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Projectors on the generalized eigenspaces for functional differential equations using integrated semigroups

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Abstract

The aim of this article is to derive explicit formulas for the projectors on the generalized eigenspaces associated to some eigenvalues for linear functional differential equations (FDE) by using integrated semigroup theory. The idea is to formulate the FDE as a non-densely defined Cauchy problem and obtain an explicit formula for the integrated solutions of the non-densely defined Cauchy problem, from which we then derive explicit formulas for the projectors on the generalized eigenspaces associated to some eigenvalues. The results are useful in studying bifurcations in some semi-linear problems.

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1. Introduction

Since the state space for functional differential equations (FDE) is infinitely dimensional, techniques and methods from functional analysis and operator theory have been further devel-

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oped and extensively used to study such equations (Hale and Verduyn Lunel [13], Diekmann et al. [9], Engel and Nagel [10]). In particular, the semigroup theory of operators on a Banach space has been successfully used to study the dynamical behavior of FDE (Adimy and Arino [3], Diekmann et al. [8], Frasson and Verduyn Lunel [12], Thieme [21], Verduyn Lunel [26], Webb [27–29]). In studying bifurcation problems, such as Hopf bifurcation, for FDE, we need to compute explicitly the flow on the center manifold (Hassard et al. [14]). To do that, we need to know detailed information about the underlying center manifold of the linearized equation. Frasson and Verduyn Lunel [12] provided explicit formulas for the spectral projection on the unstable or center subspace by using resolvent computations and Dunford calculus and studied the large time behavior of autonomous and periodic functional differential equations.

The goal of this article is to obtain explicit formulas for the projectors on the generalized eigenspaces associated to some eigenvalues for linear functional differential equations (FDE)

$$\begin{cases} \dfrac{dx(t)}{dt} = Bx(t) + \hat{L}(x_t), & \forall t \geqslant 0, \\ x_0 = \varphi \in C\big([-r, 0], \mathbb{R}^n\big) \end{cases}$$

by using integrated semigroup theory. This problem has been extensively studied since the 1970s (see Hale and Verduyn Lunel [13] and the historical remarks at the end of Chapters 6 and 7), the usual approach is based on the formal adjoint method. The method was recently further studied in the monograph of Diekmann et al. [9] using the so-called sun-star adjoint spaces, see also Kaashoek and Verduyn Lunel [15], Frasson and Verduyn Lunel [12], Diekmann et al. [8] and the references cited therein.

The main question here is to formulate the problem as an abstract Cauchy problem. In the 1970s, Webb [27], Travis and Webb [24,25] viewed the problem as a non-linear Cauchy problem and focused on many aspects of the problem by using this method. Another approach is a direct method, that is to use the variation of constant formula and work directly with the system (see Arino and Sánchez [6] and Kappel [16]).

We shall use an integrated semigroup formulation for the problem. It seems that Adimy [1,2], Adimy and Arino [3], and Thieme [21] were the first to apply such an approach in the context of FDE. This approach has been extensively developed by Arino's team in the 1990s (see Ezzinbi and Adimy [11] for a survey on this topic). Here we use a formulation of the FDE that is an intermediate between the formulations of Adimy [1,2] and Thieme [21]. In fact, compared with Adimy's approach we do not use any Radon measure to give a sense of the value of $x_t(\theta)$ at $\theta = 0$, while compared to Thieme's approach we keep only one equation. Our approach is more closely related to the one by Travis and Webb [24,25].

The rest of the paper is organized as follows. In Section 2 we demonstrate how to construct the formulations in an "intuitive manner." Then in Section 3, we recall some spectral theory and obtain an explicit formula for the integrated solutions of the non-densely defined Cauchy problem. The goal is to check that the integrated solutions of the Cauchy problem are in fact solutions of the FDE. In Section 4, which is in fact the main section of this article, we obtain an explicit formula for the projectors on the generalized eigenspaces associated to some eigenvalues. As far as we know this part is new. The projector for a simple eigenvalue is considered in Section 5. In Section 6, we discuss some applications of the results to the semi-linear problem by focusing specially on bifurcation aspects.

2. Preliminary

For $r \geqslant 0$, let $C = C([-r, 0]; \mathbb{R}^n)$ be the Banach space of continuous functions from $[-r, 0]$ to $\mathbb{R}^n$ endowed with the supremum norm

$$\|\varphi\|_C = \sup_{\theta \in [-r,0]} \|\varphi(\theta)\|_{\mathbb{R}^n}.$$

Consider the retarded functional differential equations (FDE) of the form

$$\begin{cases} \dfrac{dx(t)}{dt} = Bx(t) + \hat{L}(x_t) + f(t, x_t), & \forall t \geqslant 0, \\ x_0 = \varphi \in C, \end{cases} \tag{2.1}$$

where $x_t \in C$ satisfies $x_t(\theta) = x(t + \theta)$, $B \in M_n(\mathbb{R})$ is an $n \times n$ real matrix, $\hat{L} : C \to \mathbb{R}^n$ is a bounded linear operator given by

$$\hat{L}(\varphi) = \int_{-r}^{0} d\eta(\theta)\, \varphi(\theta),$$

here $\eta : [-r, 0] \to M_n(\mathbb{R})$ is a map of bounded variation, and $f : \mathbb{R} \times C \to \mathbb{R}^n$ is a continuous map.

In order to study the FDE (2.1) by using the integrated semigroup theory, we need to consider FDE (2.1) as an abstract non-densely defined Cauchy problem. Firstly, we regard FDE (2.1) as a PDE. Define $u \in C([0, +\infty) \times [-r, 0], \mathbb{R}^n)$ by

$$u(t, \theta) = x(t + \theta), \quad \forall t \geqslant 0,\ \forall \theta \in [-r, 0].$$

Note that if $x \in C^1([-r, +\infty), \mathbb{R}^n)$, then

$$\frac{\partial u(t, \theta)}{\partial t} = x'(t + \theta) = \frac{\partial u(t, \theta)}{\partial \theta}.$$

Hence, we must have

$$\frac{\partial u(t, \theta)}{\partial t} - \frac{\partial u(t, \theta)}{\partial \theta} = 0, \quad \forall t \geqslant 0,\ \forall \theta \in [-r, 0].$$

Moreover, for $\theta = 0$, we obtain

$$\begin{aligned} \frac{\partial u(t, 0)}{\partial \theta} &= x'(t) = Bx(t) + \hat{L}(x_t) + f(t, x_t) \\ &= Bu(t, 0) + \hat{L}\big(u(t, .)\big) + f\big(t, u(t, .)\big), \quad \forall t \geqslant 0. \end{aligned}$$

Therefore, we deduce formally that u must satisfy a PDE

$$\begin{cases} \dfrac{\partial u(t,\theta)}{\partial t} - \dfrac{\partial u(t,\theta)}{\partial \theta} = 0, \\ \dfrac{\partial u(t,0)}{\partial \theta} = Bu(t,0) + \hat{L}\big(u(t,.)\big) + f\big(t,u(t,.)\big), \quad \forall t \geqslant 0, \\ u(0,.) = \varphi \in C. \end{cases} \tag{2.2}$$

In order to rewrite the PDE (2.2) as an abstract non-densely defined Cauchy problem, we extend the state space to take into account the boundary conditions. This can be accomplished by adopting the following state space

$$X = \mathbb{R}^n \times C$$

taken with the usual product norm

$$\left\| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right\| = \|x\|_{\mathbb{R}^n} + \|\varphi\|_C.$$

Define the linear operator $A : D(A) \subset X \to X$ by

$$A\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) + B\varphi(0) \\ \varphi' \end{pmatrix}, \quad \forall \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in D(A), \tag{2.3}$$

with

$$D(A) = \{0_{\mathbb{R}^n}\} \times C^1\big([-r,0],\mathbb{R}^n\big).$$

Note that A is non-densely defined because

$$\overline{D(A)} = \{0_{\mathbb{R}^n}\} \times C \neq X.$$

We also define $L : \overline{D(A)} \to X$ by

$$L\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} \hat{L}(\varphi) \\ 0_C \end{pmatrix}$$

and $F : \mathbb{R} \times \overline{D(A)} \to X$ by

$$F\left(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}\right) = \begin{pmatrix} f(t,\varphi) \\ 0_C \end{pmatrix}.$$

Set

$$v(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ u(t) \end{pmatrix}.$$

Now we can consider the PDE (2.2) as the following non-densely defined Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + L\big(v(t)\big) + F\big(t, v(t)\big), \quad t \geqslant 0, \qquad v(0) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in \overline{D(A)}. \tag{2.4}$$

3. Some results on integrated solutions and spectrums

In this section we will first study the integrated solutions of the Cauchy problem (2.4) in the special case

$$\frac{dv(t)}{dt} = Av(t) + \begin{pmatrix} h(t) \\ 0 \end{pmatrix}, \quad t \geqslant 0, \qquad v(0) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in \overline{D(A)}, \tag{3.1}$$

where $h \in L^1((0,\tau), \mathbb{R}^n)$. Recall that $v \in C([0,\tau], X)$ is an integrated solution of (3.1) if and only if

$$\int_0^t v(s)\,ds \in D(A), \quad \forall t \in [0,\tau], \tag{3.2}$$

and

$$v(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} + A\int_0^t v(s)\,ds + \int_0^t \begin{pmatrix} h(s) \\ 0 \end{pmatrix} ds. \tag{3.3}$$

In the sequel, we will use the integrated semigroup theory to define such an integrated solution. We refer to Arendt [4], Thieme [22], Kellermann and Hieber [17], and the book of Arendt et al. [5] for further details on this subject. We also refer to Magal and Ruan [19] for more results and update references.

