On Semilinear Cauchy Problems with Non-dense Domain

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Abstract

We are interested in studying semilinear Cauchy problems in which the closed linear operator is not Hille-Yosida and its domain is not densely defined. Using intergrated semigroup theory, we study the positivity of solutions to the semilinear problem, the Lipschitz perturbation of the problem, differentiability of the solutions with respect to the state variable, time differentiability of the solutions, and the stability of equilibria. The obtained results can be used to study several types of differential equations, including delay differential equations, age-structure models in population dynamics, and evolution equations with nonlinear boundary conditions.

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Contents

1 Introduction

The main purpose of this paper is to present a comprehensive semilinear theory (Cazenave and Haraux [5], Davies and Pang [8], Hieber [15, 16], Xiao and Liang [45], etc.) that will allow us to study the dynamics of the non-densely defined Cauchy problems, such as asymptotic behavior of solutions and bifurcations. Consider the Cauchy problem:

$$\frac{du}{dt} = Au + F(t,u), \;\; t \geq 0, \;\; u(0) = u_0 \in \overline{D(A)}, \tag{1.1}$$

where $A : D(A) \subset X \to X$ is a linear operator in a Banach space X and $F : [0,+\infty) \times \overline{D(A)} \to X$ is a continuous map. We are interested in studying the problem when $\overline{D(A)}$ is not dense in X and A is not a Hille-Yosida operator. Several types of differential equations, such as delay differential equations, age-structure models in population dynamics, some partial differential equations, evolution equations with nonlinear boundary conditions, can be written as semilinear Cauchy problems with non-dense domain (see Da Prato and Sinestrari [7], Thieme [35, 36], Liu et al. [19], Magal and Ruan [22]).

When A is a Hille-Yosida operator (i.e., if there exist two constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that $(\omega,+\infty) \subset \rho(A)$ and $\| (\lambda I - A)^{-k} \|_{\mathcal{L}(X)} \leq \frac{M}{(\lambda-\omega)^k}, \forall \lambda > \omega, \forall k \geq 1$, where $\mathcal{L}(X)$ the space of bounded linear operators from X into X and $\rho(A)$ is the resolvent set of A) and is densely defined (i.e., $\overline{D(A)} = X$), the problem has been extensively studied (see Segal [33], Weissler [44], Martin [25], Pazy [30], Hirsch and Smith [17]). When A is a Hille-Yosida operator but its domain is non-densely defined, Da Prato and Sinestrari [7] investigated the existence of several types of solutions for (1.1). Thieme [35] investigated the semilinear Cauchy problem with a Lipschitz perturbation of the closed linear operator A which is non-densely defined but is Hille-Yosida. Integrated semigroup theory was used to obtain a variation of constants formula which allows to transform the integrated solutions of the evolution equation to solutions of an abstract semilinear Volterra integral equation, which in turn was used to find integrated solutions to the Cauchy problem. Moreover, sufficient and necessary conditions for the invariance of closed convex sets under the solution flow were found. Conditions for the regularity of the solution flow in time and initial state were derived. The steady states of the solution flow were characterized and sufficient conditions for local stability and instability were given. See also Thieme [36, 38].

In this paper, we attempt to extend Thieme's results [35] to the case when the operator A is not Hille-Yosida. The perturbations of the operator are only Lipschitz on bounded sets. In order to do such extensions, we need to obtain an estimate for nonhomogeneous equation which is obtained in Proposition 2.14. Then we are able to develop and extend most of his results (except the positivity result) in [35]. We obtain some weak and classical conditions for positivity of the solutions by considering the problem in a more general setting; that is, we

only impose that the integrated solutions exist when the nonlinear interaction term F is continuous.

We would like to make some comments about the assumption that the linear operator A is not a Hille-Yosida operator. We first assume that the resolvent set $\rho(A)$ of A is non-empty and that A_0, the part of A in $\overline{D(A)}$, is the infinitesimal generator of a strongly continuous semigroup $\{T_{A_0}(t)\}_{t\geq 0}$ of bounded linear operators on $\overline{D(A)}$. Then A generates an integrated semigroup $\{S_A(t)\}_{t\geq 0}$ on X, defined by

$$S_A(t) = (\lambda I - A_0)\int_0^t T_{A_0}(l)dl\,(\lambda I - A)^{-1}$$

for each $\lambda \in \rho(A)$.

Then (see Magal and Ruan [22] and Thieme [38]) we need to impose an additional condition to assure the existence of integrated solutions of the non-homogeneous Cauchy problem

$$\frac{du(t)}{dt} = Au(t) + f(t) \text{ for } t \geq 0 \text{ and } u(0) = 0. \tag{1.2}$$

Here we assume that A is not a Hille-Yosida operator (Assumptioon 2.1), but for each $f \in C([0,\tau], X)$ the Cauchy problem (1.2) has an integrated solution $u_f(t)$, and there exists a map $\delta : [0,+\infty) \to [0,+\infty)$ (independent of f) such that (Assumption 2.8 and Theorem 2.9)

$$\|u_f(t)\| \leq \delta(t) \sup_{s\in[0,t]} \|f(s)\|, \tag{1.3}$$

where

$$\delta(t) \to 0 \text{ as } t \to 0.$$

In particular as in the example presented in Magal and Ruan [22], the Cauchy problem may not have an integrated solution for $f \in L^1(0,\tau;X)$. The goal of this paper is to show that under such a condition on the Cauchy problem (1.2), we still can extend the results for the classical semi-linear Cauchy problems.

In practice, it is relatively easy to verify that the linear operator A is not a Hille-Yosida operators (i.e. verifying Assumptioon 2.1). Nevertheless, for a given example, it may require some work to verify the condition (1.3) (i.e. Assumption 2.8). In the context of age-structured models the condition (1.3) has been successfully verified by Magal and Ruan [22] using some characterization on the resolvent of A. For the same class of PDEs, Thieme [38] also successfully applied the notion integrated semigroup with bounded p-semi variation to verify condition (1.3). For parabolic systems, this question has been studied by Prevost [32] and Ducrot et al. [13]. In particular, it has been proved that if A is almost sectorial, and A_0 the part of A in $D(A)$ is a sectorial operator, then condition (1.3) is satisfied. We also refer Prevost [32] for more examples in the context of parabolic equations.

The rest of the paper is organized as follows. In section 2, we recall some results on integrated semigroups and give an estimate for solutions to the non-homogeneous equation (see Proposition 2.14) which is crucial for the stability

of equilibria to the semilinear problem. Sections 3-7 are devoted to the study of the semilinear problem. In section 3, positivity of solutions to the semilinear problem is considered. Section 4 focuses on Lipschitz perturbations of the problem. Section 5 deals with differentiability of the solutions with respect to the state variable. In section 6 we are concerned with time differentiability of the solutions. The stability of equilibria is studied in section 7. In section 8, as applications we discuss transport equations with nonlinear boundary conditions and parabolic equations with nonlocal boundary conditions and show that our results apply. In particular, we verify that our main Assumptions 2.1 and 2.8 hold for these two types of equations.

Notice that Magal and Ruan [22] presented some techniques and results for integrated semigroups when the generator is not a Hille-Yosida operator and is non-densely defined, obtained necessary and sufficient conditions for the existence of mild solutions for non-densely defined non-homogeneous Cauchy problems, and applied the results to study age structured models. Recently, Magal and Ruan [23] developed the center manifold theory for non-densely defined Cauchy problem and employed the theory to establish a Hopf bifurcation theorem for age structured models. This paper complements our previous articles [22, 23] in studying semilinear Cauchy problems with non-dense domain.

2 Integrated Semigroups

In this section we recall some results about integrated semigroups. We refer to Arendt [2, 3], Neubrander [28], Kellermann and Hieber [18], Thieme [36, 38], Arendt *et al.* [4], and Magal and Ruan [22, 23] for more detailed results on the subject.

Let X and Z be two Banach spaces. Let $\mathcal{L}(X,Z)$ denote the space of bounded linear operators from X into Z and by $\mathcal{L}(X)$ the space $\mathcal{L}(X,X)$. Let $A : D(A) \subset X \to X$ be a linear operator. If A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on X, let $\{T_A(t)\}_{t\geq 0}$ denote this semigroup. The resolvent set of A is denoted by $\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is invertible}\}$. Set

$$X_0 := \overline{D(A)},$$

and A_0 the part of A in X_0, which is a linear operator on X_0 defined by

$$A_0 x = Ax,\ \forall x \in D(A_0) := \{y \in D(A) : Ay \in X_0\}.$$

Assume that $(\omega, +\infty) \subset \rho(A)$. Then it is easy to check that for each $\lambda > \omega$,

$$D(A_0) = (\lambda I - A)^{-1} X_0 \text{ and } (\lambda I - A_0)^{-1} = (\lambda I - A)^{-1} |_{X_0}.$$

Recall that A is a Hille-Yosida operator if there exist two constants, $\omega \in \mathbb{R}$ and $M \geq 1$, such that $(\omega, +\infty) \subset \rho(A)$ and

$$\left\| (\lambda I - A)^{-k} \right\|_{\mathcal{L}(X)} \leq \frac{M}{(\lambda - \omega)^k},\ \ \forall \lambda > \omega,\ \ \forall k \geq 1.$$

In the following, we assume that A satisfies some weaker conditions.

Assumption 2.1 Assume that $A : D(A) \subset X \to X$ is a linear operator on a Banach space $(X, \|.\|)$ satisfying the following properties:

(a) There exist two constants, $\omega_A \in \mathbb{R}$ and $M_A \geq 1$, such that $(\omega_A, +\infty) \subset \rho(A)$ and

$$\left\|(\lambda I - A)^{-k}\right\|_{\mathcal{L}(X_0)} \leq \frac{M_A}{(\lambda - \omega_A)^k}, \ \ \forall \lambda > \omega_A, \ \ \forall k \geq 1;$$

(b) $\lim_{\lambda \to +\infty} (\lambda I - A)^{-1} x = 0, \forall x \in X$.

By using Lemma 2.1 in Magal and Ruan [22] and Assumption 2.1-(b) we deduce that

$$\overline{D(A_0)} = X_0.$$

By Hille-Yosida theorem (see Pazy [30], Theorem 5.3 on p.20) and the fact that if $\rho(A) \neq \emptyset$ then $\rho(A) = \rho(A_0)$ (see Magal and Ruan [23, Lemma 2.4]), one obtains the following lemma.

Lemma 2.2 *Assumption 2.1 is satisfied if and only if $\rho(A) \neq \emptyset$, A_0 is the infinitesimal generator of a linear C_0-semigroup $\{T_{A_0}(t)\}_{t \geq 0}$ on X_0, and*

$$\|T_{A_0}(t)\| \leq M_A e^{\omega_A t}, \forall t \geq 0.$$

Now we give the definition of an integrated semigroup.

Definition 2.3 Let $(X, \|.\|)$ be a Banach space. A family of bounded linear operators $\{S(t)\}_{t \geq 0}$ on X is called an *integrated semigroup* if

(i) $S(0) = 0$.

(ii) The map $t \to S(t)x$ is continuous on $[0, +\infty)$ for each $x \in X$.

(iii) $S(t)$ satisfies

$$S(s)S(t) = \int_0^s (S(r+t) - S(r))\, dr, \ \ \forall t, s \geq 0. \tag{2.2}$$

An integrated semigroup $\{S(t)\}_{t \geq 0}$ is said to be *non-degenerate* if $S(t)x = 0, \forall t \geq 0$, then $x = 0$. According to Thieme [36], a linear operator $A : D(A) \subset X \to X$ is the *generator* of a non-degenerate integrated semigroup $\{S(t)\}_{t \geq 0}$ on X if and only if

$$x \in D(A), \ y = Ax \Leftrightarrow S(t)x - tx = \int_0^t S(s)y ds, \ \forall t \geq 0. \tag{2.3}$$

From [36, Lemma 2.5], we know that if A generates $\{S_A(t)\}_{t \geq 0}$, then for each $x \in X$ and $t \geq 0$,

$$\int_0^t S_A(s)x ds \in D(A) \text{ and } S(t)x = A \int_0^t S_A(s)x ds + tx.$$

An integrated semigroup $\{S(t)\}_{t\geq 0}$ is said to be *exponentially bounded* if there exist two constants, $\widehat{M} > 0$ and $\widehat{\omega} > 0$, such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq \widehat{M} e^{\widehat{\omega} t}, \forall t \geq 0.$$

When we restrict ourselves to the class of non-degenerate exponentially bounded integrated semigroups, Thieme's notion of generator is equivalent the one introduced by Arendt [3]. More precisely, combining Theorem 3.1 in Arendt [3] and Proposition 3.10 in Thieme [36], one has the following result.

Theorem 2.4 *Let $\{S(t)\}_{t\geq 0}$ be a strongly continuous exponentially bounded family of bounded linear operators on a Banach space $(X, \|.\|)$ and $A : D(A) \subset X \to X$ be a linear operator. Then $\{S(t)\}_{t\geq 0}$ is a non-degenerate integrated semigroup and A its generator if and only if there exists some $\widehat{\omega} > 0$ such that $(\widehat{\omega}, +\infty) \subset \rho(A)$ and*

$$(\lambda I - A)^{-1} x = \lambda \int_0^\infty e^{-\lambda s} S(s) x ds, \ \ \forall \lambda > \hat{\omega}.$$

The following result is well known in the context of integrated semigroups.

Proposition 2.5 *Let Assumption 2.1 be satisfied. Then A generates a uniquely determined non-degenerate exponentially bounded integrated semigroup $\{S_A(t)\}_{t\geq 0}$. Moreover, for each $x \in X$, each $t \geq 0$, and each $\mu > \omega_A$, $S_A(t)x$ is given by*

$$\begin{array}{l} S_A(t)x = (\mu I - A_0) \int_0^t T_{A_0}(s) (\mu I - A)^{-1} x ds \\ \qquad = \mu \int_0^t T_{A_0}(s) (\mu I - A)^{-1} x ds + [I - T_{A_0}(t)] (\mu I - A)^{-1} x. \end{array} \tag{2.4}$$

Furthermore, the map $t \to S_A(t)x$ is continuously differentiable if and only if $x \in X_0$ and

$$\frac{dS_A(t)x}{dt} = T_{A_0}(t)x, \ \ \forall t \geq 0, \ \forall x \in X_0.$$

From now on we define

$$(S_A * f)(t) = \int_0^t S_A(t-s) f(s) ds, \forall t \in [0, \tau],$$

whenever $f \in L^1((0, \tau), X)$.

