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Bogdanov–Takens bifurcation in a predator–prey model

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Abstract. In this paper, we investigate a class of predator–prey model with age structure and discuss whether the model can undergo Bogdanov–Takens bifurcation. The analysis is based on the normal form theory and the center manifold theory for semilinear equations with non-dense domain combined with integrated semigroup theory. Qualitative analysis indicates that there exist some parameter values such that this predator–prey model has an unique positive equilibrium which is Bogdanov–Takens singularity. Moreover, it is shown that under suitable small perturbation, the system undergoes the Bogdanov–Takens bifurcation in a small neighborhood of this positive equilibrium.

Mathematics Subject Classification. 34K18 · 37L10 · 37L15 · 35K90 · 37G10 · 92D25.

Keywords. Predator–prey model · Age structure · Normal forms · Non-densely defined · Bogdanov–Takens bifurcation.

1. Introduction

In this article, we will analyze the following predator–prey system with an age structure

$$\begin{cases} \dfrac{\partial u(t,a)}{\partial t} + \dfrac{\partial u(t,a)}{\partial a} = -\mu u(t,a), \quad \text{for } a \geq 0, \\ u(t,0) = \eta \dfrac{V(t)\int_0^{+\infty} \beta(a) u(t,a)\mathrm{d}a}{h + V^2(t)}, \\ \dfrac{\mathrm{d}V(t)}{\mathrm{d}t} = rV(t)\left(1 - \dfrac{V(t)}{K}\right) - \dfrac{V(t)\int_0^{+\infty} u(t,a)\mathrm{d}a}{h + V^2(t)}, \\ u(0,\cdot) = u_0 \in L^1_+\left((0,+\infty),\mathbb{R}\right) \text{ and } V(0) = V_0 \geq 0, \end{cases} \tag{1.1}$$

where the number μ denotes the death rate of the predator, t is time, and a is the chronological age (i.e., the age of individuals since they were born). η denotes the maximal growth rate of predator. The function $u(t,a)$ is the density of population of predators with respect to the age at time t. This means that the number of predator with age between a_1 and a_2 is

$$\int_{a_1}^{a_2} u(t,a)\mathrm{d}a.$$

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So in particular the total number of predator is

$$U(t) := \int_0^{+\infty} u(t,a)\mathrm{d}a.$$

The number $V(t)$ is the total number of prey at time t. The parameter $r = b - d$ is the intrinsic growth rate of the prey (where b and d denote the birth rate and death rate, respectively) and $K > 0$ is the carrying capacity of the prey.

The function $\beta \in L^{\infty}_{+}\left((0,+\infty),\mathbb{R}\right)$ represents a fraction of predator which can produce new individuals by eating the prey. In particular, when $\beta(a) \equiv 1$ system (1.1) becomes the ordinary differential equation

$$\begin{cases} \dfrac{\mathrm{d}U(t)}{\mathrm{d}t} = \eta \dfrac{V(t)U(t)}{h+V(t)^2} - \mu U(t) \\ \dfrac{\mathrm{d}V(t)}{\mathrm{d}t} = rV(t)\left(1 - \dfrac{V(t)}{K}\right) - \dfrac{V(t)U(t)}{h+V(t)^2} \\ U(0) = U_0 \geq 0 \quad \text{and} \quad V(0) = V_0 \geq 0. \end{cases} \tag{1.2}$$

Note that the bifurcations of system (1.2) have been analyzed in Xiao and Ruan [52] and it has been shown that system (1.2) can undergo Bogdanov–Takens bifurcation. Namely, they prove the existence of a series of bifurcations including saddle-node bifurcation, Hopf bifurcation, and homoclinic bifurcation. The functional response $\frac{V(t)U(t)}{h+V(t)^2}$ is a so-called non-monotonic functional response. We refer to Ruan and Xiao [38] for a detailed presentation of model (1.2) and an overview on this topic. We also refer to Cushing and Saleem [13] for a presentation of prey–predator systems with an age structure.

Here it makes sense to assume that the function $\beta(a)$ is not identically equal to 1 because the young predator cannot reproduce even if they eat some preys. Therefore, in order to take care of this fact we will assume that β is a step function of the form

$$\beta(a) = \begin{cases} \beta^* & \text{if} \quad a > \tau, \\ 0 & \text{if} \quad a \leq \tau. \end{cases}$$

Moreover, it will be convenient for the computation to assume that

$$\int_0^{+\infty} \beta(a) e^{-\mu a}\mathrm{d}a = 1.$$

One may observe that without loss of generality we can make this assumption. Because by replacing $\beta(a)$ by $\frac{\beta(a)}{\int_0^{+\infty}\beta(s)e^{-\mu s}\mathrm{d}s}$ and replacing η by $\eta\int_0^{+\infty}\beta(s)e^{-\mu s}\mathrm{d}s$ the system is unchanged.

Age-structured models, described by hyperbolic partial differential equations, have been studied by many researchers (see the monographs of Cushing [12], Iannelli [24], Webb [49], and the references cited therein). Various approaches have been developed to study these models. In order to analyze the qualitative properties of such a system, it is usually convenient to combine (a) integration along the characteristics combined with Volterra integral equations (Webb [49] , Iannelli [24]) and (b) integrated semigroups method (Thieme [44,46,47], Magal [31], Magal and Ruan [32,34]). The classical study on age-structured models focused on the existence, boundedness and stability of solutions. It has been shown that some age-structured models exhibit non-trivial periodic solutions induced by Hopf bifurcation (see Prüss [37], Cushing [12], Swart [40], Kostava and Li [26], and Bertoni [2]). Recently, there has been great interest in bifurcation analysis on degenerate equilibria of age-structured models. Since age-structured models can be rewritten as abstract semilinear equations with non-dense domain (Thieme [44,46,47], Magal and Ruan [32,34]), Magal and Ruan [33] developed the center manifold theory for abstract semilinear Cauchy problems with non-dense domain. Based on the center manifold theorem proved in Magal and Ruan [33], a Hopf bifurcation theorem has been presented for abstract non-densely defined Cauchy problem in Liu et al. [28]. These theorems have been successfully applied to study the existence of Hopf bifurcation for some age-/size-structured models (see [3,9–11,34,35,39,48]). Recently, much attention has been focused on high-codimensional bifurcations, since they may reveal some complex dynamical behaviors.

Bogdanov–Takens bifurcation is one of the most important high-codimensional bifurcations in nonlinear dynamics. It is a well-studied example of a bifurcation with codimension two and named after Bogdanov [4,5] and Takens [41,42], who independently and simultaneously described this bifurcation. For more results about Bogdanov–Takens bifurcation, see, for example, Chow and Hale [6], Dumortier et al. [16], Arnold [1], Chow et al. [7], Guckenheimer and Holmes [22], Kuznetsov [27], Xiao and Ruan ([38,51]), and others. Bogdanov–Takens bifurcation occurs also in infinite dimensional differential equations associated with functional differential equations and a few other partial differential equations. In the context of functional differential equations, we refer to the book by Hale et al. [23] and papers [18,25,50,52,53]. In the context of partial differential equations with delays, we refer to [19–21,30]. But to the best of our knowledge, there is no result on Bogdanov–Takens bifurcation for an age-structured model which is an hyperbolic partial differential equations with nonlinear and non-local boundary conditions.

In this article, our aim is to study whether system (1.1) also undergoes the Bogdanov–Takens bifurcation near a positive equilibrium. Hence, we first look for some conditions of parameters which guarantee system (1.1) has a positive equilibrium who is Bogdanov–Takens singularity by applying the normal form theory developed by Liu, Magal, and Ruan [29] for the non-densely defined abstract Cauchy problems, then we choose suitable small perturbation of parameters such that system (1.1) can undergo the Bogdanov–Takens bifurcation. More precisely, we obtain that there exists a unique positive equilibrium which is Bogdanov–Takens singularity of system (1.1) if $\eta = 2\sqrt{h}$ and $K = 2\sqrt{h}$, and prove the following small perturbation system

$$
\begin{cases}
\dfrac{\partial u(t,a)}{\partial t} + \dfrac{\partial u(t,a)}{\partial a} = -\mu u(t,a) \quad \text{for } a \geq 0 \\
u(t,0) = \eta \dfrac{V(t)\int_0^{+\infty}\beta(a)u(t,a)\mathrm{d}a}{h+V(t)^2} + \widetilde{\alpha}_1\left(V(t)-\sqrt{h}\right) \\
\qquad + \widetilde{\alpha}_2\left(\int\limits_0^{+\infty}\beta(a)u(t,a)\mathrm{d}a - 2hr\mu(1-\dfrac{\sqrt{h}}{K})\right) \\
\dfrac{\mathrm{d}V(t)}{\mathrm{d}t} = rV(t)\left(1-\dfrac{V(t)}{K}\right) - \dfrac{V(t)\int_0^{+\infty}u(t,a)\mathrm{d}a}{h+V(t)^2} \\
u(0,\cdot) = u_0 \in L^1\left((0,+\infty),\mathbb{R}\right) \text{ and } V(0) = V_0 \geq 0
\end{cases}
\tag{1.3}
$$

is a versal unfolding of this Bogdanov–Takens singularity, where $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ is a very small parameter vector, other parameters h, r, and μ are any fixed, and $\eta = 2\sqrt{h}$ and $K = 2\sqrt{h}$.

Here we take the parameters $\widetilde{\alpha}_1, \widetilde{\alpha}_2$ as the bifurcation parameters. Of course, the original system (1.1) corresponds to the case where $\widetilde{\alpha}_1 = 0$ and $\widetilde{\alpha}_2 = 0$. Here we will study the bifurcations of system (1.3) near the unique positive equilibrium as parameters $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ change in the small neighborhood of the origin. The aim of the present paper is to show how to apply the normal form theory in the context of age-structured models. In this paper, we will prove that Bogdanov–Takens bifurcation occurs for system (1.3). Some more biologically meaningful examples are left for further investigation.

The paper is organized as follows. In Sect. 2, we formulate an age-structured model based on system (1.1) as a non-densely defined Cauchy problem. Then we consider the existence of the positive equilibrium, linearize the system around the positive equilibrium, and study the spectral properties of the linearized equation in Sect. 3. In Sect. 4, the eigenvalue problem for the linearized system of (1.1) around the unique positive equilibrium is investigated and the normal form and center manifold theory for semilinear equations with non-dense domain is used to carry out the analysis of the Bogdanov–Takens bifurcation. A brief conclusion is provided in the last section.

2. Preliminary

We first reformulate the model (1.1) as a semilinear Cauchy problem with non-dense domain. By setting

$$V(t) := \int_0^{+\infty} v(t,a)\mathrm{d}a,$$

we can rewrite the second equation in system (1.1) as follows

$$\begin{cases} \dfrac{\partial v(t,a)}{\partial t} + \dfrac{\partial v(t,a)}{\partial a} = -\mathrm{d}v(t,a), \\ v(t,0) = G(v(t,a),\ u(t,a)), \\ v(0,a) = \rho_0 \in L^1((0,+\infty),\mathbb{R}), \end{cases}$$

where

$$G\left(v(t,.),\ u(t,.)\right) = b\int_0^{+\infty} v(t,a)\mathrm{d}a \left(1 - \frac{\int_0^{+\infty} v(t,a)\mathrm{d}a}{K}\right) + \frac{\mathrm{d}\left(\int_0^{+\infty} v(t,a)\mathrm{d}a\right)^2}{K}$$
$$- \frac{\int_0^{+\infty} v(t,a)\mathrm{d}a \int_0^{+\infty} u(t,a)\mathrm{d}a}{h + \left(\int_0^{+\infty} v(t,a)\mathrm{d}a\right)^2}.$$

Thus, by setting $w(t,a) = \begin{pmatrix} u(t,a) \\ v(t,a) \end{pmatrix}$, we obtain the equivalent system of system (1.1)

$$\begin{cases} \dfrac{\partial w(t,a)}{\partial t} + \dfrac{\partial w(t,a)}{\partial a} = -Qw(t,a), \\ w(t,0) = B(w(t,\cdot)), \\ w(0,\cdot) = w_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in L^1((0,+\infty),\mathbb{R}^2), \end{cases}$$

where

$$Q = \begin{pmatrix} \mu & 0 \\ 0 & d \end{pmatrix} \text{ and } B(w(t,\cdot)) = \begin{pmatrix} \frac{\eta \int_0^{+\infty} v(t,a)\mathrm{d}a \int_0^{+\infty} \beta(a)u(t,a)\mathrm{d}a}{h+\left(\int_0^{+\infty} v(t,a)\mathrm{d}a\right)^2} \\ G\left(\int_0^{+\infty} v(t,a)\mathrm{d}a,\ \int_0^{+\infty} u(t,a)\mathrm{d}a\right) \end{pmatrix}.$$