From (3.2) we note that if v is an integrated solution we must have

$$v(t) = \lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} v(s)\,ds \in \overline{D(A)}.$$

Hence

$$v(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ u(t) \end{pmatrix}$$

with

$$u \in C\big([0,\tau], C\big([-r,0], \mathbb{R}^n\big)\big).$$

In order to obtain the uniqueness of the integrated solutions of (3.1) we want to prove that A generates an integrated semigroup. So firstly we need to study the resolvent of A.

We introduce some notation. Let $L : D(L) \subset X \to X$ be a linear operator on a complex Banach space X. Denote by $\rho(L)$ the resolvent set of L, $N(L)$ the null space of L, and $R(L)$ the range of L, respectively. The spectrum of L is $\sigma(L) = \mathbb{C} \setminus \rho(L)$. The *point spectrum* of L is the set

$$\sigma_P(L) := \{\lambda \in \mathbb{C}:\ N(\lambda I - L) \neq \{0\}\}.$$

The *essential spectrum* (in the sense of Browder [7]) of L is denoted by $\sigma_{\mathrm{ess}}(L)$. That is, the set of $\lambda \in \sigma(L)$ such that at least one of the following holds: (i) $R(\lambda I - L)$ is not closed; (ii) λ is a limit point of $\sigma(L)$; (iii) $N_\lambda(L) := \bigcup_{k=1}^{\infty} N((\lambda I - L)^k)$ is infinite dimensional. Define

$$X_{\lambda_0} = \bigcup_{n \geqslant 0} N\big((\lambda_0 - L)^n\big).$$

Let Y be a subspace of X. Then we denote by $L_Y : D(L_Y) \subset Y \to Y$ the part of L on Y, which is defined by

$$L_Y x = Lx, \quad \forall x \in D(L_Y) := \{x \in D(L) \cap Y:\ Lx \in Y\}.$$

Definition 3.1. Let $L : D(L) \subset X \to X$ be the infinitesimal generator of a linear C^0-semigroup $\{T_L(t)\}_{t \geqslant 0}$ on a Banach space X. We define $\omega_0(L) \in [-\infty, +\infty)$ the *growth bound* of L by

$$\omega_0(L) := \lim_{t \to +\infty} \frac{\ln(\|T_L(t)\|_{\mathcal{L}(X)})}{t}.$$

The *essential growth bound* $\omega_{0,\mathrm{ess}}(L) \in [-\infty, +\infty)$ of L is defined by

$$\omega_{0,\mathrm{ess}}(L) := \lim_{t \to +\infty} \frac{\ln(\|T_L(t)\|_{\mathrm{ess}})}{t},$$

where $\|T_L(t)\|_{\mathrm{ess}}$ is the essential norm of $T_L(t)$ defined by

$$\|T_L(t)\|_{\mathrm{ess}} = \kappa\big(T_L(t) B_X(0,1)\big),$$

here $B_X(0,1) = \{x \in X:\ \|x\|_X \leqslant 1\}$, and for each bounded set $B \subset X$, $\kappa(B) = \inf\{\varepsilon > 0$: B can be covered by a finite number of balls of radius $\leqslant \varepsilon\}$ is the Kuratovsky measure of noncompactness.

We have the following result, the existence of the projector was first proved by Webb [28, 29], and the fact that there is a finite number of points of the spectrum is proved by Engel and Nagel [10].

Theorem 3.2. *Let $L : D(L) \subset X \to X$ be the infinitesimal generator of a linear C^0-semigroup $\{T_L(t)\}_{t \geqslant 0}$ on a Banach space X. Then*

$$\omega_0(L) = \max\Big(\omega_{0,\mathrm{ess}}(L), \max_{\lambda \in \sigma(L) \setminus \sigma_{\mathrm{ess}}(L)} \mathrm{Re}(\lambda)\Big).$$

Assume in addition that $\omega_{0,\mathrm{ess}}(L) < \omega_0(L)$. Then for each $\gamma \in (\omega_{0,\mathrm{ess}}(L), \omega_0(L)]$, $\{\lambda \in \sigma(L)\colon \operatorname{Re}(\lambda) \geqslant \gamma\} \subset \sigma_p(L)$ is nonempty, finite and contains only poles of the resolvent of L. Moreover, there exists a finite rank bounded linear operator of projection $\Pi : X \to X$ satisfying the following properties:

(a) $\Pi(\lambda I - L)^{-1} = (\lambda I - L)^{-1}\Pi$, $\forall\lambda \in \rho(L)$;
(b) $\sigma(L_{\Pi(X)}) = \{\lambda \in \sigma(L)\colon \operatorname{Re}(\lambda) \geqslant \gamma\}$;
(c) $\sigma(L_{(I-\Pi)(X)}) = \sigma(L) \setminus \sigma(L_{\Pi(X)})$.

In Theorem 3.2 the projector Π is the projection on the direct sum of the generalized eigenspaces of L associated to all points $\lambda \in \sigma(L)$ with $\operatorname{Re}(\lambda) \geqslant \gamma$. As a consequence of Theorem 3.2 we have the following corollary.

Corollary 3.3. *Let $L : D(L) \subset X \to X$ be the infinitesimal generator of a linear C^0-semigroup $\{T_L(t)\}_{t\geqslant 0}$ on a Banach space X, and assume that $\omega_{0,\mathrm{ess}}(L) < \omega_0(L)$. Then*

$$\bigl\{\lambda \in \sigma(L)\colon \operatorname{Re}(\lambda) > \omega_{0,\mathrm{ess}}(L)\bigr\} \subset \sigma_P(L)$$

and each $\hat{\lambda} \in \{\lambda \in \sigma(L)\colon \operatorname{Re}(\lambda) > \omega_{0,\mathrm{ess}}(L)\}$ is a pole of the resolvent of L. That is, $\hat{\lambda}$ is isolated in $\sigma(L)$, and there exists an integer $k_0 \geqslant 1$ (the order of the pole) such that the Laurent's expansion of the resolvent takes the following form

$$(\lambda I - L)^{-1} = \sum_{n=-k_0}^{\infty} (\lambda - \lambda_0)^n B_n^{\lambda_0},$$

where $\{B_n^{\lambda_0}\}$, $n \geqslant k_0$, are bounded linear operators on X, and the above series converges in the norm of operators whenever $|\lambda - \lambda_0|$ is small enough.

The following result is due to Magal and Ruan [20, Lemma 2.1 and Proposition 3.6].

Theorem 3.4. *Let $(X, \|\,.\,\|)$ be a Banach space and $L : D(L) \subset X \to X$ be a linear operator. Assume that $\rho(L) \neq \emptyset$ and L_0, the part of L in $\overline{D(L)}$, is the infinitesimal generator of a linear C^0-semigroup $\{T_{L_0}(t)\}_{t\geqslant 0}$ on a Banach space $\overline{D(L)}$. Then*

$$\sigma(L) = \sigma(L_0).$$

Let $\Pi_0 : \overline{D(L)} \to \overline{D(L)}$ be a bounded linear operator of projection. Assume that

$$\Pi_0(\lambda I - L_0)^{-1} = (\lambda I - L_0)^{-1}\Pi_0, \quad \forall\lambda > \omega,$$

and

$$\Pi_0\bigl(\overline{D(L)}\bigr) \subset D(L_0) \quad \text{and} \quad L_0|_{\Pi_0(\overline{D(L)})} \text{ is bounded.}$$

Then there exists a unique bounded linear operator of projection Π on X satisfying the following properties:

(i) $\Pi|_{\overline{D(L)}} = \Pi_0$;
(ii) $\Pi(X) \subset \overline{D(L)}$;
(iii) $\Pi(\lambda I - L)^{-1} = (\lambda I - L)^{-1}\Pi$, $\forall \lambda > \omega$.

Moreover, for each $x \in X$ we have the following approximation formula

$$\Pi x = \lim_{\lambda \to +\infty} \Pi_0 \lambda(\lambda I - L)^{-1} x.$$

Now we go back to consider the FDE (2.1). We first have the following property.

