix} \right] + \int_\theta^0 e^{\lambda(\theta-s)} \begin{pmatrix} \beta_1(s) \\ \varphi(s) \end{pmatrix} ds \end{pmatrix},$$

for any $\begin{pmatrix} \beta \\ f \\ \beta_1(\cdot) \\ \varphi(\cdot) \end{pmatrix} \in X.$

Proof. Since $\lambda \in \Omega$, from Theorem 3.1, we know that $(\lambda - A)$ is invertible. Then

$$\lambda - B_\alpha \ \text{ is invertible } \ \Leftrightarrow \ I - DH(\bar{x})(\lambda - A)^{-1} \ \text{ is invertible,}$$

and

$$(\lambda - B_\alpha)^{-1} = (\lambda - A)^{-1}[I - DH(\bar{x})(\lambda - A)^{-1}]^{-1}.$$

We also know by direct computation that

$$[I - DH(\bar{x})(\lambda - A)^{-1}] \begin{pmatrix} \hat{\beta} \\ \hat{f} \\ \hat{\beta}_1(\cdot) \\ \hat{\varphi}(\cdot) \end{pmatrix} = \begin{pmatrix} \beta \\ f \\ \beta_1(\cdot) \\ \varphi(\cdot) \end{pmatrix}$$

is equivalent to $f = \hat{f}$, $\beta_1 = \hat{\beta}_1$, $\varphi = \hat{\varphi}$, and

$$\beta = \hat{\beta} - \eta(\alpha) \int_{-\tau_1}^0 e^{\lambda\theta} d\xi(\theta) \int_0^{+\infty} \hat{\beta}\beta(a)e^{-(\mu+\lambda)a} da - \tilde{\beta}(\hat{\varphi}, \ \hat{f}),$$

where

$$\tilde{\beta}(\hat{\varphi}, \ \hat{f}) := \eta(\alpha) \int_{-\tau_1}^0 e^{\lambda\theta} d\xi(\theta) \int_0^{+\infty} \beta(a) \left(\int_0^a e^{-(\mu+\lambda)(a-l)} [\hat{\varphi}(0)(l) + \hat{f}(l)] dl \right) da$$

$$+ \eta(\alpha) \int_{-\tau_1}^0 d\xi(\theta) \int_0^{+\infty} \beta(a) \left(\int_\theta^0 e^{\lambda(\theta-s)} \hat{\varphi}(s)(a) ds \right) da.$$

We deduce that $[I - DH(\bar{x})(\lambda - A)^{-1}]$ is invertible if and only if $\Delta(\lambda, \ \alpha) \neq 0$. Moreover,

$$[I - DH(\bar{x})(\lambda - A)^{-1}]^{-1} \begin{pmatrix} \beta \\ f \\ \beta_1(\cdot) \\ \varphi(\cdot) \end{pmatrix} = \begin{pmatrix} \hat{\beta} \\ \hat{f} \\ \hat{\beta}_1(\cdot) \\ \hat{\varphi}(\cdot) \end{pmatrix}$$

is equivalent to $\hat{f} = f$, $\hat{\beta}_1 = \beta_1$, $\hat{\varphi} = \varphi$, and

$$\hat{\beta} = \Delta(\lambda \ \alpha)^{-1}[\beta + \tilde{\beta}(\varphi, \ f)].$$

Therefore,

$$(\lambda - B_\alpha)^{-1} \begin{pmatrix} \beta \\ f \\ \beta_1(\cdot) \\ \varphi(\cdot) \end{pmatrix}$$

$$= (\lambda - A)^{-1}[I - DH(\bar{x})(\lambda - A)^{-1}]^{-1} \begin{pmatrix} \beta \\ f \\ \beta_1(\cdot) \\ \varphi(\cdot) \end{pmatrix}$$

$$= (\lambda - A)^{-1} \begin{pmatrix} \Delta(\lambda \ \alpha)^{-1}[\beta + \tilde{\beta}(\varphi, \ f)] \\ f \\ \beta_1(\cdot) \\ \varphi(\cdot) \end{pmatrix}$$

$$= \begin{pmatrix} 0_Y \\ e^{\lambda\theta}(\lambda - B)^{-1} \left[\begin{pmatrix} \beta_1(0) \\ \varphi(0)(a) \end{pmatrix} + \begin{pmatrix} \beta \\ f(a) \end{pmatrix} \right] + \int_\theta^0 e^{\lambda(\theta-s)} \begin{pmatrix} \beta_1(s) \\ \varphi(s) \end{pmatrix} ds \end{pmatrix},$$

which is the desired result and the proof is completed. $\qquad\square$

Further, we have the following result:

Theorem 4.2. *For the operator $(B_\alpha,\ D(B_\alpha))$, the following statements are true.*

(i) *The linear operator B_α is a Hille-Yosida operator on X and its part $(B_\alpha)_0$ in X_0 satisfies*

$$\omega_{0,\ ess}\left((B_\alpha)_0\right) \le -\mu. \tag{4.7}$$

(ii) *There holds*

$$\sigma\left((B_\alpha)_0\right) \cap \Omega = \sigma_p\left((B_\alpha)_0\right) \cap \Omega = \{\lambda \in \Omega : \Delta(\lambda,\ \alpha) = 0\}.$$

Proof. B_α is clearly a Hille-Yosida operator since $DH(\bar{x})$ is linear and bounded. Meanwhile, as $DH(\bar{x})$ is one-dimensional, we get from ([14], Theorem 1.2) that

$$\omega_{0,\ ess}\left((B_\alpha)_0\right) \le \omega_{0,\ ess}\left(A_0\right).$$

But (see formula (3.2) of [15])

$$\omega_{0,\ ess}\left(A_0\right) \le \omega_{0,\ ess}\left(B_0\right),$$

hence

$$\omega_{0,\ ess}\left((B_\alpha)_0\right) \le \omega_{0,\ ess}\left(B_0\right) \le -\mu.$$

Assertion (ii) follows from Theorem 4.1 immediately. $\qquad\square$

5. **Stability and bifurcation results.** Based on the work of the previous section (particularly, Theorem 4.2), we may apply the center manifold Theorem 4.21 and Proposition 4.22 in Magal and Ruan [26] to reduce System (1.1) to a ODE system in finite dimension. Thus, we can establish the results on stability and Hopf bifurcation for System (1.1). First we have the local stability of the positive steady state $\bar{u}$. That is

Theorem 5.1. *The positive steady state $\bar{u}$ of System (1.1) is (locally) stable if*

$$\frac{1}{K} < \alpha \le \frac{e^2}{K}. \tag{5.1}$$

Proof. Let $\lambda \in \mathbb{C}$ with $Re\lambda \ge 0$, and $\Delta(\alpha, \lambda) = 0$. Then

$$1 = \left| \eta(\alpha) \int_{-\tau_1}^0 e^{(Re\lambda)\theta} d\xi(\theta) \int_0^{+\infty} \beta(a) e^{-(\mu+\lambda)a} da \right|$$

$$\le |\eta(\alpha)| \int_{-\tau_1}^0 e^{(Re\lambda)\theta} d\xi(\theta) \int_0^{+\infty} \beta(a) e^{-\mu a} e^{-(Re\lambda)a} da$$

$$\le |\eta(\alpha)| \int_{-\tau_1}^0 d\xi(\theta) \int_0^{+\infty} \beta(a) e^{-\mu a} da$$

$$= \frac{1}{K} \left| 1 - \ln(\alpha K) \right| \cdot K = \left| 1 - \ln(\alpha K) \right|,$$

which contradicts (5.1). So we must have $Re\lambda < 0$ and assertion is true (note that (4) implies that $\alpha > \frac{1}{K}$ is necessary for the existence of the equilibrium $\bar{u}$). $\qquad\square$

Then, by the Hopf bifurcation theorem (Hassard et al. [18]), we have the following Hopf bifurcation result.