Following the results developed in Thieme [44] and Magal [31], we consider the Banach space

$$X = \mathbb{R}^2 \times L^1((0,+\infty),\mathbb{R}^2)$$

with $\left\| \begin{pmatrix} \delta \\ \varphi \end{pmatrix} \right\| = \|\delta\|_{\mathbb{R}^2} + \|\varphi\|_{L^1((0,+\infty),\mathbb{R}^2)}$, define the linear operator

$$L : D(L) \to X$$

by

$$L\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - Q\varphi \end{pmatrix}$$

with $D(L) = \{0\} \times W^{1,1}((0,+\infty),\mathbb{R}^2)$, and the operator $F : \overline{D(L)} \to X$ by

$$F\left(\begin{pmatrix} 0 \\ \varphi \end{pmatrix}\right) = \begin{pmatrix} B(\varphi) \\ 0 \end{pmatrix}.$$

We observe that L is non-densely defined since

$$X_0 := \overline{D(L)} = \{0\} \times L^1((0,+\infty),\mathbb{R}^2).$$

Then setting

$$x(t)=\begin{pmatrix}0\\ w(t,\cdot)\end{pmatrix},$$

we can rewrite system (1.1) as the following non-densely defined abstract Cauchy problem

$$\begin{cases}\dfrac{\mathrm{d}x(t)}{\mathrm{d}t}=Lx(t)+F(x(t)), \quad t\geq 0,\\ x(0)=\begin{pmatrix}0\\ w_0\end{pmatrix}\in\overline{D(L)}.\end{cases}\tag{2.1}$$

Set

$$X_+:=\mathbb{R}^2_+\times L^1_+((0,+\infty),\mathbb{R}^2)$$

and

$$X_{0+}:=X_0\cap X_+=\{0\}\times L^1_+((0,+\infty),\mathbb{R}^2).$$

From the results of Magal [31] and Magal and Ruan [34], we obtain the global existence, uniqueness, and positive of solutions for system (2.1).

Theorem 2.1. *There exists a unique continuous semiflow $\{U(t)\}_{t\geq 0}$ on X_{0+} such that for each $x\in X_{0+}$, the map $t\to U(t)x$ is the unique integrated solution of the Cauchy problem (2.1), that is, $t\to U(t)x$ satisfies for each $t\geq 0$ that*

$$\int_0^t U(l)x\mathrm{d}l\in D(L)$$

and

$$U(t)x=x+L\int_0^t U(l)x\mathrm{d}l+\int_0^t F\left(U(l)x\right)\mathrm{d}l, \quad t\geq 0.$$

3. Equilibria and linearized equation

The equilibrium solutions of Eq. (2.1) are obtained by solving the equation

$$L\begin{pmatrix}0\\ \overline{u}\\ \overline{\rho}\end{pmatrix}+F\begin{pmatrix}0\\ \overline{u}\\ \overline{\rho}\end{pmatrix}=0, \quad \begin{pmatrix}0\\ \overline{u}\\ \overline{\rho}\end{pmatrix}\in D(L),$$

and we obtain the following lemma.

Lemma 3.1. *The system (2.1) has always the equilibria*

$$\overline{x}_1=\begin{pmatrix}0_{\mathbb{R}^2}\\ 0_{L^1}\\ 0_{L^1}\end{pmatrix} \quad \text{and} \quad \overline{x}_2=\begin{pmatrix}0_{\mathbb{R}^2}\\ 0_{L^1}\\ \mathrm{d}Ke^{-\mathrm{d}a}\end{pmatrix}.$$

Furthermore, there exists a unique positive equilibrium of system (2.1)

$$\overline{x}(a)=\begin{pmatrix}0_{\mathbb{R}^2}\\ 2hr\mu(1-\frac{\sqrt{h}}{K})e^{-\mu a}\\ \mathrm{d}\sqrt{h}e^{-\mathrm{d}a}\end{pmatrix}$$

if and only if

$$\eta=2\sqrt{h}\ \text{and}\ K\neq\sqrt{h}.$$

In the following, we assume that $\eta = 2\sqrt{h}$ and $K \neq \sqrt{h}$. Now we make the following change of variable

$$y(t) := x(t) - \overline{x}(a)$$

and obtain

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = Ly(t) + F(y(t) + \overline{x}(a)) - F(\overline{x}(a)), \quad t \geq 0, \\ y(0) = \begin{pmatrix} 0 \\ w_0 - \overline{w}(a) \end{pmatrix} =: y_0 \in \overline{D(L)}. \end{cases} \tag{3.1}$$

Therefore, the linearized equation of (3.1) around the equilibrium 0 is given by

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = Ly(t) + \mathrm{DF}(\overline{x})y(t) \quad \text{ for } t \geq 0,\ y(0) \in X_0,$$

where

$$\mathrm{DF}(\overline{x}) \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} = \begin{pmatrix} \mathrm{DB}(\overline{w})(\varphi) \\ 0 \end{pmatrix}, \text{ for all } \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} \in D(L)$$

with

$$\mathrm{DB}(\overline{\omega})(\varphi) = \begin{pmatrix} 0_{\mathbb{R}} & 0_{\mathbb{R}} \\ -\frac{1}{\eta} & b - \frac{\eta r}{K} \end{pmatrix} \int_0^{+\infty} \varphi(a)\mathrm{d}a + \begin{pmatrix} 1 & 0_{\mathbb{R}} \\ 0_{\mathbb{R}} & 0_{\mathbb{R}} \end{pmatrix} \int_0^{+\infty} \beta(a)\varphi(a)\mathrm{d}a.$$

Then (3.1) can be written as

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = Ay(t) + H(y(t)), \quad \text{ for } t \geq 0, \tag{3.2}$$

where

$$A := L + \mathrm{DF}(\overline{x})$$

is a linear operator and

$$H(y(t)) = F(y(t) + \overline{x}) - F(\overline{x}) - \mathrm{DF}(\overline{x})y(t)$$

satisfying $H(0) = 0$ and $\mathrm{DH}(0) = 0$.

Next we will study the spectral properties of the linearized equation of (3.1) in order to investigate the dynamical behavior for system (3.1).

Define the part of L in $\overline{D(L)}$ by L_0,

$$L_0 : D(L_0) \subset X \to X$$

with $L_0 x = Lx$ for $x \in D(L_0) = \{x \in D(L) : Lx \in \overline{D(L)}\}$. Then we get for $\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in D(L_0)$,

$$L_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{L}_0\varphi \end{pmatrix},$$

where $\widehat{L}_0\varphi = -\varphi' - Q\varphi$ with

$$D(\widehat{L}_0) = \{\varphi \in W^{1,1}((0, +\infty), \mathbb{R}^2) : \varphi(0) = 0\}.$$

Denote

$$\xi := \min\{d, \mu\} > 0 \text{ and } \Omega := \{\lambda \in \mathbb{C} : Re(\lambda) > -\xi\}.$$

By applying the results of Liu et al. [28], we obtain the following result.

Lemma 3.2. *If $\lambda \in \Omega$, then $\lambda \in \rho(L)$ and*

$$(\lambda I - L)^{-1}\begin{pmatrix}\delta\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ \varphi\end{pmatrix} \Leftrightarrow \varphi(a) = e^{-\int_0^a(\lambda I+Q)\mathrm{d}l}\delta + \int_0^a e^{-\int_s^a(\lambda I+Q)\mathrm{d}l}\psi(s)\mathrm{d}s$$

with $\begin{pmatrix}\delta\\ \psi\end{pmatrix} \in X$ and $\begin{pmatrix}0\\ \varphi\end{pmatrix} \in D(L)$. Moreover, L is a Hille–Yosida operator.

From the above we have the following result.

Lemma 3.3. *The linear operator L_0 is the infinitesimal generator of the strongly continuous semigroup $\{T_{L_0}(t)\}_{t\geq 0}$ of bounded linear operators on $\overline{D(L)}$, and for each $t \geq 0$ the linear operator $T_{L_0}(t)$ is defined by*

$$T_{L_0}(t)\begin{pmatrix}0\\ \varphi\end{pmatrix} = \begin{pmatrix}0\\ \widehat{T}_{L_0}(t)\varphi\end{pmatrix},$$

where

$$\widehat{T}_{L_0}(t)(\varphi)(a) = \begin{cases} e^{-\int_{a-t}^{a} Q\mathrm{d}l}\varphi(a-t), & \text{if } a \geq t,\\ 0, & \text{otherwise.}\end{cases}$$

Now it remains to precise the spectral properties of $A = L + \mathrm{DF}(\overline{x})$. For convenience, we set $C := \mathrm{DF}(\overline{x})$. Let $\lambda \in \Omega$. Since $\lambda I - L$ is invertible, it follows that $\lambda I - A = \lambda I - (L + C)$ is invertible if and only if $I - C\,(\lambda I - L)^{-1}$ is invertible. Moreover, when $I - C\,(\lambda I - L)^{-1}$ is invertible we have

$$(\lambda I - (L + C))^{-1} = (\lambda I - L)^{-1}\left(I - C(\lambda I - L)^{-1}\right)^{-1}.$$

Consider

$$(I - C(\lambda I - L)^{-1})\begin{pmatrix}\delta\\ \varphi\end{pmatrix} = \begin{pmatrix}\gamma\\ \psi\end{pmatrix}$$

or

$$\begin{pmatrix}\delta\\ \varphi\end{pmatrix} - C\begin{pmatrix}0\\ e^{-\int_0^a(\lambda I+Q)\mathrm{d}l}\delta + \int_0^a e^{-\int_s^a(\lambda I+Q)\mathrm{d}l}\varphi(s)\mathrm{d}s\end{pmatrix} = \begin{pmatrix}\gamma\\ \psi\end{pmatrix}.$$

Then we obtain the system

$$\begin{cases}\delta - \mathrm{DB}(\overline{w})\left(e^{-\int_0^a(\lambda I+Q)\mathrm{d}l}\delta + \int_0^a e^{-\int_s^a(\lambda I+Q)\mathrm{d}l}\varphi(s)\mathrm{d}s\right) = \gamma,\\ \varphi = \psi,\end{cases}$$

i.e.,

$$\begin{cases}\delta - \mathrm{DB}(\overline{w})\left(e^{-\int_0^a(\lambda I+Q)\mathrm{d}l}\delta\right) = \gamma + \mathrm{DB}(\overline{w})\left(\int_0^a e^{-\int_s^a(\lambda I+Q)\mathrm{d}l}\varphi(s)\mathrm{d}s\right),\\ \varphi = \psi.\end{cases}$$

From the formula of $\mathrm{DB}(\overline{w})$ we know that

$$\begin{aligned}&\delta-\mathrm{DB}(\overline{w})\left(e^{-\int_0^a(\lambda I+Q)\mathrm{d}l}\delta\right)\\&=\begin{bmatrix}I-\begin{pmatrix}0&0\\-\frac{1}{\eta}&b-\frac{\eta r}{K}\end{pmatrix}\int_0^{+\infty}e^{-\int_0^a(\lambda I+Q)\mathrm{d}dl}\mathrm{d}a\\-\begin{pmatrix}1&0\\0&0\end{pmatrix}\int_0^{+\infty}\beta(a)e^{-\int_0^a(\lambda I+Q)\mathrm{d}l}\mathrm{d}a\end{bmatrix}\delta.\end{aligned}$$

Denote

$$\begin{aligned}\Delta(\lambda):=I-\begin{pmatrix}0&0\\-\frac{1}{\eta}&b-\frac{\eta r}{K}\end{pmatrix}\int_0^{+\infty}e^{-\int_0^a(\lambda I+Q)\mathrm{d}l}\mathrm{d}a\\-\begin{pmatrix}1&0\\0&0\end{pmatrix}\int_0^{+\infty}\beta(a)e^{-\int_0^a(\lambda I+Q)\mathrm{d}l}\mathrm{d}a\end{aligned}\tag{3.3}$$

and

$$K(\lambda,\psi):=\mathrm{DB}(\overline{w})\left(\int_0^a e^{-\int_s^a(\lambda I+Q)\mathrm{d}l}\psi(s)\mathrm{d}s\right).\tag{3.4}$$

Then $\Delta(\lambda)\delta=\gamma+K(\lambda,\psi)$. Whenever $\Delta(\lambda)$ is invertible, we have

$$\delta=(\Delta(\lambda))^{-1}(\gamma+K(\lambda,\psi)).$$

Lemma 3.4. *The following results hold.*

(i) $\sigma(L+C)\cap\Omega=\sigma_P(L+C)\cap\Omega=\{\lambda\in\Omega:\det\Delta(\lambda)=0\}$.

