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A Uniqueness Result for Nontrivial Steady States of a Density-dependent Population Dynamics Model

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1. INTRODUCTION

In this paper we investigate the uniqueness of a nontrivial fixed point of a map preserving a positive cone. This work is motivated by application to age-structured density-dependent population dynamics models. The main result of this paper shows that in certain situations all the fixed points are comparable for the order induced by the positive cone. We then apply this result to prove the uniqueness of the nontrivial equilibrium solution for a discrete-time population dynamics model and its continuous-time analogue.

Let K be a cone of a Banach space $(X, \|\cdot\|)$; that is, K is a closed convex subset of X, satisfying $tK \subset K$, for all $t \geq 0$, and if $x \in K \setminus \{0\}$, then $-x \notin K$. Such a cone K induces a partial order on X, denoted by $\leq$, and defined by $x \leq y \Leftrightarrow y - x \in K$. In the sequel, we will also define

$$x < y \Leftrightarrow y - x \in K \setminus \{0\} \quad \text{and} \quad x \ll y \Leftrightarrow y - x \in \operatorname{Int}(K),$$

where $\operatorname{Int}(K)$ denotes the interior of K in $(X, \|\cdot\|)$.

Let (Λ, d_Λ) be a metric space, and $\lambda_0 \in \Lambda$. Let $F: \Lambda \times K \to K$ be a continuous map, such that $F(\lambda_0, \cdot)$ is right differentiable at zero (see Deimling [2, p. 225] for the corresponding definition), for each $\lambda \in \Lambda$, $F(\lambda, \cdot)$ is asymptotically smooth (see Hale [3, p. 11] for the corresponding definition), and $F(\lambda, 0) = 0$.

In the sequel, for each $\lambda \in \Lambda$, we will denote by $F_\lambda: K \to K$ the map defined for all $x \in K$ by $F_\lambda(x) = F(\lambda, x)$. Moreover, given (M, d) a metric

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space, and $T\colon M \to M$ a map, we will denote by T^m ($m \in \mathbb{N}$) the map defined by

$$T^0 = \mathrm{Id} \quad \text{and} \quad T^m = T^{m-1} \circ T \quad \forall m \geq 1.$$

Given $g\colon M \to X$ a map, and A a subset of M, we will denote by

$$\|g\|_{\mathrm{Lip}, A} = \sup_{x, y \in A : x \neq y} \|x - y\|^{-1} \|g(x) - g(y)\|.$$

We will make the following hypotheses on F.

($H1$) Zero is globally asymptotically stable for F_{λ_0}.

($H2$) There exists $v \in \mathrm{Int}_X(K)$ such that $D_+F_{\lambda_0}(0)v = v$, there exists $v^* \in K^* \setminus \{0\}$ (with $v^*(v) = 1$) such that $D_+F_{\lambda_0}(0)^* v^* = v^*$, and

$$r\big((\mathrm{Id} - P) D_+ F_{\lambda_0}(0)(\mathrm{Id} - P)\big) < 1,$$

where $D_+F_{\lambda_0}(0)$ denotes the right derivative of F_{λ_0} at zero, and $P \in \mathscr{L}(X)$ is defined by

$$P(x) = v^*(x) v, \quad \forall x \in X.$$

($H3$) We assume that

$$\lim_{\delta \to 0} \sup_{\lambda \in \Lambda : d_\Lambda(\lambda, \lambda_0) \leq \delta} \|g_\lambda\|_{\mathrm{Lip}, B_K(0, \delta)} = 0,$$

where $B_K(0, \delta) = \{x \in K \colon \|x\| \leq \delta\}$, and $g\colon \Lambda \times K \to X$ is defined by

$$g(\lambda, x) = F(\lambda, x) - D_+ F_{\lambda_0}(0) x, \quad \forall x \in K, \quad \forall \lambda \in \Lambda.$$

($H4$) For each $\lambda \in \Lambda$, there exists $\alpha_\lambda > 0$, such that $B_K(0, \alpha_\lambda)$ is positively invariant by F_λ (i.e., $F_\lambda(B_K(0, \alpha_\lambda)) \subset B_K(0, \alpha_\lambda)$), and for all $x \in K : \|x\| \geq \alpha_\lambda$, there exists $m = m(x) \in \mathbb{N}$ such that $\|F_\lambda^m(x)\| < \alpha_\lambda$.

($H5$) If $\{A_\lambda\}_{\lambda \in \Lambda}$ is a subset family of K, such that for each $\lambda \in \Lambda$, A_λ is maximal compact invariant for F_λ (see Hale [3, p. 17] for the corresponding definition), then there exists a compact subset $C \subset K$ such that

$$\lim_{R \to 0^+} \delta\Bigg(\bigcup_{\lambda \in \Lambda : d_\Lambda(\lambda, \lambda_0) \leq R} A_\lambda, C \Bigg) = 0,$$

where $\delta(B_1, B_2)$ is defined by $\delta(B_1, B_2) = \sup_{y \in B_1} \inf_{x \in B_2} \|x - y\|$.

The following theorem is the main result of this paper.

THEOREM 1.1. *Let K be a cone of a Banach space $(X, \|\cdot\|)$, such that* $\mathrm{Int}(K) \neq \emptyset$. *Let (Λ, d_Λ) be a metric space, and $\lambda_0 \in \Lambda$. Let $F: \Lambda \times K \to K$ be a continuous map, such that $F(\lambda_0, \cdot)$ is right differentiable at zero, for each $\lambda \in \Lambda$, F_λ is asymptotically smooth, $F_\lambda(0) = 0$, and F satisfies assumptions $(H1)$–$(H5)$.*

Then there exists $\delta > 0$, such that $\forall \lambda \in \Lambda \setminus \{\lambda_0\} : d_\Lambda(\lambda_0, \lambda) \leq \delta$, and if $\bar{x}_1, \bar{x}_2 \in K$ are two distinct fixed points of F_λ, then

$$\bar{x}_1 \ll \bar{x}_2 \quad \text{or} \quad \bar{x}_1 \gg \bar{x}_2.$$

As consequence of Theorem 1.1, we will obtain a uniqueness result for nontrivial equilibrium solutions of the model introduced by Liu and Cohen [4]. Consider the difference equation $\forall t \in \mathbb{N}$

$$\begin{cases} x_1(t+1) = \sum_{i=1}^{n} \left[b_i \cdot x_i(t) \cdot \exp\left(- \sum_{j=1}^{n} \tilde{\gamma}_{ij} \cdot x_j(t) \right) \right] \\ x_2(t+1) = x_1(t) \cdot \exp\left(- \left[M_1 + \sum_{j=1}^{n} \gamma_{1j} \cdot x_j(t) \right] \right) \\ \vdots \\ x_n(t+1) = x_{n-1}(t) \cdot \exp\left(- \left[M_{n-1} + \sum_{j=1}^{n} \gamma_{n-1,j} \cdot x_j(t) \right] \right). \end{cases} \tag{1}$$

with

$$x_i(0) = x_i \geq 0, \quad \forall i = 1, \dots, n,$$

where $b_i \geq 0$, $\tilde{\gamma}_{ij} \geq 0$, and $M_k \geq 0$, $\gamma_{kl} \geq 0$, $\forall i, j, l = 1, \dots, n$, $\forall k = 1, \dots, n-1$.