Theorem 5.2. *Assume that there is a number $\alpha^* > \frac{e^2}{K}$ such that*

(i) *if $\lambda \in \Omega$ and $\Delta(\alpha^*,\ \lambda) = 0$, then $Re\left(\frac{\partial \Delta(\alpha^*, \lambda)}{\partial \lambda}\right) \ne 0$;*

(ii) *there exists* $\theta^* > 0$ *such that* $\Delta(\alpha^*,\ i\theta^*) = 0$ *and* $\Delta(\alpha^*,\ i\theta) \neq 0$ *for any* $\theta \in (0,, +\infty) \setminus \{\theta^*\}$.

Then for this $\alpha^* > \frac{e^2}{K}$, *System (1.1) undergoes a Hopf bifurcation at the equilibrium* $\bar{u}$. *Particularly, a non-trivial periodic solution bifurcates from the equilibrium* $\bar{u}$.

6. **Hopf bifurcation.** In this section, as an application of Theorem 5.2, we study the bifurcation problem for System (1.1) under the following assumptions:

(i) The function $\xi(\cdot)$ is given by

$$\xi(\theta) = \left\{ \begin{array}{ll} 0, & \theta \in (-\tau_1, 0], \\ -1, & \theta = -\tau_1. \end{array} \right.$$

(ii) The birth function $\beta(\cdot)$ is given by

$$\beta(a) = \left\{ \begin{array}{ll} \beta_0(a - \tau_2)^n e^{\chi(a-\tau_2)}, & a \geq \tau_2, \\ 0, & a < \tau_2, \end{array} \right. \tag{6.1}$$

where $\beta_0 > 0$, $\chi \geq 0$ and $n \in \mathbb{N}$.

By Theorem 5.1 we already knew that the positive equilibrium $\bar{u}$ of the system (1.1) is locally asymptotically stable if $\frac{1}{K} < \alpha \leq \frac{e^2}{K}$, hence we will study the existence of a bifurcation value $\alpha > \frac{e^2}{K}$. Recalling (4.4), we must have

$$\eta(\alpha) < 0. \tag{6.2}$$

Since the function $\beta(a)$ must be bounded, we consider the following two cases:

(a) $n = 0$, $\chi = 0$;

(b) $n \geq 1$, $\chi > 0$.

As a first step, we prove the existence of purely imaginary eigenvalues.

Case a. Under Assumption (i), (ii) and (a) the characteristic equation (4.6) turns out to be

$$\Delta(\lambda,\ \alpha) = 1 - \beta_0 \eta(\alpha) e^{-\lambda\tau_1} \frac{e^{-(\lambda+\mu)\tau_2}}{\lambda + \mu} = 0, \tag{6.3}$$

or

$$\lambda + \mu = \beta_0 \eta(\alpha) e^{-\mu\tau_2} e^{-(\tau_1+\tau_2)\lambda}. \tag{6.4}$$

Let $\hat{\lambda} = (\tau_1 + \tau_2)\lambda$, then (6.4) becomes

$$\hat{\lambda} + \mu(\tau_1 + \tau_2) = \beta_0 \eta(\alpha)(\tau_1 + \tau_2) e^{-\mu\tau_2} e^{-\hat{\lambda}}. \tag{6.5}$$

Thus, from ([17], Theorem A.5) we get a sufficient and necessary condition for local asymptotic stability as follows.

Theorem 6.1. *For case (a), The positive equilibrium* $\bar{u}$ *of the system (1.1) is local asymptotically stable if and only if*

$$-\beta_0 \eta(\alpha)(\tau_1 + \tau_2) e^{-\mu\tau_2} < \phi \sin \phi - \mu(\tau_1 + \tau_2) \cos \phi, \tag{6.6}$$

where ϕ *satisfies that*

$$\phi = \mu(\tau_1 + \tau_2) \tan \phi. \tag{6.7}$$

Let $\hat{\lambda} = \omega i$ in (6.5) with $\omega > 0$, then

$$[\omega i + \mu(\tau_1 + \tau_2)] e^{\omega i} = \beta_0 \eta(\alpha)(\tau_1 + \tau_2) e^{-m\tau_2}.$$