(ii) *If* $\lambda\in\rho(L+C)\cap\Omega$, *we have the following formula for the resolvent*

$$\begin{aligned}&(\lambda I-(L+C))^{-1}\begin{pmatrix}\delta\\\varphi\end{pmatrix}=\begin{pmatrix}0\\\psi\end{pmatrix}\Longleftrightarrow\\&\psi(a)=e^{-\int_0^a(\lambda I+Q)\mathrm{d}l}(\Delta(\lambda))^{-1}(\delta+K(\lambda,\varphi))+\int_0^a e^{-\int_s^a(\lambda I+Q)\mathrm{d}l}\varphi(s)\mathrm{d}s,\end{aligned}\tag{3.5}$$

where $\Delta(\lambda)$ *and* $K(\lambda,\varphi)$ *are defined in* (3.3) *and* (3.4).

Proof. Assume that $\lambda\in\Omega$ and $\det(\Delta(\lambda))\neq0$. From the above discussion, we have

$$(I-C(\lambda I-L)^{-1})^{-1}\begin{pmatrix}\delta\\\varphi\end{pmatrix}=\begin{pmatrix}(\Delta(\lambda))^{-1}(\delta+K(\lambda,\varphi))\\\varphi\end{pmatrix}.$$

By Lemma 3.2 we obtain (3.5) and (ii) follows. Therefore, we have $\{\lambda\in\Omega:\det(\Delta(\lambda))\neq0\}\subset\rho(L+C)\cap\Omega$, and

$$\sigma(L+C)\cap\Omega\subset\{\lambda\in\Omega:\det(\Delta(\lambda))=0\}.\tag{3.6}$$

Conversely, assume that $\lambda\in\Omega$ and $\det(\Delta(\lambda))=0$. We claim that we can find $\begin{pmatrix}0\\\psi\end{pmatrix}\in D(L)\setminus\{0\}$ such that

$$(L+C)\begin{pmatrix}0\\\psi\end{pmatrix}=\lambda\begin{pmatrix}0\\\psi\end{pmatrix}.\tag{3.7}$$

In fact, set

$$\begin{pmatrix} \delta \\ \varphi \end{pmatrix} := (\lambda I - L) \begin{pmatrix} 0 \\ \psi \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 0 \\ \psi \end{pmatrix} = (\lambda I - L)^{-1} \begin{pmatrix} \delta \\ \varphi \end{pmatrix}.$$

We can find a nonzero solution of (3.7) if and only if we can find $\begin{pmatrix} \delta \\ \varphi \end{pmatrix} \in X \setminus \{0\}$ satisfying

$$[I - C(\lambda I - L)^{-1}] \begin{pmatrix} \delta \\ \varphi \end{pmatrix} = 0.$$

From the above arguments, we can obtain $\begin{pmatrix} \delta \\ \varphi \end{pmatrix} \neq 0$ satisfying

$$\begin{cases} \Delta(\lambda)\delta = 0, \\ \varphi = 0. \end{cases}$$

Since by assumption $\det(\Delta(\lambda)) = 0$, we can find $\delta \neq 0$ such that $\Delta(\lambda)\delta = 0$. So we can find $\begin{pmatrix} 0 \\ \psi \end{pmatrix} \in D(L) \setminus \{0\}$ satisfying (3.7), and thus $\lambda \in \sigma_P(L + C)$. Hence,

$$\{\lambda \in \Omega : \det(\Delta(\lambda)) = 0\} \subset \sigma_P(L + C) \subset \sigma(L + C) \tag{3.8}$$

From (3.6) and (3.8), (i) follows. □

From the explicit formula for the resolvent, we deduce the following lemma.

Lemma 3.5. *For each $\lambda_0 \in \sigma(A) \cap \Omega$, λ_0 is a pole of the resolvent of order k_0 if and only if λ_0 is a root of order k_0 of $\Delta(\lambda)$.*

Since L is a Hille–Yosida operator, and $\mathrm{DF}(\overline{x})$ is bounded, A is also a Hille–Yosida operator. Consequently, A_0 generates a strongly continuous semigroup $\{T_{A_0}(t)\}$ on X_0. In order to apply the center manifold theorem and normal form theory, we need to study the essential growth bound of A. The **essential growth bound** $\omega_{0,\mathrm{ess}}(A) \in [-\infty, +\infty)$ of A is defined by

$$\omega_{0,\mathrm{ess}}(A) := \lim_{t \to +\infty} \frac{\ln(\|T_A(t)\|_{\mathrm{ess}})}{t},$$

where $\|T_A(t)\|_{\mathrm{ess}}$ is the essential norm of $T_A(t)$ defined by

$$\|T_A(t)\|_{\mathrm{ess}} = \kappa(T_A(t)B_X(0,1)),$$

here $B_X(0,1) = \{x \in X : \|x\|_X \leq 1\}$, and for each bounded set $B \subset X$,

$$\kappa(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by a finite number of balls of radius } \leq \varepsilon\}$$

is the Kuratovsky measure of non-compactness. By using a perturbation result we obtain the following estimation.

Proposition 3.6. *The essential growth rate of the strongly continuous semigroup generated by A_0 is strictly negative, that is,*

$$\omega_{0,ess}(A_0) \leq -\xi < 0.$$

Proof. From lemma 3.4, we obtain

$$\|T_{L_0}(t)\| \leq e^{-\xi t}, \quad \text{for all } t \geq 0.$$

Thus, we have

$$\omega_{0,\mathrm{ess}}(L_0) \leq \omega_0(L_0) \leq -\xi.$$

Since $\mathrm{DF}(\overline{x})$ is a compact bounded linear operator, we can apply the perturbation results in Thieme [43] or Ducrot et al. [14] to deduce that

$$\omega_{0,\mathrm{ess}}(A_0) \leq -\xi < 0.$$

□

4. Bogdanov–Takens singularity

In this section, we apply the normal form theory in [32,34] to study equilibrium $O(0,0)$ of system (3.2) and prove that the equilibrium $O(0,0)$ of (3.2) is a Bogdanov–Takens singularity under some assumptions. Last we consider a small perturbation system of (3.2) which corresponds to system (1.3) and show the perturbation system is a versal unfolding of the Bogdanov–Takens singularity. Hence, system (1.3) can undergo Bogdanov–Takens bifurcation. For clearity, we rewrite system (3.2) as

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = Ay(t) + H(y(t)), \quad \text{for } t \geq 0, \tag{4.1}$$

where $A = L + \mathrm{DF}(\overline{x})$, $H(0) = 0$, and $\mathrm{DH}(0) = 0$.

Notice that $\sigma(A) \cap \Omega = \sigma_P(A) \cap \Omega = \{\lambda \in \Omega : \det \Delta(\lambda) = 0\}$. By computation, we obtain that

$$\Delta(\lambda) = \begin{pmatrix} 1 - \dfrac{\beta^* e^{-(\lambda+\mu)\tau}}{\lambda+\mu} & 0 \\ \frac{1}{(\lambda+\mu)\eta} & 1 - \dfrac{(bK - \eta r)}{K(\lambda + d)} \end{pmatrix}$$

and

$$\begin{aligned} &\det(\Delta(\lambda)) \\ &= \left[\frac{(\lambda+\mu) - \beta^* e^{-(\lambda+\mu)\tau}}{(\lambda+\mu)}\right]\left[\frac{K(\lambda+d) - (bK - \eta r)}{K(\lambda+d)}\right] \\ &= \frac{\left(\lambda + \mu - \beta^* e^{-(\lambda+\mu)\tau}\right)[K\lambda - rK + \eta r]}{K(\lambda+\mu)(\lambda+d)} \\ &= \frac{\kappa_1\lambda^2 + \kappa_2\lambda + \kappa_3 e^{-\lambda\tau}\lambda + \kappa_4 e^{-\lambda\tau} + \kappa_5}{K(\lambda+\mu)(\lambda+d)} \\ &=: \frac{f(\lambda)}{g(\lambda)} = 0, \ \lambda \in \Omega \end{aligned}$$

with

$$\begin{aligned} &\kappa_1 = K, \ \kappa_2 = \mu K + r(\eta - K), \ \kappa_3 = -K\mu, \\ &\kappa_4 = -r\mu(\eta - K), \ \kappa_5 = r\mu(\eta - K). \end{aligned}$$

It is obvious that $\{\lambda \in \Omega : \det \Delta(\lambda) = 0\} = \{\lambda \in \Omega : f(\lambda) = 0\}$.

Assumption 4.1. *Assume that* $K = \eta = 2\sqrt{h}$.

Under Assumption 4.1,

$$f(\lambda) = \kappa_1 \lambda^2 + \kappa_2 \lambda + \kappa_3 e^{-\lambda\tau}\lambda = 0,\ \ \lambda \in \Omega \tag{4.2}$$

with

$$\kappa_1 = K > 0,\ \ \kappa_2 = \mu K > 0,\ \ \kappa_3 = -\mu K < 0.$$

Since $f(0) = 0$, $f'(0) = \kappa_2 + \kappa_3 = 0$, and $f''(0) = 2(\kappa_1 - \tau\kappa_3) \neq 0$, $\lambda = 0$ is a root of (4.2) with multiplicity 2. Next we claim that $\lambda = 0$ is the unique root of (4.2) with zero real parts.

In fact, let $\lambda = i\omega$ $(\omega > 0)$ be a purely imaginary root of (4.2). Then we obtain

$$-\kappa_1\omega + i\kappa_2 + \kappa_3\left(\cos(\omega\tau) - i\sin(\omega\tau)\right) i = 0.$$

Separating real and imaginary parts in the above equation, we obtain

$$\begin{cases} \kappa_2 = -\kappa_3\cos(\omega\tau), \\ -\kappa_1\omega = -\kappa_3\sin(\omega\tau). \end{cases} \tag{4.3}$$

Thus, we have

$$\kappa_2^2 + (\kappa_1\omega)^2 = \kappa_3^2$$

and since $\kappa_2 = -\kappa_3$ we obtain

$$\kappa_1^2\omega^2 = 0.$$

Thus, (4.2) has no purely imaginary root $\lambda = i\omega$ $(\omega > 0)$.