We now consider the non-homogeneous Cauchy problem

$$\frac{du}{dt} = Au(t) + f(t), \ t \in [0, \tau], \ \ u(0) = x \in \overline{D(A)} \tag{2.5}$$

and assume that f belongs to some appropriate subspace of $L^1((0, \tau), X)$.

Definition 2.6 A continuous map $u \in C([0, \tau], X)$ is called an *integrated solution* (or *mild solution*) of (2.5) if and only if

$$\int_0^t u(s) ds \in D(A), \ \ \forall t \in [0, \tau]$$

and

$$u(t) = x + A\int_0^t u(s)ds + \int_0^t f(s)ds,\ \forall t \in [0,\tau]\,.$$

Since A generates a non-degenerate integrated semigroup on X, we can apply Theorem 3.7 in Thieme [36] and obtain the following result.

Lemma 2.7 *Let Assumption 2.1 be satisfied. Then for each $x \in \overline{D(A)}$ and each $f \in L^1((0,\tau_0),X)$, (2.5) has at most one integrated solution.*

Denote

$$(S_A \diamond f)(t) = \frac{d}{dt}(S_A * f)(t)$$

whenever the map $t \to (S_A * f)(t)$ is continuously differentiable.

We will say that $\{S_A(t)\}_{t\geq 0}$ has *a bounded semi-variation on* $[0,t]$ if

$$V^{\infty}(S_A,0,t) := \sup\Big\{\Big\|\sum_{i=1}^{n}\big(S_A(t_i) - S_A(t_{i-1})\big)x_i\Big\|\Big\} < +\infty,$$

where the supremum is taken over all partitions $0 = t_0 < .. < t_n = t$ of the interval $[a,b]$ and over any $(x_1,..,x_n) \in X^n$ with $\|x_i\|_X \leq 1$, $\forall i = 1,..,n$.

In the sequel, we will only assume that (2.5) has an integrated solution whenever $f \in C([0,\tau],X)$, so we will make the following assumption.

Assumption 2.8 Let $\tau_0 > 0$ be fixed. Assume that $\{S_A(t)\}_{t\geq 0}$ has a bounded semi-variation on $[0,\tau_0]$ and

$$\lim_{t(>0)\to 0} V^{\infty}(S_A,0,t) = 0.$$

The following theorem, proved by Thieme [38], provides an equivalent condition to Assumption 2.8 and is very helpful in application. We refer to Magal and Ruan [22], Prevost [32] and Ducrot et al. [13] for verifying Assumption 2.8 for age-structured models and parabolic equations.

Theorem 2.9 *Let Assumption 2.1 be satisfied. Then Assumption 2.8 is satisfied if and only if for each $f \in C([0,\tau_0],X)$, the map $t \to (S_A * f)(t)$ is continuously differentiable on $[0,\tau_0]$, and*

$$\|(S_A \diamond f)(t)\| \leq \delta(t) \sup_{s\in[0,t]} \|f(s)\|, \forall t \in [0,\tau_0],$$

where $\delta : [0,\tau_0] \to [0,+\infty)$ is a non-decreasing map satisfying

$$\lim_{t(>0)\to 0} \delta(t) = 0.$$

Proof. Assume first that Assumption 2.8 is satisfied. By using Lemma 3.1 in [38], we have for each $t \in [0,\tau_0]$ and each $f \in C([0,\tau_0],X)$,

$$\|(S_A \star f)(t)\| \leq V^\infty(S_A,0,t) \sup_{s\in[0,t]} \|f(s)\|,$$

where $(S_A \star f)(t)$ is defined as a Stieltjes convolution (see [38]). Moreover, by Theorem 4.2 in [38], we deduce that for each $f \in C([0,\tau_0],X)$, the map $t \to (S_A * f)(t)$ is continuously differentiable and

$$(S_A \star f)(t) = (S_A \diamond f)(t) := \frac{d}{dt}(S_A * f)(t).$$

So by fixing $\delta(t) = V^\infty(S_A,0,t)$ we obtain the desired estimate.

Conversely, by using the same arguments as in the proof of Theorem 3.4 in [38], one deduces that

$$V^\infty(S_A(.),0,t) \leq \delta(t), \forall t \in [0,\tau_0],$$

and the result follows. ∎

We have

$$S_A(\tau+h) = S_A(\tau) + T_{A_0}(\tau)S_A(h), \forall h \geq 0,$$

so by using Assumption 2.8 we deduce that $t \to S_A(t)$ has a bounded semi-variation on $[0,2\tau_0]$. Now using induction arguments, we deduce that $t \to S_A(t)$ has a bounded semi-variation on $[0,\tau]$ for each $\tau \geq 0$, and by using Theorem 2.9 we obtain the following result.

Theorem 2.10 *Let Assumptions 2.1 and 2.8 be satisfied. Then for each $\tau > 0$, $t \to S_A(t)$ has a bounded semi-variation on $[0,\tau]$. Moreover, for each $f \in C([0,\tau],X)$, the map $t \to (S_A * f)(t)$ is continuously differentiable, $(S_A * f)(t) \in D(A), \forall t \in [0,\tau]$, and $u(t) = (S_A \diamond f)(t)$ satisfies*

$$u(t) = A\int_0^t u(s)ds + \int_0^t f(s)ds, \ \forall t \in [0,\tau]$$

and

$$\|u(t)\| \leq V^\infty(S_A,0,t) \sup_{s\in[0,t]} \|f(s)\|, \ \forall t \in [0,\tau].$$

Furthermore, for each $\lambda \in (\omega,+\infty)$, we have

$$(\lambda I - A)^{-1}(S_A \diamond f)(t) = \int_0^t T_{A_0}(t-s)(\lambda I - A)^{-1} f(s)ds. \tag{2.6}$$

Corollary 2.11 *Let Assumptions 2.1 and 2.8 be satisfied. Then for each $x \in X_0$ and each $f \in C([0,\tau],X_0)$, the Cauchy problem (2.5) has a unique integrated solution $u \in C([0,\tau],X_0)$ given by*

$$u(t) = T_{A_0}(t)x + (S_A \diamond f)(t), \ \forall t \in [0,\tau].$$

Moreover, we have

$$\|u(t)\| \leq M_A e^{\omega t}\|x\| + V^\infty(S_A,0,t) \sup_{s\in[0,t]} \|f(s)\|, \ \forall t \in [0,\tau].$$

We now consider a bounded perturbation of A. The following result was proved in Magal and Ruan [22, Theorem 3.1], which is also closely related to Desch and Schappacher's theorem (see [10] or Engel and Nagel [14, Theorem 3.1, p. 183]).

Theorem 2.12 *Let Assumptions 2.1 and 2.8 be satisfied. Let $L \in \mathcal{L}(X_0, X)$. Then $A+L : D(A) \subset X \to X$ satisfies Assumptions 2.1 and 2.8. More precisely, if $\{S_{A+L}(t)\}_{t\geq 0}$ denotes the integrated semigroup generated by $A+L$, and $\tau_1 \in (0, \tau_0]$ is chosen such that $\|L\|_{\mathcal{L}(X_0,X)} V^\infty(S_A, 0, \tau_1) < 1$, then for each $f \in C([0,\tau_1], X)$,*

$$\|(S_{A+L} \diamond f)(t)\| \leq \frac{V^\infty(S_A,0,t)}{1-\|L\|_{\mathcal{L}(X_0,X)} V^\infty(S_A,0,\tau_1)} \sup_{s\in[0,t]} \|f(s)\|, \forall t \in [0,\tau_1].$$

We have the following lemma.

Lemma 2.13 *Let Assumptions 2.1 and 2.8 be satisfied. Then $t \to S_A(t)$ is continuous from $[0,+\infty)$ into $\mathcal{L}(X)$ and*

$$\lim_{\lambda\to+\infty} \left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(X)} = 0.$$

Proof. By Proposition 3.10 in Thieme [36], we have for each $\lambda > \max(0, \omega_A)$ that

$$(\lambda I - A)^{-1} x = \lambda \int_0^{+\infty} e^{-\lambda t} S_A(t) x dt.$$

Note that

$$S_A(t)x = \frac{d}{dt}\int_0^t S_A(t-s)x ds,$$

so by Assumption 2.8, we have

$$\|S_A(t)x\| \leq V^\infty(S_A,0,t)\|x\|, \quad \forall t \geq 0.$$

But

$$S_A(t+h) - S_A(t) = T_{A_0}(t) S_A(h),$$

it follows that $t \to S_A(t)$ is operator norm continuous. Let $\varepsilon > 0$ be fixed and let $\tau_\varepsilon > 0$ be such that $V^\infty(S_A, 0, \tau_\varepsilon) \leq \varepsilon$. Choose $\gamma > \max(0, \omega_A)$ and $M_\gamma > 0$ such that

$$\|S_A(t)x\| \leq M_\gamma e^{\gamma t}, \forall t \geq 0.$$

Then we have for each $\lambda > \gamma$ that

$$\left\|(\lambda I - A)^{-1} x\right\| \leq \lambda \left[M_\gamma \int_{\tau_\varepsilon}^{+\infty} e^{(\gamma-\lambda)t} dt + \varepsilon \int_0^{\tau_\varepsilon} e^{-\lambda t} dt\right] \|x\|.$$

Thus,

$$\limsup_{\lambda\to+\infty} \left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(X)} \leq \varepsilon.$$

This proves the lemma. ∎

By using the fact that $(S_A \diamond f)(t) \in X_0, \forall t \in [0,\tau]$, and formula (2.6), we have for each $f \in C([0,\tau], X)$ that

$$(S_A \diamond f)(t) = \lim_{\mu \to +\infty} \int_0^t T_{A_0}(t-l)\mu (\mu I - A)^{-1} f(l) dl, \ \forall t \in [0,\tau]. \tag{2.7}$$

This approximation formula was already observed by Thieme [35] in the classical context of integrated semigroups generated by a Hille-Yosida operator. From this approximation formulation, we then deduce that for each pair $t, s \in [0,\tau]$ with $s \leq t$, and $f \in C([0,\tau], X)$,

$$(S_A \diamond f)(t) = T_{A_0}(t-s)(S_A \diamond f)(s) + (S_A \diamond f(s+.))(t-s). \tag{2.8}$$

Let $|.|$ denote the norm on X_0 defined by

$$|x| = \sup_{l \geq 0} e^{-\omega_A l} \|T_{A_0}(l) x\|, \ \ \forall x \in X_0.$$

Clearly, we have

$$|T_{A_0}(t)|_{\mathcal{L}(X_0)} \leq e^{\omega_A t}, \ \ \forall t \geq 0,$$

$$\|x\| \leq |x| \leq M_A \|x\|, \ \ \forall x \in X_0,$$

and by Assumption 2.8, for each $f \in C([0,\tau], X)$ and each $t \in [0,\tau]$,

$$|(S_A \diamond f)(t)| \leq M_A \|(S_A \diamond f)(t)\| \leq M_A V^{\infty}(S_A, 0, t) \sup_{s \in [0,t]} \|f(s)\|. \tag{2.9}$$

The following proposition is one of the main tools for studying semilinear problems (see next section and also Magal and Ruan [23] for another class of applications of this result).

Proposition 2.14 *Let Assumptions 2.1 and 2.8 be satisfied. Let $\varepsilon > 0$ be fixed. Then for each $\tau_\varepsilon > 0$ satisfying $M_A V^{\infty}(S_A, 0, \tau_\varepsilon) \leq \varepsilon$, we have*

$$\|(S_A \diamond f)(t)\| \leq C(\varepsilon, \gamma) \sup_{s \in [0,t]} e^{\gamma(t-s)} \|f(s)\|, \ \ \forall t \geq 0$$

whenever $\gamma \in (\omega_A, +\infty)$, $f \in C(\mathbb{R}_+, X)$, and with

$$C(\varepsilon, \gamma) := \frac{2\varepsilon \max(1, e^{-\gamma \tau_\varepsilon})}{1 - e^{(\omega_A - \gamma)\tau_\varepsilon}}.$$

Proof. Let $\varepsilon > 0$, $f \in C(\mathbb{R}_+, X)$, and $\gamma > \omega_A$ be fixed. Let $\tau_\varepsilon = \tau_\varepsilon(\varepsilon) \in (0,\tau]$ be such that $M_A V^{\infty}(S_A, 0, \tau_\varepsilon) \leq \varepsilon$. By (2.9), we have

$$|(S_A \diamond f)(t)| \leq \varepsilon \sup_{s \in [0,t]} \|f(s)\|, \forall t \in [0, \tau_\varepsilon]. \tag{2.10}$$

Let $\gamma > \omega_A$ be fixed. Set

$$\varepsilon_1 = \varepsilon \max\left(1, e^{-\gamma \tau_\varepsilon}\right).$$

Let $k \in \mathbb{N}$ and $t \in [k\tau_\varepsilon, (k+1)\tau_\varepsilon]$ be fixed. First, notice that if $\gamma \geq 0$, we have

$$\begin{aligned}
\varepsilon \sup_{s\in[k\tau_\varepsilon,t]} \|f(s)\| &= \varepsilon \sup_{s\in[k\tau_\varepsilon,t]} e^{\gamma s}e^{-\gamma s}\|f(s)\| \\
&\leq \varepsilon e^{\gamma t} \sup_{s\in[k\tau_\varepsilon,t]} e^{-\gamma s}\|f(s)\| \\
&= \varepsilon_1 e^{\gamma t} \sup_{s\in[k\tau_\varepsilon,t]} e^{-\gamma s}\|f(s)\|.
\end{aligned}$$

Moreover, if $\gamma < 0$, we have

$$\begin{aligned}
\varepsilon \sup_{s\in[k\tau_\varepsilon,t]} \|f(s)\| &= \varepsilon \sup_{s\in[k\tau_\varepsilon,t]} e^{\gamma s}e^{-\gamma s}\|f(s)\| \\
&\leq \varepsilon e^{\gamma k\tau_\varepsilon} \sup_{s\in[k\tau_\varepsilon,t]} e^{-\gamma s}\|f(s)\| \\
&= \varepsilon e^{\gamma t}e^{-\gamma t}e^{\gamma k\tau_\varepsilon} \sup_{s\in[k\tau_\varepsilon,t]} e^{-\gamma s}\|f(s)\| \\
&= e^{\gamma t}\varepsilon e^{-\gamma(t-k\tau_\varepsilon)} \sup_{s\in[k\tau_\varepsilon,t]} e^{-\gamma s}\|f(s)\| \\
&\leq e^{\gamma t}\varepsilon e^{-\gamma\tau_\varepsilon} \sup_{s\in[k\tau_\varepsilon,t]} e^{-\gamma s}\|f(s)\| \\
&= e^{\gamma t}\varepsilon_1 \sup_{s\in[k\tau_\varepsilon,t]} e^{-\gamma s}\|f(s)\|.
\end{aligned}$$