Now we can fix (note (6.2))

$$\eta(\alpha) = -\frac{c_1 e^{\mu\tau_2}}{\beta_0(\tau_1 + \tau_2)},$$

for some constant $c_1 > 0$, to obtain

$$\omega i + \mu(\tau_1 + \tau_2) = -c_1 e^{-\omega i} = -c_1[\cos\omega - i\sin\omega].$$

We must have

$$c_1 = \sqrt{\mu^2(\tau_1 + \tau_2)^2 + \omega^2},$$

and

$$\tan\omega = -\frac{\omega}{\mu(\tau_1 + \tau_2)}$$

and impose that

$$\sin\omega = \frac{\omega}{c_1} > 0.$$

From the above computation we obtain the following proposition.

Proposition 5. *Let τ_1, $\tau_2 > 0$ and $\mu > 0$ be fixed. Then the characteristic equation* (6.3) *has a pair of imaginary solutions $\pm i\omega$ with $\omega > 0$ if and only if there exists $\omega > 0$ which is a solution of equation*

$$\tan\omega = -\frac{\omega}{\mu(\tau_1 + \tau_2)} \tag{6.8}$$

with

$$\sin\omega = \frac{\omega}{c_1} > 0, \tag{6.9}$$

and

$$\eta(\alpha) = -\frac{c_1 e^{\mu\tau_2}}{\beta_0(\tau_1 + \tau_2)}$$

with

$$c_1 = \sqrt{\mu^2(\tau_1 + \tau_2)^2 + \omega^2}.$$

Moreover, for each $k \in \mathbb{N}$, there exists a unique $\omega_k \in ((2k + \frac{1}{2})\pi, (2k + 1)\pi)$ (which is a function of τ_1, τ_2 and μ) satisfying (6.8) *and* (6.9).

Case (b). If $n \geq 1$ and $\chi > 0$, then

$$\Delta(\lambda, \alpha) = 1 - \eta(\alpha)e^{-\lambda\tau_1}\int_{\tau_2}^{+\infty}\beta_0(a - \tau_2)^n e^{-\chi(a-\tau_2)}e^{-(\mu+\lambda)a}da$$

$$= 1 - \eta(\alpha)e^{-\lambda\tau_1}e^{-(\mu+\lambda)\tau_2}\beta_0\int_0^{+\infty}a^n e^{-(\mu+\lambda+\chi)a}da$$

$$= 1 - \beta_0\eta(\alpha)e^{-\mu\tau_2}\frac{n!}{(\mu+\lambda+\chi)^{n+1}}e^{-(\tau_1+\tau_2)\lambda}$$

$$= 0.$$

It is equivalent to

$$\lambda + \mu + \chi = \beta_0\eta(\alpha)n!e^{-\mu\tau_2}\frac{e^{-(\tau_1+\tau_2)\lambda}}{(\lambda+\mu+\chi)^n}.$$

Let $\lambda = \omega i$ with $\omega > 0$, then

$$\omega i + \mu + \chi$$

$$= \beta_0 \eta(\alpha) n! e^{-\mu \tau_2} \frac{e^{-(\tau_1 + \tau_2)\omega i}}{(\omega i + \mu + \chi)^n}$$

$$= \beta_0 \eta(\alpha) n! e^{-\mu \tau_2} \frac{\cos[-(\tau_1 + \tau_2)\omega] + i \sin[-(\tau_1 + \tau_2)]\omega}{\left[\sqrt{\omega^2 + (\mu + \chi)^2} (\cos \theta + i \sin \theta) \right]^n}$$

$$= \frac{\beta_0 \eta(\alpha) n! e^{-\mu \tau_2}}{\left(\sqrt{\omega^2 + (\mu + \chi)^2} \right)^n} (\cos[-(\tau_1 + \tau_2)\omega - n\theta] + i \sin[-(\tau_1 + \tau_2)\omega - n\theta]),$$

where

$$\theta = \arctan \frac{\omega}{\mu + \chi}.$$

Now we fix

$$\eta(\alpha) = -c_2 \frac{\left(\sqrt{\omega^2 + (\mu + \chi)^2} \right)^n}{\beta_0 n! e^{-\mu \tau_2}} \tag{6.10}$$