Theorem 3.5. *For the operator A defined in* (3.1)*, the resolvent set of A satisfies*

$$\rho(A) = \rho(B),$$

where B is an $n \times n$ matrix defined in (2.1)*. Moreover, for each $\lambda \in \rho(A)$, we have the following explicit formula for the resolvent of A*:

$$(\lambda I - A)^{-1}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix} = \begin{pmatrix}0_{\mathbb{R}^n}\\ \psi\end{pmatrix}$$

$$\Leftrightarrow \quad \psi(\theta) = e^{\lambda\theta}(\lambda I - B)^{-1}\big[\varphi(0) + \alpha\big] + \int_\theta^0 e^{\lambda(\theta - s)}\varphi(s)\, ds. \tag{3.4}$$

Proof. Let us first prove that $\rho(A) \subset \rho(B)$. We only need to show that $\sigma(B) \subset \sigma(A)$. Let $\lambda \in \sigma(B)$. Then, there exists $x \in \mathbb{C}^n \setminus \{0\}$ such that $Bx = \lambda x$. If we consider

$$\varphi(\theta) = e^{\lambda\theta}x,$$

we have

$$A\begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi\end{pmatrix} = \begin{pmatrix}-\varphi'(0) + B\varphi(0)\\ \varphi'\end{pmatrix} = \begin{pmatrix}-\lambda x + Bx\\ \lambda\varphi\end{pmatrix} = \begin{pmatrix}0_{\mathbb{R}^n}\\ \lambda\varphi\end{pmatrix}.$$

Thus $\lambda \in \sigma(A)$. This implies that $\sigma(B) \subset \sigma(A)$. On the other hand, if $\lambda \in \rho(B)$, for $\binom{\alpha}{\varphi} \in X$ we must have $\binom{0_{\mathbb{R}^n}}{\psi} \in D(A)$ such that

$$\begin{aligned}
&(\lambda I - A)\begin{pmatrix}0_{\mathbb{R}^n}\\ \psi\end{pmatrix} = \begin{pmatrix}\alpha\\ \varphi\end{pmatrix}\\
&\quad\Leftrightarrow \quad \begin{cases}\psi'(0) - B\psi(0) = \alpha,\\ \lambda\psi - \psi' = \varphi\end{cases}\\
&\quad\Leftrightarrow \quad \begin{cases}(\lambda I - B)\psi(0) = \alpha + \varphi(0),\\ \lambda\psi - \psi' = \varphi\end{cases}\\
&\quad\Leftrightarrow \quad \begin{cases}(\lambda I - B)\psi(0) = \alpha + \varphi(0),\\ \psi(\theta) = e^{\lambda(\theta - \hat\theta)}\psi(\hat\theta) + \int_{\hat\theta}^{\theta} e^{\lambda(\theta - l)}\varphi(l)\, dl, \ \forall \theta \geqslant \hat\theta,\end{cases}
\end{aligned}$$

$$\Leftrightarrow \quad \begin{cases} (\lambda I - B)\psi(0) = \alpha + \varphi(0), \\ \psi(\hat\theta) = e^{\lambda\hat\theta}\psi(0) - \int_0^{\hat\theta} e^{\lambda(\hat\theta - l)}\varphi(l)\,dl, \ \forall\hat\theta \in [-r,0], \end{cases}$$

$$\Leftrightarrow \quad \psi(\hat\theta) = e^{\lambda\hat\theta}(\lambda I - B)^{-1}\big[\alpha + \varphi(0)\big] - \int_0^{\hat\theta} e^{\lambda(\hat\theta - l)}\varphi(l)\,dl, \quad \forall\hat\theta \in [-r,0].$$

Therefore, we obtain that $\lambda \in \rho(A)$ and the formula in (3.4) holds. □

Since B is a matrix on $\mathbb{R}^n$, we have $\omega_0(B) := \sup_{\lambda\in\sigma(B)} \operatorname{Re}(\lambda)$ and the following lemma.

Lemma 3.6. *The linear operator $A : D(A) \subset X \to X$ is a Hille–Yosida operator. More precisely, for each $\omega_A > \omega_0(B)$, there exists $M_A \geqslant 1$ such that*

$$\big\|(\lambda I - A)^{-n}\big\|_{\mathcal{L}(X)} \leqslant \frac{M_A}{(\lambda - \omega_A)^n}, \quad \forall n \geqslant 1, \ \forall \lambda > \omega_A. \tag{3.5}$$

Proof. Let $\omega_A > \omega_0(B)$. We can define the equivalent norm on $\mathbb{R}^n$

$$|x| := \sup_{t\geqslant 0} e^{-\omega_A t}\big\|e^{Bt}x\big\|.$$

Then we have

$$\big|e^{Bt}x\big| \leqslant e^{\omega_A t}|x|, \quad \forall t \geqslant 0,$$

and

$$\|x\| \leqslant |x| \leqslant M_A\|x\|,$$

where

$$M_A := \sup_{t\geqslant 0}\big\|e^{(B-\omega_A I)t}\big\|_{M_n(\mathbb{R})}.$$

Moreover, for each $\lambda > \omega_A$, we have

$$\big|(\lambda I - B)^{-1}x\big| = \left|\int_0^{+\infty} e^{-\lambda s}e^{Bs}x\,ds\right| \leqslant \frac{|x|}{\lambda - \omega_A}.$$

We define $\|.\|$ the equivalent norm on X by

$$\left|\begin{pmatrix}\alpha \\ \varphi\end{pmatrix}\right| = |\alpha| + \|\varphi\|_{\omega_A},$$

where

$$\|\varphi\|_{\omega_A} := \sup_{\theta\in[-r,0]}\big|e^{-\omega_A\theta}\varphi(\theta)\big|.$$

Using (3.4) and the above results, we obtain

$$
\begin{aligned}
&\left|(\lambda I - A)^{-1}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}\right| \\
&\quad\leqslant \sup_{\theta\in[-r,0]}\left[e^{-\omega_A\theta}e^{\lambda\theta}\left|(\lambda I - B)^{-1}\left[\varphi(0)+\alpha\right]\right| + e^{-\omega_A\theta}\int_\theta^0 e^{\lambda(\theta-s)}\left|\varphi(s)\right|ds\right] \\
&\quad\leqslant \sup_{\theta\in[-r,0]}\left[e^{-\omega_A\theta}e^{\lambda\theta}\frac{1}{\lambda-\omega_A}\left[\left|\varphi(0)\right|+|\alpha|\right] + e^{-\omega_A\theta}e^{\lambda\theta}\int_\theta^0 e^{-(\lambda-\omega_A)s}\,ds\,\|\varphi\|_{\omega_A}\right] \\
&\quad= \frac{1}{\lambda-\omega_A}|\alpha| + \sup_{\theta\in[-r,0]}\left[\frac{e^{-\omega_A\theta}e^{\lambda\theta}}{\lambda-\omega_A}\left|\varphi(0)\right| + \frac{e^{-\omega_A\theta}e^{\lambda\theta}[e^{-(\lambda-\omega_A)\theta}-1]}{\lambda-\omega_A}\|\varphi\|_{\omega_A}\right] \\
&\quad\leqslant \frac{1}{\lambda-\omega_A}\left[|\alpha|+\|\varphi\|_{\omega_A}\right] \\
&\quad= \frac{1}{\lambda-\omega_A}\left|\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}\right|.
\end{aligned}
$$

Therefore, (3.5) holds and the proof is completed. □

Since A is a Hille–Yosida operator, A generates a non-degenerate integrated semigroup $\{S_A(t)\}_{t\geqslant 0}$ on X. It follows from Thieme [22] and Kellerman and Hieber [17] that the abstract Cauchy problem (3.1) has at most one integrated solution.