The difference equation (1) is investigated by Liu and Cohen [4]; they obtain an existence and uniqueness result for nontrivial equilibrium solutions. Here we will obtain a new kind of condition for the uniqueness of the nontrivial fixed point. Let us denote by

$$\lambda = (b_1, b_2, \dots, b_n)^T \in \mathbb{R}_+^n, \text{ and } R(\lambda) = \sum_{j=1}^{n} b_j l_j,$$

where

$$l_1 = 1, \text{ and } l_i = \prod_{j=1}^{i-1} \exp(-M_j), \quad \forall i = 2, \dots, n. \tag{2}$$

Let $\lambda_0 = (b_1^0, b_2^0, \dots, b_n^0)^T \in \mathbb{R}_+^n$ be fixed such that $R(\lambda_0) = 1$, and assume that

(i) $\forall i = 1, \dots, n (b_i^0 > 0) \Rightarrow (\tilde{\gamma}_{ii} > 0)$.

(ii) $$L_1 = \begin{pmatrix} b_1^0 & b_2^0 & \cdots & \cdots & b_{n_0}^0 \\ p_1 & 0 & \cdots & \cdots & 0 \\ 0 & p_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & p_{n_0-1} & 0 \end{pmatrix} \in M_{n_0}(\mathbb{R}) \text{ is primitive,}$$

where $p_i = \exp(-M_i)$, $\forall i = 1, \dots, n-1$, and $n_0 = \max\{k \in \{1, 2, \dots, n\} : b_k^0 > 0\}$. We denote by $F: \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+^n$, the map associated to Eq. (1), which is defined by

$$F(\lambda, x) = \begin{bmatrix} \sum_{i=1}^{n} \left[b_i \cdot x_i \exp\left(-\sum_{j=1}^{n} \tilde{\gamma}_{ij} \cdot x_j \right) \right] \\ x_1 \exp\left(-\left[M_1 + \sum_{j=1}^{n} \gamma_{1j} \cdot x_j \right] \right) \\ \vdots \\ x_{n-1} \exp\left(-\left[M_{n-1} + \sum_{j=1}^{n} \gamma_{n-1,j} \cdot x_j \right] \right) \end{bmatrix},$$

$$\forall \lambda \in \mathbb{R}_+^n, \quad \forall x \in \mathbb{R}_+^n.$$

Then by defining Λ as

$$\Lambda = \left\{ \lambda \in \mathbb{R}_+^n : \lambda_0 \leq \lambda \leq C_1 \lambda_0 \right\}, \text{ for some } C_1 > 1,$$

one has the following theorem.

THEOREM 1.2. *Under assumptions* (i)–(ii), *there exists* $\delta > 0$, *such that for all* $\lambda \in \Lambda \setminus \{\lambda_0\}$: $\|\lambda_0 - \lambda\|_{R_+^n} \leq \delta$, *and* F_λ *has at most one non-null fixed point*.

The model of Liu and Cohen [4] is obtained by discretizing the following partial derivative equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} = -\left(\mu(a) + \int_0^m \gamma(a,s) u(t,s)\, ds \right) u(t,a), \\ u(t,0) = \int_0^m f(a) \exp\left(-\int_0^m \tilde{\gamma}(a,s) u(t,s)\, ds \right) u(t,a)\, da. \end{cases} \tag{3}$$

We now investigate the uniqueness of the nontrivial steady state of Eq. (3). Such equations are extensively studied in the book by Webb [9], and we refer to this book for a survey on this subject. Integrating over age, the problem of finding a steady state of Eq. (3) can be rewritten as the following fixed point problem: To find $u \in C_+^0[0,m]$, satisfying

$$u(a) = H_f(u)\sigma(a)\exp\left[-\int_0^a V(u)(s)\, ds \right], \quad \forall a \in [0,m], \tag{4}$$

where for all $u \in C_+^0[0,m]$, for all $a \in [0,m]$,

$$V(u)(a) = \int_0^m \gamma(a,s) u(s)\, ds, \; \tilde{V}(u)(a) = \int_0^m \tilde{\gamma}(a,s) u(s)\, ds,$$

$$H_f(u) = \int_0^m f(s) \exp\left[-\tilde{V}(u)(s) \right] u(s)\, ds,$$

and

$$\sigma(a) = \exp\left[-\int_0^a \mu(s)\, ds \right].$$

In this example $X = C^0[0,m]$, endowed with the norm $\|v\|_\infty = \sup_{a\in[0,m]}|v(a)|$, $K = C_+^0[0,m]$, and $\lambda = f \in L_+^\infty[0,m]$. Let $\lambda_0 = f_0 \in L_+^\infty[0,m]$, such that $\int_0^m f_0(a)\sigma(a)\, da = 1$, and there exists $\varepsilon \in]0,m[$, such that

$$\operatorname{supp}(f_0) \subset [\varepsilon, m].$$

We will make the following assumptions,

(iii) $\forall a \in [0,m[$, $\int_0^a \mu(s)\, ds < +\infty$, and $\lim_{a \to m^-} \int_0^a \mu(s)\, ds = +\infty$.

(iv) $\gamma, \tilde{\gamma} \in C^0([0,m] \times [0,m], \mathbb{R}_+)$.

(v) There exists $C_0 > 0$, such that for all $a \in [0,\varepsilon]$, $\gamma(a,s) \geq C_0 f_0(s)$ for almost every $s \in [0,m]$.

Then by defining Λ as

$$\Lambda = \left\{ f \in L_+^\infty[0,m]:\ f_0 \leq f \leq C_1 f_0 \right\}, \text{ for some } C_1 > 1,$$

one has the following theorem.

THEOREM 1.3. *Under assumptions* (iii)–(v), *there exists* $\delta > 0$, *such that for all* $f \in \Lambda$: $\|f_0 - f\|_{L^\infty[0,m]} < \delta$, *Eq.* (4) *has at most one non-null solution.*

2. ATTRACTORS EXISTENCE

In this section we recall some results proved in Magal [6] using results on dissipative discrete time dynamical systems of the book by Hale [3].

The following proposition is proved in Magal [6, Proposition 2.2].

PROPOSITION 2.1. *Let* K *be a cone of a Banach space* $(X, \|\cdot\|)$, *and let* (Λ, d_Λ) *be a metric space. Let* F: $\Lambda \times K \to K$ *be a continuous map, such that for* $\lambda \in \Lambda$, F_λ *is asymptotically smooth. Assume in addition that* F *satisfies Assumption* $(H4)$.