with $c_2 > 0$, then we obtain that

$$\mu + \chi + \omega i = -c_2 \left(\cos[-(\tau_1 + \tau_2)\omega - n\theta] + i \sin[-(\tau_1 + \tau_2)\omega - n\theta] \right),$$

and we have

$$c_2 = \sqrt{\omega^2 + (\mu + \chi)^2},$$

$$\tan[-(\tau_1 + \tau_2)\omega - n\theta] = \frac{\omega}{\mu + \chi},$$

or

$$\tan[(\tau_1 + \tau_2)\omega + n\theta] = -\frac{\omega}{\mu + \chi}.$$

We must impose that

$$\sin[(\tau_1 + \tau_2)\omega + n\theta] = \frac{\omega}{\sqrt{\omega^2 + (\mu + \chi)^2}} > 0.$$

From the above computations we obtain the following proposition.

Proposition 6. *Let τ_1, $\tau_2 > 0$, $\mu > 0$, $\beta_0 > 0$, and $n \in \mathbb{N}$ be fixed. Then the characteristic equation has a pair of purely imaginary solutions $\pm i\omega$ with $\omega > 0$ if and only if $\omega > 0$ is a solution of equation*

$$\tan(\Theta(\omega)) = -\frac{\omega}{\mu + \chi} \tag{6.11}$$

with

$$\sin(\Theta(\omega)) > 0, \tag{6.12}$$

where

$$\Theta(\omega) = (\tau_1 + \tau_2)\omega + n\theta, \ \theta = \arctan \frac{\omega}{\mu + \chi}.$$

And $\eta(\alpha)$ is given by (4.4). Moreover, there exists a sequence $\omega_k \to +\infty$ as $k \to +\infty$, $k \in \mathbb{N}$ (which is a function of τ_1, τ_2, μ, β_0, and n) satisfying (6.11) and (6.12). In particular, for each k, there exists a unique $\omega_k \in \left(\Theta^{-1} \left(2k\pi + \frac{\pi}{2} \right), \ \Theta^{-1} \left((2k+1)\pi \right) \right)$ satisfying (6.11) and (6.12), where Θ^{-1} is the inverse function of $\Theta(\omega)$ on $(0, +\infty)$.

Proof. Note that if $\omega > 0$ solves (6.11), then

$$\tan\left(\Theta(\omega)\right) < 0,$$

so

$$\Theta(\omega) \in \left(m\pi + \frac{\pi}{2}, (m+1)\pi\right), \ m \in \mathbb{Z}.$$

Moreover, in order to ensure

$$\sin\Theta(\omega) > 0,$$

we must take $m = 2k$, $k \in \mathbb{Z}$. Now since $\Theta(\omega)$ is a continuous function of ω, and

$$\Theta(0) = 0, \ \Theta(+\infty) = +\infty,$$

for any $k \in \mathbb{N}$, there exist $\hat{\omega}_{k_1}$, $\hat{\omega}_{k_2} > 0$ such that $\Theta(\hat{\omega}_{k_1}) = 2k\pi + \frac{\pi}{2}$, $\Theta(\hat{\omega}_{k_2}) = (2k+1)\pi$. Observe that the right-hand side of Eq. (6.11) is a strictly monotone decreasing function of ω, and since the function $\tan(\Theta(\omega))$ can take any value from $-\infty$ to 0 when $\omega \in (\hat{\omega}_{k_1},\ \hat{\omega}_{k_2})$, we deduce that Eq. (6.11) has a solution $\omega_k \in (\hat{\omega}_{k_1},\ \hat{\omega}_{k_2})$. Thus there exists a sequence of $\omega_k \to +\infty$ satisfying (6.11) and (6.12).