From Sect. 3, we know that under Assumption 4.1

$$\sigma(A) \cap i\mathbb{R} = \{0\} \ \text{ and } \omega_{0,\mathrm{ess}}(A_0) < 0.$$

Now we compute the projectors on the generalized eigenspace associated to eigenvalue 0 of A. From the above discussion, we already knew that 0 is a pole of $(\lambda I - A)^{-1}$of finite order 2. This means that 0 is isolated in $\sigma(A) \cap \Omega$, and the Laurent's expansion of the resolvent around 0 takes the following form

$$(\lambda I - A)^{-1} = \sum_{n=-2}^{+\infty} \lambda^n B_{n,0}^A.$$

The bounded linear operator $B_{-1,0}^A$ is the projector on the generalized eigenspace of A associated to 0. We remark that

$$\lambda^2 (\lambda I - A)^{-1} = \sum_{m=0}^{+\infty} \lambda^m B_{m-2,0}^A.$$

So we have the following approximation formula

$$B_{-1,0}^A = \lim_{\lambda \to 0} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\lambda^2 (\lambda I - A)^{-1}\right).$$

Lemma 4.2. *Let Assumption 4.1 be satisfied. Then* 0 *is a pole of* $(\lambda I - A)^{-1}$ *of order* 2*, and the projector on the generalized eigenspace of* A *associated to the eigenvalue* 0 *is given by*

$$B_{-1,0}^A \begin{pmatrix} \delta \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \psi \end{pmatrix},$$

where

$$\begin{aligned}\psi(a) = &\begin{pmatrix} e^{-\mu a} & 0 \\ 0 & e^{-da} \end{pmatrix} \begin{pmatrix} \frac{\mu}{1+\tau\mu} & 0 \\ -\frac{2(1+\tau\mu)+\mathrm{d}\tau^2\mu}{2K(1+\tau\mu)^2} + \frac{ad}{(1+\tau\mu)K} & d \end{pmatrix} \\ &\left[\delta + \mathrm{DB}(\overline{w}) \left(\int_0^a \begin{pmatrix} e^{-\mu(a-s)} & 0 \\ 0 & e^{-d(a-s)} \end{pmatrix} \varphi(s)\mathrm{d}s\right)\right] \\ &+ \begin{pmatrix} e^{-\mu a} & 0 \\ 0 & e^{-da} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\frac{d}{(1+\tau\mu)K} & 0 \end{pmatrix} \\ &DB(\overline{w}) \left(\int_0^a \begin{pmatrix} (s-a)e^{-\mu(a-s)} & 0 \\ 0 & (s-a)e^{-d(a-s)} \end{pmatrix} \varphi(s)\mathrm{d}s\right).\end{aligned} \tag{4.4}$$

Proof.

$$\lambda^2(\lambda I - A)^{-1} \begin{pmatrix} \delta \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{\psi} \end{pmatrix} \Longleftrightarrow$$

$$\overline{\psi}(a) = \lambda^2 e^{-\int_0^a (\lambda I + Q)\mathrm{d}l} (\Delta(\lambda))^{-1} (\delta + K(\lambda, \varphi)) + \lambda^2 \int_0^a e^{-\int_s^a (\lambda I + Q)\mathrm{d}l} \varphi(s)\mathrm{d}s,$$

where $K(\lambda,\varphi) = \mathrm{DB}(\overline{w}) \left(\int_0^a e^{-\int_s^a (\lambda I+Q)\mathrm{d}l} \varphi(s)\mathrm{d}s\right)$. We need to differentiate with respect to λ the above formula. We first have

$$(\Delta(\lambda))^{-1} = \begin{pmatrix} \frac{(\lambda+\mu)}{\lambda+\mu-\mu e^{-\lambda\tau}} & 0 \\ -\frac{(\lambda+d)}{(\lambda+\mu-\mu e^{-\lambda\tau})\lambda K} & \frac{\lambda+d}{\lambda} \end{pmatrix}$$

and

$$\frac{d(\Delta(\lambda))^{-1}}{d\lambda} = \begin{pmatrix} \frac{\left(\lambda+\mu-\mu e^{-\lambda\tau}\right)-(\lambda+\mu)\left(1+\tau\mu e^{-\lambda\tau}\right)}{(\lambda+\mu-\mu e^{-\lambda\tau})^2} & 0 \\ -\frac{\left(\lambda+\mu-\mu e^{-\lambda\tau}\right)\lambda-(\lambda+d)\left[\left(\lambda+\mu-\mu e^{-\lambda\tau}\right)+\left(1+\tau\mu e^{-\lambda\tau}\right)\lambda\right]}{K(\lambda+\mu-\mu e^{-\lambda\tau})^2\lambda^2} & \frac{-\mathrm{d}}{\lambda^2} \end{pmatrix}.$$

Then

$$\lim_{\lambda\to 0}\left[2\lambda(\Delta(\lambda))^{-1} + \lambda^2 \frac{\mathrm{d}(\Delta(\lambda))^{-1}}{\mathrm{d}\lambda}\right] = \begin{pmatrix} \frac{\mu}{1+\tau\mu} & 0 \\ -\frac{2(1+\tau\mu)+d\tau^2\mu}{2K(1+\tau\mu)^2} & d \end{pmatrix},$$

and

$$\lim_{\lambda\to 0}\left(\lambda^2(\Delta(\lambda))^{-1}\right) = \begin{pmatrix} 0 & 0 \\ -\frac{\mathrm{d}}{(1+\tau\mu)K} & 0 \end{pmatrix}.$$

Furthermore,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{e^{-\int_0^a (\lambda I + D)\mathrm{d}l} \lambda^2 (\Delta(\lambda))^{-1} (\delta + K(\lambda,\varphi))\right\} \\ &= \begin{pmatrix} -\mathrm{a}e^{-(\lambda+\mu)a} & 0 \\ 0 & -\mathrm{a}e^{-(\lambda+d)a} \end{pmatrix} \\ &\lambda^2(\Delta(\lambda))^{-1}\left[\delta + \mathrm{DB}(\overline{w})\left(\int_0^a \begin{pmatrix} e^{-(\lambda+\mu)(a-s)} & 0 \\ 0 & e^{-(\lambda+d)(a-s)} \end{pmatrix} \varphi(s)\mathrm{d}s\right)\right] \\ &\quad + \begin{pmatrix} e^{-(\lambda+\mu)a} & 0 \\ 0 & e^{-(\lambda+d)a} \end{pmatrix} \lambda^2 (\Delta(\lambda))^{-1}\end{aligned}$$

$$
\begin{aligned}
&\mathrm{DB}(\overline{w})\left(\int_0^a \begin{pmatrix} (s-a)e^{-(\lambda+\mu)(a-s)} & 0 \\ 0 & (s-a)e^{-(\lambda+d)(a-s)} \end{pmatrix} \varphi(s)\mathrm{d}s\right) \\
&\quad + \begin{pmatrix} e^{-(\lambda+\mu)a} & 0 \\ 0 & e^{-(\lambda+d)a} \end{pmatrix} \left[2\lambda\left(\Delta(\lambda)\right)^{-1} + \lambda^2 \frac{d\left(\Delta(\lambda)\right)^{-1}}{d\lambda}\right] \\
&\quad \times \left[\delta + \mathrm{DB}(\overline{w})\left(\int_0^a \begin{pmatrix} e^{-(\lambda+\mu)(a-s)} & 0 \\ 0 & e^{-(\lambda+d)(a-s)} \end{pmatrix} \varphi(s)\mathrm{d}s\right)\right].
\end{aligned}
$$

Then

$$
\begin{aligned}
&\lim_{\lambda\to 0} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{e^{-\int_0^a (\lambda I + D)\mathrm{d}l} \lambda^2 \left(\Delta(\lambda)\right)^{-1}\left(\delta + K(\lambda,\varphi)\right)\right\} \\
&\quad = \begin{pmatrix} e^{-\mu a} & 0 \\ 0 & e^{-da} \end{pmatrix} \begin{pmatrix} \frac{\mu}{1+\tau\mu} & 0 \\ -\frac{2(1+\tau\mu)+d\tau^2\mu}{2K(1+\tau\mu)^2} + \frac{\mathrm{ad}}{(1+\tau\mu)K} & d \end{pmatrix} \\
&\quad\quad \times \left[\delta + \mathrm{DB}(\overline{w})\left(\int_0^a \begin{pmatrix} e^{-\mu(a-s)} & 0 \\ 0 & e^{-d(a-s)} \end{pmatrix} \varphi(s)\mathrm{d}s\right)\right] \\
&\quad\quad + \begin{pmatrix} e^{-\mu a} & 0 \\ 0 & e^{-da} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\frac{\mathrm{d}}{(1+\tau\mu)K} & 0 \end{pmatrix} \\
&\quad \mathrm{DB}(\overline{w})\left(\int_0^a \begin{pmatrix} (s-a)e^{-\mu(a-s)} & 0 \\ 0 & (s-a)e^{-d(a-s)} \end{pmatrix} \varphi(s)\mathrm{d}s\right).
\end{aligned}
$$

Thus, we obtain

$$
B_{-1,0}^{A}\begin{pmatrix} \delta \\ \varphi \end{pmatrix} = \lim_{\lambda\to 0}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\lambda^2\left(\lambda I - A\right)^{-1}\right]\begin{pmatrix} \delta \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \psi \end{pmatrix},
$$

where ψ is defined in (4.4). □

From the above results, we obtain a state space decomposition with respect to the spectral properties of the linear operator A. More precisely, the projector on the linear center manifold is defined by

$$
\Pi_c^A\begin{pmatrix} \delta \\ \varphi \end{pmatrix} = B_{-1,0}^{A}\begin{pmatrix} \delta \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \psi \end{pmatrix},
$$

where $\psi(a)$ is defined in (4.4). Set

$$
\Pi_h^A := I - \Pi_c^A.
$$

We denote by

$$
X_c := \Pi_c^A\left(X\right), \quad X_h := \Pi_h^A\left(X\right), \quad A_c := A\mid_{X_c}, \; A_h := A\mid_{X_h}.
$$

Now we have the decomposition

$$
X = X_c \oplus X_h.
$$

Since

$$
B_{-1,0}^{A}\begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0_{L^1} \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_1 \end{pmatrix}
$$

with

$$\psi_1(a) = \begin{pmatrix} \frac{\mu e^{-\mu a}}{1+\tau\mu} \\ e^{-da}\left(\frac{ad}{(1+\tau\mu)K} - \frac{2(1+\tau\mu)+d\tau^2\mu}{2K(1+\tau\mu)^2}\right) \end{pmatrix}, \tag{4.5}$$

and

$$B^A_{-1,0}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \atop 0_{L^1}\right) = \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix}$$

with

$$\psi_2(a) = \begin{pmatrix} 0 \\ de^{-da} \end{pmatrix}. \tag{4.6}$$

we obtain the basis $\{\chi_1, \chi_2\}$ of X_c defined by

$$\chi_1(a) = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \begin{pmatrix} 0_{L^1} \\ -de^{-da} \end{pmatrix} \end{pmatrix}, \quad \chi_2(a) = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \begin{pmatrix} K\mu e^{-\mu a} \\ dae^{-da} \end{pmatrix} \end{pmatrix}.$$

Note that

$$A\chi_1(a) = \begin{pmatrix} 0_{\mathbb{R}^2} \\ 0_{L^1} \end{pmatrix}, \quad A\chi_2(a) = \chi_1(a).$$

The matrix of A_c in the basis $\{\chi_1, \chi_2\}$ of X_c is given by

$$A\,[\chi_1, \chi_2] = [0, \chi_1]$$

or by using a usual symbolic matrix computation formula

$$A\,[\chi_1, \chi_2] = [\chi_1, \chi_2]\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{4.7}$$

We now consider the normal form of system (4.1) on the 2-dimensional center manifold of equilibrium $O(0,0)$. Let $\mathcal{L}_s(X_c^2, D(A))$ be the space of bounded 2-linear symmetric maps from $X_c^2 = X_c \times X_c$ into $D(A)$ and $V^2(X_c, D(A))$ be the linear space of homogeneous polynomials of degree 2. More precisely, given the basis $\{\chi_1, \chi_2\}$ of X_c, $V^2(X_c, D(A))$ is the space of finite linear combination of maps of the form

$$x_c = x_1\chi_1 + x_2\chi_2 \to x_1^{n_1} x_2^{n_2} V, \quad x_c \in X_c$$

with

$$n_1 + n_2 = 2, \text{ and } V \in D(A).$$

Define a map $\mathcal{G} : \mathcal{L}_s(X_c^2, D(A)) \to V^2(X_c, D(A))$ by $\mathcal{G}(L)(x_c) = L(x_c, x_c), \quad \forall L \in \mathcal{L}_s(X_c^2, D(A))$ and define $\Theta_2^c : V^2(X_c, X_c) \to V^2(X_c, X_c)$ by

$$\Theta_2^c\,(\Upsilon_c) := [A_c, \Upsilon_c], \quad \forall \Upsilon_c \in V^2(X_c, X_c), \tag{4.8}$$

where $[,]$ is the Lie bracket

$$[A_c, \Upsilon_c]\,(x_c) = D\Upsilon_c\,(x_c)\,(A_c x_c) - A_c\Upsilon_c\,(x_c)\,, \quad \forall x_c \in X_c.$$

We decompose $V^2(X_c, X_c)$ into the direct sum

$$V^2(X_c, X_c) = \mathcal{R}_2^c \oplus \mathcal{C}_2^c, \tag{4.9}$$

where

$$\mathcal{R}_2^c := R(\Theta_2^c)$$

is the range of Θ_2^c, and $\mathcal{C}_2^c$ is some complementary space of $\mathcal{R}_2^c$ into $V^2(X_c, X_c)$.