Therefore, for each $k \in \mathbb{N}$, each $t \in [k\tau_\varepsilon, (k+1)\tau_\varepsilon]$, and each $\gamma \in \mathbb{R}$, we obtain

$$\varepsilon \sup_{s\in[k\tau_\varepsilon,t]} \|f(s)\| \leq e^{\gamma t}\varepsilon_1 \sup_{s\in[k\tau_\varepsilon,t]} e^{-\gamma s}\|f(s)\|. \tag{2.11}$$

It follows from (2.10) and (2.11) that for all $t \in [0, \tau_\varepsilon]$,

$$|(S_A \diamond f)(t)| \leq \varepsilon \sup_{s\in[0,t]} \|f(s)\| = e^{\gamma t}\varepsilon_1 \sup_{s\in[0,t]} e^{-\gamma s}\|f(s)\|. \tag{2.12}$$

Using (2.8) with $s = \tau_\varepsilon$, we have for all $t \in [\tau_\varepsilon, 2\tau_\varepsilon]$ that

$$(S_A \diamond f)(t) = T_0(t-\tau_\varepsilon)(S_A \diamond f)(\tau_\varepsilon) + (S_A \diamond f(\tau_\varepsilon + .))(t-\tau_\varepsilon).$$

Using (2.10), (2.11), and (2.12), we have

$$\begin{aligned}
|(S_A \diamond f)(t)| &\leq e^{\omega_A(t-\tau_\varepsilon)}|(S_A \diamond f)(\tau_\varepsilon)| + |(S_A \diamond f(\tau_\varepsilon + .))(t-\tau_\varepsilon)| \\
&\leq e^{\omega_A(t-\tau_\varepsilon)}e^{\gamma\tau_\varepsilon}\varepsilon_1 \sup_{s\in[0,\tau_\varepsilon]} e^{-\gamma s}\|f(s)\| + \varepsilon \sup_{s\in[\tau_\varepsilon,t]} \|f(s)\| \\
&\leq e^{\omega_A(t-\tau_\varepsilon)}e^{\gamma\tau_\varepsilon}\varepsilon_1 \sup_{s\in[0,\tau_\varepsilon]} e^{-\gamma s}\|f(s)\| + e^{\gamma t}\varepsilon_1 \sup_{s\in[\tau_\varepsilon,t]} e^{-\gamma s}\|f(s)\| \\
&\leq \varepsilon_1 e^{\gamma t}\left(e^{(\omega_A-\gamma)(t-\tau_\varepsilon)} + 1\right) \sup_{s\in[0,t]} e^{-\gamma s}\|f(s)\|.
\end{aligned}$$

Similarly, for all $t \in [2\tau_\varepsilon, 3\tau_\varepsilon]$,

$$(S_A \diamond f)(t) = T_{A_0}(t-2\tau_\varepsilon)(S_A \diamond f)(2\tau_\varepsilon) + (S_A \diamond f(2\tau_\varepsilon + .))(t-2\tau_\varepsilon)$$

and

$$\begin{aligned}
|(S_A \diamond f)(t)| &\leq e^{\omega_A(t-2\tau_\varepsilon)}\varepsilon_1 e^{\gamma 2\tau_\varepsilon}\left(e^{(\omega_A-\gamma)\tau_\varepsilon}+1\right)\sup_{s\in[0,2\tau_\varepsilon]} e^{-\gamma s}\|f(s)\| \\
&\quad +\varepsilon \sup_{s\in[2\tau_\varepsilon,t]}\|f(s)\| \\
&\leq e^{\omega_A(t-2\tau_\varepsilon)}\varepsilon_1 e^{\gamma 2\tau_\varepsilon}\left(e^{(\omega_A-\gamma)\tau_\varepsilon}+1\right)\sup_{s\in[0,2\tau_\varepsilon]} e^{-\gamma s}\|f(s)\| \\
&\quad +\varepsilon_1 e^{\gamma t}\sup_{s\in[2\tau_\varepsilon,t]} e^{-\gamma s}\|f(s)\| \\
&\leq \varepsilon_1 e^{\gamma t}\left[e^{(\omega_A-\gamma)(t-2\tau_\varepsilon)}\left(e^{(\omega_A-\gamma)\tau_\varepsilon}+1\right)+1\right]\sup_{s\in[0,t]} e^{-\gamma s}\|f(s)\|.
\end{aligned}$$

By induction, we obtain $\forall k\in\mathbb{N}$ with $k\geq 1, \forall t\in[k\tau_\varepsilon,(k+1)\tau_\varepsilon]$, and for each $\gamma>\omega_A$ that

$$\begin{aligned}
|(S_A \diamond f)(t)| &\leq \varepsilon_1 e^{\gamma t}\sup_{s\in[0,t]} e^{-\gamma s}\|f(s)\|\left[e^{(\omega_A-\gamma)(t-k\tau_\varepsilon)}\sum_{n=0}^{k-1}\left(e^{(\omega_A-\gamma)\tau_\varepsilon}\right)^n+1\right] \\
&\leq \varepsilon_1 e^{\gamma t}\sup_{s\in[0,t]} e^{-\gamma s}\|f(s)\|\left[\sum_{n=0}^{\infty}\left(e^{(\omega_A-\gamma)\tau_\varepsilon}\right)^n+1\right].
\end{aligned}$$

Since $\gamma>\omega_A$, we have for each $t\geq 0$ that

$$e^{-\gamma t}\|(S_A \diamond f)(t)\|\leq e^{-\gamma t}|(S_A \diamond f)(t)|\leq \frac{2\varepsilon_1}{1-e^{(\omega_A-\gamma)\tau_\varepsilon}}\sup_{s\in[0,t]} e^{-\gamma s}\|f(s)\|.$$

This completes the proof. ∎

Let $I\subset[0,+\infty)$ be an interval. Set $s:=\inf I\geq 0$. For each $\gamma\geq 0$, define

$$BC^\gamma(I,Y):=\left\{\varphi\in C(I,Y):\sup_{l\in I} e^{-\gamma(l-s)}\|\varphi(l)\|_Y<+\infty\right\}$$

and

$$\|\varphi\|_{BC^\gamma(I,Y)}:=\sup_{l\in I} e^{-\gamma(l-s)}\|\varphi(l)\|_Y.$$

It is well known that $BC^\gamma(I,Y)$ endowed with the norm $\|.\|_{BC^\gamma(I,Y)}$ is a Banach space.

By using Proposition 2.14 we obtain the following result.

Lemma 2.15 *Let Assumptions 2.1 and 2.8 be satisfied. For each $s\geq 0$ and each $\sigma\in(s,+\infty]$, define a linear operator $\mathcal{L}_s: C([s,\sigma),X)\to C([s,\sigma),X_0)$ by*

$$\mathcal{L}_s(\varphi)(t)=(S_A\diamond\varphi(.+s))(t-s),\ \ \forall t\in[s,\sigma),\ \ \forall\varphi\in C([s,\sigma),X).$$

Then for each $\gamma>\omega_A$, $\mathcal{L}_s$ is a bounded linear operator from $BC^\gamma([s,\sigma),X)$ into $BC^\gamma([s,\sigma),X_0)$. Moreover, for each $\varepsilon>0$ and each $\tau_\varepsilon>0$ such that $M_A V^\infty(S_A,0,\tau_\varepsilon)\leq\varepsilon$,

$$\|\mathcal{L}_s(\varphi)\|_{\mathcal{L}(BC^\gamma([s,\sigma),X),BC^\gamma([s,\sigma),X_0))}\leq C(\varepsilon,\gamma).$$

Proof. Let $\varphi \in BC^{\gamma}([s,\sigma),X)$ be fixed. By using Proposition 2.14, we have for $t \in [s,\sigma)$ that

$$\begin{aligned} e^{-\gamma(t-s)}\left\|(S_A \diamond \varphi(.+s))(t-s)\right\| &\leq C(\varepsilon,\gamma)\sup_{l\in[0,t-s]} e^{-\gamma l}\left\|\varphi(l+s)\right\| \\ &= C(\varepsilon,\gamma)\sup_{r\in[s,t]} e^{-\gamma(r-s)}\left\|\varphi(r)\right\| \\ &\leq C(\varepsilon,\gamma)\sup_{r\in[s,\sigma)} e^{-\gamma(r-s)}\left\|\varphi(r)\right\| \end{aligned}$$

and the result follows. ∎

3 Semilinear Problems – Positivity

Start from this section, we consider the semilinear Cauchy problem

$$\frac{dU(t,s)x}{dt} = AU(t,s)x + F(t,U(t,s)x),\ \ t \geq s,\ \ U(s,s)x = x \in X_0.$$

We shall investigate the properties of the non-autonomous semiflow generated by the following problem

$$U(t,s)x = x + A\int_s^t U(l,s)xdl + \int_s^t F(l,U(l,s)x)dl,\ \ t \geq s \geq 0, \tag{3.1}$$

or equivalently

$$U(t,s)x = T_{A_0}(t-s)x + (S_A \diamond F(.+s,U(.+s,s)x))(t-s),\quad t \geq s \geq 0. \tag{3.2}$$

Definition 3.1 Consider two maps $\chi : [0,+\infty)\times X_0 \to (0,+\infty]$ and $U : D_\chi \to X_0$, where

$$D_\chi = \left\{(t,s,x) \in [0,+\infty)^2 \times X_0 : s \leq t < s + \chi(s,x)\right\}.$$

We say that U is *a maximal non-autonomous semiflow* on X_0 if U satisfies the following properties:

(i) $\chi(r,U(r,s)x) + r = \chi(s,x) + s,\ \forall s \geq 0, \forall x \in X_0,\ \forall r \in [s, s+\chi(s,x))$.

(ii) $U(s,s)x = x,\ \forall s \geq 0,\ \forall x \in X_0$.

(iii) $U(t,r)U(r,s)x = U(t,s)x,\ \forall s \geq 0, \forall x \in X_0,\ \forall t,r \in [s, s+\chi(s,x))$ with $t \geq r$.

(iv) If $\chi(s,x) < +\infty$, then

$$\lim_{t\to(s+\chi(s,x))^-}\left\|U(t,s)x\right\| = +\infty.$$

Set

$$D = \left\{ (t, s, x) \in [0, +\infty)^2 \times X_0 : t \geq s \right\}.$$

In order to present a theorem on the existence and uniqueness of solutions to equation (3.1), we make the following assumption.

Assumption 3.2 Assume that $F : [0, +\infty) \times \overline{D(A)} \to X$ is a continuous map, and for each $\sigma > 0$ and each $\xi > 0$ there exists $K(\sigma, \xi) > 0$ such that

$$\|F(t, x) - F(t, y)\| \leq K(\sigma, \xi) \|x - y\|$$

whenever $t \in [0, \sigma]$, $y, x \in X_0$ with $\|x\| \leq \xi$ and $\|y\| \leq \xi$.

The following theorem is proved in Magal and Ruan [22, Theorem 5.2].

Theorem 3.3 *Let Assumptions 2.1-2.8 and 3.2 be satisfied. Then there exist a map $\chi : [0, +\infty) \times X_0 \to (0, +\infty]$ and a maximal non-autonomous semi-flow $U : D_\chi \to X_0$, such that for each $x \in X_0$ and each $s \geq 0$, $U(., s)x \in C([s, s + \chi(s, x)), X_0)$ is a unique maximal solution of (3.1) (or equivalently a unique maximal solution of (3.2)). Moreover, D_χ is open in D and the map $(t, s, x) \to U(t, s)x$ is continuous from D_χ into X_0.*

We are now interested in the positivity of the solutions of equation (3.1). Let $X_+ \subset X$ be a cone of X. That is, X_+ is a closed convex subset of X satisfying the following two properties:

(i) $\lambda x \in X_+, \forall x \in X_+, \forall \lambda \geq 0$.

(ii) $x \in X_+$ and $-x \in X_+ \Rightarrow x = 0$.

It is clear that $X_{0+} = X_0 \cap X_+$ is also a cone of X_0. Recall that such a cone defines a partial order on the Banach space X which is defined by

$$x \geq y \quad \text{if and only if} \quad x - y \in X_+.$$

We need the following assumption to prove the positivity of solutions of equation (3.1).

Assumption 3.4 Assume that there exists a linear operator $B \in \mathcal{L}(X_0, X)$ such that

(a) For each $\gamma > 0$, $A - \gamma B$ is resolvent positive (i.e., $(\lambda I - (A - \gamma B))^{-1} X_+ \subset X_+$ for all $\lambda > \omega_A$ large enough).

(b) For each $\xi > 0$ and each $\sigma > 0$, there exists $\gamma = \gamma(\xi, \sigma) > 0$, such that

$$F(t, x) + \gamma B x \in X_+$$

whenever $x \in X_{0+}$, $\|x\| \leq \xi$ and $t \in [0, \sigma]$.