Furthermore, $\Theta(\omega)$ is clearly a strictly monotone increasing function of ω. So for any$k \in \mathbb{N}$, we have $\hat{\omega}_{k_1} = \Theta^{-1}(2k\pi + \frac{\pi}{2})$, $\hat{\omega}_{k_2} = \Theta^{-1}(K(2k+1)\pi)$, and the function $\tan(\Theta(\omega))$ is increasing when $\omega \in (\hat{\omega}_{k_1},\ \hat{\omega}_{k_2})$. Thus there exists a unique $\omega_k \in (2k\pi + \frac{\pi}{2}),\ K(2k+1)\pi)$ satisfying (6.11) and (6.12), and the result follows. $\square$

Next we verify the transversality conditions for the model with for the above two cases. Since Case (a) is a special situation of (b), we only investigate the transversality condition under Assumption (b).

Lemma 6.2. *Let (b) be satisfied. If $\alpha > \frac{e^2}{K}$, $\lambda \in \Omega$ and $\Delta(\lambda,\ \alpha) = 0$, then*

$$\frac{\partial\Delta(\lambda,\ \alpha)}{\partial\lambda} \neq 0.$$

Proof. Under Assumption (b) we have

$$\Delta(\lambda,\ \alpha) = 1 - \beta_0\eta(\alpha)e^{-\mu\tau_2}\frac{n!}{(\lambda + \mu + \chi)^{n+1}}e^{-(\tau_1 + \tau_2)\lambda}$$

and

$$\begin{aligned}
&\frac{\partial\Delta(\lambda,\ \alpha)}{\partial\lambda} \\
&= \beta_0\eta(\alpha)e^{-\mu\tau_2}n! \\
&\quad \times \frac{(\tau_1 + \tau_2)e^{-(\tau_1+\tau_2)\lambda}(\lambda+\mu+\chi)^{n+1} + (n+1)e^{-(\tau_1+\tau_2)\lambda}(\lambda+\mu+\chi)^n}{(\lambda+\mu+\chi)^{2(n+1)}} \\
&= \beta_0\eta(\alpha)e^{-\mu\tau_2}n!\left[\frac{(\tau_1+\tau_2)e^{-(\tau_1+\tau_2)\lambda}}{(\lambda+\mu+\chi)^{n+1}} + \frac{(n+1)e^{-(\tau_1+\tau_2)\lambda}}{(\lambda+\mu+\chi)^{n+2}}\right].
\end{aligned}$$

Since $\Delta(\lambda,\ \alpha) = 0$, it yields that (note $\lambda \in \Omega$)

$$\frac{\partial\Delta(\lambda,\ \alpha)}{\partial\lambda} = (\tau_1 + \tau_2) + \frac{(n+1)}{\lambda + \mu + \chi} \neq 0.$$

$\square$

Theorem 6.3. *Let Assumptions (b) be satisfied. For each $k > 0$ large enough, let $\lambda_k = i\omega_k$ be the purely imaginary root of the characteristic equation associated to $\alpha_k > \frac{e^2}{K}$ (defined in (6.10)), then there exists $l\delta_k > 0$ (small enough) and a C^1-map.$f\hat{\lambda}_k : (\alpha_k - \delta_k, \ \alpha_k + \delta_k) \to \mathbb{C}$ such that*

$$\hat{\lambda}_k(\alpha_k) = i\omega_k, \ \ \Delta(\hat{\lambda}_k(\alpha), \ \alpha) = 0, \tag{6.13}$$

for each $\alpha \in (\alpha_k - \delta_k, \ \alpha_k + \delta_k)$, satisfying the transversality condition

$$Re\left(\frac{d\hat{\lambda}_k(\alpha_k)}{d\alpha}\right) > 0.$$

Proof. By virtue of the implicit function theorem and Lemma 6.2, we obtain that there is a C^1- map $\hat{\lambda}_k(\cdot)$ verifying (6.13) and