Let

$$G_2 \in V^2(X_c, D(A)). \tag{4.10}$$

By applying Theorem 4.4 in Liu, Magal, and Ruan [29], we obtain that after the following change of variable locally around 0

$$y = \overline{y} + G_2(\Pi_c^A \overline{y}), \tag{4.11}$$

system (4.1) becomes

$$\frac{\mathrm{d}\overline{y}(t)}{\mathrm{d}t} = A\overline{y}(t) + \widehat{H}(\overline{y}(t)), \quad \text{for } t \geq 0, \tag{4.12}$$

and the reduced equation of system (4.12) has the following form which is in the normal form at the order 2,

$$\frac{\mathrm{d}\overline{y}_c(t)}{\mathrm{d}t} = A_c\overline{y}_c(t) + \frac{1}{2!}\Pi_c^A D^2\widehat{H}\left(0\right)\left(\overline{y}_c(t), \overline{y}_c(t)\right) + R_c\left(\overline{y}_c(t)\right), \tag{4.13}$$

where

$$\begin{aligned}
&\mathcal{G}\left(\frac{1}{2!}\Pi_c^A D^2\widehat{H}\left(0\right)|_{X_c\times X_c}\right) \in \mathcal{C}_2^c,\\
&\frac{1}{2!}\Pi_c^A D^2\widehat{H}\left(0\right)\left(\overline{y}_c(t), \overline{y}_c(t)\right)\\
&\quad = \frac{1}{2!}\Pi_c^A D^2 H\left(0\right)\left(\overline{y}_c(t), \overline{y}_c(t)\right) - \left[A_c, \Pi_c^A G_2\right]\left(\overline{y}_c(t)\right)
\end{aligned}$$

and the remainder term $R_c \in C^3(X_c, X_c)$ is a rest of order 3, that is to say that $D_j R_c(0) = 0$ for each $j = 1, 2$.

In the following, we will compute

$$\frac{1}{2!}\Pi_c^A D^2 H(0)\left(\overline{y}_c(t), \overline{y}_c(t)\right).$$

Set

$$\overline{y} := \begin{pmatrix} 0 \\ \widehat{\varphi} \end{pmatrix} \in \overline{D(A)} \quad \text{with } \widehat{\varphi} = \begin{pmatrix} \widehat{\varphi}^1 \\ \widehat{\varphi}^2 \end{pmatrix}$$

$$\overline{y}_c := \Pi_c^A\overline{y} = \Pi_c^A\begin{pmatrix} 0 \\ \widehat{\varphi} \end{pmatrix} = B_{-1,0}^A\begin{pmatrix} 0 \\ \widehat{\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ \widetilde{\psi} \end{pmatrix}$$

with

$$\begin{aligned}
\widetilde{\psi}(a) = &\begin{pmatrix} e^{-\mu a} & 0 \\ 0 & e^{-da} \end{pmatrix}\begin{pmatrix} \frac{\mu}{1+\tau\mu} & 0 \\ -\frac{2(1+\tau\mu)+d\tau^2\mu}{2K(1+\tau\mu)^2} + \frac{ad}{(1+\tau\mu)K} & d \end{pmatrix}\\
&\left[\mathrm{DB}(\overline{w})\left(\int_0^a \begin{pmatrix} e^{-\mu(a-s)} & 0 \\ 0 & e^{-d(a-s)} \end{pmatrix}\widehat{\varphi}(s)\mathrm{d}s\right)\right]\\
&+\begin{pmatrix} e^{-\mu a} & 0 \\ 0 & e^{-da} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -\frac{d}{(1+\tau\mu)K} & 0 \end{pmatrix}\\
&\mathrm{DB}(\overline{w})\left(\int_0^a \begin{pmatrix} (s-a)e^{-\mu(a-s)} & 0 \\ 0 & (s-a)e^{-d(a-s)} \end{pmatrix}\widehat{\varphi}(s)\mathrm{d}s\right)
\end{aligned}$$

and

$$\overline{y}_h := \Pi_h^A\overline{y} = (I - \Pi_c^A)\begin{pmatrix} 0 \\ \widehat{\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{\varphi} - \widetilde{\psi} \end{pmatrix}.$$

We observe that for each

$$\overline{y}_1 := \begin{pmatrix} 0 \\ \varphi_1 \end{pmatrix}, \ \overline{y}_2 := \begin{pmatrix} 0 \\ \varphi_2 \end{pmatrix} \in \overline{D(A)}$$

with

$$\varphi_1 = \begin{pmatrix} \varphi_1^1 \\ \varphi_1^2 \end{pmatrix}, \varphi_2 = \begin{pmatrix} \varphi_2^1 \\ \varphi_2^2 \end{pmatrix},$$

$$D^2H(0)\left(\overline{y}_1, \overline{y}_2\right) = D^2F(\overline{x})\left(\overline{y}_1, \overline{y}_2\right) = \left(\begin{pmatrix} -\frac{2hr\mu(K-\sqrt{h})\int_0^{+\infty}\varphi_1^2(a)da\int_0^{+\infty}\varphi_2^2(a)da}{Kh} \\ -\frac{r\int_0^{+\infty}\varphi_2^2(a)da\int_0^{+\infty}\varphi_1^2(a)da}{K} \end{pmatrix} \atop 0 \right).$$

Then

$$\frac{1}{2!}D^2H\left(0\right)\left(\overline{y}\right)^2 = \left(\begin{pmatrix} -\frac{\left(\int_0^{+\infty}\widehat{\varphi}^2(a)da\right)^2 2hr\mu(K-\sqrt{h})}{Kh} \\ -\frac{r\left(\int_0^{+\infty}\widehat{\varphi}^2(a)da\right)^2}{K} \end{pmatrix} \atop 0 \right).$$

By projecting on X_c, we obtain

$$\frac{1}{2!}\Pi_c^A D^2H(0)\left(\overline{y}\right)^2 = \frac{1}{2!}\begin{pmatrix} 0 \\ \overline{\psi} \end{pmatrix},$$

where

$$\overline{\psi}(a) = \begin{pmatrix} -\frac{\mu e^{-\mu a}\left(\int_0^{+\infty}\widehat{\varphi}^2(a)da\right)^2 2hr\mu(K-\sqrt{h})}{Kh(1+\tau\mu)} \\ -\frac{\left(\int_0^{+\infty}\widehat{\varphi}^2(a)da\right)^2\left[2hr\mu(K-\sqrt{h})\left(\frac{ad}{(1+\tau\mu)K}-\frac{2(1+\tau\mu)+d\tau^2\mu}{2K(1+\tau\mu)^2}\right)+hrd\right]e^{-da}}{hK} \end{pmatrix}. \tag{4.14}$$

Set

$$\overline{y}_c = x_1\chi_1 + x_2\chi_2.$$

We shall compute (4.12) expressed in terms of the basis $\{\chi_1, \chi_2\}$. Consider $V^2(\mathbb{R}^2, \mathbb{R}^2)$ which denotes the linear space of the homogeneous polynomials of degree 2 in two real variables, x_1, x_2 with coefficients in $\mathbb{R}^2$. The operators Θ_2^c considered in (4.8) now act in the spaces $V^2(\mathbb{R}^2, \mathbb{R}^2)$ and satisfies

$$\Theta_2^c(G_{2,c})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = [A_c, G_{2,c}]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = D_xG_{2,c}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}A_c\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - A_cG_{2,c}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$
$$\text{for all } G_{2,c} \in V^2(\mathbb{R}^2, \mathbb{R}^2)$$

with

$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is known [7,22,27,42] that a normal form for Bogdanov–Takens singularity which gives the flow on the center manifold is

$$\begin{aligned} \dot{x}_1 &= x_2 + O(|(x_1, x_2)|^3) \\ \dot{x}_2 &= A_1x_1^2 + A_2x_1x_2 + O(|(x_1, x_2)|^3). \end{aligned}$$

In order to obtain from equation (4.13) the second-order terms in the above normal form for Bogdanov–Takens singularity, we need to choose a complementary space for $R(\Theta_2^c)$ in $V^2(\mathbb{R}^2, \mathbb{R}^2)$. The canonical basis of $V^2(\mathbb{R}^2, \mathbb{R}^2)$ has six elements:

$$\begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix}.$$

Their images under Θ_2^c are, respectively,

$$\begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2x_1x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -x_1x_2 \\ x_2^2 \end{pmatrix}, \begin{pmatrix} -x_1^2 \\ 2x_1x_2 \end{pmatrix}, \begin{pmatrix} -x_2^2 \\ 0 \end{pmatrix}.$$

We choose the following complementary space of $R(\Theta_2^c)$ in $V^2(\mathbb{R}^2, \mathbb{R}^2)$

$$\mathcal{C}_2^c = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix} \right\}.$$

Note that

$$\begin{aligned} &\frac{1}{2!}\Pi_c^A D^2H(0)\left(x_1\chi_1 + x_2\chi_2\right)^2 \\ &= (\chi_1, \chi_2)\frac{1}{2!}\begin{pmatrix} -\frac{2r\mu(K-\sqrt{h})\left(2(1+\tau\mu)+d\tau^2\mu\right)(x_2-dx_1)^2}{2d^2K^2(1+\tau\mu)^2} + \frac{r(x_2-dx_1)^2}{d^2K} \\ -\frac{2r\mu(K-\sqrt{h})(x_2-dx_1)^2}{d^2K^2(1+\tau\mu)} \end{pmatrix} \end{aligned}$$

in the above formula we are using a matrix symbolic computation.

Remember we assumed that $K = 2\sqrt{h}$; therefore, we obtain that (4.13) expressed in terms of the basis $\{\chi_1, \chi_2\}$ becomes

$$\begin{aligned} \dot{x}_1 &= x_2 + O(|(x_1, x_2)|^3) \\ \dot{x}_2 &= A_1x_1^2 + A_2x_1x_2 + O(|(x_1, x_2)|^3), \end{aligned} \tag{4.15}$$

where

$$\begin{aligned} A_1 &= -\frac{r\mu}{2K\left(1+\tau\mu\right)} < 0, \\ A_2 &= \frac{r}{K}\left(1 - \frac{\tau^2\mu^2}{2(1+\tau\mu)^2}\right) > 0. \end{aligned}$$

From the above analysis, we obtain the following theorem.

Theorem 4.3. *Suppose that Assumption 4.1 holds. Then the unique positive equilibrium* $E^* = (hr\mu e^{-\mu a}, \sqrt{h})$ *of (1.1) has a Bogdanov–Takens singularity, whose local dynamics on the center manifold of* E^* *are determined by (4.15).*

5. Bogdanov–Takens bifurcation

To observe whether system (4.1) can undergo Bogdanov–Takens bifurcation under a small perturbation, we rewrite the linear operator of system (4.1)

$$\mathrm{DB}(\overline{\omega})(\varphi) = \begin{pmatrix} 0_{\mathbb{R}} & \alpha_1 \\ -\frac{1}{\eta} & b - \frac{\eta r}{K} \end{pmatrix} \int_0^{+\infty} \varphi(a)\mathrm{d}a + \begin{pmatrix} \beta_1 & 0_{\mathbb{R}} \\ 0_{\mathbb{R}} & 0_{\mathbb{R}} \end{pmatrix} \int_0^{+\infty} \beta(a)\varphi(a)\mathrm{d}a$$

with

$$\alpha_1 = 0,\ \beta_1 = 1.$$

We introduce two small parameters $\widetilde{\alpha} = (\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ by setting $\widetilde{\alpha}_1 = \alpha_1$, $\widetilde{\alpha}_2 = \beta_1 - 1$ to system (4.1). Then the following system is a small perturbation system of system (4.1)

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \widetilde{A}y(t) + H(y(t)), \quad \text{for } t \geq 0, \tag{5.1}$$

which corresponds to the model (1.3), where $\widetilde{A} = L + S$ with

$$S\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \widetilde{S}(\varphi) \\ 0 \end{pmatrix} \tag{5.2}$$

and

$$\begin{aligned}\widetilde{S}(\varphi) = &\begin{pmatrix} 0 & \widetilde{\alpha}_1 \\ -\frac{1}{\eta} & b - \frac{\eta r}{K} \end{pmatrix} \int\limits_0^{+\infty} \varphi(a)\mathrm{d}a \\ &+ \begin{pmatrix} 1+\widetilde{\alpha}_2 & 0 \\ 0 & 0 \end{pmatrix} \int\limits_0^{+\infty} \beta(a)\varphi(a)\mathrm{d}a\end{aligned} \tag{5.3}$$

with $\|\widetilde{\alpha}\| \ll 1$.