Lemma 3.7. *Let $h \in L^1((0,\tau),\mathbb{R}^n)$ and $\varphi \in C([-r,0],\mathbb{R}^n)$. Then there exists a unique integrated solution, $t \to v(t)$, of the Cauchy problem* (3.1) *which can be expressed explicitly by the following formula*

$$v(t) = \begin{pmatrix}0_{\mathbb{R}^n}\\ u(t)\end{pmatrix}$$

with

$$u(t)(\theta) = x(t+\theta), \quad \forall t \in [0,\tau],\ \forall \theta \in [-r,0], \tag{3.6}$$

where

$$x(t) = \begin{cases}\varphi(t), & t \in [-r,0],\\ e^{Bt}\varphi(0) + \int_0^t e^{B(t-s)}h(s)\,ds, & t\in[0,\tau].\end{cases}$$

Proof. Since A is a Hille–Yosida operator, there is at most one integrated solution of the Cauchy problem (3.1). So it is sufficient to prove that u defined by (3.6) satisfies for each $t \in [0,\tau]$ the following

$$\begin{pmatrix}0_{\mathbb{R}^n}\\ \int_0^t u(l)\,dl\end{pmatrix} \in D(A) \tag{3.7}$$

and

$$\begin{pmatrix} 0_{\mathbb{R}^n} \\ u(t) \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} + A\begin{pmatrix} 0_{\mathbb{R}^n} \\ \int_0^t u(l)\,dl \end{pmatrix} + \begin{pmatrix} \int_0^t h(l)\,dl \\ 0 \end{pmatrix}. \tag{3.8}$$

Since

$$\int_0^t u(l)(\theta)\,dl = \int_0^t x(l+\theta)\,dl = \int_\theta^{t+\theta} x(s)\,ds$$

and $x \in C([-r,\tau],\mathbb{R}^n)$, $\int_0^t u(l)\,dl \in C^1([-r,0],\mathbb{R}^n)$. Therefore, (3.7) follows. Moreover,

$$A\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) + B\varphi(0) \\ \varphi' \end{pmatrix}$$

whenever $\varphi \in C^1([-r,0],\mathbb{R}^n)$. Hence

$$\begin{aligned} A\begin{pmatrix} 0 \\ \int_0^t u(l)\,dl \end{pmatrix} &= \begin{pmatrix} -[x(t) - x(0)] + B\int_0^t x(s)\,ds \\ x(t+.) - x(.) \end{pmatrix} \\ &= -\begin{pmatrix} 0 \\ \varphi \end{pmatrix} + \begin{pmatrix} -[x(t) - \varphi(0)] + B\int_0^t x(s)\,ds \\ x(t+.) \end{pmatrix}. \end{aligned}$$

Therefore, (3.8) is satisfied if and only if

$$x(t) = \varphi(0) + B\int_0^t x(s)\,ds + \int_0^t h(s)\,ds. \tag{3.9}$$

By using the usual variation of constant formula, we deduce that (3.9) is equivalent to

$$x(t) = e^{Bt}\varphi(0) + \int_0^t e^{B(t-s)}h(s)\,ds.$$

The proof is completed. □

Recall that $A_0 : D(A_0) \subset \overline{D(A)} \to \overline{D(A)}$, the part of A in $\overline{D(A)}$, is defined by

$$A_0\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = A\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}, \quad \forall \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in D(A_0),$$

where

$$D(A_0) = \left\{ \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in D(A):\ A\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in \overline{D(A)} \right\}.$$

From the definition of A in (2.3) and the fact that $\overline{D(A)} = \{0_{\mathbb{R}^n}\} \times C([-r,0],\mathbb{R}^n)$, we know that A_0 is the linear operator defined by

$$A_0\begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi\end{pmatrix} = \begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi'\end{pmatrix}, \quad \forall\begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi\end{pmatrix} \in D(A_0),$$

where

$$D(A_0) = \left\{\begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi\end{pmatrix} \in \{0_{\mathbb{R}^n}\} \times C^1\big([-r,0],\mathbb{R}^n\big)\colon\ -\varphi'(0) + B\varphi(0) = 0\right\}.$$

Now by using the fact that A is a Hille–Yosida operator, we deduce that A_0 is the infinitesimal generator of a strongly continuous semigroup $\{T_{A_0}(t)\}_{t\geqslant 0}$ and

$$v(t) = T_{A_0}(t)\begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi\end{pmatrix}$$

is an integrated solution of

$$\frac{dv(t)}{dt} = Av(t), \quad t \geqslant 0, \qquad v(0) = \begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi\end{pmatrix} \in \overline{D(A)}.$$

Using Lemma 3.7 with $h = 0$, we obtain the following result.

Lemma 3.8. *The linear operator A_0 is the infinitesimal generator of a strongly continuous semigroup $\{T_{A_0}(t)\}_{t\geqslant 0}$ of bounded linear operators on $\overline{D(A)}$ which is defined by*

$$T_{A_0}(t)\begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi\end{pmatrix} = \begin{pmatrix}0_{\mathbb{R}^n}\\ \hat{T}_{A_0}(t)(\varphi)\end{pmatrix}, \tag{3.10}$$

where

$$\hat{T}_{A_0}(t)(\varphi)(\theta) = \begin{cases} e^{B(t+\theta)}\varphi(0), & t+\theta \geqslant 0,\\ \varphi(t+\theta), & t+\theta \leqslant 0.\end{cases}$$

Since A is a Hille–Yosida operator, we know that A generates an integrated semigroup $\{S_A(t)\}_{t\geqslant 0}$ on X, and $t \to S_A(t)\binom{x}{\varphi}$ is an integrated solution of

$$\frac{dv(t)}{dt} = Av(t) + \begin{pmatrix}x\\ \varphi\end{pmatrix}, \quad t \geqslant 0, \qquad v(0) = 0.$$

Since $S_A(t)$ is linear we have

$$S_A(t)\begin{pmatrix}x\\ \varphi\end{pmatrix} = S_A(t)\begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi\end{pmatrix} + S_A(t)\begin{pmatrix}x\\ 0\end{pmatrix},$$

where

$$S_A(t)\begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi\end{pmatrix}=\int_0^t T_{A_0}(l)\begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi\end{pmatrix}dl$$

and $S_A(t)\binom{x}{0}$ is an integrated solution of

$$\frac{dv(t)}{dt}=Av(t)+\begin{pmatrix}x\\0\end{pmatrix},\quad t\geqslant 0,\qquad (0)=0.$$

Therefore, by using Lemma 3.7 with $h(t)=x$ and the above results, we obtain the following result.

Lemma 3.9. *The linear operator A generates an integrated semigroup $\{S_A(t)\}_{t\geqslant 0}$ on X. Moreover,*

$$S_A(t)\begin{pmatrix}x\\ \varphi\end{pmatrix}=\begin{pmatrix}0_{\mathbb{R}^n}\\ \hat{S}_A(t)(x,\varphi)\end{pmatrix},\quad \begin{pmatrix}x\\ \varphi\end{pmatrix}\in X,$$

where $\hat{S}_A(t)$ is the linear operator defined by

$$\hat{S}_A(t)(x,\varphi)=\hat{S}_A(t)(0,\varphi)+\hat{S}_A(t)(x,0)$$

with

$$\hat{S}_A(t)(0,\varphi)(\theta)=\int_0^t \hat{T}_{A_0}(s)(\varphi)(\theta)\,ds=\int_{-\theta}^t e^{B(s+\theta)}\varphi(0)\,ds+\int_0^{-\theta}\varphi(s+\theta)\,ds$$

and

$$\hat{S}_A(t)(x,0)(\theta)=\begin{cases}\int_0^{t+\theta}e^{Bs}x\,ds, & t+\theta\geqslant 0,\\ 0, & t+\theta\leqslant 0.\end{cases}$$

Now we focus on the spectrums of A and $A+L$. Since A is a Hille–Yosida operator, so is $A+L$. Moreover, $(A+L)_0 : D((A+L)_0)\subset\overline{D(A)}\to\overline{D(A)}$, the part of $A+L$ in $\overline{D(A)}$, is a linear operator defined by

$$(A+L)_0\begin{pmatrix}0\\ \varphi\end{pmatrix}=\begin{pmatrix}0\\ \varphi'\end{pmatrix},\quad \forall\begin{pmatrix}0\\ \varphi\end{pmatrix}\in D\big((A+L)_0\big),$$

where

$$D\big((A+L)_0\big)=\left\{\begin{pmatrix}0\\ \varphi\end{pmatrix}\in\{0_{\mathbb{R}^n}\}\times C^1\big([-r,0],\mathbb{R}^n\big):\ \varphi'(0)=B\varphi(0)+\hat{L}(\varphi)\right\}.$$

From Theorems 3.4 and 3.5, we know that

$$\sigma(B) = \sigma(A) = \sigma(A_0) \quad \text{and} \quad \sigma(A+L) = \sigma\big((A+L)_0\big).$$

From (3.10), we have

$$\hat{T}_{A_0}(t)(\varphi)(\theta) = e^{B(r+\theta)} e^{B(t-r)} \varphi(0), \quad t \geqslant r,\ \theta \in [-r, 0].$$

Therefore,

$$\hat{T}_{A_0}(t) = L_2 L_1,$$

where $L_1 : C \to \mathbb{R}^n$ and $L_2 : \mathbb{R}^n \to C$ are linear operators defined by

$$L_1 \varphi = e^{B(t-r)} \varphi(0), \quad \varphi \in C,\ t \geqslant r,$$

and

$$L_2(x)(\theta) = e^{B(r+\theta)} x, \quad x \in \mathbb{R}^n,\ \theta \in [-r, 0],$$

respectively. Clearly L_1 is compact. Hence, we have

$$\omega_{0,\mathrm{ess}}(A_0) = -\infty \quad \text{and} \quad \sigma(B) = \sigma(A) = \sigma_P(A_0) = \sigma(A_0).$$

Therefore,

$$\omega_0(A_0) = \sup_{\lambda \in \sigma_P(A_0)} \mathrm{Re}(\lambda).$$

In the following lemma, we specify the point spectrum of $(A+L)_0$.