$$
\begin{aligned}
\frac{d\hat{\lambda}_k(\alpha_k)}{d\alpha} &= -\frac{\partial\Delta(i\omega_k, \ \alpha_k)}{\partial\alpha}\left[\frac{\partial\Delta(i\omega_k, \ \alpha_k)}{\partial\lambda}\right]^{-1} \\
&= \frac{\eta'(\alpha_k)}{\eta(\alpha_k)} \cdot \frac{\omega_k + \mu + \chi}{(\tau_1 + \tau_2)(i\omega_k + \mu + \chi) + (n+1)} \\
&= -\frac{1}{\alpha_k(1 - \ln(\alpha_k K))} \cdot \frac{1}{\tau_1 + \tau_2} \\
&\quad \cdot \frac{\omega_k^2 + (\mu + \chi)\left[\mu + \chi + \frac{n+1}{\tau_1+\tau_2}\right] - \left[2(\mu + \chi) + \frac{n+1}{\tau_1+\tau_2}\right]\omega_k i}{\omega_k^2 + \left[\mu + \chi + \frac{n+1}{\tau_1+\tau_2}\right]^2},
\end{aligned}
$$

which implies

$$Re\left(\frac{d\hat{\lambda}_k(\alpha_k)}{d\alpha}\right) > 0.$$

$\square$

By combining the results on the essential growth rate of the linearized equations (Eq. (4.3)), the simplicity of the imaginary eigenvalues (Lemmas 6.2), the existence of purely imaginary eigenvalues (Proposition 5 or Proposition 6), and the transversality condition (Theorem 6.3), we are in a position to apply Theorem 5.2 to obtain the following Hopf bifurcation results.

In the case (a), we obtain that

Theorem 6.4 (Hopf bifurcation). *Let Assumption (a) be satisfied. Then for any $k \in \mathbb{N}$, the number α_k (defined in Proposition 5) is a Hopf bifurcation point for system (1.1) parameterized by α, and around the positive equilibrium point $\bar{u}$ given in (4.1).*

For the case (b) we have

Theorem 6.5 (Hopf bifurcation). *Let Assumptions (b) be satisfied. Then, there exists $k_0 \in \mathbb{N}$ (large enough) such that for each $k \geq k_0$, the number α_k (defined in Proposition 6) is a Hopf bifurcation point for System (1.1) parameterized by α, around the equilibrium point $\bar{u}$ given in (4.1).*

7. **Numerical simulations.** In this section, we present some numerical simulations to illustrate the results obtained in the previous sections.

We take here the coefficients as

$$\gamma = \beta_0 = 0.5, \ \mu = 0.05, \ \tau_1 = 2, \ \tau_2 = 8.$$

Then $K = \frac{\beta_0}{\mu} e^{\mu \tau_2} = 14.9182$.

I. From (6.7) we get $\phi = 1.1656$, and then we choose $\alpha = 12.1222$ to solve the inequality (6.6). Thus by virtue of Theorem 6.1 we infer that the equilibrium of System (1.1) is local asymptotically stable, see Fig 7.1 below.
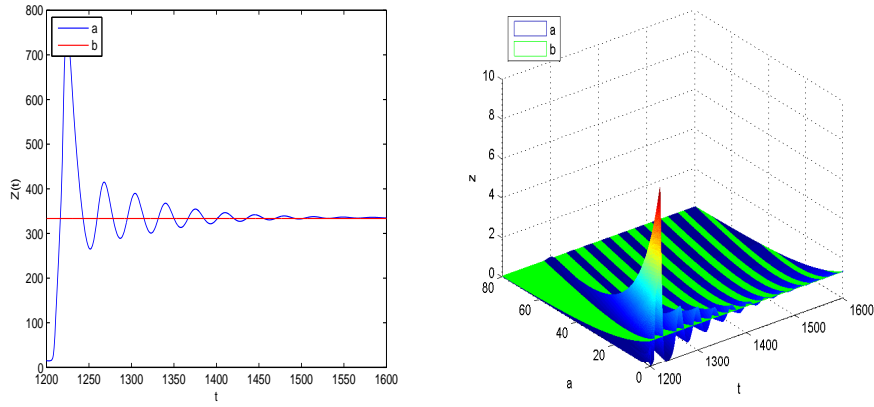

FIGURE 7.1. The case for $\alpha = 12.1222$. "a" represents the stationary solution.