Note that $O(0,0)$ is an equilibrium of system (5.1) and

$$\sigma(\widetilde{A}) \cap \Omega = \sigma_P(\widetilde{A}) \cap \Omega = \{\lambda \in \Omega : \det \Delta^*(\lambda) = 0\}$$

with

$$\det\left(\Delta^*(\lambda)\right) = \frac{\widetilde{\kappa}_1\lambda^2 + \widetilde{\kappa}_2\lambda + \widetilde{\kappa}_3 e^{-\lambda\tau}\lambda + \widetilde{\kappa}_4 e^{-\lambda\tau} + \widetilde{\kappa}_5}{K\eta\left(\lambda+\mu\right)\left(\lambda+d\right)}, \qquad \lambda \in \Omega$$

and

$$\begin{aligned}&\widetilde{\kappa}_1 = K\eta,\ \widetilde{\kappa}_2 = \eta r(\eta - K) + \eta K\mu,\ \widetilde{\kappa}_3 = -\eta\mu K\left(1+\widetilde{\alpha}_2\right), \\ &\widetilde{\kappa}_4 = -\eta r\mu\left(1+\widetilde{\alpha}_2\right)\left(\eta - K\right),\ \widetilde{\kappa}_5 = \eta r\mu\left(\eta - K\right) - K\widetilde{\alpha}_1.\end{aligned}$$

It is easy to check that when $\widetilde{\alpha}_1 = 0$ and $\widetilde{\alpha}_2 = 0$, $\lambda = 0$ is a root of $\det \Delta^*(\lambda) = 0$ with multiplicity 2. From Theorem 4.3, we obtain that the equilibrium $O(0,0)$ of system (5.1) is a cusp of codimension 2.

To determine that system (5.1) is the versal unfolding of system (4.1) with Bogdanov–Takens singularity, we consider the following suspension system

$$\begin{cases} \dfrac{\mathrm{d}\widetilde{\alpha}(t)}{\mathrm{d}t} = 0, \quad \text{for } t \geq 0, \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = \widetilde{A}y(t) + H(y(t)), \quad \text{for } t \geq 0, \\ \widetilde{\alpha}(0) = \widetilde{\alpha} \in \mathbb{R}^2, \\ y(0) = y_0 \in \overline{D(\widetilde{A})}. \end{cases} \tag{5.4}$$

In order to rewrite (5.4) as an abstract Cauchy problem, we set

$$\mathcal{X} = \mathbb{R}^2 \times X.$$

We consider the linear operator $\mathcal{A} : D\left(\mathcal{A}\right) \subset \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{A}\begin{pmatrix} \widetilde{\alpha} \\ \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^2} \\ A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \end{pmatrix}$$

with $D\left(\mathcal{A}\right) = \mathbb{R}^2 \times D(A)$. Then

$$\overline{D\left(\mathcal{A}\right)} = \mathbb{R}^2 \times \overline{D(A)} := \mathcal{X}_0.$$

Since A is a Hille–Yosida operator, we can prove that $\mathcal{A}$ is also a Hille–Yosida operator.

Consider $\mathcal{F}:\overline{D(\mathcal{A})}\to\mathcal{X}$ the nonlinear map defined by

$$\mathcal{F}\left(\begin{pmatrix}\widetilde{\alpha}\\ \begin{pmatrix}0\\ \varphi\end{pmatrix}\end{pmatrix}\right)=\begin{pmatrix}0_{\mathbb{R}^2}\\ W\left(\begin{pmatrix}\widetilde{\alpha}\\ \begin{pmatrix}0\\ \varphi\end{pmatrix}\end{pmatrix}\right)\end{pmatrix},$$

where $W:\overline{D(\mathcal{A})}\to X$ is defined by

$$W\left(\begin{pmatrix}\widetilde{\alpha}\\ \begin{pmatrix}0\\ \varphi\end{pmatrix}\end{pmatrix}\right)=H\begin{pmatrix}0\\ \varphi\end{pmatrix}+\widetilde{H}\left(\widetilde{\alpha},\begin{pmatrix}0\\ \varphi\end{pmatrix}\right)$$

with

$$\widetilde{H}\left(\widetilde{\alpha},\begin{pmatrix}0\\ \varphi\end{pmatrix}\right)=\begin{pmatrix}\widetilde{\widetilde{H}}(\widetilde{\alpha},\varphi)\\ 0\end{pmatrix}$$

and

$$\widetilde{\widetilde{H}}(\widetilde{\alpha},\varphi)=\begin{pmatrix}0 & \widetilde{\alpha}_1\\ 0 & 0\end{pmatrix}\int_0^{+\infty}\varphi(a)\mathrm{d}a+\begin{pmatrix}\widetilde{\alpha}_2 & 0\\ 0 & 0\end{pmatrix}\int_0^{+\infty}\beta(a)\varphi(a)\mathrm{d}a.$$

Then we have

$$\mathcal{F}\begin{pmatrix}0\\ 0\end{pmatrix}=0\quad\text{and}\quad D\mathcal{F}\begin{pmatrix}0\\ 0\end{pmatrix}=0.$$

Now we can reformulate system (5.4) as the following system

$$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t}=\mathcal{A}\xi(t)+\mathcal{F}(\xi(t)),\quad \xi(0)=\xi_0\in\overline{D(\mathcal{A})}. \tag{5.5}$$

Note that

$$\sigma(A)\cap i\mathbb{R}=\{0\}\ \text{and}\ \omega_{0,\text{ess}}\left((A)_0\right)<0.$$

Lemma 5.1. *Let Assumption 4.1 be satisfied. Then*

$$\sigma(\mathcal{A})=\sigma(\mathcal{A}_0)=\sigma(A_0)=\sigma(A),$$

and for each $\lambda\in\rho(\mathcal{A})$,

$$(\lambda I-\mathcal{A})^{-1}\left(\begin{pmatrix}\beta\\ \begin{pmatrix}\delta\\ \psi\end{pmatrix}\end{pmatrix}\right)=\begin{pmatrix}\lambda^{-1}\beta\\ (\lambda I-A)^{-1}\begin{pmatrix}\delta\\ \psi\end{pmatrix}\end{pmatrix}.$$

Proof. Let $\lambda\in\mathbb{C}\diagdown\sigma(A)$. Then

$$\begin{aligned}&(\lambda I-\mathcal{A})\left(\begin{pmatrix}\widehat{\beta}\\ \begin{pmatrix}0\\ \varphi\end{pmatrix}\end{pmatrix}\right)=\left(\begin{pmatrix}\beta\\ \begin{pmatrix}\delta\\ \psi\end{pmatrix}\end{pmatrix}\right)\\ \Leftrightarrow&\begin{pmatrix}\lambda\widehat{\beta}\\ \lambda\begin{pmatrix}0\\ \varphi\end{pmatrix}-A\begin{pmatrix}0\\ \varphi\end{pmatrix}\end{pmatrix}=\left(\begin{pmatrix}\beta\\ \begin{pmatrix}\delta\\ \psi\end{pmatrix}\end{pmatrix}\right)\\ \Leftrightarrow&\begin{cases}\widehat{\beta}=\lambda^{-1}\beta,\\ \begin{pmatrix}0\\ \varphi\end{pmatrix}=(\lambda I-A)^{-1}\begin{pmatrix}\delta\\ \psi\end{pmatrix}.\end{cases}\end{aligned}$$

It follows that

$$(\lambda I-\mathcal{A})^{-1}\begin{pmatrix}\beta\\ \begin{pmatrix}\delta\\ \psi\end{pmatrix}\end{pmatrix}=\begin{pmatrix}\lambda^{-1}\beta\\ (\lambda I-A)^{-1}\begin{pmatrix}\delta\\ \psi\end{pmatrix}\end{pmatrix}.$$

Then

$$\rho(\mathcal{A})\supset\mathbb{C}\diagdown\sigma(A).$$

Moreover, if $\lambda\in\sigma(A)$, we have

$$(\lambda I-\mathcal{A})\begin{pmatrix}0\\ \begin{pmatrix}0\\ \varphi\end{pmatrix}\end{pmatrix}=\begin{pmatrix}0\\ \begin{pmatrix}\delta\\ \psi\end{pmatrix}\end{pmatrix}$$
$$\Leftrightarrow(\lambda I-A)\begin{pmatrix}0\\ \varphi\end{pmatrix}=\begin{pmatrix}\delta\\ \psi\end{pmatrix}.$$

So $\lambda\in\sigma(\mathcal{A})$. □

Lemma 5.2. *Let Assumption 4.1 be satisfied. Then we have*

$$T_{\mathcal{A}_0}(t)\begin{pmatrix}\beta\\ x\end{pmatrix}:=\begin{pmatrix}\beta\\ T_{A_0}(t)x\end{pmatrix}\tag{5.6}$$

and

$$S_{\mathcal{A}}(t)\begin{pmatrix}\beta\\ x\end{pmatrix}:=\begin{pmatrix}t\beta\\ S_A(t)x\end{pmatrix}.\tag{5.7}$$

Furthermore,

$$\omega_{0,ess}(\mathcal{A}_0)=\omega_{0,ess}(A_0).$$

Proof. Recall that

$$(\lambda I-A_0)^{-1}x=\int_0^{+\infty}e^{-\lambda t}T_{A_0}(t)x\mathrm{d}t$$

and

$$(\lambda I-A)^{-1}x=\lambda\int_0^{+\infty}e^{-\lambda t}S_A(t)x\mathrm{d}t.$$

Thus, for each $\lambda>0$ large enough,

$$\int_0^{+\infty}e^{-\lambda t}\begin{pmatrix}\beta\\ T_{A_0}(t)x\end{pmatrix}\mathrm{d}t=\begin{pmatrix}\lambda^{-1}\beta\\ (\lambda I-A_0)^{-1}x\end{pmatrix}$$

and

$$\lambda\int_0^{+\infty}e^{-\lambda t}\begin{pmatrix}t\beta\\ S_A(t)x\end{pmatrix}\mathrm{d}t=\begin{pmatrix}\lambda^{-1}\beta\\ (\lambda I-A)^{-1}x\end{pmatrix}.$$

It follows that $T_{\mathcal{A}_0}(t)$ and $S_{\mathcal{A}}(t)$ are defined, respectively, by (5.6) and (5.7).

By using formula (5.6), we deduce that

$$\|T_{\mathcal{A}_0}(t)\|_{\text{ess}}=\|T_{A_0}(t)\|_{\text{ess}},\quad\text{for all } t\geq 0,$$

and it follows that

$$\omega_{0,\text{ess}}\left(\mathcal{A}_0\right)=\lim_{t\to+\infty}\frac{\ln\left(\|T_{\mathcal{A}_0}(t)\|_{\text{ess}}\right)}{t}=\lim_{t\to+\infty}\frac{\ln\left(\|T_{A_0}(t)\|_{\text{ess}}\right)}{t}=\omega_{0,\text{ess}}\left(A_0\right).$$

This completes the proof. □

Now we compute the projectors on the generalized eigenspace associated to eigenvalue 0 of $\mathcal{A}$. Note that

$$B_{-1,0}^{\mathcal{A}}:=\frac{1}{2\pi i}\int_{S_{\mathbb{C}}(0,\varepsilon)^+}(\lambda I-\mathcal{A})^{-1}\,\mathrm{d}\lambda,$$

where $S_{\mathbb{C}}(0,\varepsilon)^+$ is the counterclockwise oriented circumference $|\lambda|=\varepsilon$ for sufficiently small $\varepsilon>0$ such that $|\lambda|\le\varepsilon$ does not contain other point of the spectrum than 0. Since

$$(\lambda I-\mathcal{A})^{-1}\begin{pmatrix}\mu\\x\end{pmatrix}=\begin{pmatrix}\lambda^{-1}\mu\\(\lambda I-A)^{-1}x\end{pmatrix}=\begin{pmatrix}\lambda^{-1}\mu\\\sum\limits_{k=-2}\lambda^k B_{k,0}^{A}x\end{pmatrix},$$

it follows that

$$B_{-1,0}^{\mathcal{A}}\begin{pmatrix}\mu\\x\end{pmatrix}=\begin{pmatrix}\mu\\B_{-1,0}^{A}x\end{pmatrix}.$$