Proposition 3.5 *Let Assumptions 2.1-2.8 and 3.2-3.4 be satisfied. Then for each $x \in X_{0+}$ and each $s \geq 0$, we have*

$$U(t,s)x \in X_{0+}, \ \ \forall t \in [s, s+\chi(s,x)).$$

Proof. Without loss of generality we can assume that $s = 0$ and $x \in X_{0+}$. Moreover, using the semiflow property, it is sufficient to prove that there exists $\varepsilon \in (0, \chi(0,x))$ such that $U(t,0)x \in X_{0+}$, $\forall t \in [0,\varepsilon]$. Let $x \in X_{0+}$ be fixed. We set $\xi := 2(\|x\|+1)$. Let $\gamma > 0$ such that

$$F(t,x) + \gamma Bx \in X_+$$

whenever $x \in X_{0+}$, $\|x\| \leq \xi$ and $t \in [0,1]$. Fix $\tau_\gamma > 0$ such that $\gamma V^\infty(S_A, 0, \tau_\gamma) < 1$. For each $\sigma \in (0, \tau_\gamma)$, define

$$E^\sigma = \{\varphi \in C([0,\sigma], X_{0+}) : \|\varphi(t)\| \leq \xi, \ \ \forall t \in [0,\sigma]\}.$$

Then it is sufficient to consider the fixed point problem

$$u(t) = T_{(A-\gamma B)_0}(t)x + (S_{A-\gamma B} \diamond F(.,u(.)) + \gamma Bu(.))(t) =: \Psi(u)(t), \forall t \in [0,\sigma].$$

Since $A - \gamma B$ is resolvent positive, we have $T_{(A-\gamma B)_0}(t)X_{0+} \subset X_{0+}, \forall t \geq 0$. Using the approximation formula (2.7), we also deduce that for each $\tau > 0$,

$$(S_{A-\gamma B} \diamond \varphi)(t) \in X_{0+}, \ \ \forall t \in [0,\tau], \ \ \forall \varphi \in C([0,\tau], X_+).$$

Moreover, by using Theorem 2.12, for each $\varphi \in E^\sigma$ and each $t \in [0,\sigma]$, we deduce that

$$\begin{aligned}
\|\Psi(\varphi)(t)\| &= \left\|T_{(A-\gamma B)_0}(t)x + (S_{A-\gamma B} \diamond F(.,\varphi(.)) + \gamma\varphi(.))(t)\right\| \\
&\leq \left\|T_{(A-\gamma B)_0}(t)x\right\| + \frac{V^\infty(S_A,0,t)}{1-\gamma V^\infty(S_A,0,\tau_\gamma)} \sup_{s\in[0,t]} \|F(s,\varphi(s)) + \gamma\varphi(s)\| \\
&\leq \sup_{t\in[0,\sigma]} \left\|T_{(A-\gamma B)_0}(t)x\right\| \\
&\quad + \frac{V^\infty(S_A,0,\sigma)}{1-\gamma V^\infty(S_A,0,\tau_\gamma)} \left[\sup_{s\in[0,\sigma]} \|F(s,0)\| + [K(1,\xi)+\gamma]\,\xi\right].
\end{aligned}$$

Hence, there exists $\sigma_1 \in (0,1)$ such that

$$\Psi(E^\sigma) \subset E^\sigma, \ \ \forall \sigma \in (0,\sigma_1].$$

Therefore, for each $\sigma \in (0,\sigma_1]$ and each pair $\varphi, \psi \in E^\sigma$, we have for $t \in [0,\sigma]$ that

$$\begin{aligned}
\|\Psi(\varphi)(t) - \Psi(\psi)(t)\| &= \|(S_{A-\gamma B} \diamond [F(.,\varphi(.)) - F(.,\psi(.)) + \gamma(\varphi-\psi)(.)])(t)\| \\
&\leq \frac{V^\infty(S_A,0,\sigma)}{1-\gamma V^\infty(S_A,0,\tau_\gamma)} [K(1,\xi)+\gamma] \sup_{s\in[0,\sigma]} \|(\varphi-\psi)(s)\|.
\end{aligned}$$

Thus, there exists $\sigma_2 \in (0,\sigma_1]$ such that $\Psi(E^{\sigma_2}) \subset E^{\sigma_2}$ and Ψ is a contraction strict on E^{σ_2}. The result then follows. ∎

Example 3.6 (1) We refer to Thieme [35] and Hirsch and Smith [17] for more results on the positivity of semiflows.

(2) Usually Proposition 3.5 is applied with $B = I$. But the case $B \neq I$ can also be useful. Consider the following functional differential equation:

$$\begin{cases} \dfrac{dx(t)}{dt} = f(x_t), \forall t \geq 0, \\ x_0 = \varphi \in C([-\tau, 0], \mathbb{R}^n), \end{cases} \tag{3.3}$$

where $f : C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is Lipschitz continuous on bounded sets of $C([-\tau, 0], \mathbb{R}^n)$. In order to obtain the positivity of solutions, it is sufficient to assume that for each $M \geq 0$ there exists $\gamma = \gamma(M) > 0$ such that

$$f(\varphi) + \gamma\varphi(0) \geq 0$$

whenever $\|\varphi\|_\infty \leq M$ and $\varphi \in C\left([-\tau, 0], \mathbb{R}^n_+\right)$. It is well known that this condition is sufficient to obtain the positivity of solutions (see Martin and Smith [26, 27]). In order to prove this, one may also apply Proposition 3.5. By identifying x_t with $u(t) = \begin{pmatrix} 0 \\ x_t \end{pmatrix}$, the system (3.3) can be rewritten as a non-densely defined Cauchy problem (see Liu, Magal and Ruan [19] for more details)

$$\frac{dv(t)}{dt} = Av(t) + F(v(t)), \ \forall t \geq 0, \text{ and } v(0) = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$

with $X = \mathbb{R}^n \times C([-\tau, 0], \mathbb{R}^n)$, $D(A) = \{0_{\mathbb{R}^n}\} \times C^1([-\tau, 0], \mathbb{R}^n)$, where $A : D(A) \subset X \to X$ is defined by

$$A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) \\ \varphi' \end{pmatrix}$$

and $F : \overline{D(A)} \to X$ by

$$F\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} f(\varphi) \\ 0_C \end{pmatrix}.$$

Then Proposition 3.5 applies with

$$B\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \varphi(0) \\ 0_C \end{pmatrix}.$$

Recall that a cone X_+ of a Banach space $(X, \|.\|)$ is *normal* if there exists a norm $\|.\|_1$ equivalent to $\|.\|$, which is monotone, that is,

$$\forall x, y \in X, \ 0 \leq x \leq y \Rightarrow \|x\|_1 \leq \|y\|_1.$$

Corollary 3.7 *Let Assumptions 2.1-2.8 and 3.2-3.4 be satisfied. Assume in addition that*

(a) *X_+ is a normal cone of $(X, \|.\|)$.*

(b) *There exist a continuous map* $G : [0,+\infty) \times X_{0+} \to X_+$ *and two real numbers* $k_1 \geq 0$ *and* $k_2 \geq 0$, *such that for each* $t \geq 0$ *and each* $x \in X_{0+}$,

$$F(t,x) \leq G(t,x) \text{ and } \|G(t,x)\| \leq k_1 \|x\| + k_2.$$

Then

$$\chi(s,x) = +\infty, \ \ \forall s \geq 0, \ \ \forall x \in X_{0+}.$$

Moreover, we have the following estimate: for each $\gamma > 0$ *large enough, there exist* $C_1 > 0$ *and* $C_2 > 0$ *such that*

$$\|U(t,s)x\| \leq e^{\gamma(t-s)} \left[C_1 \|x\| + C_2\right].$$

Proof. Without loss of generality, we can assume that $s = 0$ and the norm $\|.\|$ is monotone. Let $\varepsilon \in \left(0, \dfrac{1}{2k_1}\right)$ and $\tau_\varepsilon > 0$ such that $M_A V^\infty(S_A, 0, \tau_\varepsilon) \leq \varepsilon$. Let $x \in X_{0+}$ be fixed. Then by Proposition 3.5, we have for each $t \in [0, \chi(0,x))$ that

$$\begin{aligned} 0 &\leq U(t,0)x = T_{A_0}(t)x + (S_A \diamond F(.,U(.,0)x))(t) \\ &\leq T_{A_0}(t)x + (S_A \diamond G(.,U(.,0)x))(t). \end{aligned}$$

Hence, for each $\gamma > \max(\omega_A, 0)$, we have for each $t \in [0, \chi(0,x))$ that

$$\begin{aligned} e^{-\gamma t}\|U(t,0)x\| &\leq e^{-\gamma t}\|T_{A_0}(t)x\| + e^{-\gamma t}\|(S_A \diamond G(.,U(.,0)x))(t)\| \\ &\leq M_A e^{(-\gamma+\omega_A)t}\|x\| + C(\varepsilon,\gamma) \sup_{s\in[0,t]} e^{-\gamma s}\|G(s,U(s,0)x)\| \\ &\leq M_A\|x\| + C(\varepsilon,\gamma) \sup_{s\in[0,t]} e^{-\gamma s}\left[k_1\|U(s,0)x\| + k_2\right] \\ &\leq M_A\|x\| + k_2 C(\varepsilon,\gamma) + k_1 C(\varepsilon,\gamma) \sup_{s\in[0,t]} e^{-\gamma s}\|U(s,0)x\|. \end{aligned}$$

Since $2k_1\varepsilon < 1$, for $\gamma > \max(\omega_A, 0)$ sufficiently large, we obtain $k_1 C(\varepsilon,\gamma) = \frac{2k_1\varepsilon}{\left(1-e^{(\omega_A-\gamma)\tau_\varepsilon}\right)} < 1$ and the result follows. ∎

4 Global Lipschitz Perturbation

We now consider the case where the map $x \to F(t,x)$ is Lipschitz continuous. Let $(Y, \|.\|_Y)$ and $(Z, \|.\|_Z)$ be two Banach spaces. Let E be a subset of Y and $G : Y \to Z$ be a map. Define

$$\|G\|_{Lip(E,Z)} = \sup_{x,y\in E: x\neq y} \frac{\|G(x) - G(y)\|_Z}{\|x-y\|_Y}.$$

For each $x \in Y$ and $r > 0$, set

$$B_Y(x,r) = \{y \in Y : \|x-y\|_Y < r\}, \ \overline{B}_Y(x,r) = \{y \in Y : \|x-y\|_Y \leq r\}.$$

The main results of this section are on the existence and uniqueness of a solution to the integral equation (3.1) and its estimate when $x \to F(t,x)x$ is globally Lipschitz continuous.

Proposition 4.1 *Let Assumptions 2.1-2.8 be satisfied. Let $F : [0,+\infty)\times\overline{D(A)} \to X$ be a continuous map and $\sigma \in (0,+\infty]$ be a fixed constant. Assume that*

$$\Gamma_F(\sigma) := \sup_{t\in[0,\sigma)} \|F(t,.)\|_{Lip(X_0,X)} < +\infty.$$

Then for each $x \in X_0$ and each $s \in [0,\sigma)$, there exists a unique solution $U(.,s)x \in C([s,\sigma), X_0)$ of

$$U(t,s)x = x + A\int_s^t U(l,s)xdl + \int_s^t F(l,U(l,s)x)dl, \quad \forall t \in [s,\sigma).$$

Moreover, there exists $\gamma_0 > \max(0,\omega_A)$ such that for each $\gamma \geq \gamma_0$, each pair $t,s \in [0,\sigma)$ with $t \geq s$, and each pair $x,y \in X_0$, we have

$$\|U(t,s)x\| \leq e^{\gamma(t-s)}\left[2M_A\|x\| + \sup_{l\in[s,\sigma)} e^{-\gamma(l-s)}\|F(l,0)\|\right]$$

and

$$\|U(t,s)x - U(t,s)y\| \leq e^{\gamma(t-s)}2M_A\|x-y\|.$$

Proof. Fix $s,t \in [0,\sigma)$ with $s<t$. Let $\varepsilon > 0$ such that

$$\varepsilon\max(\Gamma_F(\sigma),1) < 1/8.$$

Let $\tau_\varepsilon > 0$ such that $M_A V^\infty(S_A,0,\tau_\varepsilon) \leq \varepsilon$. Then by Lemma 2.15 we have for each $\gamma > \omega_A$ that

$$\|\mathcal{L}_s(\varphi)\|_{\mathcal{L}(BC^\gamma([s,+\infty),X),BC^\gamma([s,+\infty),X_0))} \leq C(\gamma,\varepsilon) = \frac{2\varepsilon\max(1,e^{-\gamma\tau_\varepsilon})}{\left(1-e^{(\omega_A-\gamma)\tau_\varepsilon}\right)}.$$

Let $\gamma_0 \geq \max(0,\omega_A)$ such that

$$\frac{1}{\left(1-e^{(\omega_A-\gamma)\tau_\varepsilon}\right)} < 2, \ \ \forall\gamma \geq \gamma_0.$$

To prove the proposition it is sufficient to prove that the following fixed point problem

$$U(.,s)x = T_{A_0}(.-s)x + \mathcal{L}_s \circ \Psi(U(.,s)x) \tag{3.4}$$

admits a solution $U(.,s)x \in BC^\gamma([s,\sigma),X_0)$, where $\Psi : BC^\gamma([s,\sigma),X_0) \to BC^\gamma([s,\sigma),X)$ is a nonlinear operator defined by

$$\Psi(\varphi)(l) = F(l,\varphi(l)), \ \ \forall l \in [s,t], \ \forall\varphi \in BC^\gamma([s,\sigma),X_0).$$

We have

$$\|T_{A_0}(.-s)\|_{\mathcal{L}(X_0,BC^\gamma([s,\sigma),X_0))} \leq M_A,$$
$$\|\mathcal{L}_s\|_{\mathcal{L}(BC^\gamma([s,\sigma),X),BC^\gamma([s,\sigma),X_0))} \leq 4\varepsilon,$$

and

$$\|\Psi\|_{Lip(BC^\gamma([s,\sigma),X_0),BC^\gamma([s,\sigma),X))} \leq \Gamma_F(\sigma).$$

From this we deduce that

$$\|\mathcal{L}_s \circ \Psi\|_{Lip(BC^\gamma([s,\sigma),X_0),BC^\gamma([s,\sigma),X_0))} \leq 4\varepsilon\Gamma_F(\sigma) \leq 1/2.$$

Thus, the fixed point problem (3.4) has a unique solution. Moreover, for each $x \in X_0$, there exists a unique solution in $BC^\gamma([s,\sigma),X_0)$.

$$\begin{aligned}\|U(.,s)x\|_{BC^\gamma([s,t],X_0)} &\leq M_A\|x\| + \|\mathcal{L}_s(\Psi(0))\| + \|\mathcal{L}_s(\Psi(U(.,s)x) - \Psi(0))\| \\ &\leq M_A\|x\| + 4\varepsilon\|\Psi(0)\|_{BC^\gamma([s,\sigma),X)} + \frac{1}{2}\|U(.,s)x\|_{BC^\gamma([s,\sigma),X_0)},\end{aligned}$$

which implies that

$$\|U(.,s)x\|_{BC^\gamma([s,\sigma),X_0)} \leq 2M_A\|x\| + 8\varepsilon\|\Psi(0)\|_{BC^\gamma([s,\sigma),X)}.$$

Since by construction $\varepsilon \leq \frac{1}{8}$, we have

$$\sup_{l\in[s,\sigma)} e^{-\gamma(l-s)}\|U(.,s)x\| \leq 2M_A\|x\| + \sup_{l\in[s,\sigma)} e^{-\gamma(l-s)}\|F(l,0)\|.$$

Similarly, we have for each pair $x, y \in X_0$ that

$$U(.,s)x - U(.,s)y = T_{A_0}(.-s)(x-y) + \mathcal{L}_s[\Psi(U(.,s)x) - \Psi(U(.,s)y)].$$

Therefore,

$$\begin{aligned}&\|U(.,s)x - U(.,s)y\|_{BC^\gamma([s,\sigma),X_0)} \\ &\quad\leq M_A\|x-y\| + \frac{1}{2}\|U(.,s)x - U(.,s)y\|_{BC^\gamma([s,\sigma),X_0)}.\end{aligned}$$

This completes the proof. ∎

5 Differentiability with Respect to the State Variable

In this section we investigate the differentiability of solutions with respect to the state variable.