Lemma 3.10. *The point spectrum of* $(A+L)_0$ *is the set*

$$\sigma_P\big((A+L)_0\big) = \big\{\lambda \in \mathbb{C}:\ \det\big(\Delta(\lambda)\big) = 0\big\},$$

where

$$\Delta(\lambda) = \lambda I - B - \hat{L}\big(e^{\lambda \cdot} I\big) = \lambda I - B - \int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta). \tag{3.11}$$

Proof. Let $\lambda \in \mathbb{C}$. Then $\lambda \in \sigma_P((A+L)_0)$ if and only if there exists $\binom{0_{\mathbb{R}^n}}{\varphi} \in D((A+L)_0) \setminus \{0\}$ such that

$$(A+L)_0 \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \lambda \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}.$$

That is, $\lambda \in \sigma_P((A+L)_0)$ if and only if there exists $\varphi \in C^1([-r,0],\mathbb{C}^n) \setminus \{0\}$ such that

$$\varphi'(\theta) = \lambda\varphi(\theta), \quad \forall \theta \in [-r,0], \tag{3.12}$$

and

$$\varphi'(0) = B\varphi(0) + \hat{L}(\varphi). \tag{3.13}$$

Eq. (3.12) is equivalent to

$$\varphi(\theta) = e^{\lambda\theta}\varphi(0), \quad \forall \theta \in [-r,0]. \tag{3.14}$$

Therefore,

$$\varphi \neq 0 \quad \Leftrightarrow \quad \varphi(0) \neq 0.$$

By combining (3.13) and (3.14), we obtain

$$\lambda\varphi(0) = B\varphi(0) + \hat{L}\big(e^{\lambda\cdot}\varphi(0)\big).$$

The proof is completed. □

From the discussion in this section, we obtain the following proposition.

Proposition 3.11. *The linear operator $A+L : D(A) \to X$ is a Hille–Yosida operator. $(A+L)_0$ is the infinitesimal generator of a strongly continuous semigroup $\{T_{(A+L)_0}(t)\}_{t \geqslant 0}$ of bounded linear operators on $\overline{D(A)}$. Moreover,*

$$T_{(A+L)_0}(t)\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \hat{T}_{(A+L)_0}(t)(\varphi) \end{pmatrix} \tag{3.15}$$

with

$$\hat{T}_{(A+L)_0}(t)(\varphi)(\theta) = x(t+\theta), \quad \forall t \geqslant 0,\ \forall \theta \in [-r,0],$$

where

$$x(t) = \begin{cases} \varphi(t), & \forall t \in [-r,0], \\ e^{Bt}\varphi(0) + \int_0^t e^{B(t-s)}\hat{L}(x_s)\,ds, & \forall t \geqslant 0. \end{cases}$$

Furthermore

$$\omega_{0,\mathrm{ess}}\big((A+L)_0\big) = -\infty \quad \text{and} \quad \omega_0\big((A+L)_0\big) = \max_{\lambda \in \sigma_P((A+L)_0)} \mathrm{Re}(\lambda),$$

$$\sigma(A+L) = \sigma\big((A+L)_0\big) = \sigma_P\big((A+L)_0\big) = \big\{\lambda \in \mathbb{C}\colon \det\big(\Delta(\lambda)\big) = 0\big\},$$

and each $\lambda_0 \in \sigma((A+L)_0) = \sigma(A+L)$ is a pole of $(\lambda I - (A+L))^{-1}$. For each $\gamma \in \mathbb{R}$, the subset $\{\lambda \in \sigma((A+L)_0)\colon \mathrm{Re}(\lambda) \geqslant \gamma\}$ is either empty or finite.

Proof. The first part of the result follows immediately from Lemma 3.7 applied with $h(t) = \hat{L}(x_t)$. So it remains to prove that $\omega_{0,\mathrm{ess}}((A+L)_0) = -\infty$. But this property follows from the fact that $T_{(A+L)_0}(t)$ is compact for each t large enough. This is an immediate consequence of Theorem 3 in Thieme [23] (which applies because $LT_{A_0}(t)$ is compact for each $t > 0$, and $T_{A_0}(t)$ is compact for $t \geqslant r$). $\square$

4. Projectors on the eigenspaces

Let $\lambda_0 \in \sigma(A+L)$. From the above discussion we already knew that λ_0 is a pole of $(\lambda I - (A+L))^{-1}$ of finite order $k_0 \geqslant 1$. This means that λ_0 is isolated in $\sigma(A+L)$ and the Laurent's expansion of the resolvent around λ_0 takes the following form

$$\big(\lambda I - (A+L)\big)^{-1} = \sum_{n=-k_0}^{+\infty} (\lambda - \lambda_0)^n B_n^{\lambda_0}. \tag{4.1}$$

The bounded linear operator $B_{-1}^{\lambda_0}$ is the projector on the generalized eigenspace of $(A+L)$ associated to λ_0. The goal of this section is to provide a method to compute $B_{-1}^{\lambda_0}$. We remark that

$$(\lambda - \lambda_0)^{k_0}\big(\lambda I - (A+L)\big)^{-1} = \sum_{m=0}^{+\infty} (\lambda - \lambda_0)^m B_{m-k_0}^{\lambda_0}.$$

So we have the following approximation formula

$$B_{-1}^{\lambda_0} = \lim_{\lambda \to \lambda_0} \frac{1}{(k_0 - 1)!} \frac{d^{k_0-1}}{d\lambda^{k_0-1}} \big((\lambda - \lambda_0)^{k_0}\big(\lambda I - (A+L)\big)^{-1}\big). \tag{4.2}$$

In order to give an explicit formula for $B_{-1}^{\lambda_0}$, we need the following results.

Lemma 4.1. *For each $\lambda \in \rho(A+L)$, we have the following explicit formula for the resolvent of $A+L$:*

$$\begin{aligned} &\big(\lambda I - (A+L)\big)^{-1}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix} = \begin{pmatrix}0_{\mathbb{R}^n}\\ \psi\end{pmatrix} \\ &\quad\Leftrightarrow\quad \psi(\theta) = \int_\theta^0 e^{\lambda(\theta - s)}\varphi(s)\,ds + e^{\lambda\theta}\Delta(\lambda)^{-1}\left[\alpha + \varphi(0) + \hat{L}\left(\int_{\cdot}^0 e^{\lambda(\cdot - s)}\varphi(s)\,ds\right)\right]. \end{aligned} \tag{4.3}$$

Proof. We consider the linear operator $A_\gamma : D(A) \subset X \to X$ defined by

$$A_\gamma\begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi\end{pmatrix} = \begin{pmatrix}-\varphi'(0) + (B - \gamma I)\varphi(0)\\ \varphi'\end{pmatrix}, \quad \forall\begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi\end{pmatrix} \in D(A),$$

and

$$L_\gamma \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} \hat{L}(\varphi) + \gamma\varphi(0) \\ 0_C \end{pmatrix}.$$

Then we have

$$A + L = A_\gamma + L_\gamma.$$

Moreover,

$$\omega_0(B - \gamma I) = \max_{\lambda \in \sigma(B-\gamma I)} \mathrm{Re}(\lambda) = \max_{\lambda \in \sigma(B)} \mathrm{Re}(\lambda) - \gamma = \omega_0(B) - \gamma.$$

Hence by Theorem 3.5, for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \omega_0(B) - \gamma$, we have $\lambda \in \rho(A_\gamma)$ and

$$\begin{aligned} &(\lambda I - A_\gamma)^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} \\ &\Leftrightarrow \quad \psi(\theta) = e^{\lambda\theta}\big(\lambda I - (B - \gamma I)\big)^{-1}\big[\varphi(0) + \alpha\big] + \int_\theta^0 e^{\lambda(\theta - s)}\varphi(s)\,ds. \end{aligned} \tag{4.4}$$

Therefore, for each $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \omega_0(B) - \gamma$, we deduce that $[\lambda I - (A_\gamma + L_\gamma)]$ is invertible if and only if $I - L_\gamma(\lambda I - A_\gamma)^{-1}$ is invertible, and

$$\big(\lambda I - (A_\gamma + L_\gamma)\big)^{-1} = (\lambda I - A_\gamma)^{-1}\big[I - L_\gamma(\lambda I - A_\gamma)^{-1}\big]^{-1}. \tag{4.5}$$