Then, for each $\lambda \in \Lambda$, *there exists a subset* $A_\lambda \subset B_K(0, \alpha_\lambda)$ *maximal compact and invariant by* F_λ, *which is stable and attracts the compact sets of* K *by* F_λ.

The following proposition is proved in Magal [6, Proposition 2.5].

PROPOSITION 2.2. *Let* K *be a cone of a Banach space* $(X, \|\cdot\|)$, *and let* (Λ, d_Λ) *be a metric space. Let* F: $\Lambda \times K \to K$ *be a continuous map, such that* $\lambda \in \Lambda$, F_λ *is asymptotically smooth,* $F_\lambda(0) = 0$, *and* F *satisfies assumptions* $(H1)(H4)$ *and* $(H5)$.

Then $\forall \varepsilon > 0$, $\exists \eta > 0$ *such that* $A_\lambda \subset B_K(0, \varepsilon)$, $\forall \lambda \in \Lambda$: $d_\Lambda(\lambda, \lambda_0) \leq \eta$.

3. A REDUCTION RESULT

In this section we adapt to invariant bounded sets the method developed by Vanderbauwhede [7]. For a survey of this question, we also refer to the paper by Vanderbauwhede [8] and the book by Chow *et al.* [1, p. 1–48].

We consider a Banach space $(X, \|\cdot\|)$, M a bounded subset of X, and T: $M \to M$ a continuous map, and we assume that M is invariant by T (i.e., $T(M) = M$). The problem is then to reduce the following system:

$$\begin{cases} x(t+1) = T(x(t)), & \forall t \in \mathbb{N} \\ \text{with} & \\ x(0) = x_0 \in M. & \end{cases} \tag{5}$$

Here, to reduce the system (5) means that given $P \in \mathscr{L}(X)$ a bounded linear operator of projection, we look for a map ϕ: $P(M) \to (\mathrm{Id} - P)(M)$

such that $\forall x \in M$

$$x = P(x) + \phi(P(x)).$$

In other words, we look for a map ϕ: $P(M) \to (\mathrm{Id} - P)(M)$ such that the graph of ϕ (i.e., $\mathrm{Gr}(\phi) = \{y + \phi(y)$: $y \in P(M)\}$ is equal to M. Then each solution of system (5) corresponds to a solution of the following system (and conversely):

$$\begin{cases} y(t+1) = P\big(T\big[y(t) + \phi(P[y(t)])\big]\big), \quad \forall t \in \mathbb{N} \\ \text{with} \\ y(0) = y_0 \in P(M). \end{cases} \tag{6}$$

We will make the following assumptions.

($\tilde{H}1$) M is invariant by T (i.e., $T(M) = M$).

($\tilde{H}2$) $\forall x \in M$, $T(x) = A(x) + g(x)$ where $A \in \mathscr{L}(X)$, and $g \in \mathrm{Lip}(M, X)$.

($\tilde{H}3$) X has a decomposition $X = X_1 \oplus X_2$, where X_1 and X_2 are closed subspaces X which are positively invariant A and

$$a = \sup_{\lambda \in \sigma(A_1)} |\lambda| < b = \inf_{\lambda \in \sigma(A_2)} |\lambda| \leq 1$$

where $A_i = A|_{X_i} \in \mathscr{L}(X_i)$, for $i = 1, 2$.

In the sequel, we will denote by $P \in \mathscr{L}(X)$ the linear bounded operator projection satisfying $\mathrm{Im}(P) = X_2$, and $\mathrm{Ker}(P) = X_1$, and for each $\eta > 0$, we will denote by $Y_\eta^-(X)$ the Banach space of all the sequences $y = \{y_{-p} \in X$: $p \in \mathbb{N}\}$ satisfying $\|y\|_{Y_\eta^-(X)} = \sup\{\eta^p \|y_{-p}\|$: $p \in \mathbb{N}\} < +\infty$, and we denote $Y_\eta^-(M)$ the subset of all sequence $y = \{y_{-p} \in X$: $p \in \mathbb{N}\} \subset M$.

The following lemma can be found in the paper by Vanderbauwhede [7, Lemma 1, p. 410].

LEMMA 3.1. *Let $A \in \mathscr{L}(X)$ be bounded linear operator satisfying assumption* ($\tilde{H}3$). *Then* $\forall \varepsilon > 0$, $\exists M = M(\varepsilon) > 0$ *such that* $\forall m \in \mathbb{N}$

$$\|A_1^m\| \leq M(\varepsilon)(a + \varepsilon)^m \quad \textit{and} \quad (b - \varepsilon)^m \|A_2^{-m}\| \leq M(\varepsilon).$$

The proof of the following lemma uses similar of arguments as in the proof of Lemma 2, p. 411 in Vanderbauwhede [7].

LEMMA 3.2. *Let* $(X, \|\cdot\|)$ *be a Banach space, M be a bounded subset of X, and T: $M \to M$ be a continuous map satisfying* ($\tilde{H}1$), ($\tilde{H}2$), *and* ($\tilde{H}3$).

Let $y = \{y_{-p}: p \in \mathbb{N}\} \subset M$ *be a negative* T*-orbit, then* $\forall p \in \mathbb{N}$,

$$y_{-p} = A_2^{-p} P y_0 - \sum_{l=1}^{p}{}^{(+)} A_2^{l-1-p} P g(y_{-l}) + \sum_{l=1}^{\infty} A_1^{l-1} (\mathrm{Id} - P) g(y_{-p-l}), \tag{7}$$

where the notation $\sum^{(+)}$ *indicate that the corresponding sum is only present if* $p \geq 1$.

Equation (7) can be rewritten in the more compact form $y \in Y_\eta^-(M)$,

$$y = SP(y_0) + KG(y), \tag{8}$$

where for each $\eta \in]a, b[$, the operators $S \in \mathscr{L}(X_2, Y_\eta^-(X))$, $K \in \mathscr{L}(Y_\eta^-(X))$, and $G \in C^0(Y_\eta^-(M), Y_\eta^-(X))$ are defined by

$$(Sx_2)_{-p} = A_2^{-p} x_2, \quad \forall x_2 \in X_2, \quad \forall p \in \mathbb{N},$$

$$(Ky)_{-p} = -\sum_{l=1}^{p}{}^{(+)} A_2^{l-1-p} P y_{-l} + \sum_{l=1}^{\infty} A_1^{l-1} (\mathrm{Id} - P) y_{-p-l}, \quad \forall y \in Y_\eta^-(X), \quad \forall p \in \mathbb{N},$$

and

$$(Gy)_{-p} = g(y_{-p}), \quad \forall y \in Y_\eta^-(M), \quad \forall p \in \mathbb{N}.$$

The fact that S and K are bounded linear operators follows from Lemma 3.1. Moreover, one has for all $y, \tilde{y} \in Y_\eta^-(X)$,

$$\|G(y) - G(\tilde{y})\|_{Y_\eta^-(X)} \leq \|g\|_{\mathrm{Lip}} \|y - \tilde{y}\|_{Y_\eta^-(X)}.$$