Proposition 5.1 *Let Assumptions 2.1-2.8 and 3.2 be satisfied. Assume in addition that:*

(a) For each $t \geq 0$ the map $x \to F(t,x)$ is continuously differentiable from X_0 into X.

(b) The map $(t,x) \to D_x F(t,x)$ is continuous from $[0,+\infty)\times X_0$ into $\mathcal{L}(X_0,X)$.

Let $x_0 \in X_0$, $s \geq 0$, $\tau \in [0,\chi(s,x_0))$, and $\gamma \in (0,\chi(s,x_0)-\tau)$. Let $\eta > 0$ (there exists such a constant since D_χ is open in D) such that

$$\chi(s,y) > \tau+\gamma, \forall y \in B_{X_0}(x_0,\eta).$$

Then for each $t \in [s, s+\tau+\gamma]$, the map $x \to U(t,s)x$ is defined from $B_{X_0}(x,\eta)$ into X_0 and is differentiable at x_0. Moreover, if we set

$$V(t,s)y = D_x U(t,s)(x)(y), \forall y \in X_0,$$

then $t \to V(t,s)y$ is an integrated solution of the Cauchy problem

$$\begin{array}{rcl} \dfrac{dV(t,s)y}{dt} & = & AV(t,s)y + D_x F(t,U(t,s)x_0)V(t,s)y,\ t \in [s, s+\chi(s,x_0)), \\ V(s,s)y & = & y \end{array}$$

or equivalently $t \to V(t,s)y$ is a solution of

$$V(t,s)y = T_{A_0}(t-s)y + (S_A \diamond D_x F(.,U(.,s)x_0)V(.,s)y)(t-s), \forall t \in [s, s+\chi(s,x_0)).$$

Proof. First by using the result in the subsection about the Lipschitz case, it clear that $W(t,s)$ is well defined. Set

$$R(t)(y) = U(t,s)(x_0+y) - U(t,s)(x_0) - V(t,s)y.$$

Then

$$R(t)(y) = (S_A \diamond [F(.,U(.,s)(x_0+y)) - F(.,U(.,s)(x_0)) - D_x F(.,U(.,s)x_0)V(.,s)y])(t-s).$$

But

$$\begin{array}{rl} & F(t,U(t,s)(x_0+y)) - F(t,U(t,s)(x_0)) \\ = & \displaystyle\int_0^1 D_x F(t, rU(t,s)(x_0+y) + (1-r)U(t,s)(x_0))(U(t,s)(x_0+y) - U(t,s)(x_0))\,dr \\ = & \displaystyle\int_0^1 \Psi_1(t,r,y)(U(t,s)(x_0+y) - U(t,s)(x_0))\,dr \\ & + D_x F(t,U(t,s)(x_0))(U(t,s)(x_0+y) - U(t,s)(x_0)), \end{array}$$

where

$$\Psi_1(t,r,y) = D_x F(t, rU(t,s)(x_0+y) + (1-r)U(t,s)(x_0)) - D_x F(t,U(t,s)(x_0)).$$

Thus,

$$R(t)y = (S_A \diamond [K(.) + D_x F(.,U(.,s)x_0)R(.)y])(t-s),$$

where

$$K(t) = \int_0^1 \Psi_2(t,r,y)\left(U(t,s)\left(x_0+y\right) - U(t,s)\left(x_0\right)\right)dr$$

and

$$\Psi_2(t,r,y) = D_xF(t,rU(t,s)\left(x_0+y\right)+(1-r)U(t,s)\left(x_0\right)) - D_xF(t,U(t,s)\left(x_0\right)).$$

The result follows from Proposition 2.14 and the continuity of $(t,x) \to U(t,s)x$.
■

6 Time Differentiability

In this subsection, we study the time differentiability of the solutions. Consider a solution $u \in C([0,\tau],\overline{D(A)})$ of

$$u(t) = x_0 + A\int_0^t u(s)ds + \int_0^t F(s,u(s))ds,\ t \in [0,\tau].$$

Assume that $x_0 \in D(A)$ and $F : [0,T] \times \overline{D(A)} \to X$ is a C^1 map. When the domain of A is dense, it is well known (see Pazy [30], Theorem 6.1.5, p. 187) that for each $x \in D(A)$, the map $t \to u(t)$ is a classical solution. That is, the map $t \to u(t)$ is continuously differentiable, $u(t) \in D(A)$ for all $t \in [0,\tau]$, and satisfies

$$\begin{cases} u'(t) = Au(t) + f(t,u(t)),\ \forall t \in [0,\tau], \\ u(0) = x_0. \end{cases}$$

In this subsection, we consider the same problem but in the the context of non-densely defined Cauchy problems. When A satisfies the Hille-Yosida condition, this problem has been studied by Thieme [35] and Magal [21]. So the goal is to extend these results to the non-Hille-Yosida case. This problem turns to be more difficult. For each $\tau > 0$, set

$$C^{1,+}([0,\tau],X) = \left\{ f \in C\left([0,\tau],X\right) : \frac{d^+f}{dt} \in C\left([0,\tau),X\right) \text{ and } \lim_{t \nearrow \tau} \frac{d^+f}{dt}(t) \text{ exists} \right\}.$$

The following lemma is a variant of a result due to Da Prato and Sinestrari [7].

Lemma 6.1 *Let $A : D(A) \subset X \to X$ be a closed linear operator. Let $\tau > 0$, $f \in C([0,\tau],X)$, and $x \in X_0$ be fixed. Assume that $u \in C\left([0,\tau],X\right)$ is a solution of*

$$u(t) = x + A\int_0^t u(s)ds + \int_0^t f(s)ds,\ \forall t \in [0,\tau].$$

Assume in addition that u belongs to $C^{1,+}([0,\tau],X)$ or $C([0,\tau],D(A))$. Then

$$u \in C^1([0,\tau],X) \cap C([0,\tau],D(A))$$

and

$$u'(t) = Au(t) + f(t), \forall t \in [0,\tau].$$

Proof. If $u \in C([0,T], D(A))$, since A is closed, we have

$$u(t) = x + \int_0^t Au(s)ds + \int_0^t f(s)ds, \ \ \forall t \in [0,\tau].$$

So $u \in C^1([0,\tau], X)$ and $u'(t) = Au(t) + f(t), \forall t \in [0,\tau]$.

If $u \in C^{1,+}([0,\tau], X)$, then we have for each $t \in [0,\tau)$ and $h > 0$ that

$$\frac{u(t+h) - u(t)}{h} - \frac{\int_t^{t+h} f(s)ds}{h} = A\frac{\int_t^{t+h} u(s)ds}{h}.$$

Since A is closed, we deduce that $u(t) \in D(A)$ and $Au(t) = \frac{d^+u}{dt}(t) + f(t), \forall t \in [0,\tau)$. Since $u \in C^{1,+}([0,\tau], X)$, we then deduce that $u \in C([0,\tau], D(A))$ and complete the proof. ∎

Lemma 6.2 *Let Assumptions 2.1 and 2.8 be satisfied. Assume that $g \in C^1([0,T], X)$ and $g(0) \in \overline{D(A)}$, then $t \to (S_A \diamond g)(t)$ is continuously differentiable and*

$$\frac{d}{dt}(S_A \diamond g)(t) = T_{A_0}(t)g(0) + (S_A \diamond g')(t), \ \ \forall t \in [0,T].$$

Proof. Since g is continuously differentiable, the map $t \to (S_A * g)(t)$ is continuously differentiable,

$$\frac{d}{dt}(S_A * g)(t) = S_A(t)g(0) + (S_A * g')(t), \forall t \in [0,T],$$

Since $g(0) \in \overline{D(A)}$, we have $S_A(t)g(0) = \int_0^t T_{A_0}(l)g(0)dl, \forall t \in [0,T]$, and the result follows. ∎

The following theorem is due to Vanderbauwhede [39, Theorem 3.5].

Lemma 6.3 *(Fibre Contraction Theorem) Let M_1 and M_2 be two complete metric spaces and $\Psi : M_1 \times M_2 \to M_1 \times M_2$ a mapping of the form*

$$\Psi(x,y) = (\Psi_1(x), \Psi_2(x,y)), \forall (x,y) \in M_1 \times M_2$$

satisfying the following properties:

(i) *Ψ_1 has a fixed point $\overline{x} \in M_1$ such that for each $x \in M_1$,*

$$\Psi_1^n(x) \to \overline{x} \text{ as } n \to +\infty.$$

(ii) *There exists $k \in [0,1)$ such that for each $x \in M_1$ the map $y \to \Psi_2(x,y)$ is k-Lipschitz continuous.*

(iii) *The map $x \to \Psi_2(x,\overline{y})$ is continuous, where $\overline{y} \in M_2$ is a fixed point of the map $y \to \Psi_2(\overline{x},y)$.*

Then for each $(x,y) \in M_1 \times M_2$,

$$\Psi^n(x,y) \to (\overline{x},\overline{y}) \text{ as } n \to +\infty.$$

The key result of this section is the following lemma.

Lemma 6.4 *Let Assumptions 2.1 and 2.8 be satisfied. Let $\tau > 0$ be fixed and $F : [0,\tau] \times \overline{D(A)} \to X$ be continuously differentiable. Assume that there exists an integrated solution $u \in C([0,\tau], X)$ of the Cauchy problem*

$$\begin{aligned} &\frac{du(t)}{dt} = Au(t) + F(t,u(t)),\ t \in [0,\tau], \\ &u(0) = x \in X_0. \end{aligned}$$

Assume in addition that

$$x \in D(A_0) \text{ and } F(0,x) \in \overline{D(A)}.$$

Then there exists $\varepsilon > 0$, such that $u \in C^1([0,\varepsilon], X) \cap C([0,\varepsilon], D(A))$ and

$$u'(t) = Au(t) + F(t,u(t)),\ \ \forall t \in [0,\varepsilon].$$

Proof. Since F is continuously differentiable, there exist $\varepsilon_0 > 0$, $K_1 > 0$, and $K_2 > 0$ such that

$$\|\partial_t F(t,y)\| \le K_1 \text{ and } \|\partial_x F(t,y)\|_{\mathcal{L}(X_0,X)} \le K_2$$

whenever $\|x - y\| \le \varepsilon_0$ and $0 \le t \le \varepsilon_0$. For each $\varepsilon \in (0,\varepsilon_0]$, set

$$\begin{aligned} M_1^\varepsilon &= \{\varphi \in C([0,\varepsilon], X_0) : \varphi(0) = x,\ \|\varphi(t) - x\| \le \varepsilon_0, \forall t \in [0,\varepsilon]\}, \\ M_2^\varepsilon &= \{\varphi \in C([0,\varepsilon], X_0) : \varphi(0) = A_0 x + F(0,x), \\ &\qquad \|\varphi(t) - A_0 x + F(0,x)\| \le \varepsilon_0,\ \forall t \in [0,\varepsilon]\}. \end{aligned}$$

From now on, we assume that for each $i = 1, 2$, M_i^ε is endowed with the metric $d(\varphi,\widehat{\varphi}) = \|\varphi - \widehat{\varphi}\|_{\infty,[0,\varepsilon]}$ and $M_1^\varepsilon \times M_2^\varepsilon$ is endowed with the usual product distance $d((\varphi,\psi),(\widehat{\varphi},\widehat{\psi})) = d(\varphi,\widehat{\varphi}) + d(\psi,\widehat{\psi})$.

For each $\varepsilon \in (0,\varepsilon_0]$, set

$$E^\varepsilon = \left\{(\varphi_1,\varphi_2) \in M_1^\varepsilon \times M_2^\varepsilon : \varphi_1(t) = x + \int_0^t \varphi_2(s)ds, \forall t \in [0,\varepsilon]\right\}.$$

Then it is clear that E^ε is a closed subset of $M_1^\varepsilon \times M_2^\varepsilon$.

We consider a map $\Psi : M_1^\varepsilon \times M_2^\varepsilon \to C([0,\varepsilon], X_0) \times C([0,\varepsilon], X_0)$ defined by

$$\Psi(\varphi_1,\varphi_2) = (\Psi_1(\varphi_1), \Psi_2(\varphi_1,\varphi_2)),\ \ \forall (\varphi_1,\varphi_2) \in M_1^\varepsilon \times M_2^\varepsilon,$$

where for each $t \in [0,\varepsilon]$,

$$\begin{aligned} \Psi_1(\varphi_1)(t) &= T_{A_0}(t)x + (S_A \diamond F(.,\varphi_1(.)))(t), \\ \Psi_2(\varphi_1,\varphi_2)(t) &= T_{A_0}(t)[A_0 x + F(0,x)] \\ &\quad + (S_A \diamond \partial_t F(.,\varphi_1(.)) + \partial_x F(.,\varphi_1(.))\varphi_2(.))(t). \end{aligned}$$

One can easily check that Ψ is a continuous map. We now prove that for some $\varepsilon > 0$ small enough, $\Psi\left(M_1^\varepsilon \times M_2^\varepsilon\right) \subset M_1^\varepsilon \times M_2^\varepsilon$, and

$$\Psi_1\left(\varphi_1\right)(0) = x, \quad \Psi_2\left(\varphi_1, \varphi_2\right)(0) = \left[A_0 x + F(0,x)\right].$$

For each $\varepsilon \in (0, \varepsilon_0]$, each $t \in [0,\varepsilon]$, and each $\varphi \in M_1^\varepsilon$, we have

$$\begin{aligned}
&\left\|\Psi_1(\varphi)(t) - x\right\| \\
&\quad \le \left\|T_{A_0}(t)x - x\right\| + \left\|\left(S_A \diamond F(.,\varphi(.))\right)(t)\right\| \\
&\quad \le \left\|T_{A_0}(t)x - x\right\| + V^\infty(S_A,0,t) \sup_{s\in[0,t]} \left\|F(s,\varphi(s))\right\| \\
&\quad \le \left\|T_{A_0}(t)x - x\right\| + V^\infty(S_A,0,\varepsilon)\left[\sup_{s\in[0,t]} \left\|F(s,x)\right\| + K_2 \sup_{s\in[0,t]} \left\|\varphi(s) - x\right\|\right] \\
&\quad \le \sup_{t\in[0,\varepsilon]} \left\|T_{A_0}(t)x - x\right\| + V^\infty(S_A,0,\varepsilon)\left[\sup_{s\in[0,\varepsilon]} \left\|F(s,x)\right\| + K_2\varepsilon_0\right].
\end{aligned}$$

Thus, there exists $\varepsilon_1 \in (0,\varepsilon_0]$ such that for each $\varepsilon \in (0,\varepsilon_1]$, $\Psi_1\left(M_1^\varepsilon\right) \subset M_1^\varepsilon$.