We also know that $[I - L_\gamma(\lambda I - A_\gamma)^{-1}]\binom{\alpha}{\varphi} = \binom{\hat{\alpha}}{\hat{\varphi}}$ is equivalent to $\varphi = \hat{\varphi}$ and

$$\begin{aligned} &\alpha - \big[\hat{L}\big(e^{\lambda\cdot}\big(\lambda I - (B - \gamma I)\big)^{-1}\alpha\big) + \gamma\big(\lambda I - (B - \gamma I)\big)^{-1}\alpha\big] \\ &\quad = \hat{\alpha} + \begin{bmatrix} \hat{L}(e^{\lambda\cdot}(\lambda I - (B - \gamma I))^{-1}\hat{\varphi}(0) + \int_\cdot^0 e^{\lambda(.-s)}\hat{\varphi}(s)\,ds) \\ + \gamma(\lambda I - (B - \gamma I))^{-1}\hat{\varphi}(0) \end{bmatrix}. \end{aligned}$$

Because

$$\begin{aligned} &\alpha - \hat{L}\big(e^{\lambda\cdot}\big(\lambda I - (B - \gamma I)\big)^{-1}\alpha\big) - \gamma\big(\lambda I - (B - \gamma I)\big)^{-1}\alpha \\ &\quad = \big[\lambda I - (B - \gamma I) - \hat{L}\big(e^{\lambda\cdot} I\big) - \gamma I\big]\big(\lambda I - (B - \gamma I)\big)^{-1}\alpha \\ &\quad = \big[\lambda I - B - \hat{L}\big(e^{\lambda\cdot} I\big)\big]\big(\lambda I - (B - \gamma I)\big)^{-1}\alpha \\ &\quad = \Delta(\lambda)\big(\lambda I - (B - \gamma I)\big)^{-1}\alpha, \end{aligned}$$

we deduce that $[I - L_\gamma(\lambda I - A_\gamma)^{-1}]$ is invertible if and only if $\Delta(\lambda) = [\lambda I - B - \hat{L}(e^{\lambda\cdot} I)]$ is invertible. Moreover,

$$\big[I - L_\gamma(\lambda I - A_\gamma)^{-1}\big]^{-1} \begin{pmatrix} \hat{\alpha} \\ \hat{\varphi} \end{pmatrix} = \begin{pmatrix} \alpha \\ \varphi \end{pmatrix}$$

is equivalent to $\varphi = \hat{\varphi}$ and

$$\alpha = \big(\lambda I - (B - \gamma I)\big)\Delta(\lambda)^{-1} \begin{bmatrix} \hat{\alpha} + \hat{L}(e^{\lambda \cdot}(\lambda I - (B - \gamma I))^{-1}\hat{\varphi}(0) + \int_{\cdot}^{0} e^{\lambda(\cdot - s)}\hat{\varphi}(s)\,ds) \\ +\gamma(\lambda I - (B - \gamma I))^{-1}\hat{\varphi}(0) \end{bmatrix}. \tag{4.6}$$

Note that $A + L = A_\gamma + L_\gamma$. By using (4.4), (4.5) and (4.6), we obtain for each $\gamma > 0$ large enough that

$$\begin{aligned} &\big(\lambda I - (A + L)\big)^{-1}\begin{pmatrix}\alpha \\ \varphi\end{pmatrix} = \begin{pmatrix}0_{\mathbb{R}^n} \\ \psi\end{pmatrix} \\ &\quad \Leftrightarrow \quad \psi(\theta) = e^{\lambda\theta}\big(\lambda I - (B - \gamma I)\big)^{-1}\varphi(0) + \int_\theta^0 e^{\lambda(\theta - s)}\varphi(s)\,ds \\ &\qquad\qquad + e^{\lambda\theta}\Delta(\lambda)^{-1}\begin{bmatrix} \alpha + \hat{L}(e^{\lambda \cdot}(\lambda I - (B - \gamma I))^{-1}\varphi(0) + \int_{\cdot}^{0} e^{\lambda(\cdot - s)}\varphi(s)\,ds) \\ +\gamma(\lambda I - (B - \gamma I))^{-1}\varphi(0) \end{bmatrix}. \end{aligned}$$

Now by taking the limit when $\gamma \to +\infty$, the result follows. $\square$

The map $\lambda \to \Delta(\lambda)$ from $\mathbb{C}$ into $M_n(\mathbb{C})$ is differentiable and

$$\Delta^{(1)}(\lambda) := \frac{d\Delta(\lambda)}{d\lambda} = I - \int_{-r}^{0} d\eta(\theta)\,\theta e^{\lambda\theta}.$$

So the map $\lambda \to \Delta(\lambda)$ is analytic and

$$\Delta^{(n)}(\lambda) := \frac{d^n\Delta(\lambda)}{d\lambda^n} = -\int_{-r}^{0} d\eta(\theta)\,\theta^n e^{\lambda\theta}, \quad n \geqslant 2.$$

We know that the inverse function

$$\psi : L \to L^{-1}$$

of a linear operator $L \in \mathrm{Isom}(X)$ is differentiable, and

$$D\psi(L)\hat{L} = -L^{-1} \circ \hat{L} \circ L^{-1}.$$

Applying this result, we deduce that $\lambda \to \Delta(\lambda)^{-1}$ from $\rho(A + L)$ into $M_n(\mathbb{C})$ is differentiable, and $\frac{d}{d\lambda}\Delta(\lambda)^{-1} = -\Delta(\lambda)^{-1}(\frac{d}{d\lambda}\Delta(\lambda))\Delta(\lambda)^{-1}$. Therefore, we obtain that the map $\lambda \to \Delta(\lambda)^{-1}$ is analytic and has a Laurent's expansion around λ_0

$$\Delta(\lambda)^{-1} = \sum_{n=-\hat{k}_0}^{+\infty} (\lambda - \lambda_0)^n \Delta_n.$$

From the following lemma we know that $\hat{k}_0 = k_0$.

Lemma 4.2. *Let $\lambda_0 \in \sigma(A+L)$. Then the following statements are equivalent*

(a) λ_0 *is a pole of order* k_0 *of* $(\lambda I - (A+L))^{-1}$;
(b) λ_0 *is a pole of order* k_0 *of* $\Delta(\lambda)^{-1}$;
(c) $\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^{k_0}\Delta(\lambda)^{-1} \neq 0$ *and* $\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^{k_0+1}\Delta(\lambda)^{-1} = 0$.

Proof. The proof follows from the explicit formula of the resolvent of $A+L$ obtained in Lemma 4.1. □

Lemma 4.3. *The matrices* $\Delta_{-1},\ldots,\Delta_{-k_0}$ *satisfy*

$$\Delta_{k_0}(\lambda_0)\begin{pmatrix}\Delta_{-1}\\ \Delta_{-2}\\ \vdots\\ \Delta_{-k_0+1}\\ \Delta_{-k_0}\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$$

and

$$(\,\Delta_{-k_0}\quad \Delta_{-k_0+1}\quad \cdots\quad \Delta_{-2}\quad \Delta_{-1}\,)\,\Delta_{k_0}(\lambda_0) = (\,0\quad \cdots\quad 0\,),$$

where

$$\Delta_{k_0}(\lambda_0) = \begin{pmatrix}\Delta(\lambda_0) & \Delta^{(1)}(\lambda_0) & \Delta^{(2)}(\lambda_0)/2! & \cdots & \Delta^{(k_0-1)}(\lambda_0)/(k_0-1)!\\ 0 & \ddots & \ddots & \ddots & \vdots\\ \vdots & 0 & \ddots & \ddots & \Delta^{(2)}(\lambda_0)/2!\\ \vdots & & \ddots & \ddots & \Delta^{(1)}(\lambda_0)\\ 0 & \cdots & \cdots & 0 & \Delta(\lambda_0)\end{pmatrix}.$$

Proof. We have

$$(\lambda-\lambda_0)^{k_0} I = \Delta(\lambda)\left(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\Delta_{n-k_0}\right) = \left(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\Delta_{n-k_0}\right)\Delta(\lambda).$$

Hence,

$$\begin{aligned}(\lambda-\lambda_0)^{k_0} I &= \left(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\frac{\Delta^{(n)}(\lambda_0)}{n!}\right)\left(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\Delta_{n-k_0}\right)\\ &= \sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\sum_{k=0}^{n}\frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!}\Delta_{k-k_0}\end{aligned}$$

and

$$(\lambda-\lambda_0)^{k_0} I = \sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\sum_{k=0}^{n}\Delta_{k-k_0}\frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!}.$$

By the uniqueness of the Taylor's expansion for analytic maps, we obtain that for $n \in \{0, \dots, k_0 - 1\}$,

$$0 = \sum_{k=0}^{n} \Delta_{k-k_0} \frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!} = \sum_{k=0}^{n} \frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!} \Delta_{k-k_0}.$$