THEOREM 3.3. *Let* $(X, \|\cdot\|)$ *be a Banach space, let* M *be a bounded subset of* X, *and let* $T: M \to M$ *be a continuous map satisfying Assumptions* $(\tilde{H}1)$, $(\tilde{H}2)$, *and* $(\tilde{H}3)$ *and* $\eta \in]a, b[$. *Then there exists* $C_1(A, \eta) > 0$, *such that if*

$$C_1(A, \eta) \|g\|_{\mathrm{Lip}} < 1,$$

there exists a map $\phi \in \mathrm{Lip}(P(M), (\mathrm{Id} - P)(M))$ *such that* $\forall x \in M$

$$x = P(x) + \phi(P(x)),$$

and there exists $C_2(A, \eta) > 0$ *such that*

$$\|\phi\|_{\mathrm{Lip}} \leq \frac{C_2(A,\eta)}{1 - C_1(A,\eta)\|g\|_{\mathrm{Lip}}}\|g\|_{\mathrm{Lip}}.$$

Moreover, $\forall x \in M$, *there exists one and only one negative T-orbit* $y \in Y_\eta^-(M)$ *through* x.

Proof. In the sequel we will take $C_1(A,\eta) = \|K\|_{\mathscr{L}(Y_\eta^-(X))}$. We start by proving that there exists a map ϕ: $P(M) \to (\mathrm{Id} - P)(M)$ such that for all $x \in M$, $(\mathrm{Id} - P)(x) = \phi(Px)$. Let $y_0, z_0 \in M$ such that $Py_0 = Pz_0$. We now show that necessarily $(\mathrm{Id} - P)y_0 = (\mathrm{Id} - P)z_0$. Indeed, since M is invariant by T, we may find two negative T-orbits $y, z \in Y_\eta^-(M)$ such that $(y)_0 = y_0$, and $(z)_0 = z_0$. Then by Lemma 3.2, one has

$$y = SP(y_0) + KG(y), \text{ and } z = SP(z_0) + KG(z).$$

But, since $P(y_0) = P(z_0)$ one has

$$y - KG(y) = z - KG(z).$$

Let Ψ: $Y_\eta^-(M) \to Y_\eta^-(X)$ be the map defined by $\Psi(x) = x - KG(x)$, $\forall x \in Y_\eta^-(M)$. Let us show that Ψ is injective. Let $x_1, x_2 \in Y_\eta^-(M)$; then

$$x_1 - x_2 = \Psi(x_1) - \Psi(x_2) + KG(x_1) - KG(x_2),$$

thus

$$\begin{aligned}&\|x_1 - x_2\|_{Y_\eta^-(X)} \\ &\qquad \leq \|\Psi(x_1) - \Psi(x_2)\|_{Y_\eta^-(X)} + \|K\|_{\mathscr{L}(Y_\eta^-(X))}\|g\|_{\mathrm{Lip}}\|x_1 - x_2\|_{Y_\eta^-(X)},\end{aligned}$$

and by stating that

$$C = 1 - \|K\|_{\mathscr{L}(Y_\eta^-(X))}\|g\|_{\mathrm{Lip}} > 0,$$

one has

$$d_{Y_\eta^-(M)}(x_1, x_2) = \|x_1 - x_2\|_{Y_\eta^-(X)} \leq C^{-1}\|\Psi(x_1) - \Psi(x_2)\|_{Y_\eta^-(X)}. \quad (9)$$

From Eq. (9), and since $\Psi(y) = \Psi(z)$ we deduce that $y = z$. Finally we obtain

$$(\mathrm{Id} - P)y_0 = (\mathrm{Id} - P)z_0.$$

So there exists a map ϕ: $P(M) \to (\mathrm{Id} - P)(M)$ such that for all $x \in M$

$$(\mathrm{Id} - P)(x) = \phi(Px).$$

The previous part of the proof also shows that given $x \in M$, there exists one and only one negative T-orbit through x. To prove that ϕ is Lipschitzian, let us first come back to the map Ψ: $Y_\eta^-(M) \to Y_\eta^-(X)$, and let us denote by $\tilde{\Psi}$: $Y_\eta^-(M) \to \Psi(Y_\eta^-(M))$ the map defined by

$$\tilde{\Psi}(x) = \Psi(x), \quad \forall x \in Y_\eta^-(M).$$

Then by construction of $\tilde{\Psi}$, and from Eq. (9), $\tilde{\Psi}$ is invertible. Moreover, one has from Eq. (9) for all $x_1, x_2 \in \Psi(Y_\eta^-(M))$

$$\|\tilde{\Psi}^{-1}(x_1) - \tilde{\Psi}^{-1}(x_2)\|_{Y_\eta^-(X)} \leq C^{-1}\|x_1 - x_2\|_{Y_\eta^-(X)}. \tag{10}$$

Let $x_2, y_2 \in P(M)$, and $x, y \in Y_\eta^-(M)$ be the negative T-orbits through $x_2 + \phi(x_2)$ and $y_2 + \phi(y_2)$, respectively. Then we have from Lemma 3.2 that

$$x = S(x_2) + KG(x) \quad \text{and} \quad y = S(y_2) + KG(y),$$

so

$$x = \tilde{\Psi}^{-1}(S(x_2)) \quad \text{and} \quad y = \tilde{\Psi}^{-1}(S(y_2)),$$

and

$$\phi(x_2) = (\mathrm{Id} - P)\big((S(x_2) + KG(x))_0\big) \text{ and}$$
$$\phi(y_2) = (\mathrm{Id} - P)\big((S(y_2) + KG(y))_0\big).$$

From this we deduce that

$$\phi(x_2) = \sum_{l=1}^{\infty} A_1^{l-1}(\mathrm{Id} - P)g(x_{-l}) \quad \text{and}$$
$$\phi(y_2) = \sum_{l=1}^{\infty} A_1^{l-1}(\mathrm{Id} - P)g(y_{-l}),$$

so

$$\begin{aligned} &\|\phi(x_2) - \phi(y_2)\| \\ &\quad \leq \|g\|_{\mathrm{Lip}}\|(\mathrm{Id} - P)\|_{L(X)}\left[\sum_{l=0}^{+\infty}\|A_1^l\|_{L(X_1)}\eta^{-l}\right] \\ &\qquad \times \|\tilde{\Psi}^{-1}(Sx_2) - \tilde{\Psi}^{-1}(Sy_2)\|_{Y_\eta^-(X)}. \end{aligned}$$

Finally, one has

$$\|\phi\|_{\text{Lip}} \leq \frac{\|(\text{Id} - P)\|_{L(X)}\left[\sum_{l=0}^{+\infty} \|A_1^l\|_{L(X_1)}\eta^{-l}\right]\|S\|_{L(X_2, Y_\eta^-(X))}}{1 - \|K\|_{L(Y_\eta^-(X))}\|g\|_{\text{Lip}}}\|g\|_{\text{Lip}}.$$