Moreover, for each $\varepsilon \in (0,\varepsilon_1]$, each $t \in [0,\varepsilon]$, and each $(\varphi_1,\varphi_2) \in M_1^\varepsilon \times M_2^\varepsilon$, we have

$$\begin{aligned}
&\left\|\Psi_2\left(\varphi_1,\varphi_2\right)(t) - \left[A_0x + F(0,x)\right]\right\| \\
&\quad \le \left\|T_{A_0}(t)\left[A_0x + F(0,x)\right] - \left[A_0x + F(0,x)\right]\right\| \\
&\qquad + \left\|\left(S_A \diamond \partial_t F(.,\varphi_1(.)) + \partial_x F(.,\varphi_1(.))\varphi_2(.)\right)(t)\right\| \\
&\quad \le \sup_{t\in[0,\varepsilon]} \left\|T_{A_0}(t)\left[A_0x + F(0,x)\right] - \left[A_0x + F(0,x)\right]\right\| \\
&\qquad + V^\infty(S_A,0,\varepsilon) \sup_{s\in[0,\varepsilon]} \left\|\partial_t F(s,\varphi_1(s))\right\| \\
&\qquad + V^\infty(S_A,0,\varepsilon) \sup_{s\in[0,\varepsilon]} \left\|\partial_x F(s,\varphi_1(s))\right\| \left\|\varphi_2(.)\right\| \\
&\quad \le \sup_{t\in[0,\varepsilon]} \left\|T_{A_0}(t)\left[A_0x + F(0,x)\right] - \left[A_0x + F(0,x)\right]\right\| \\
&\qquad + V^\infty(S_A,0,\varepsilon)\left\{K_1 + K_2\left[\left\|A_0x + F(0,x)\right\| + \varepsilon_0\right]\right\}.
\end{aligned}$$

Therefore, there exists $\varepsilon_2 \in (0,\varepsilon_1]$ such that for each $\varepsilon \in (0,\varepsilon_2]$, $\Psi_2\left(M_1^\varepsilon \times M_2^\varepsilon\right) \subset M_2^\varepsilon$.

Similarly, for each $\varepsilon \in (0,\varepsilon_2]$, $\Psi\left(M_1^\varepsilon \times M_2^\varepsilon\right) \subset M_1^\varepsilon \times M_2^\varepsilon$. Now we check that $\Psi\left(E^\varepsilon\right) \subset E^\varepsilon$. Let $(\varphi_1,\varphi_2) \in E^\varepsilon$. Then $\varphi_1 \subset C^1\left([0,\varepsilon], X_0\right)$ and $\varphi_1'(t) = \varphi_2(t), \forall t \in [0,\varepsilon]$. Notice that

$$\Psi_1\left(\varphi_1\right)(t) = T_{A_0}(t)x + \left(S_A \diamond F(.,\varphi_1(.))\right)(t),$$

using Lemma 6.2 and the fact that $x \in D(A_0)$ and $F(0,x) \in \overline{D(A)}$, we have

$$\begin{aligned}
\frac{d\Psi_1\left(\varphi_1\right)(t)}{dt} &= A_0 T_{A_0}(t)x + T_{A_0}(t)F(0,x) + \left(S_A \diamond \frac{d}{dt}F(.,\varphi_1(.))\right)(t) \\
&= T_{A_0}(t)\left[A_0x + F(0,x)\right] + \left(S_A \diamond \partial_t F(.,\varphi_1(.)) + \partial_x F(.,\varphi_1(.))\varphi_2(.)\right)(t).
\end{aligned}$$

Thus,

$$\frac{d\Psi_1(\varphi_1)(t)}{dt} = \Psi_2(\varphi_1,\varphi_2)(t)$$

and

$$\Psi(E^\varepsilon) \subset E^\varepsilon.$$

Next we apply Lemma 6.3. It remains to verify i) and ii) for some $\varepsilon \in (0,\varepsilon_2]$ small enough. Let $(\varphi_1,\varphi_2),(\widehat{\varphi}_1,\widehat{\varphi}_2) \in M_1^\varepsilon \times M_2^\varepsilon$ be fixed. We have for each $\varepsilon \in (0,\varepsilon_2]$ that

$$\begin{aligned}\|\Psi_1(\varphi_1)(t) - \Psi_1(\widehat{\varphi}_1)(t)\| &= \|(S_A \diamond F(.,\varphi_1(.)) - F(.,\widehat{\varphi}_1(.)))(t)\| \\ &\leq V^\infty(S_A,0,\varepsilon)\|F(s,\varphi_1(s)) - F(s,\widehat{\varphi}_1(s))\| \\ &\leq V^\infty(S_A,0,\varepsilon)K_2 \sup_{s\in[0,\varepsilon]}\|\varphi_1(s) - \widehat{\varphi}_1(s)\|.\end{aligned}$$

So there exists $\varepsilon_3 \in (0,\varepsilon_2]$ such that $\delta_1 := V^\infty(S_A,0,\varepsilon_3)K_2 \in (0,1)$, we have for each $\varepsilon \in (0,\varepsilon_3]$ that

$$\|\Psi_1(\varphi_1) - \Psi_1(\widehat{\varphi}_1)\|_{\infty,[0,\varepsilon]} \leq \delta_1 \|\varphi_1 - \widehat{\varphi}_1\|_{\infty,[0,\varepsilon]}.$$

Moreover,

$$\begin{aligned}\|\Psi_2(\varphi_1,\varphi_2)(t) - \Psi_2(\varphi_1,\widehat{\varphi}_2)(t)\| &= \|(S_A \diamond \partial_x F(.,\varphi_1(.))(\varphi_2(.) - \widehat{\varphi}_2))(t)\| \\ &\leq V^\infty(S_A,0,\varepsilon)K_2 \sup_{s\in[0,\varepsilon]}\|\varphi_2(s) - \widehat{\varphi}_2(s)\| \\ &\leq \delta_1 \sup_{s\in[0,\varepsilon]}\|\varphi_2(s) - \widehat{\varphi}_2(s)\|,\end{aligned}$$

which implies that

$$\|\Psi_2(\varphi_1,\varphi_2)(t) - \Psi_2(\varphi_1,\widehat{\varphi}_2)\|_{\infty,[0,\varepsilon]} \leq \delta_1 \|\varphi_2 - \widehat{\varphi}_2\|_{\infty,[0,\varepsilon]}.$$

Hence, for $\varepsilon = \varepsilon_3$ we have $\Psi(M_1^\varepsilon \times M_2^\varepsilon) \subset M_1^\varepsilon \times M_2^\varepsilon$, $\Psi(E^\varepsilon) \subset E^\varepsilon$ and Ψ satisfies the assumptions of Lemma 6.3. We deduce that there exists $(u,v) \in M_1^\varepsilon \times M_2^\varepsilon$ such that for each $(\varphi_1,\varphi_2) \in M_1^\varepsilon \times M_2^\varepsilon$,

$$\Psi^n(\varphi_1,\varphi_2) \to (u,v) \text{ as } n \to +\infty.$$

Since $\Psi(E^\varepsilon) \subset E^\varepsilon$ and E^ε is closed, we deduce that $(u,v) \in E^\varepsilon$. In particular, $u \in C^1([0,\varepsilon],X)$, and the result follows. ∎

Lemma 6.5 *Let Assumptions 2.1 and 2.8 be satisfied. Let $\tau > 0$ be fixed and $F : [0,\tau] \times \overline{D(A)} \to X$ be continuously differentiable. Assume that there exists an integrated solution $u \in C([0,\tau],X)$ of the Cauchy problem*

$$\begin{aligned}&\frac{du(t)}{dt} = Au(t) + F(t,u(t)),\ t \in [0,\tau], \\ &u(0) = x \in X_0.\end{aligned}$$

Assume in addition that

$$x \in D(A_0) \text{ and } F(0,x) \in \overline{D(A)}.$$

Then $u \in C^1([0,\tau],X) \cap C([0,\tau],D(A))$ *and*

$$u'(t) = Au(t) + F(t,u(t)), \quad \forall t \in [0,\tau].$$

Proof. Let $w \in C([0,\tau],\overline{D(A)})$ be a solution of the equation

$$\begin{aligned} w(t) &= Ax + F(0,x) + A\int_0^t w(s)ds \\ &\quad + \int_0^t \frac{\partial}{\partial t}F(s,u(s)) + D_xF(s,u(s))w(s)ds, \forall t \in [0,\tau]. \end{aligned}$$

From the subsection concerning globally Lipschitz perturbation, it is clear that the solution $w(t)$ exists and is uniquely determined. Since $u(t)$ exists on $[0,\tau]$, let $t \in [0,\tau)$ be fixed. We have for each $h \in (0,\tau - t)$ that

$$\begin{aligned} &\frac{u(t+h)-u(t)}{h} \\ &= \frac{1}{h}A\left[\int_0^{t+h} u(s)ds - \int_0^t u(s)ds\right] \\ &\quad + \frac{1}{h}\left[\int_0^{t+h} F(s,u(s))ds - \int_0^t F(s,u(s))ds\right] \\ &= A\left[\int_0^t \frac{u(s+h)-u(s)}{h}ds\right] + \frac{1}{h}A\int_0^h u(s)ds \\ &\quad + \int_0^t \frac{F(s+h,u(s+h)) - F(s,u(s))}{h}ds + \frac{1}{h}\int_0^h F(s,u(s))ds. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{u(t+h)-u(t)}{h} - w(t) \\ &= A\int_0^t \left[\frac{u(s+h)-u(s)}{h} - w(s)\right]ds \\ &\quad + \frac{1}{h}A\int_0^h u(s)ds + \frac{1}{h}\int_0^h F(s,u(s))ds - Ax - F(0,x) \\ &\quad + \int_0^t \left[\frac{F(s+h,u(s+h)) - F(s+h,u(s))}{h} - D_xF(s,u(s))w(s)\right]ds \\ &\quad + \int_0^t \left[\frac{F(s+h,u(s)) - F(s,u(s))}{h} - \frac{\partial}{\partial t}F(s,u(s))\right]ds. \end{aligned}$$

Denote

$$v_h(t) := \frac{u(t+h)-u(t)}{h} - w(t)$$

and

$$x_h := \frac{1}{h}A\int_0^h u(s)ds + \frac{1}{h}\int_0^h F(s,u(s))ds - Ax - F(0,x).$$

We have

$$\begin{aligned} v_h(t) &= x_h + A\int_0^t v_h(s)ds \\ &\quad + \int_0^t\int_0^1 D_xF\left(l\left(u(s+h)-u(s)\right)+u(s)\right)\left(\frac{u(s+h)-u(s)}{h} - w(s)\right)dlds \\ &\quad + \int_0^t\int_0^1 \left[D_xF\left(l\left(u(s+h)-u(s)\right)+u(s)\right) - D_xF\left(u(s)\right)\right]w(s)dlds \\ &\quad + \int_0^t \left[\frac{F(s+h,u(s))-F(s,u(s))}{h} - \frac{\partial}{\partial t}F(s,u(s))\right]ds. \end{aligned}$$

Set

$$K = \sup_{l\in[0,1],s\in[0,\tau],h\in[0,\tau-s]} \left\|D_xF\left(l\left(u(s+h)-u(s)\right)+u(s)\right)\right\|_{\mathcal{L}(X_0,X)} < +\infty.$$

Let $\widehat{\tau} > 0$ such that

$$MV^{\infty}(S_A,0,t) \leq \frac{1}{8(K+1)}, \forall t \in [0,\widehat{\tau}].$$

Choose $\gamma > \max(0,\omega_A)$ so that

$$\frac{1}{4\left(1-e^{(\omega_A-\gamma)\widehat{\tau}}\right)} < \frac{1}{2}.$$

Then by Proposition 2.14, we have for all $\gamma > \max(0,\omega_A)$ that

$$\begin{aligned} e^{-\gamma t}\left\|v_h(t)\right\| &\leq M\left\|x_h\right\| + \frac{1}{2}\sup_{s\in[0,\tau]} e^{-\gamma s}\left\|v_h(s)\right\| \\ &\quad + \sup_{s\in[0,\tau]} e^{-\gamma s}\left\|\int_0^1 \left[D_xF\left(l\left(u(s+h)-u(s)\right)+u(s)\right) - D_xF\left(u(s)\right)\right]w(s)dl\right\| \\ &\quad + \sup_{s\in[0,\tau]} e^{-\gamma s}\left\|\int_0^1 \left[\frac{F(s+h,u(s))-F(s,u(s))}{h} - \frac{\partial}{\partial t}F(s,u(s))\right]dl\right\|, \end{aligned}$$

which implies that

$$\begin{aligned} &e^{-\gamma t}\left\|v_h(t)\right\| \\ &\quad \leq 2M\left\|x_h\right\| \\ &\qquad +2\sup_{s\in[0,\tau]} e^{-\gamma s}\left\|\int_0^1 \left[D_xF\left(l\left(u(s+h)-u(s)\right)+u(s)\right) - D_xF\left(u(s)\right)\right]w(s)dl\right\| \\ &\qquad +2\sup_{s\in[0,\tau]} e^{-\gamma s}\left\|\int_0^1 \left[\frac{F(s+h,u(s))-F(s,u(s))}{h} - \frac{\partial}{\partial t}F(s,u(s))\right]dl\right\|. \end{aligned}$$

We now claim that

$$\lim_{h \searrow 0} x_h = 0.$$

Indeed, we have

$$\frac{u(h) - u(0)}{h} = \frac{1}{h} A \int_0^h u(s) ds + \frac{1}{h} \int_0^h F(s, u(s)) ds$$

and by Lemma 6.4, we have

$$\lim_{h \to 0^+} \frac{u(h) - u(0)}{h} = Ax + F(0, x),$$

so

$$\lim_{h \searrow 0} x_h = 0.$$

We conclude that for each $t \in [0, \tau)$, we will have

$$\lim_{h \to 0^+} \frac{u(t+h) - u(t)}{h} = w(t).$$

Since $w \in C([0, \tau], X)$, we deduce that $u \in C^{1,+}([0, \tau], X)$. By using Lemma 6.1, we obtain the result. ∎

To extend the differentiability result to the case where $F(0, x_0) \notin \overline{D(A)}$, we notice that, since $u(t) \in \overline{D(A)}$ for all $t \in [0, T]$, a necessary condition for the differentiability is

$$Ax + F(0, x) \in \overline{D(A)}.$$

In fact, this condition is also sufficient. Indeed, taking any bounded linear operator $B \in \mathcal{L}(X)$, if u satisfies

$$u(t) = x + A \int_0^t u(s) ds + \int_0^t F(s, u(s)) ds, \ \forall t \in [0, T],$$

then we have

$$u(t) = x + (A + B) \int_0^t u(s) ds + \int_0^t (F(s, u(s)) - Bu(s)) ds, t \in [0, T].$$

So to prove the differentiability of $u(t)$ it is sufficient to find B such that $(A + B)x \in \overline{D(A)}$. Take $B(\varphi) = -x^*(\varphi)Ax$, where $x^* \in X^*$ is a continuous linear form with $x^*(x) = 1$ if $x \neq 0$, which is possible by the Hahn-Banach theorem. We then have

$$x \in D(A) = D(A + B) \text{ and } (A + B)x \in \overline{D(A)} = \overline{D(A + B)}.$$

Moreover, assuming that $Ax + F(0, x) \in \overline{D(A)}$, we obtain $F(0, x_0) - Bx \in \overline{D(A)}$. By using Theorem 2.12, we deduce that $A + B$ satisfies Assumptions 2.1 and 2.8 and have the following theorem.