We have

$$B_{-1,0}^{\mathcal{A}}\begin{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\0\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\0\end{pmatrix},$$

$$B_{-1,0}^{\mathcal{A}}\begin{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\\0\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\\0\end{pmatrix},$$

$$B_{-1,0}^{\mathcal{A}}\begin{pmatrix}0_{\mathbb{R}^2}\\\begin{pmatrix}1\\0\end{pmatrix}\\0\end{pmatrix}=\begin{pmatrix}0_{\mathbb{R}^2}\\B_{-1,0}^{A}\begin{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\0\end{pmatrix}\end{pmatrix}=\begin{pmatrix}0_{\mathbb{R}^2}\\0\\\psi_1\end{pmatrix}$$

and

$$B_{-1,0}^{\mathcal{A}}\begin{pmatrix}0_{\mathbb{R}^2}\\\begin{pmatrix}0\\1\end{pmatrix}\\0\end{pmatrix}=\begin{pmatrix}0_{\mathbb{R}^2}\\B_{-1,0}^{A}\begin{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\\0\end{pmatrix}\end{pmatrix}=\begin{pmatrix}0_{\mathbb{R}^2}\\0\\\psi_2\end{pmatrix},$$

where ψ_1 and ψ_2 are defined in (4.5) and (4.6). Set

$$e_1(a)=\begin{pmatrix}0_{\mathbb{R}^2}\\0_{\mathbb{R}^2}\\\begin{pmatrix}0\\-de^{-da}\end{pmatrix}\end{pmatrix},\quad e_2(a)=\begin{pmatrix}0_{\mathbb{R}^2}\\0_{\mathbb{R}^2}\\\begin{pmatrix}K\mu e^{-\mu a}\\dae^{-da}\end{pmatrix}\end{pmatrix},$$

$$e_3(a)=\begin{pmatrix}1\\0\\\begin{pmatrix}0_{\mathbb{R}^2}\\0\end{pmatrix}\end{pmatrix},\quad e_4(a)=\begin{pmatrix}0\\1\\\begin{pmatrix}0_{\mathbb{R}^2}\\0\end{pmatrix}\end{pmatrix}.$$

Then

$$\mathcal{A}e_1(a)=0,\qquad \mathcal{A}e_2(a)=e_1(a),\qquad \mathcal{A}e_3(a)=\mathcal{A}e_4(a)=0.$$

The projector on the generalized eigenspace of $\mathcal{A}$ associated to 0 is given in the following lemma.

Lemma 5.3. *0 is a pole of order 2 of the resolvent of $\mathcal{A}$, and the projector on the generalized eigenspace of $\mathcal{A}$ associated to 0 is given by*

$$B_{-1,0}^{\mathcal{A}}\begin{pmatrix}\mu\\x\end{pmatrix}=\begin{pmatrix}\mu\\B_{-1,0}^{A}x\end{pmatrix}.$$

From the above results, we obtain a state space decomposition with respect to the spectral properties of the linear operator $\mathcal{A}$. More precisely, the projector on the linear center manifold is defined by

$$\Pi_c^{\mathcal{A}}\begin{pmatrix}\mu\\x\end{pmatrix}=B_{-1,0}^{\mathcal{A}}\begin{pmatrix}\mu\\x\end{pmatrix}=\begin{pmatrix}\mu\\B_{-1,0}^{A}x\end{pmatrix}.$$

Set

$$\Pi_h^{\mathcal{A}}:=I-\Pi_c^{\mathcal{A}}.$$

We denote by

$$\mathcal{X}_c:=\Pi_c(\mathcal{X}),\quad \mathcal{X}_h:=\Pi_h(\mathcal{X}),\quad \mathcal{A}_c:=\mathcal{A}\mid_{\mathcal{X}_c},\ \mathcal{A}_h:=\mathcal{A}\mid_{\mathcal{X}_h}.$$

Now we have the decomposition

$$\mathcal{X}=\mathcal{X}_c\oplus\mathcal{X}_h.$$

Define the basis of $\mathcal{X}_c$ by $\{e_1,e_2,e_3,e_4\}$ and the matrix of $\mathcal{A}_c$ in the basis $\{e_1,e_2,e_3,e_4\}$ of $\mathcal{X}_c$ is given by

$$\mathcal{A}[e_1,e_2,e_3,e_4]=[e_1,e_2,e_3,e_4]\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

Set

$$\xi:=\begin{pmatrix}\widetilde{\alpha}\\w\end{pmatrix}=\begin{pmatrix}\widetilde{\alpha}\\\begin{pmatrix}0\\\widehat{\varphi}\end{pmatrix}\end{pmatrix}\in\overline{D(\mathcal{A})},$$

$$\xi_c:=\Pi_c^{\mathcal{A}}\xi=\begin{pmatrix}\widetilde{\alpha}\\B_{-1,0}^{A}w\end{pmatrix}=\begin{pmatrix}\widetilde{\alpha}\\\begin{pmatrix}0\\\widetilde{\psi}\end{pmatrix}\end{pmatrix}$$

with

$$\begin{aligned}\widetilde{\psi}(a)=&\begin{pmatrix}e^{-\mu a}&0\\0&e^{-da}\end{pmatrix}\begin{pmatrix}\frac{\mu}{1+\tau\mu}&0\\-\frac{2(1+\tau\mu)+d\tau^2\mu}{2K(1+\tau\mu)^2}+\frac{ad}{(1+\tau\mu)K}&d\end{pmatrix}\\&\left[\mathrm{DB}(\overline{w})\left(\int_0^a\begin{pmatrix}e^{-\mu(a-s)}&0\\0&e^{-d(a-s)}\end{pmatrix}\widehat{\varphi}(s)\mathrm{d}s\right)\right]\\&+\begin{pmatrix}e^{-\mu a}&0\\0&e^{-da}\end{pmatrix}\begin{pmatrix}0&0\\-\frac{\mathrm{d}}{(1+\tau\mu)K}&0\end{pmatrix}\\&\mathrm{DB}(\overline{w})\left(\int_0^a\begin{pmatrix}(s-a)e^{-\mu(a-s)}&0\\0&(s-a)e^{-d(a-s)}\end{pmatrix}\widehat{\varphi}(s)\mathrm{d}s\right)\end{aligned}$$

and

$$\xi_h:=\Pi_h^{\mathcal{A}}\xi=(I-\Pi_c^{\mathcal{A}})\xi=\begin{pmatrix}0_{\mathbb{R}^2}\\0_{\mathbb{R}^2}\\\widehat{\varphi}-\widetilde{\psi}\end{pmatrix}.$$

We observe that for each

$$\xi_1 := \begin{pmatrix} \widehat{\alpha}_1 \\ w_1 \end{pmatrix}, \quad \xi_2 := \begin{pmatrix} \widehat{\alpha}_2 \\ w_2 \end{pmatrix} \in \overline{D(\mathcal{A})}$$

with $w_i = \begin{pmatrix} 0_{\mathbb{R}} \\ \varphi_i \end{pmatrix}$, $i = 1, 2$,

$$\begin{aligned} &D^2\mathcal{F}(0)\,(\xi_1, \xi_2) \\ &\quad = \begin{pmatrix} 0_{\mathbb{R}^2} \\ D^2H(0)(w_1, w_2) + \begin{pmatrix} \widehat{\alpha}_1^1 \int_0^{+\infty} \varphi_2^2 \mathrm{d}a + \widehat{\alpha}_2^1 \int_0^{+\infty} \varphi_1^2 \mathrm{d}a + \widehat{\alpha}_1^2 \int_0^{+\infty} \beta\varphi_2^1 \mathrm{d}a + \widehat{\alpha}_2^2 \int_0^{+\infty} \beta\varphi_1^1 \mathrm{d}a \\ 0 \end{pmatrix} \end{pmatrix}. \end{aligned}$$

Then

$$\frac{1}{2!} D^2\mathcal{F}\,(0)\,(\xi)^2 = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \frac{1}{2!} D^2H(0)(w)^2 + \begin{pmatrix} \widetilde{\alpha}_1 \int_0^{+\infty} \widehat{\varphi}^2 da + \widetilde{\alpha}_2 \int_0^{+\infty} \beta\widehat{\varphi}^1 da \\ 0 \end{pmatrix} \end{pmatrix}.$$

By projecting on $\mathcal{X}_c$, we obtain

$$\begin{aligned} &\frac{1}{2!} \Pi_c^{\mathcal{A}} D^2\mathcal{F}(0) \left(\begin{pmatrix} \widetilde{\alpha} \\ \begin{pmatrix} 0 \\ \widehat{\varphi} \end{pmatrix} \end{pmatrix} \right)^2 \\ &\quad = \begin{pmatrix} 0_{\mathbb{R}^2} \\ B_{-1,0}^{A} \left(\frac{1}{2!} D^2H(0)(w)^2 + \begin{pmatrix} \begin{pmatrix} \widetilde{\alpha}_1 \int_0^{+\infty} \widehat{\varphi}^2 da + \widetilde{\alpha}_2 \int_0^{+\infty} \beta\widehat{\varphi}^1 da \\ 0 \end{pmatrix} \\ 0 \end{pmatrix} \right) \end{pmatrix} \\ &\quad = \begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ \frac{1}{2}\overline{\psi} + \widehat{\psi} \end{pmatrix} \end{pmatrix}, \end{aligned}$$

where $\overline{\psi}$ is defined in (4.14) and

$$\widehat{\psi}(a) = \begin{pmatrix} \frac{\mu}{1+\tau\mu} \left(\widetilde{\alpha}_1 \int_0^{+\infty} \widehat{\varphi}^2 da + \widetilde{\alpha}_2 \int_0^{+\infty} \beta\widehat{\varphi}^1 da \right) e^{-\mu a} \\ \left(-\frac{2(1+\tau\mu)+d\tau^2\mu}{2K(1+\tau\mu)^2} + \frac{ad}{(1+\tau\mu)K} \right) \left(\widetilde{\alpha}_1 \int_0^{+\infty} \widehat{\varphi}^2 da + \widetilde{\alpha}_2 \int_0^{+\infty} \beta\widehat{\varphi}^1 da \right) e^{-da} \end{pmatrix}.$$

Now we compute $\frac{1}{2!}\Pi_c^{\mathcal{A}} D^2\mathcal{F}(0)\,(\xi_c)^2$ expressed in terms of the basis $\{e_1, e_2, e_3, e_4\}$. Note that $\xi_c = \begin{pmatrix} \widetilde{\alpha} \\ \begin{pmatrix} 0 \\ \widetilde{\psi} \end{pmatrix} \end{pmatrix}$ can be expressed by

$$\begin{aligned} \xi_c = \begin{pmatrix} \widetilde{\alpha} \\ \begin{pmatrix} 0 \\ \widetilde{\psi} \end{pmatrix} \end{pmatrix} &= x_1 e_1 + x_2 e_2 + \widetilde{\alpha}_1 e_3 + \widetilde{\alpha}_2 e_4 \\ &= \begin{pmatrix} \widetilde{\alpha}_1 \\ \widetilde{\alpha}_2 \\ 0_{\mathbb{R}^2} \\ \begin{pmatrix} x_2 K\mu e^{-\mu a} \\ -x_1 d e^{-da} + x_2 da e^{-da} \end{pmatrix} \end{pmatrix}. \end{aligned}$$