II. Take $\alpha = 0.2896$ which satisfies $(0.067 =) \frac{1}{K} < \alpha \leq \frac{e^2}{K} (= 0.4953)$. it follows from Theorem 5.1 that the equilibrium of System (1.1) is local asymptotically stable, see Fig 7.2.
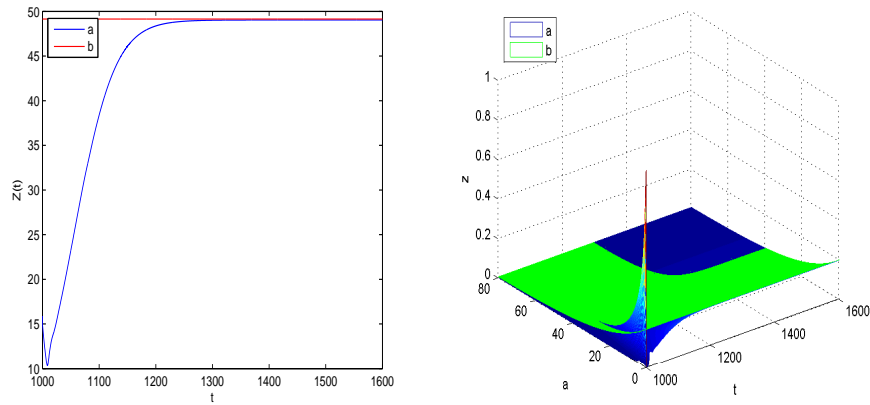

FIGURE 7.2. The case for $\alpha = 0.2896$. "a" represents the stationary solution.

III. Finally we determine $\omega = 1.8366 \in \left(\frac{\pi}{2}, \pi\right)$ and $c_1 = 1.9034$ such that (6.8) and (6.9) are verified, then we have $\alpha = 870.805$. According to Proposition 5 and Theorem 6.4 we deduce that for this α System (1.1) undergoes a Hopf bifurcation at the equilibrium $\bar{u}$, see Fig. 7.3.
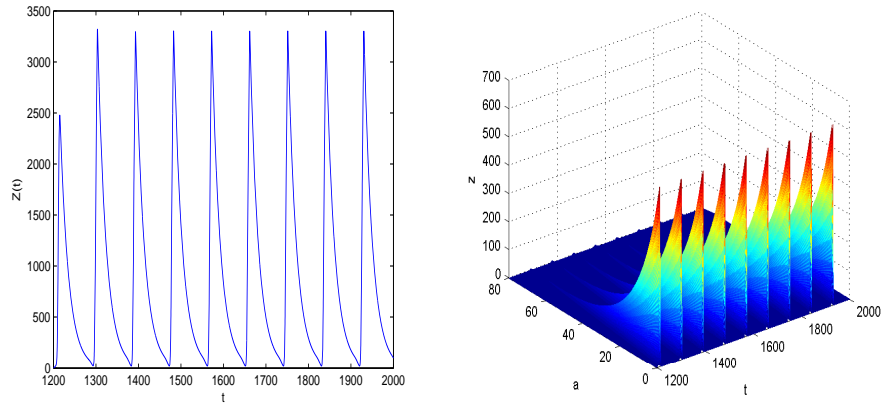

FIGURE 7.3. The case of Hopf bifurcation (with $\alpha = 870.805$).

8. **Conclusions.** Age-structured models with delayed process have rich background in practical fields. In this study we have investigated the existence of a positive equilibrium for System (1.1), then applying the center manifold theorem founded in [26] we have established the results of locally asymptotic stability of this equilibrium (Theorem 5.1 and Theorem 6.1) and theorems of the Hopf bifurcation at this equilibrium (Theorem 5.2, Theorem 6.4 and Theorem 6.5). Particularly, Formulas (5.1), (6.6), (6.8) and (6.11) reflect clearly the influences of the delays τ_1 and τ_2 on the stability and Hopf bifurcations. The numerical simulations manifest explicitly the obtained results on stability and Hopf bifurcation of System (1.1) (see Fig 7.1-7.3). As we know, age-structured models have been used to study many biological and epidemiological problems, such as the evolutionary epidemiology of type A influenza, the epidemics of schistosomiasisin human hosts and population dynamics. We expect that our methods here can be applied to those practical models with time lags to achieve some more interesting and meaningful results for their asymptotic behaviors (than the cases without delay).

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