Theorem 6.6 *Let Assumptions 2.1 and 2.8 be satisfied. Let $\tau > 0$ be fixed and $F : [0,\tau] \times \overline{D(A)} \to X$ be continuously differentiable. Assume that there exists an integrated solution $u \in C([0,\tau], X)$ of the Cauchy problem*

$$\begin{aligned} &\frac{du(t)}{dt} = Au(t) + F(t,u(t)),\ t \in [0,\tau], \\ &u(0) = x \in X_0. \end{aligned}$$

Assume in addition that

$$x \in D(A) \text{ and } Ax + F(0,x) \in \overline{D(A)}.$$

Then $u \in C^1([0,\tau], X) \cap C([0,\tau], D(A))$ and

$$u'(t) = Au(t) + F(t,u(t)),\ \ \forall t \in [0,\tau].$$

We now consider the nonlinear generator

$$A_N\varphi = A\varphi + F(0,\varphi),\ \varphi \in D(A_N) = D(A).$$

As in the linear case, one may define $A_{N,0}$ (the part A_N in $\overline{D(A)}$) as follows

$$A_{N,0} = A_N \text{ on } D(A_{N,0}) = \left\{ y \in D(A) : A_N y \in \overline{D(A)} \right\}.$$

Of course, one may ask about the density of the domain $D(A_{N,0})$ in $\overline{D(A)}$.

Lemma 6.7 *Under Assumptions 2.1-2.8 and 3.2, the domain $D(A_{N,0})$ is dense in $X_0 = \overline{D(A)}$. Assume in addition that X has a positive cone X_+ and that Assumption 3.4 is satisfied. Then $D(A_{N,0}) \cap X_{0+}$ is dense in X_{0+}.*

Proof. Let $y \in \overline{D(A)}$ be fixed. Consider the following fixed point problem: $x_\lambda \in D(A)$ satisfies

$$(\lambda I - A - F)x_\lambda = \lambda y \Leftrightarrow x_\lambda = \lambda(\lambda I - A)^{-1}y + (\lambda I - A)^{-1}F(0,x_\lambda).$$

Denote

$$\Phi_\lambda(x) = \lambda(\lambda I - A)^{-1}y + (\lambda I - A)^{-1}F(0,x_\lambda),\ \ \forall x \in X_0.$$

Let $r > 0$ be fixed. Since $y \in \overline{D(A)}$, by Lemma 2.13, $\lim_{\lambda\to+\infty} \left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(X)} = 0$, we deduce that there exists $\lambda_0 > \omega_A$ such that

$$\Phi_\lambda(B_{X_0}(y,r)) \subset B_{X_0}(y,r),\ \forall \lambda \geq \lambda_0,$$

where $B(y,r)$ denotes the ball centered at y with radius r in X_0. Moreover, there exists $\lambda_1 \geq \lambda_0$, such that for each $\lambda \geq \lambda_1$, Φ_λ is a strict contraction on $B(y,r)$. Hence, $\forall \lambda \geq \lambda_1$, there exists $x_\lambda \in B(y,r)$ such that $\Phi_\lambda(x_\lambda) = x_\lambda$. Finally, using the fact that $y \in \overline{D(A)}$, we have $\lim\limits_{\lambda\to+\infty} \lambda(\lambda I - A)^{-1}y = y$, so

$$\lim_{\lambda\to+\infty} x_\lambda = y.$$

The proof of the positive case is similar. ∎

7 Stability of Equilibria

In this section we first investigate the local stability of an equilibrium.

Proposition 7.1 *Let Assumptions 2.1 and 2.8 be satisfied. Let $F : \overline{D(A)} \to X$ be a continuous map. Assume that*

(a) *There exists $\overline{x} \in D(A)$ such that $A\overline{x} + F(\overline{x}) = 0$.*

(b) *There exist $\widehat{M} \geq 1$, $\widehat{\omega} < 0$, and $L \in \mathcal{L}(X_0, X)$ such that*

$$\left\| T_{(A+L)_0}(t) \right\|_{\mathcal{L}(X_0)} \leq \widehat{M} e^{\widehat{\omega} t}, \ \ \forall t \geq 0.$$

(c) $\|F - L\|_{Lip\left(\overline{B}_{X_0}(\overline{x}, r), X\right)} \to 0$ *as* $r \to 0$.

Then for each $\gamma \in (\widehat{\omega}, 0)$ there exists $\varepsilon > 0$, such that for each $x \in \overline{B}_{X_0}(\overline{x}, \varepsilon)$, there exists a unique solution $U(.)x \in C([0, +\infty), X_0)$ of

$$U(t)x = x + A \int_0^t U(s)x ds + \int_0^t F(U(s)x)\, ds, \ \ \forall t \geq 0$$

which satisfies

$$\|U(t)x - \overline{x}\| \leq e^{\gamma t} 2\widehat{M} \|x - \overline{x}\|, \ \ \forall t \geq 0, \forall x \in X_0.$$

Proof. Without loss of generality we can assume that $\overline{x} = 0$, $L = 0$, $\omega_A < 0$, and

$$\|F\|_{Lip\left(\overline{B}_{X_0}(\overline{x}, \eta), X\right)} \to 0 \text{ as } \eta \to 0.$$

Choose $\eta_0 > 0$ such that

$$\|F\|_{Lip\left(\overline{B}_{X_0}(\overline{x}, \eta_0), X\right)} < +\infty.$$

Let $\phi : (-\infty, +\infty) \to [0, +\infty)$ be a Lipschitz continuous map such that

$$\phi(\alpha) \begin{cases} = 0, & \text{if } 2 \leq |\alpha| \\ \in [0, 1], & \text{if } 1 \leq |\alpha| \leq 2 \\ = 1, & \text{if } |\alpha| \leq 1. \end{cases}$$

Set

$$F_r(x) = \phi(r \|x\|) F(x), \forall x \in X_0, \forall r > 0.$$

Then

$$F_r(x) = \begin{cases} 0, & \text{if } \frac{2}{r} \leq \|x\|, \\ F(x), & \text{if } \|x\| \leq \frac{1}{r}. \end{cases}$$

Choose $\eta \in (0, \eta_0]$ and fix $r = \dfrac{2}{\eta}$. Let $x, y \in X_0$. Define $\varphi : [0, 1] \to \mathbb{R}$ by

$$\varphi(t) = \|F_r(t(x - y) + y) - F_r(y)\|, \ \forall t \in [0, 1].$$

Since $\|F\|_{Lip\left(\overline{B}_{X_0}(\overline{x},\eta),X\right)} < +\infty$, the map φ is Lipschitz continuous, we have for each pair $t, s \in [0,1]$ that

$$|\varphi(t) - \varphi(s)| \leq \|F\|_{Lip\left(\overline{B}_{X_0}(\overline{x},\eta),X\right)} \left(2\|\phi\|_{Lip} + 1\right) \|x - y\| \, |t - s| \, .$$

In particular, for $t = 1$ and $s = 0$, we deduce that

$$\|F_r(x) - F_r(y)\| \leq \|F\|_{Lip\left(\overline{B}_{X_0}(\overline{x},\frac{2}{r}),X\right)} \left(2\|\phi\|_{Lip} + 1\right) \|(x - y)\| \, .$$

So for all $r \geq \dfrac{2}{\eta_0}$, $F_r \in Lip(X_0, X)$ and

$$\|F_r\|_{Lip(X_0,X)} \leq \|F\|_{Lip\left(\overline{B}_{X_0}(\overline{x},\frac{2}{r}),X\right)} \left(2\|\phi\|_{Lip} + 1\right) \to 0 \text{ as } r \to +\infty. \quad (3.5)$$

For each $r \geq \dfrac{2}{\eta_0}$, we consider the nonlinear semigroup $\{U_r(t)\}$ which is a solution of

$$U_r(t)x = x + A\int_0^t U_r(s)xds + \int_0^t F_r(U_r(s)x)\, ds, \ \ \forall t \geq 0.$$

Let $\gamma \in (\widehat{\omega}, 0)$ be fixed. By Proposition 4.1 and (3.5), there exists $r_0 = r_0(\gamma) \geq \dfrac{2}{\eta_0}$ such that

$$\|U_{r_0}(t)x\| \leq e^{\gamma t} 2M \|x\| \, , \ \ \forall t \geq 0, \ \forall x \in X_0.$$

Let $\varepsilon \in \left(0, \frac{1}{2r_0}\frac{1}{2M}\right)$. Then for each $x \in B_{X_0}(0,\varepsilon)$,

$$\|U_{r_0}(t)x\| \leq e^{\gamma t} 2M \|x\| \leq \frac{1}{2r_0}.$$

On the other hand, since $F = F_r$ on $B_{X_0}\left(0, \frac{1}{2r_0}\right)$, we deduce that for each $x \in B_{X_0}(0,\varepsilon)$, $U_{r_0}(.)x$ is a solution of

$$U_r(t)x = x + A\int_0^t U_r(s)xds + \int_0^t F(U_r(s)x)\, ds, \ \ \forall t \geq 0.$$

The uniqueness of the solution with initial value x in $B_{X_0}(0,\varepsilon)$ follows from the fact that F is locally Lipschitz continuous around 0 and by using the arguments of Lemma 3.3 in Magal and Ruan [22]. ∎

Remark 7.2 (1) If F is continuously differentiable in $B_{X_0}(\overline{x}, r_0)$, we set $L = DF(\overline{x})$. Then by the formula

$$F(x) - F(y) = \int_0^1 DF(s(x-y)+y)(x-y)ds, \ \ \forall x, y \in B_{X_0}(\overline{x}, \varepsilon) \, ,$$

it is clear that

$$\|F - DF(\overline{x})\|_{Lip\left(\overline{B}_{X_0}(\overline{x},r),X\right)} \to 0 \text{ as } r \to 0.$$

So if $\overline{x}$ is an equilibrium (i.e. assertion (a) is satisfied) and

$$\left\|T_{(A+DF(\overline{x}))_0}(t)\right\|_{\mathcal{L}(X_0)} \leq \widehat{M}e^{\widehat{\omega}t}, \ \ \forall t \geq 0$$

for some $\widehat{M} \geq 1$ and $\widehat{\omega} < 0$, the conclusion of the proposition holds.

(2) In order to see an example where the condition (c) is more appropriate than the usual differentiability condition, we consider the following case. Assume that F is quasi-linear, that is $F(x) = L(x)x$, where $L : X_0 \to \mathcal{L}(X_0, X)$ is a Lipschitz continuous map (but not necessarily differentiable in a neighborhood of 0). Then

$$\begin{aligned} \|(F - L(0))\, x - (F - L(0))\, y\| &= \|(L(x) - L(0))\, x - (L(y) - L(0))\, y\| \\ &\leq \|(L(x) - L(0))\, x - (L(y) - L(0))\, x\| \\ &\quad + \|(L(y) - L(0))\, x - (L(y) - L(0))\, y\| \\ &\leq [\|x\| + \|y\|]\, \|L\|_{Lip}\, \|x - y\|. \end{aligned}$$

So

$$\|(F - L(0))\|_{Lip\left(B_{X_{0+}}(0,\varepsilon),X\right)} \leq 2\varepsilon \|L\|_{Lip} \to 0 \text{ as } \varepsilon \to 0.$$

Thus, in this case we can apply the condition (c), but F is not differentiable.

We now investigate the global asymptotic stability of an equilibrium.

Proposition 7.3 *Let Assumptions 2.1 and 2.8 be satisfied. Let $F : \overline{D(A)} \to X$ be a Lipschitz continuous map. Assume that:*

(a) *There exists $\overline{x} \in D(A)$ such that $A\overline{x} + F(\overline{x}) = 0$.*

(b) *There exist $\widehat{M} > 0$, $\widehat{\omega} < 0$, and $L \in \mathcal{L}(X_0, X)$, such that*

$$\left\|T_{(A+L)_0}(t)\right\|_{\mathcal{L}(X_0)} \leq \widehat{M}e^{\widehat{\omega}t}, \ \ \forall t \geq 0.$$

Consider $\{U(t)\}_{t\geq 0}$ the C_0-semigroup of nonlinear operators on X_0 which is solution of

$$U(t)x = x + A\int_0^t U(s)xds + \int_0^t F(U(s)x)\, ds, \ \ \forall t \geq 0.$$

Then for each $\gamma \in (\widehat{\omega}, 0)$, there exists $\delta_0 = \delta_0(\gamma) > 0$, such that

$$\|F - L\|_{Lip(X_0,X)} \leq \delta_0 \Rightarrow \|U(t)x - \overline{x}\| \leq e^{\gamma t} 2\widehat{M} \|x - \overline{x}\|, \ \forall t \geq 0, \ \forall x \in X_0.$$

So $\overline{x}$ is a globally exponentially stable equilibrium of $\{U(t)\}_{t\geq 0}$.

Proof. Replacing $U(t)x$ by $V(t)x = U(t)(x + \overline{x}) - \overline{x}$ and $F(.)$ by $G(.) = F(. + \overline{x}) - F(\overline{x})$, respectively. Without loss of generality we can assume that $\overline{x} = 0$. Moreover, using Theorem 2.12 and replacing M by $\widehat{M}$, ω_A by $\widehat{\omega}$, A by

$A+L$, and F by $F-L$, respectively. We can further assume that $L=0$ and $\omega_A<0$.