∎

4. PROOF OF THEOREM 1.1

Under Assumption $(H4)$ Proposition 2.1 applies, and we deduce that for each $\lambda \in \Lambda$, there exists $A_\lambda \subset Y_\lambda$, which is compact maximal invariant by F_λ, stable for F_λ, and attracts all the compacts subsets of K by F_λ. Under Assumptions $(H1)$, $(H4)$, and $(H5)$ Proposition 2.2 applies, and one has $\forall \varepsilon > 0$, $\exists \delta_0 > 0$ such that

$$A_\lambda \subset B_K(0, \varepsilon) = \{x \in K: \|x\| \leq \varepsilon\}, \quad \forall \lambda \in B_\Lambda(\lambda_0, \delta_0).$$

Consider now the projection operator $P \in \mathscr{L}(X)$, the bounded linear operator defined by $P(x) = v^*(x)v$, $\forall x \in X$ introduced in Assumption $(H2)$. Then, since $v \in \text{Int}_X(K)$, there exists $\varepsilon > 0$ such that

$$v + B_X(0, \varepsilon) \subset \text{Int}_X(K). \tag{11}$$

where $B_X(0, \varepsilon) = \{x \in X: \|x\| \leq \varepsilon\}$.

Let $\eta \in]a, 1[$, where $a = r((\text{Id} - P)D_+F_{\lambda_0}(0)(\text{Id} - P)) < 1$. Then, under Assumption $(H3)$, and by upper semicontinuity of the family $\{A_\lambda\}_{\lambda \in \Lambda}$ at $\lambda = \lambda_0$, there exists $\delta_1 > 0$ such that $\forall \lambda \in \Lambda$: $d_\Lambda(\lambda_0, \lambda) \leq \delta_1$

$$C_1\big(D_+F_{\lambda_0}(0), \eta\big)\|g_\lambda\|_{\text{Lip}, A_\lambda} < 1 \quad \text{and}$$

$$\frac{C_2\big(D_+F_{\lambda_0}(0), \eta\big)}{1 - C_1\big(D_+F_{\lambda_0}(0), \eta\big)\|g\|_{\text{Lip}, A_\lambda}}\|g\|_{\text{Lip}, A_\lambda} \leq \frac{\varepsilon}{\|v\|}.$$

Thus we can apply Theorem 3.3 to $F_\lambda|_{A_\lambda}$, and we deduce that for all $\lambda \in \Lambda$: $d_\Lambda(\lambda_0, \lambda) \leq \delta_1$ there exists a map $\phi_\lambda \in \text{Lip}(P(A_\lambda), (\text{Id} - P)(A_\lambda))$ such that $\forall x \in A_\lambda$

$$x = P(x) + \phi_\lambda(P(x))$$

and

$$\|\phi_\lambda\|_{\text{Lip}} \leq \frac{\varepsilon}{\|v\|}.$$

Let $\lambda^* \in \Lambda$: $d_\Lambda(\lambda_0, \lambda^*) \leq \delta_1$ be fixed, and let $x_1, x_2 \in A_{\lambda^*}$ be fixed. Then $x_1 - x_2 = P(x_1 - x_2) + \phi_\lambda(P(x_1)) - \phi_\lambda(P(x_2)) = v^*(x_1 - x_2)v + \phi_\lambda(P(x_1)) - \phi_\lambda(P(x_2))$, and

$$\begin{aligned}&\|\phi_\lambda(P(x_1)) - \phi_\lambda(P(x_2))\| \\ &\qquad \leq \|\phi_\lambda\|_{\text{Lip}}\|P(x_1) - P(x_2)\| \leq \frac{\varepsilon}{\|v\|}\|v^*(x_1 - x_2)v\|.\end{aligned}$$

Thus

$$\|\phi_\lambda(P(x_1)) - \phi_\lambda(P(x_2))\| \leq \varepsilon v^*(x_1 - x_2).$$

From this we deduce using Eq. (11) that

$$x_1 - x_2 \in v^*(x_1 - x_2)\left[v + B_X(0, \varepsilon)\right] \subset v^*(x_1 - x_2)\text{Int}(K),$$

and the proof is complete. ∎

5. PROOF OF THEOREM 1.2

We now prove that Assumptions $(H1)$ to $(H5)$ are satisfied for F. Here $K = \mathbb{R}^n_+$, and $X = \mathbb{R}^n$ is endowed with the norm $\|\cdot\|_\infty$ defined by $\|x\|_\infty = \max\{|x_i|: i = 1, \dots, n\}$, $\forall x \in \mathbb{R}^n_+$. One may note that

$$F(\lambda, x) = L(\lambda, x)x, \quad \forall x \in \mathbb{R}^n_+, \quad \forall \lambda \in \Lambda,$$

where the map L: $\Lambda \times \mathbb{R}^n_+ \to M_n(\mathbb{R}_+)$ is a continuous map from $\Lambda \times \mathbb{R}^n_+$ into the set of nonnegative matrices, defined by

$$L(\lambda, x) = \begin{bmatrix} L_1(\lambda, x) & 0 \\ L_2(\lambda, x) & L_3(\lambda, x) \end{bmatrix},$$

where (under assumption (ii)), and the fact that $\lambda \geq \lambda_0$ $\forall \lambda \in \Lambda$, the block matrix $L_1(\lambda, x) \in M_{n_0}(\mathbb{R})$ is primitive $\forall \lambda \in \Lambda$, $\forall x \in \mathbb{R}^n_+$, with $r(L_1(\lambda_0, 0)) = 1$ (because $L_1(\lambda_0, 0) = L_1$), and $r(L_1(\lambda, 0)) > 1$, $\forall \lambda \in \Lambda$, and the block matrix $L_3(\lambda_0, 0)$ satisfies $r(L_3(\lambda_0, 0)) = 0$.

Assumption $(H1)$ can be verified using the Lyapunov method with the Lyapunov function

$$V(x) = \max\left\{\frac{x_i}{l_i}: i = 1, \dots, n\right\},$$

where l_i is defined in Eq. (2). Using this Lyapunov function with assumption (i) one may verify assumption $(H1)$. Moreover, $F(\lambda_0, \cdot)$ is clearly right differentiable at zero, and by using Assumption (ii), one deduces from non-negative matrix theory that $r(D_+F(\lambda_0,0)) = 1$, 1 is a simple eigenvalue of

$$D_+F(\lambda_0,0) = \begin{bmatrix} L_1(\lambda_0,0) & 0 \\ L_2(\lambda_0,0) & L_3(\lambda_0,0) \end{bmatrix},$$

and there is no other eigenvalue into the peripheral spectrum of $D_+F(\lambda_0,0)$. Moreover, by a direct computation, if we denote by $v \in \mathbb{R}^n_+$ the eigenvector of $D_+F(\lambda_0,0)$ associated to the eigenvalue 1, one has

$$v = (l_1 \ \ l_2 \ \ \cdots \ \ l_n)^T \in \mathrm{Int}(\mathbb{R}^n_+),$$

where l_i is defined in Eq. (2), and Assumption $(H2)$ is satisfied.