We first obtain that

$$
\begin{aligned}
&\frac{1}{2!}\Pi_c^{\mathcal{A}} D^2\mathcal{F}(0)\left(x_1e_1+x_2e_2+\widetilde{\alpha}_1e_3+\widetilde{\alpha}_2e_4\right)^2\\
&\quad=(e_1,e_2,e_3,e_4)\\
&\quad\times\begin{pmatrix}
-\frac{\left(2(1+\tau\mu)+d\tau^2\mu\right)\left(\frac{x_2}{d}-x_1\right)^2\int_0^{+\infty}\beta(a)\overline{u}(a)da}{4dhK(1+\tau\mu)^2}+\frac{r\left(\frac{x_2}{d}-x_1\right)^2}{2K}+\frac{2(1+\tau\mu)+d\tau^2\mu}{2dK(1+\tau\mu)^2}\vartheta\\
-\frac{\left(\frac{x_2}{d}-x_1\right)^2\int_0^{+\infty}\beta(a)\overline{u}(a)da}{2hK(1+\tau\mu)}+\frac{1}{(1+\tau\mu)K}\vartheta\\
0\\
0
\end{pmatrix}
\end{aligned}
$$

in the above formula we are using a matrix symbolic computation with $\vartheta=-\widetilde{\alpha}_1x_1+\frac{\widetilde{\alpha}_1x_2}{d}+\widetilde{\alpha}_2x_2K\mu$. In the following we will compute the normal form of system (5.5) by using the same procedure as for system (4.1). As before the normal form up to second-order terms on the local center manifold is given by the ordinary differential equation on $\mathcal{X}_c$:

$$
\frac{\mathrm{d}\xi_c(t)}{\mathrm{d}t}=\mathcal{A}_c\xi_c(t)+\frac{1}{2!}\Pi_c^{\mathcal{A}}D^2\widehat{\mathcal{F}}(0)\left(\xi_c(t),\xi_c(t)\right)+R_c\left(\xi_c(t)\right),
$$

where

$$
\begin{aligned}
&\frac{1}{2!}\Pi_c^{\mathcal{A}}D^2\widehat{\mathcal{F}}(0)\left(\xi_c,\xi_c\right)\\
&\quad=\frac{1}{2!}\Pi_c^{\mathcal{A}}D^2\mathcal{F}(0)\left(\xi_c,\xi_c\right)-[\mathcal{A}_c,\Pi_c^{\mathcal{A}}G_2](\xi_c).
\end{aligned}
$$

Set

$$
\xi_c=x_1e_1+x_2e_2+\widetilde{\alpha}_1e_3+\widetilde{\alpha}_2e_4.
$$

Now we compute the normal form expressed in terms of the basis $\{e_1,e_2,e_3,e_4\}$. Consider $V^2(\mathbb{R}^4,\mathbb{R}^4)$ which denote the linear space of the homogeneous polynomials of degree 2 in 4 real variables, $x_1,x_2,\widetilde{\alpha}_1,\widetilde{\alpha}_2$ with coefficients in $\mathbb{R}^4$. The operators Θ_2^c satisfies

$$
\begin{aligned}
\Theta_2^c(G_{2,c})\begin{pmatrix}x_1\\x_2\\\widetilde{\alpha}_1\\\widetilde{\alpha}_2\end{pmatrix}&=[\mathcal{A}_c,G_{2,c}]\begin{pmatrix}x_1\\x_2\\\widetilde{\alpha}_1\\\widetilde{\alpha}_2\end{pmatrix}\\
&=D_xG_{2,c}\begin{pmatrix}x_1\\x_2\\\widetilde{\alpha}_1\\\widetilde{\alpha}_2\end{pmatrix}\mathcal{A}_c\begin{pmatrix}x_1\\x_2\\\widetilde{\alpha}_1\\\widetilde{\alpha}_2\end{pmatrix}-\mathcal{A}_cG_{2,c}\begin{pmatrix}x_1\\x_2\\\widetilde{\alpha}_1\\\widetilde{\alpha}_2\end{pmatrix}\\
&=\begin{pmatrix}\left(D_x\begin{pmatrix}G_{2,c}^1\\G_{2,c}^2\end{pmatrix}M_c\begin{pmatrix}x_1\\x_2\end{pmatrix}-M_c\begin{pmatrix}G_{2,c}^1\\G_{2,c}^2\end{pmatrix}\right)\\D_x\begin{pmatrix}G_{2,c}^3\\G_{2,c}^4\end{pmatrix}M_c\begin{pmatrix}x_1\\x_2\end{pmatrix}\end{pmatrix},\\
\text{for all } G_{2,c}(\cdot)&=\begin{pmatrix}G_{2,c}^1(\cdot)\\G_{2,c}^2(\cdot)\\G_{2,c}^3(\cdot)\\G_{4,c}^4(\cdot)\end{pmatrix}\in V^2(\mathbb{R}^4,\mathbb{R}^4)
\end{aligned}
$$

with

$$\mathcal{A}_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ M_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We define $\overline{\Theta}_2^c : V^2(\mathbb{R}^4, \mathbb{R}^2) \to V^2(\mathbb{R}^4, \mathbb{R}^2)$ by

$$\overline{\Theta}_2^c \begin{pmatrix} G_{2,c}^1 \\ G_{2,c}^2 \end{pmatrix} = D_x \begin{pmatrix} G_{2,c}^1 \\ G_{2,c}^2 \end{pmatrix} M_c \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - M_c \begin{pmatrix} G_{2,c}^1 \\ G_{2,c}^2 \end{pmatrix},$$

$$\text{for all } \begin{pmatrix} G_{2,c}^1 \\ G_{2,c}^2 \end{pmatrix} \in V^2(\mathbb{R}^4, \mathbb{R}^2).$$

The canonical basis of $V^2(\mathbb{R}^4, \mathbb{R}^2)$ has 20 elements:

$$\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \widetilde{\alpha}_1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \widetilde{\alpha}_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} x_2 \widetilde{\alpha}_1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \widetilde{\alpha}_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \widetilde{\alpha}_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} \widetilde{\alpha}_1 \widetilde{\alpha}_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \widetilde{\alpha}_2^2 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 \widetilde{\alpha}_1 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 \widetilde{\alpha}_2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ x_2 \widetilde{\alpha}_1 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2 \widetilde{\alpha}_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \widetilde{\alpha}_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ \widetilde{\alpha}_1 \widetilde{\alpha}_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \widetilde{\alpha}_2^2 \end{pmatrix}$$

Their images under $\overline{\Theta}_2^c$ are, respectively,

$$\begin{pmatrix} 2x_1 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \widetilde{\alpha}_1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \widetilde{\alpha}_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} -x_1^2 \\ 2x_1 x_2 \end{pmatrix}, \begin{pmatrix} -x_1 x_2 \\ x_2^2 \end{pmatrix}, \begin{pmatrix} -x_1 \widetilde{\alpha}_1 \\ x_2 \widetilde{\alpha}_1 \end{pmatrix}, \begin{pmatrix} -x_1 \widetilde{\alpha}_2 \\ x_2 \widetilde{\alpha}_2 \end{pmatrix}, \begin{pmatrix} -x_2^2 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} -x_2 \widetilde{\alpha}_1 \\ 0 \end{pmatrix}, \begin{pmatrix} -x_2 \widetilde{\alpha}_2 \\ 0 \end{pmatrix}, \begin{pmatrix} -\widetilde{\alpha}_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} -\widetilde{\alpha}_1 \widetilde{\alpha}_2 \\ 0 \end{pmatrix}, \begin{pmatrix} -\widetilde{\alpha}_2^2 \\ 0 \end{pmatrix}.$$

A complementary space of $R(\overline{\Theta}_2^c)$ in $V^2(\mathbb{R}^4, \mathbb{R}^2)$ is

$$R(\overline{\Theta}_2^c)^c = \text{span} \left\{ \begin{matrix} \begin{pmatrix} 0 \\ x_2 \widetilde{\alpha}_1 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 \widetilde{\alpha}_1 \end{pmatrix}, \\ \begin{pmatrix} 0 \\ x_1 \widetilde{\alpha}_2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2 \widetilde{\alpha}_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \widetilde{\alpha}_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ \widetilde{\alpha}_1 \widetilde{\alpha}_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \widetilde{\alpha}_2^2 \end{pmatrix} \end{matrix} \right\}.$$

Thus, we obtain the normal form up to second-order term as follows

$$\begin{aligned} \dot{x}_1 &= x_2 + O(|(x_1, x_2)|^3) \\ \dot{x}_2 &= \lambda_1 x_1 + \lambda_2 x_2 + A_1(\widetilde{\alpha}_1, \widetilde{\alpha}_2) x_1^2 + A_2(\widetilde{\alpha}_1, \widetilde{\alpha}_2) x_1 x_2 + O(|(x_1, x_2)|^3), \end{aligned} \tag{5.8}$$

where $A_1(0,0)$ and $A_2(0,0)$ are defined in (4.15) and

$$\lambda_1 = -\frac{1}{(1+\tau\mu)K} \tilde{\alpha}_1, \quad \lambda_2 = \left(-\frac{\tau^2 \mu}{2K(1+\tau\mu)^2} \right) \tilde{\alpha}_1 + \frac{\mu}{(1+\tau\mu)} \tilde{\alpha}_2.$$

Let

$$\begin{aligned} x_1 &\to x_1, \\ x_2 &\to x_2 + O(|(x_1, x_2)|^3). \end{aligned}$$

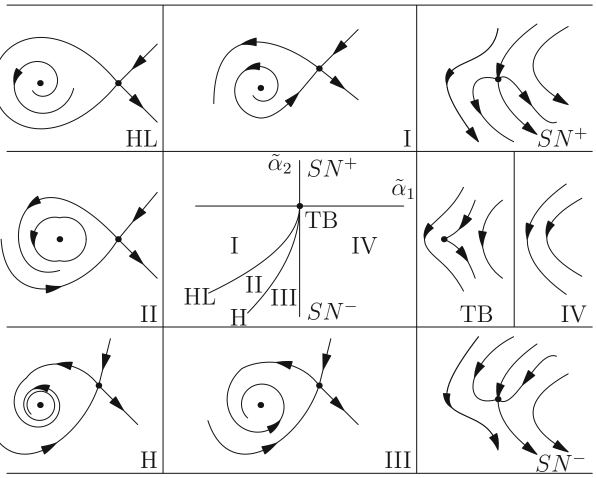

Fig. 1. Bogdanov–Takens bifurcation diagram of system (5.9) in the small neighborhood of $(\tilde{\alpha}_1, \tilde{\alpha}_2) = (0, 0)$

Then

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \lambda_1 x_1 + \lambda_2 x_2 + A_1(\tilde{\alpha}_1, \tilde{\alpha}_2)x_1^2 + A_2(\tilde{\alpha}_1, \tilde{\alpha}_2)x_1x_2 + O(|(x_1, x_2)|^3).\end{aligned}$$

After rescalings and reparameterizations (see [7, page 181]), we can transform the above system into

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \mu_1 + \mu_2 x_2 + x_1^2 - x_1x_2 + O(|(x_1, x_2)|^3),\end{aligned} \tag{5.9}$$

with $\mu_1 = -\frac{A_2^4\lambda_1^2}{4A_1^4}, \mu_2 = \left|\frac{A_2}{A_1}\right| \left(\lambda_2 - \frac{A_2\lambda_1}{2A_1}\right)$. From the above analysis, we obtain the following theorem.

Theorem 5.4. *Assume that Assumption 4.1 holds. Then system (1.3) can undergo Bogdanov–Takens bifurcation in a small neighborhood of the unique positive equilibrium as the bifurcating parameters $(\tilde{\alpha}_1, \tilde{\alpha}_2)$ vary in a small neighborhood of $(0, 0)$. More precisely, there exist four bifurcation curves: saddle-node bifurcation curves SN^+ and SN^-, Hopf bifurcation curve H, and homoclinic bifurcation curve HL, in the small neighborhood of $(0, 0)$ of parameter plane $(\tilde{\alpha}_1, \tilde{\alpha}_2)$, such that system (5.1) has a unique stable limit cycle as $(\tilde{\alpha}_1, \tilde{\alpha}_2)$ lies between H and HL, and no limit cycle for system (5.1) outside this region. The corresponding bifurcation diagram is shown in Fig. 1.*

6. Conclusion

Codimension-two bifurcations are important phenomenon in nonlinear dynamics. However, compared with codimension-one bifurcations in age-structured models, there has been little work done on the codimension-two bifurcations which produce rich dynamics phenomenon. To the best of our knowledge, we haven't found works that have been published on the Bogdanov–Takens bifurcation for age-structured models. In this paper, a predator–prey model with age structure in predator population is investigated. By performing the bifurcation analysis, we prove that the model has a positive equilibrium which is Bogdanov–Takens singularity, and we can choose two parameters as the bifurcation parameters such that this model undergoes Bogdanov–Takens bifurcation in the small neighborhood of the positive equilibrium as bifurcation parameters vary in a small neighborhood of the bifurcation values (see Theorem 5.4). More precisely, on the curve SN, the model has only a unique equilibrium which is a saddle-node; when the parameters $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ lie between the curve SN and the curve H, the model has a saddle and a stable focus and no periodic orbit; on the HL curve, the model has an unstable focus and a stable homoclinic loop; the model has a saddle, an unstable focus, and no periodic orbits when the parameters lie between the curve HL and the curve SN; however, when the parameters lie between the curve H and the curve HL, the model has a unique stable limit cycle (see Fig. 1).

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