Therefore, the result follows. $\square$

Now we look for an explicit formula for the projector $B_{-1}^{\lambda_0}$ on the generalized eigenspace associated to λ_0. Set

$$\Psi_1(\lambda)(\varphi)(\theta) := \int_\theta^0 e^{\lambda(\theta - s)} \varphi(s)\, ds$$

and

$$\Psi_2(\lambda)\left(\begin{pmatrix} \alpha \\ \varphi \end{pmatrix}\right)(\theta) := e^{\lambda\theta}\left[\alpha + \varphi(0) + \hat{L}\left(\int_{.}^{0} e^{\lambda(.-s)} \varphi(s)\, ds\right)\right].$$

Then both maps are analytic and

$$\big(\lambda I - (A+L)\big)^{-1}\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \Psi_1(\lambda)(\varphi)(\theta) + \Delta(\lambda)^{-1}\Psi_2(\lambda)\binom{\alpha}{\varphi}(\theta) \end{pmatrix}.$$

We observe that the only singularity in the last expression is $\Delta(\lambda)^{-1}$. Since Ψ_1 and Ψ_2 are analytic, we have for $j = 1, 2$ that

$$\Psi_j(\lambda) = \sum_{n=0}^{+\infty} \frac{(\lambda - \lambda_0)^n}{n!} L_n^j(\lambda_0),$$

where $|\lambda - \lambda_0|$ is small enough and $L_n^j(.) := \frac{d^n \Psi_j(.)}{d\lambda^n}$, $\forall n \geqslant 0, \forall j = 1, 2$. Hence we get

$$\begin{aligned}
&\lim_{\lambda \to \lambda_0} \frac{1}{(k_0 - 1)!} \frac{d^{k_0 - 1}}{d\lambda^{k_0 - 1}} \big((\lambda - \lambda_0)^{k_0} \Psi_1(\lambda)\big) \\
&\quad = \lim_{\lambda \to \lambda_0} \frac{1}{(k_0 - 1)!} \sum_{n=0}^{+\infty} \frac{(n + k_0)!}{(n+1)!} \frac{(\lambda - \lambda_0)^{n+1}}{n!} L_n^1(\lambda_0) \\
&\quad = 0
\end{aligned}$$

and

$$\lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!}\frac{d^{k_0-1}}{d\lambda^{k_0-1}}\big[(\lambda-\lambda_0)^{k_0}\Delta(\lambda)^{-1}\Psi_2(\lambda)\big]$$
$$= \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!}\frac{d^{k_0-1}}{d\lambda^{k_0-1}}\left[\left(\sum_{n=-k_0}^{+\infty}(\lambda-\lambda_0)^{n+k_0}\Delta_n\right)\left(\sum_{n=0}^{+\infty}\frac{(\lambda-\lambda_0)^n}{n!}L_n^2(\lambda_0)\right)\right]$$
$$= \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!}\frac{d^{k_0-1}}{d\lambda^{k_0-1}}\left[\left(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^{n}\Delta_{n-k_0}\right)\left(\sum_{n=0}^{+\infty}\frac{(\lambda-\lambda_0)^n}{n!}L_n^2(\lambda_0)\right)\right]$$
$$= \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!}\frac{d^{k_0-1}}{d\lambda^{k_0-1}}\left[\sum_{n=0}^{+\infty}\sum_{j=0}^{n}(\lambda-\lambda_0)^{n-j}\Delta_{n-j-k_0}\frac{(\lambda-\lambda_0)^j}{j!}L_j^2(\lambda_0)\right]$$
$$= \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!}\frac{d^{k_0-1}}{d\lambda^{k_0-1}}\left[\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^{n}\sum_{j=0}^{n}\Delta_{n-j-k_0}\frac{1}{j!}L_j^2(\lambda_0)\right] = \sum_{j=0}^{k_0-1}\frac{1}{j!}\Delta_{-1-j}L_j^2(\lambda_0).$$

From the above results we can obtain the explicit formula for the projector $B_{-1}^{\lambda_0}$ on the generalized eigenspace associated to λ_0, which is given in the following proposition.

Proposition 4.4. *Each $\lambda_0 \in \sigma((A+L))$ is a pole of $(\lambda I-(A+L))^{-1}$ of order $k_0 \geqslant 1$. Moreover, k_0 is the only integer such that there exists $\Delta_{-k_0} \in M_n(\mathbb{R})$ with $\Delta_{-k_0} \neq 0$, such that*

$$\Delta_{-k_0} = \lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^{k_0}\Delta(\lambda)^{-1}.$$

Furthermore, the projector $B_{-1}^{\lambda_0}$ on the generalized eigenspace of $(A+L)$ associated to λ_0 is defined by the following formula

$$B_{-1}^{\lambda_0}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix} = \begin{bmatrix} 0_{\mathbb{R}^n} \\ \sum_{j=0}^{k_0-1}\frac{1}{j!}\Delta_{-1-j}L_j^2(\lambda_0)\binom{\alpha}{\varphi}\end{bmatrix}, \tag{4.7}$$

where

$$\Delta_{-j} = \lim_{\lambda\to\lambda_0}\frac{1}{(k_0-j)!}\frac{d^{k_0-j}}{d\lambda^{k_0-j}}\big((\lambda-\lambda_0)^{k_0}\Delta(\lambda)^{-1}\big), \quad j=1,\dots,k_0,$$
$$L_0^2(\lambda)\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}(\theta) = e^{\lambda\theta}\left[\alpha+\varphi(0)+\hat{L}\left(\int_{\cdot}^{0} e^{\lambda(.-s)}\varphi(s)\,ds\right)\right],$$

and

$$L_j^2(\lambda)\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}(\theta) = \frac{d^j}{d\lambda^j}\left[L_0^2(\lambda)\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}(\theta)\right]$$
$$= \sum_{k=0}^{j}C_j^k\theta^k e^{\lambda\theta}\frac{d^{j-k}}{d\lambda^{j-k}}\left[\alpha+\varphi(0)+\hat{L}\left(\int_{\cdot}^{0} e^{\lambda(.-s)}\varphi(s)\,ds\right)\right], \quad j\geqslant 1,$$

here

$$\frac{d^i}{d\lambda^i}\left[\alpha+\varphi(0)+\hat{L}\left(\int_{.}^{0} e^{\lambda(.-s)}\varphi(s)\,ds\right)\right]=\hat{L}\left(\int_{.}^{0}(.-s)^i e^{\lambda(.-s)}\varphi(s)\,ds\right),\quad i\geqslant 1.$$

5. Projector for a simple eigenvalue

In studying Hopf bifurcation it usually requires to consider the projector for a simple eigenvalue. In this section we study the case when λ_0 is a simple eigenvalue of $(A+L)$. That is, λ_0 is pole of order 1 of the resolvent of $(A+L)$ and the dimension of the eigenspace of $(A+L)$ associated to the eigenvalue λ_0 is 1.

We know that λ_0 is a pole of order 1 of the resolvent of $(A+L)$ if and only if there exists $\Delta_{-1}\neq 0$, such that

$$\Delta_{-1}=\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)\Delta(\lambda)^{-1}.$$

From Lemma 4.3, we have $\Delta_{-1}\Delta(\lambda_0)=\Delta(\lambda_0)\Delta_{-1}=0$. Hence

$$\Delta_{-1}\big[B+\hat{L}\big(e^{\lambda_0\cdot}I\big)\big]=\big[B+\hat{L}\big(e^{\lambda_0\cdot}I\big)\big]\Delta_{-1}=\lambda_0\Delta_{-1}.$$

From the proof of Lemma 3.10, it can be checked that λ_0 is simple if and only if $\dim[N(\Delta(\lambda_0))]=1$. In that case, there exist $V_{\lambda_0}, W_{\lambda_0}\in\mathbb{C}^n\setminus\{0\}$ such that

$$W_{\lambda_0}^T\Delta(\lambda_0)=0\quad\text{and}\quad\Delta(\lambda_0)V_{\lambda_0}=0.\tag{5.1}$$

Hence, by Lemma 4.3 (replacing $V_{\lambda_0}W_{\lambda_0}^T$ by $\delta V_{\lambda_0}W_{\lambda_0}^T$ for some $\delta\neq 0$ if necessary), we can always assume that

$$\Delta_{-1}=V_{\lambda_0}W_{\lambda_0}^T.\tag{5.2}$$

Then we can see that $B_{-1}^{\lambda_0}B_{-1}^{\lambda_0}=B_{-1}^{\lambda_0}$ if and only if

$$\Delta_{-1}=\Delta_{-1}\left[I+\hat{L}\left(\int_{.}^{0}e^{\lambda_0\cdot}\,ds\right)\right]\Delta_{-1}.\tag{5.3}$$

Therefore, we obtain the following corollary.