Assumption $(H3)$ is an immediate consequence of the regularity of the exponential map. We are now interested in Assumption $(H4)$. From Assumption (i) the constant

$$C_2 = \min\left\{ \frac{\tilde{\gamma}_{ii}}{b_i^0} : i = 1,\ldots,n, \quad \text{and} \quad b_i^0 > 0 \right\} > 0$$

is well defined, and one has

$$C_1 x \exp(C_2 x) \leq \frac{C_1}{C_2 e}, \quad \forall x \geq 0.$$

The ball

$$B_{R^n_+}(0,\tilde{M}) = \left\{ x \in \mathbb{R}^n_+ : \|x\|_\infty \leq \tilde{M} \right\}$$

(with $\tilde{M} = (C_1/C_2 e) + 1$) is then positively invariant by $F(\lambda,\cdot)$, and $\forall x \in \mathbb{R}^n_+$ one also has

$$F_\lambda^n(x) \in B_{R^n_+}\left(0, \frac{C_1}{C_2 e}\right) \subset \mathrm{Int}_{R^n_+}\left[B_{R^n_+}(0,\tilde{M}) \right],$$

and by taking $\alpha_\lambda = \tilde{M}$, Assumption $(H4)$ is satisfied.

Assumption $(H5)$ is then automatically satisfied, because from Proposition 2.1, we know that under Assumption $(H4)$, if A_λ is compact, maximal, and invariant by $F(\lambda,\cdot)$, then $A_\lambda \subset B_{R^n_+}(0,\tilde{M})$. So by taking $C = B_{R^n_+}(0,\tilde{M})$ in Assumption $(H5)$, we deduce that Assumption $(H5)$ is satisfied.

One may now apply Theorem 1.1, and we deduce that there exists $\delta > 0$ such that $\forall \lambda \in \Lambda \setminus \{\lambda_0\}$: $d_\Lambda(\lambda_0,\lambda) \leq \delta$, if $\bar{x}_1, \bar{x}_2 \in \mathbb{R}^n_+$ are two distinct fixed points of F_λ, then $\bar{x}_1 \ll \bar{x}_2$ or $\bar{x}_1 \gg \bar{x}_2$.

Assume that there exists $\lambda \in \Lambda \setminus \{\lambda_0\}$: $d_\Lambda(\lambda_0, \lambda) \leq \delta$, and there exist $\bar{x}_1, \bar{x}_2 \in \mathbb{R}_+^n \setminus \{0\}$, two distinct fixed points of F_λ. Then from Theorem 1.1 one has

$$0 \ll \bar{x}_1 \ll \bar{x}_2 \quad \text{or} \quad 0 \ll \bar{x}_2 \ll \bar{x}_1.$$

But then since

$$F(\lambda, x) = L(\lambda, x)x, \quad \forall x \in \mathbb{R}_+^n, \quad \forall \lambda \in \Lambda,$$

where

$$L(\lambda, x) = \begin{bmatrix} L_1(\lambda, x) & 0 \\ L_2(\lambda, x) & L_3(\lambda, x) \end{bmatrix}, \quad \forall x \in \mathbb{R}_+^n,$$

and $L_1(\lambda, x)$ is an irreducible matrix $\forall x \in \mathbb{R}_+^n$, we deduce that

$$r\big(L_1(\lambda, \bar{x}_1)\big) = r\big(L_1(\lambda, \bar{x}_2)\big) = 1.$$

On the other hand, assume, for example, that $0 \ll \bar{x}_1 \ll \bar{x}_2^*$, then by construction one has

$$L_1(\lambda, \bar{x}_1) > L_1(\lambda, \bar{x}_2),$$

and since $L_1(\lambda, x)$ is an irreducible matrix $\forall x \in \mathbb{R}_+^n$, this implies that

$$r\big(L_1(\lambda, \bar{x}_1)\big) > r\big(L_1(\lambda, \bar{x}_2)\big),$$

and we obtain a contradiction. From this contradiction, we deduce that there exists $\delta > 0$, such that $\forall \lambda \in \Lambda \setminus \{\lambda_0\}$: $d_\Lambda(\lambda_0, \lambda) \leq \delta$, and F_λ has at most one nontrivial fixed point. The proof is complete. ∎

6. PROOF OF THEOREM 1.3

We first remark that to find a solution of Eq. (4) is equivalent to solving the following fixed point problem: Find $v \in C_+^0[0, m]$, satisfying for all $a \in [0, m]$

$$v(a) = H_f(\sigma v)\exp\left[-\int_0^a V(\sigma v)(s)\, ds\right]. \tag{12}$$

Let F: $L_+^\infty[0, m] \times C^0[0, m] \to C^0[0, m]$, the map defined for each $f \in L_+^\infty[0, m]$, for all $v \in C^0[0, m]$, by

$$F_f(v)(a) = H_f(\sigma v)\exp\left[-\int_0^a V(\sigma v)(s)\, ds\right], \quad \forall a \in [0, m],$$

where

$$H_f(\sigma v) = \int_0^m f(s)\exp\left[-\tilde{V}(\sigma v)(s)\right]\sigma(s)v(s)\,ds.$$

To prove Theorem 1.3, we start by proving the following proposition.

PROPOSITION 6.1. *Under the assumptions of Theorem 1.3, there exists* $\delta > 0$, *such that for all* $f \in \Lambda$: $\|f_0 - f\|_{L^\infty[0,m]} < \delta$, *if* $v_1, v_2 \in C_+^0[0, m]$ *are two distinct fixed points of* F_f, *then*

$$v_1 \ll v_2 \quad or \quad v_2 \ll v_1.$$

To prove Proposition 6.1, we will apply Theorem 1.1 of F_f^2. We start by proving some preliminary lemmas.

LEMMA 6.2. *Under the assumptions of Theorem* 1.3, *there exists* $M > 0$, *such that*

$$\|F_f^2(v)\|_\infty \le M, \quad \forall f \in \Lambda, \quad \forall v \in C_+^0[0, m].$$

Proof. Let $v \in C_+^0[0, m]$, and $f \in \Lambda$. Then for all $a \in [0, m]$,

$$F_f^2(v)(a) = H_f\big(F_f(v)\big)\exp\left[-\int_0^a V\big(\sigma F_f(v)\big)(s)\,ds\right];$$

thus

$$\|F_f^2(v)\|_\infty \le \textstyle\int_0^m f(s)\sigma(s)v(s)\,ds \int_0^m f(a)\sigma(a)\exp\left[-\int_0^a V(\sigma v)(s)\,ds\right]da,$$

$$\|F_f^2(v)\|_\infty \le \textstyle\int_\varepsilon^m f(s)\sigma(s)v(s)\,ds \int_\varepsilon^m f(a)\sigma(a)\,da\,\exp\left[-\int_0^\varepsilon V(\sigma v)(s)\,ds\right].$$

Under assumption made on f_0, and by construction of Λ, one has

$$\begin{aligned}\int_0^\varepsilon V(\sigma v)(s)\,ds &= \int_0^\varepsilon \int_0^m \gamma(s,\xi)\sigma(\xi)v(\xi)\,d\xi\,ds\\ &\ge \frac{C_0}{C_1}\int_0^\varepsilon \int_0^m f(\xi)\sigma(\xi)v(\xi)\,d\xi\,ds;\end{aligned}$$

thus

$$\int_0^\varepsilon V(\sigma v)(s)\,ds \ge \frac{C_0}{C_1}\varepsilon \int_\varepsilon^m f(\xi)\sigma(\xi)v(\xi)\,d\xi.$$

So, we obtain, using in addition that

$$\|F_f^2(v)\|_\infty \leq \int_\varepsilon^m f(a)\sigma(a)\,da \int_\varepsilon^m f(\xi)\sigma(\xi)v(\xi)\,d\xi$$
$$\times \exp\left[-\frac{C_0}{C_1}\varepsilon\int_\varepsilon^m f(\xi)\sigma(\xi)v(\xi)\,d\xi\right],$$
$$\|F_f^2(v)\|_\infty \leq C_1 \int_\varepsilon^m f(\xi)\sigma(\xi)v(\xi)\,d\xi$$
$$\times \exp\left[-\frac{C_0}{C_1}\varepsilon\int_\varepsilon^m f(\xi)\sigma(\xi)v(\xi)\,d\xi\right];$$

thus

$$\|F_f^2(v)\|_\infty \leq \frac{C_1^2}{C_0\,\varepsilon}.$$

∎

LEMMA 6.3. *Under assumptions of Theorem 1.3, for each* $f \in \Lambda$, $F_f|_{C_+^0[0,m]}$ *is completely continuous*.

Proof. The proof is a direct consequence of the Ascoli−Arzela criteria of compactness in $C^0[0,m]$, and we will not detail further the proof. ∎

LEMMA 6.4. *Under assumptions of Theorem 1.3, Assumption* $(H1)$ *is satisfied for* $F_{f_0}^2|_{C_+^0[0,m]}$.

Proof. From the proof of Lemma 6.2, we already know that for all $v \in C_+^0[0,m]$,

$$\|F_{f_0}^2(v)\|_\infty \leq \int_\varepsilon^m f_0(a)\sigma(a)\,da \int_\varepsilon^m f_0(\xi)\sigma(\xi)v(\xi)\,d\xi$$
$$\times \exp\left[-\frac{C_0}{C_1}\varepsilon\int_\varepsilon^m f_0(\xi)\sigma(\xi)v(\xi)\,d\xi\right],$$

and as $\int_\varepsilon^m f_0(a)\sigma(a)\,da = 1$, one has

$$\|F_{f_0}^2(v)\|_\infty \leq \int_\varepsilon^m f_0(\xi)\sigma(\xi)v(\xi)\,d\xi \exp\left[-\frac{C_0}{C_1}\varepsilon\int_\varepsilon^m f_0(\xi)\sigma(\xi)v(\xi)\,d\xi\right].$$

From this we deduce that for all $v \in C_+^0[0,m]$,

$$\|F_{f_0}^2(v)\|_\infty \leq \|v\|_\infty,$$

and zero is a stable fixed point of F_f^2. Moreover, since for all $v \in C_+^0[0,m]$,

$$\begin{aligned} \textstyle\int_\varepsilon^m f_0(\xi)\sigma(\xi)F_f^2(v)\,d\xi &\le \|F_{f_0}^2(v)\|_\infty \\ &\le \textstyle\int_\varepsilon^m f_0(\xi)\sigma(\xi)v(\xi)\,d\xi \\ &\qquad \times \exp\left[-\frac{C_0}{C_1}\varepsilon \textstyle\int_\varepsilon^m f_0(\xi)\sigma(\xi)v(\xi)\,d\xi\right], \end{aligned}$$

one also has global asymptotic stability using the Lyapunov function

$$L(v) = \int_\varepsilon^m f_0(a)\sigma(a)v(a)\,da, \quad \forall v \in C_+^0[0,m].$$

∎

LEMMA 6.5. *Under the assumptions of Theorem 1.3, Assumption $(H2)$ is satisfied for $F_{f_0}^2|_{C_+^0[0,m]}$.*

Proof. $F_{f_0}|_{C_+^0[0,m]}$, $F_{f_0}^2|_{C_+^0[0,m]}$, are clearly right differentiable at zero, and one has for each $v \in C_+^0[0,m]$

$$D_+F_{f_0}(0)(v)(s) = \int_0^m f_0(a)\sigma(a)v(a)\,da, \quad \forall s \in [0,m].$$

$D_+F_{f_0}(0)$ is a projection operator, so the spectrum of $D_+F_{f_0}(0)$ is $\{0,1\}$,

$$D_+F_{f_0}(0)v = v \quad \text{and} \quad \|v\|_\infty = 1 \Leftrightarrow v(a) = 1, \quad \forall a \in [0,m],$$

and by taking

$$v^*(y) = \int_0^m f_0(a)\sigma(a)y(a)\,da, \quad \forall y \in C_+^0[0,m],$$

Assumption $(H2)$ is satisfied. ∎

LEMMA 6.6. *Under the assumptions of Theorem 1.3, Assumption $(H3)$ is satisfied for $F^2\colon \Lambda \times C_+^0[0,m] \to C_+^0[0,m]$, the map defined by*

$$F^2(f,v) = F_f^2(v), \quad \forall (f,v) \in \Lambda \times C_+^0[0,m].$$

Proof. To prove the lemma, it is sufficient to remark that for each $f \in \Lambda$, and each $v \in C_+^0[0,m]$, $F_f^2\colon C^0[0,m] \to C^0[0,m]$ is differentiable at v, and $DF_f^2(v)$ depends continuously on f and v. Indeed, by definition

$$g_f(v) = F_f^2(v) - D_+F_{f0}^2(0)v, \quad \forall v \in C_+^0[0,m].$$

Let $\varepsilon > 0$, $v_1, v_2 \in C^0_+[0, m]$; then

$$\begin{aligned} g_f(v_1) - g_f(v_2) &= \int_0^1 DF_f^2(sv_1 + (1-s)v_2)(s[v_1 - v_2]) \\ &\quad - D_+F_{f_0}^2(0)(s[v_1 - v_2])\, ds, \end{aligned}$$

and one has

$$\|g_f\|_{\mathrm{Lip},\, B_K(0,\,\delta)} \le \sup_{v \in B_K(0,\,\delta)} \left\| DF_f^2(v) - D_+F_{f_0}^2(0) \right\|.$$

Using now the continuity of $DF_f^2(v)$ in v and f, one has

$$\lim_{\delta \to 0} \sup_{\lambda \in \Lambda:\ d_\Lambda(\lambda,\, \lambda_0) \le \delta} \|g_\lambda\|_{\mathrm{Lip},\, B_K(0,\,\delta)} = 0.$$

∎

LEMMA 6.7. *Under the assumptions of Theorem 1.3, Assumption* $(H4)$ *is satisfied for* $F^2\colon \Lambda \times C^0_+[0, m] \to C^0_+[0, m]$, *with* $\alpha_f = M + 1$, $\forall f \in \Lambda$.