Corollary 5.1. $\lambda_0\in\sigma((A+L))$ *is a simple eigenvalue of* $(A+L)$ *if and only if*

$$\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^2\Delta(\lambda)^{-1}=0$$

and

$$\dim\big[N\big(\Delta(\lambda_0)\big)\big]=1.$$

Moreover, the projector on the eigenspace associated to λ_0 *is*

$$B_{-1}^{\lambda_0}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}=\begin{bmatrix}0_{\mathbb{R}^n}\\ e^{\lambda_0\theta}\Delta_{-1}[\alpha+\varphi(0)+\hat{L}(\int_{.}^{0}e^{\lambda_0(.-s)}\varphi(s)\,ds)]\end{bmatrix}, \tag{5.4}$$

where

$$\Delta_{-1}=V_{\lambda_0}W_{\lambda_0}^T,$$

in which $V_{\lambda_0}, W_{\lambda_0}\in\mathbb{C}^n\setminus\{0\}$ *are two vectors satisfying* (5.1) *and*

$$\Delta_{-1}=\Delta_{-1}\left[I+\hat{L}\left(\int_{.}^{0}e^{\lambda_0\cdot}\,ds\right)\right]\Delta_{-1}.$$

Remark 5.2. Eq. (5.4) can be rewritten as

$$B_{-1}^{\lambda_0}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}=\begin{bmatrix}0_{\mathbb{R}^n}\\ e^{\lambda_0\theta}V_{\lambda_0}[W_{\lambda_0}^T\alpha+\langle\langle e^{-\lambda_0\cdot}W_{\lambda_0}^T,\varphi\rangle\rangle]\end{bmatrix},$$

where $\langle\langle\cdot,\cdot\rangle\rangle$ is the formal adjoint product defined by

$$\langle\langle\chi,\varphi\rangle\rangle=\chi(0)\varphi(0)-\int_{-r}^{0}\int_{0}^{\theta}\chi(\xi-\theta)\,d\eta(\theta)\,\varphi(\xi)\,d\xi$$

with $\chi\in C([0,r],\mathbb{C}^{n*})$ and $\varphi\in C([-r,0],\mathbb{C}^n)$.

6. Comments on the semi-linear problem

In this section we give a few comments and remarks concerning the results obtained in this paper. In order to study the semi-linear FDE

$$\begin{cases}\dfrac{dx(t)}{dt}=Bx(t)+\hat{L}(x_t)+f(x_t), & \forall t\geqslant 0,\\ x_0^{\varphi}=\varphi\in C\big([-r,0],\mathbb{R}^n\big),\end{cases} \tag{6.1}$$

we considered the associated abstract Cauchy problem

$$\frac{dv(t)}{dt}=Av(t)+L\big(v(t)\big)+F\big(v(t)\big),\quad t\geqslant 0,\quad\text{and}\quad v(0)=\begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi\end{pmatrix}\in\overline{D(A)}, \tag{6.2}$$

where

$$F\begin{pmatrix}0\\ \varphi\end{pmatrix}=\begin{pmatrix}f(\varphi)\\ 0\end{pmatrix}.$$

By using Lemma 3.7 we can check that the integrated solutions of (6.2) are the usual solutions of the FDE (6.1).

Now we are in the position to investigate the properties of the semiflows generated by the FDE by using the known results on non-densely defined semi-linear Cauchy problems. In particular when f is Lipschitz continuous, from the results of Thieme [21], for each $\varphi \in C$ we obtain a unique solution $t \to x^{\varphi}(t)$ on $[-r, +\infty)$ of (6.1), and we can define a non-linear C^0-semigroup $\{U(t)\}_{t \geqslant 0}$ on C by

$$U(t)\varphi = x_t^{\varphi}.$$

From the results in Magal [18], one may also consider the case where f is Lipschitz on bounded sets of C. The non-autonomous case has also been considered in Thieme [21] and Magal [18]. We refer to Ezzinbi and Adimy [11] for more results about the existence of solutions using integrated semigroups.

In order to describe the local asymptotic behavior around some equilibrium, we assume that $\bar{x} \in \mathbb{R}^n$ is an equilibrium of the FDE (6.1), that is,

$$0 = B\bar{x} + L(\bar{x}1_{[-r,0]}) + f(\bar{x}1_{[-r,0]}).$$

Then by the stability result of Thieme [21], we obtain the following stability results for FDE.

Theorem 6.1 *(Exponential stability). Assume that $f : C \to \mathbb{R}^n$ is continuously differentiable in some neighborhood of $\bar{x}1_{[-r,0]}$, and that $Df(\bar{x}1_{[-r,0]}) = 0$. Assume in addition that each solution of the characteristic equation $\Delta(\lambda) = 0$ has strictly negative real part. Then there exist $\eta, M, \gamma \in [0, +\infty)$, such that for each $\varphi \in C$ with $\|\varphi - \bar{x}1_{[-r,0]}\|_C \leqslant \eta$, the FDE* (6.1) *has a unique solution $t \to x^{\varphi}(t)$ on $[-r, +\infty)$, which satisfies*

$$\left\| x_t^{\varphi} - \bar{x}1_{[-r,0]} \right\|_C \leqslant M e^{-\gamma t} \|\varphi - \bar{x}1_{[-r,0]}\|_C, \quad \forall t \geqslant 0.$$

The above theorem is well known in the context of FDE (see Hale and Verduyn Lunel [13]). So here we do not need to prove such a result again. Nevertheless, as noticed first by Ovide Arino in the early 1990s, the non-densely defined operator approach can be very useful in investigating bifurcation problem in the context of FDE. Such an approach was extensively studied by Arino's team (see Adimy [1,2], Adimy and Arino [3], Ezzinbi and Adimy [11] and references therein). More recently, the existence and smoothness of the center manifolds was also investigated for abstract non-densely defined Cauchy problems by Magal and Ruan [20]. More precisely, if we denote $\Pi_c : X \to X$, the bounded linear operator of projection

$$\Pi_c = B_{-1}^{\lambda_1} + \cdots + B_{-1}^{\lambda_m},$$

where $\{\lambda_1, \lambda_2, \ldots, \lambda_m\} = \sigma_C(A + L) := \{\lambda \in \sigma(A + L)\colon \operatorname{Re}(\lambda) = 0\}$, then $X_c = \Pi_c(X)$ is the direct sum of the generalized eigenspaces associated to the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$. Moreover,

$$\Pi_c(X) \subset X_0,$$

and Π_c commutes with the resolvent of $(A+L)$. Set $X_h = R(I-\Pi_c)$ ($\nsubseteq X_0$). Then we can split the original abstract Cauchy problem (6.2) into the following system

$$\begin{cases} \dfrac{du_c(t)}{dt} = (A+L)_c u_c(t) + \Pi_c F\big(u_c(t)+u_h(t)\big), \\ \dfrac{du_h(t)}{dt} = (A+L)_h u_h(t) + \Pi_h F\big(u_c(t)+u_h(t)\big), \end{cases} \tag{6.3}$$

where $(A+L)_c$, the part of $A+L$ in X_c, is a bounded linear operator (since $\dim(X_c) < +\infty$), and $(A+L)_h$, the part of $A+L$ in X_h, is a non-densely defined Hille–Yosida operator. So the first equation of (6.3) is an ordinary differential equation and the second equation of (6.3) is a new non-densely defined Cauchy problem, with

$$\sigma\big((A+L)_h\big) = \sigma\big((A+L)\big) \setminus \sigma_C(A+L).$$

If we assume that F is C^k in some neighborhood of the equilibrium, we can find (see [20]) a manifold

$$M = \big\{x_c + \psi(x_c)\colon\ x_c \in X_c\big\},$$

where $\psi : X_c \to X_h \cap \overline{D(A)}$ is C^k, and M is local invariant by the semiflow generated by (6.2). Consequently, we obtain the reduced system on the center manifold

$$\frac{du_c(t)}{dt} = (A+L)_c u_c(t) + \Pi_c F\big(u_c(t) + \psi\big(u_c(t)\big)\big),$$

which allows us, for example, to prove the classical Hopf bifurcation result for the FDE.

Let us finally consider the following class of functional partial differential equations

$$\begin{cases} \dfrac{dx(t)}{dt} = Bx(t) + \hat{L}(x_t) + f(x_t), \quad \forall t \geqslant 0, \\ x_0 = \varphi \in C\big([-r,0], Y\big), \end{cases}$$

where $B : D(B) \subset Y \to Y$ is a linear operator on a Banach space Y, $\hat{L} : C([-r,0],Y) \to Y$ is a bounded linear operator, and $f : C([-r,0],Y) \to Y$ is a continuous map. Assume that B is the infinitesimal generator of compact linear C^0-semigroup $\{T_B(t)\}_{t\geqslant 0}$ on Y. Then we fall down into the context of the book of Wu [30], and all the results from Sections 3 to 6 can be adapted to such a context.

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