Proof. Using Lemma 6.2, we know that there exists $M > 0$, such that

$$\|F_f^2(v)\|_\infty \le M, \quad \forall f \in \Lambda, \quad \forall v \in C^0_+[0, m].$$

So to verify Assumption $(H4)$, it is sufficient to take

$$\alpha_f = M + 1,\ \forall f \in \Lambda.$$

∎

To prove Assumption $(H5)$, we will use the following proposition, which is proved in Magal [5, Corollary 3.6.2, p. 90].

PROPOSITION 6.8. *Let* (Λ, d_Λ) *be a metric space,* $\lambda_0 \in \Lambda$, *let* (M, d) *be a metric space, and let* $T\colon \Lambda \times M \to M$ *be continuous map. Let* $\{A_\lambda\}_{\lambda \in \Lambda}$ *be a family of subsets of* M, *such that for each* $\lambda \in \Lambda$, A_λ *is maximal compact invariant for* T_λ. *Assume in addition that*

(i) T_{λ_0} *is completely continuous.*

(ii) *There exists a bounded subset* $U \subset M$, *such that* $\bigcup_{\lambda \in \Lambda} A_\lambda \subset U$.

(iii) $T_\lambda(y) \to T_{\lambda_0}(y)$ *as* $\lambda \to \lambda_0$ *uniformly on* y *in* U.

Then there exists a compact subset $C \subset M$ *such that*

$$\lim_{R\to 0^+} \delta\left(\bigcup_{\lambda\in\Lambda:\ d_\Lambda(\lambda,\lambda_0)\le R} A_\lambda, C\right) = 0,$$

where $\delta(B_1, B_2)$ *is defined by* $\delta(B_1, B_2) = \sup_{y\in B_1} \inf_{x\in B_2} d(x, y)$.

LEMMA 6.9. *Under the assumptions of Theorem 1.3, Assumption (H5) is satisfied for* $F^2\colon \Lambda \times C^0_+[0, m] \to C^0_+[0, m]$.

Proof. From Lemma 6.7, we know that Assumption $(H4)$ is satisfied for $F^2\colon \Lambda \times C^0_+[0, m] \to C^0_+[0, m]$, with $\alpha_f = M + 1$, $\forall f \in \Lambda$. So, from Proposition 2.1 we deduce that if $\{A_f\}_{f\in\Lambda}$ is a subset family of $C^0_+[0, m]$, such that for each $f \in \Lambda$, A_f is maximal compact invariant for F^2_f, then $A_f \subset B_{C^0_+[0,m]}(0, M+1) = \{v \in C^0_+[0, m]\colon \|v\|_\infty \le M + 1\}$. To prove Lemma 6.9, it remains to verify Assumption (iii) of Proposition 6.8. But one can verify that there exists $C > 0$, such that

$$\|F^2_f(v) - F^2_{f_0}(v)\|_\infty \le C\|f - f_0\|_{L^\infty[0,m]},$$
$$\forall v \in B_{C^0_+[0,m]}(0, M+1), \quad \forall f \in \Lambda.$$

∎

Proof (of Proposition 6.1). From Lemmas 6.4, 6.5, 6.6, 6.7, and 6.9, Theorem 1.1 applies to F^2, and we deduce that there exists $\delta > 0$, such that $\forall f \in \Lambda \setminus \{f_0\}$: $\|f_0 - f\|_{L^\infty[0,m]} \le \delta$, if $\bar{v}_1, \bar{v}_2 \in C^0_+[0, m]$ are two distinct fixed points of F^2_λ, then

$$\bar{v}_1 \ll \bar{v}_2 \quad \text{or} \quad \bar{v}_1 \gg \bar{v}_2.$$

As every fixed point of F_λ is a fixed point F^2_λ, the same conclusion hold for F_λ. ∎

Proof (of Theorem 1.3). By Proposition 6.1, we know that there exists $\delta > 0$, such that $\forall f \in \Lambda \setminus \{f_0\}$: $\|f_0 - f\|_{L^\infty[0,m]} \le \delta$, if $\bar{v}_1, \bar{v}_2 \in C^0_+[0, m]$ are two distinct fixed points of F_λ, then

$$\bar{v}_1 \ll \bar{v}_2 \quad \text{or} \quad \bar{v}_1 \gg \bar{v}_2.$$

On the other hand, for each $f \in \Lambda$, F_f has the form

$$F_f(v) = L_f(v)v, \quad \forall v \in C^0_+[0, m],$$

where $L\colon \Lambda \times C^0_+[0, m] \to \mathscr{L}(C^0[0, m])$, is defined for each $f \in \Lambda$, $v \in$

$C_+^0[0,m]$, $y \in C^0[0,m]$, and each $a \in [0,m]$, by

$$L_f(v)(y)(a) = \int_0^m f(s) \exp\left[-\tilde{V}(\sigma v)(s)\right] \sigma(s) y(s)\, ds \exp\left[-\int_0^a V(\sigma v)(s)\, ds\right].$$

One may note that, for each $f \in \Lambda$, $v \in C_+^0[0,m]$,

$$L_f(v) = r(f,v) Q_f(v),$$

where $Q_f(v)$ is a projection operator, and

$$r(f,v) = \int_0^m f(a) \exp\left[-\tilde{V}(\sigma v)(a)\right] \sigma(a) \exp\left[-\int_0^a V(\sigma v)(s)\, ds\right] da.$$

Let $f \in \Lambda \setminus \{f_0\}$: $\|f_0 - f\|_{L^\infty[0,m]} \leq \delta$. Suppose that there exists $\bar{v}_1, \bar{v}_2 \in C_+^0[0,m] \setminus \{0\}$, two distinct fixed points of F_f; then

$$0 \ll \bar{v}_1 \ll \bar{v}_2 \quad \text{or} \quad \bar{v}_1 \gg \bar{v}_2 \gg 0.$$

Assume for example that $0 \ll \bar{v}_1 \ll \bar{v}_2$, then

$$\bar{v}_i = r(f, \bar{v}_i) Q_f(\bar{v}_i), \quad \forall i = 1, 2,$$

so

$$r(f, \bar{v}_1) = r(f, \bar{v}_2) = 1.$$

On the other hand, by using Assumption (vi) of Theorem 1.3, one has

$$r(f, \bar{v}_1) > r(f, \bar{v}_2),$$

and we obtain a contradiction. The proof is complete. ∎

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