Fix $\tau>0$ and set $\varepsilon:=M\delta(\tau)$. Let $\gamma\in(\omega_A,0)$ be fixed. Choose $\delta_0=\delta_0(\gamma)>0$ such that

$$\delta_0\frac{2\varepsilon e^{-\gamma\tau_\varepsilon}}{\left(1-e^{(\omega_A-\gamma)\tau_\varepsilon}\right)}\leq\frac{1}{2}.$$

Then by Lemma 2.15 we have

$$\left\|\mathcal{L}_0(\varphi)\right\|_{\mathcal{L}(BC^\gamma([0,+\infty),X),BC^\gamma([0,+\infty),X_0))}\leq\frac{2\varepsilon e^{-\gamma\tau_\varepsilon}}{\left(1-e^{(\omega_A-\gamma)\tau_\varepsilon}\right)}\leq\frac{1}{2\delta_0}.$$

It is sufficient to consider the problem $U(.)x\in BC^\gamma([0,+\infty),X_0)$,

$$U(t)x=T_{A_0}(t)x+\mathcal{L}_0\left(\Psi\left(U(.)x\right)\right)(t),\forall t\in[0,+\infty),$$

where $\Psi:BC^\gamma([0,+\infty),X_0)\to BC^\gamma([0,+\infty),X)$ is defined by

$$\Psi(\varphi)(t)=F(\varphi(t)),\forall t\in[0,+\infty).$$

If $\|F\|_{Lip(X_0,X)}\leq\delta_0$, we have $\|\mathcal{L}_0\circ\Psi\|_{Lip(BC^\gamma([0,+\infty),X_0),BC^\gamma([0,+\infty),X_0))}\leq1/2$, so for each $t\geq0$

$$\|U(.)x\|_{BC^\gamma([0,+\infty),X_0)}\leq M\|x\|+\frac{1}{2}\|U(.)x\|_{BC^\gamma([0,+\infty),X_0)}$$

and the result follows. ∎

Let $L:X_0\to X$ be a bounded linear operator. In order to apply the Stability Theorem, one need to prove that there exist two constant, $\widehat{M}\geq1$ and $\widehat{\omega}<0$, such that

$$\left\|T_{(A+L)_0}(t)\right\|_{\mathcal{L}(X_0)}\leq\widehat{M}e^{\widehat{\omega}t},\ \ \forall t\geq0,$$

which is also equivalent to

$$\omega_0\left((A+L)_0\right):=\lim_{t\to+\infty}\frac{\ln\left(\left\|T_{(A+L)_0}(t)\right\|_{\mathcal{L}(X_0)}\right)}{t}<\widehat{\omega}.$$

This condition can also be expressed by using spectral properties of the linear operator $A+L$. In fact, by applying the results obtained by Webb [43] (see also Engel and Nagel [14] for more results), it is necessary to verify the following two properties:

(a) **(Point spectrum condition)**

$$\sigma_p(A+L)\subset\{\lambda\in\mathbb{C}:Re(\lambda)<\widehat{\omega}\}$$

where

$$\sigma_p(A+L)=\{\lambda\in\mathbb{C}:N(\lambda I-(A+L))\neq\{0\}\}.$$

(b) **(Essential growth rate condition)**

$$\omega_{0,ess}\left((A+L)_0\right) := \lim_{t\to+\infty} \frac{\ln\left(\left\|T_{(A+L)_0}(t)\right\|_{ess}\right)}{t} < \widehat{\omega},$$

where

$$\left\|T_{(A+L)_0}(t)\right\|_{ess} = \kappa\left(T_{(A+L)_0}(t)B_{X_0}(0,1)\right)$$

with $B_{X_0}(0,1) = \{x \in X_0 : \|x\| \leq 1\}$, and for each bounded set $B \subset X$, $\kappa(B) = \inf\{\varepsilon > 0 : B$ can be covered by a finite number of balls of radius $\leq \varepsilon\}$ is the Kuratovsky measure of non-compactness.

In practice the essential growth rate condition can be studied by using perturbation technics. When $R(L) \subset X_0$ this question has been investigated by Webb [41, 42, 43]. When $R(L) \not\subseteq X_0$, this question has been investigated in the Hille-Yosida case by Thieme [37] and extended to the non-Hille-Yosida case by Ducrot et al. [12].

As a consequence of Theorem 2.2 in Desch and Schappacher [9] and Proposition 5.1, we have the following result on the instability of an equilibrium (see Thieme [35, Corollary 4.3]).

Proposition 7.4 *Let Assumptions 2.1 and 2.8 be satisfied. Let $F : \overline{D(A)} \to X$ be a Lipschitz continuous map. Assume that there exists $\overline{x} \in D(A)$ such that $A\overline{x} + F(\overline{x}) = 0$. Assume that $\omega_{0,ess}\left((A + DF(\overline{x}))_0\right) < 0$ and there exists $\lambda \in \sigma_p\left((A + DF(\overline{x}))_0\right)$ with $\mathrm{Re}(\lambda) > 0$. Then $\overline{x}$ is an unstable equilibrium in the following sense: There exist a constant $\varepsilon > 0$ and a sequence $x_n\ (\in X_0) \to \overline{x}$ as $t_n \to +\infty$, such that*

$$\|U(t_n)x_n - \overline{x}\| \geq \varepsilon \text{ for all } n \geq 0.$$

Remark 7.5 When the property of an equilibrium changes from stability to instability, interesting and complex dynamics, such as Hopf bifurcation, can occur. We refer to Magal and Ruan [23] for detailed results on this aspect.

8 Applications

As we mentioned in the Introduction, the key point in applying our results is to verify the main Assumptions 2.1 and 2.8. In this section, as examples we consider transport equations and parabolic equations and verify that the Assumptions 2.1 and 2.8 hold, so that the theory developed in this article applies.

8.1 Transport equations

First consider the transport equation with a nonlinear (and nonlocal) boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = M(u(t,.))(x), \quad t > 0, \quad a > 0 \\ u(t,0) = G(u(t,.)) \\ u(0,.) = u_0 \in L^p\left((0,+\infty),\mathbb{R}\right), \end{cases} \tag{8.1}$$

where $M : L^p((0,+\infty),\mathbb{R}) \to L^p((0,+\infty),\mathbb{R})$ and $G : L^p((0,+\infty),\mathbb{R}) \to \mathbb{R}$ are continuous maps.

In order to take into account the nonlinear boundary condition we consider the extended state space

$$X = \mathbb{R} \times L^p((0,+\infty),\mathbb{R})$$

endowed with the usual product norm and a linear operator $A : D(A) \subset X \to X$ defined by

$$A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' \end{pmatrix}$$

with

$$D(A) = \{0_{\mathbb{R}}\} \times W^{1,p}((0,+\infty),\mathbb{R}).$$

Notice that $X_0 = \overline{D(A)} \neq X$. So A is non-densely defined in X. Also, note that A is Hille-Yosida if and only if $p = 1$. So A is not a Hille-Yosida operator if we assume $p > 1$.

Now we consider a nonlinear operator $F : \overline{D(A)} \to X$ defined by

$$F(\begin{pmatrix} 0 \\ \varphi \end{pmatrix}) = \begin{pmatrix} G(\varphi) \\ M(\varphi) \end{pmatrix}.$$

Then by identifying $u(t,.)$ to $v(t) = \begin{pmatrix} 0 \\ u(t,.) \end{pmatrix}$, the PDE (8.1) can be formulated as a non-densely defined Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + F(v(t)) \text{ for } t \geq 0 \text{ with } u(0) = x \in \overline{D(A)}. \tag{8.2}$$

Lemma 8.1 *The operator A satisfies Assumption 2.1.*

Proof. It is readily to check that the resolvent is given by the formula

$$\begin{aligned} &(\lambda I - A)^{-1}\begin{pmatrix} y \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \\ &\Leftrightarrow \varphi(a) = e^{-\lambda a}y + \int_0^a e^{-\lambda(a-s)}\psi(s)ds. \end{aligned}$$

It follows that

$$\left\|(\lambda I - A)^{-1}\begin{pmatrix} y \\ 0 \end{pmatrix}\right\| = \left(\frac{1}{p\lambda}\right)^{1/p}|y|.$$

By using Young's inequality we also have

$$\left\|(\lambda I - A)^{-1}\begin{pmatrix} 0 \\ \psi \end{pmatrix}\right\| = \frac{1}{\lambda}\|\psi\|_{L^p}.$$

So A_0, the part of A in $X_0 = \overline{D(A)}$, is a Hille-Yosida operator and we have the estimate

$$0 < \lim_{\lambda \to +\infty} \lambda^{1/p} \left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(X)} < +\infty.$$

This completes the proof. ∎

Now we can claim that A_0 (the part of B in X_0) generates a C_0-semigroup $\{T_{A_0}(t)\}_{t\geq 0}$ and A generates an integrated semigroup $S_A(t)$.

Lemma 8.2 $\{T_{A_0}(t)\}_{t\geq 0}$, *the* C_0*-semigroup generated by* A_0 *(the part of* A *in* X_0*), is defined by*

$$T_{A_0}(t)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{T}_{A_0}(t)\varphi \end{pmatrix}$$

with

$$\widehat{T}_{A_0}(t)(\varphi)(a) = \begin{cases} e^{-\lambda a}\varphi(a-t) & \text{if } a \geq t, \\ 0 & \text{if } a < t. \end{cases}$$

Moreover, $\{S_A(t)\}_{t\geq 0}$, *the integrated semigroup generated by* A, *is defined by*

$$S_A(t)\begin{pmatrix} y \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ W(t)y + \int_0^t \widehat{T}_{B_0}(s)\varphi ds \end{pmatrix}$$

with

$$W(t)(y)(a) = \begin{cases} e^{-\lambda a}y & \text{if } a \leq t, \\ 0 & \text{if } a > t. \end{cases}$$

Proof. For $T_{A_0}(t)$ and $S_A(t)$ defined by the above formulas, we have

$$\frac{d}{dt}(\lambda I - A)^{-1} T_{A_0}(t)x = \lambda(\lambda I - A)^{-1} T_{A_0}(t)x - T_{A_0}(t)x$$

and

$$\frac{d}{dt}(\lambda I - A)^{-1} S_A(t)x = \lambda(\lambda I - A)^{-1} S_A(t)x - S_A(t)x + (\lambda I - A)^{-1} x.$$

The result follows from Lemma 2.10 of Magal and Ruan [22]. ∎

Lemma 8.3 *For each* τ, *each* $f \in L^p((0,\tau), X_0)$ *and each* $x \in X_0$, *there exists a unique integrated solution* $u \in C([0,\tau], X_0)$ *of the Cauchy problem*

$$\frac{du}{dt} = Au(t) + f(t), \quad t \in [0,\tau], \ u(0) = x, \tag{8.3}$$

given by

$$u(t) = T_{A_0}(t)x + (S_A \diamond f)(t), \quad \forall t \in [0,\tau]. \tag{8.4}$$

Moreover, there is a constant M_τ *such that*

$$\|(S_A \diamond f)(t)\| \leq M_\tau \left(\int_0^t \|f(s)\|^p ds\right)^{1/p}, \quad \forall t \in [0,\tau]. \tag{8.5}$$

Proof. Let $\psi \in C_c^\infty((0,+\infty), Y^*)$ be fixed. We define $x^* \in X_0^*$ by

$$x^*\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \int_0^{+\infty} \psi(s)(\varphi(s))ds.$$

Let $x = \begin{pmatrix} y \\ \varphi \end{pmatrix} \in X$. For each $\lambda > \omega$, we have

$$x^*\left((\lambda I - A)^{-1}\begin{pmatrix} y \\ \varphi \end{pmatrix}\right) = \int_0^{+\infty} e^{(-\lambda+\omega)t} W_{x^*}(t)(y)\, dt$$

with

$$W_{x^*}(t)(y) = e^{-(\lambda+\omega)t}\psi(t)y.$$

Therefore,

$$\begin{aligned} x^*\left((\lambda I - A)^{-n}\begin{pmatrix} y \\ \varphi \end{pmatrix}\right) &= \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}}{d\lambda^{n-1}}x^*\left((\lambda I - A)^{-1}\begin{pmatrix} y \\ \varphi \end{pmatrix}\right) \\ &= \frac{1}{(n-1)!}\int_0^{+\infty} t^{n-1}e^{(-\lambda+\omega)t}W_{x^*}(t)(y)\,dt \end{aligned}$$

and

$$\left|x^*\left((\lambda I - A)^{-n}\begin{pmatrix} y \\ \varphi \end{pmatrix}\right)\right| \leq \frac{1}{(n-1)!}\int_0^{+\infty} t^{n-1}e^{-\lambda t}M\left\|\psi(t)\right\|_{Y^*}dt\left\|y\right\|_Y.$$

Hence, Proposition 4.1 and Theorem 4.7 in Magal and Ruan [22] imply that $u(t)$ defined by (8.4) is an integrated solution of (8.3). Moreover, by using Lemma 2.10 in Magal and Ruan [22], we the inequality in (8.5). ∎

Now combine Theorem 2.9 and Lemma 8.3, we know that Assumption 2.8 is satisfied. Thus, the theory developed in sections 2-7 applies to the transport equation (8.1).

8.2 Parabolic equations

In the parabolic equations, due to the boundary conditions it is well known that elliptic operators are not in general Hille-Yosida operators but almost sectorial operators (see Da Prato [6], Okazawa [29], Periago and Straub [31] and references therein). To illustrate this property we consider the parabolic system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + M(u(t,.))(x), \quad t>0, \quad x>0 \\ -\dfrac{\partial u(t,0)}{\partial x} = G(u(t,.)) \\ u(0,.) = u_0 \in L^p\left((0,+\infty),\mathbb{R}\right), \end{cases} \tag{8.6}$$

As before we consider a linear operator $A : D(A) \subset X \to X$ defined by

$$A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \varphi'(0) \\ \varphi'' \end{pmatrix}$$

with

$$D(A) = \{0_{\mathbb{R}}\} \times W^{2,p}\left((0,+\infty),\mathbb{R}\right).$$

One may observe that A_0, the part of A in $\overline{D(A)} = \{0_{\mathbb{R}}\} \times L^p\left((0,+\infty),\mathbb{R}\right)$, is the generator of the strongly continuous semigroup of bounded linear operators associated to

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \quad t>0, \quad x>0 \\ -\dfrac{\partial u(t,0)}{\partial x} = 0 \\ u(0,.) = u_0 \in L^p\left((0,+\infty),\mathbb{R}\right). \end{cases}$$

That is,

$$A_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi'' \end{pmatrix}$$

with

$$D(A_0) = \left\{ \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in \{0_{\mathbb{R}}\} \times W^{2,p}\left((0,+\infty), \mathbb{R}\right) : \varphi'(0) = 0 \right\}.$$

In particular it is well known that A_0 is the infinitesimal generator of an analytic semigroup on $\overline{D(A)}$. The resolvent of A is defined by the formula

$$\begin{aligned} &(\lambda I - A)^{-1} \begin{pmatrix} \alpha \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \\ &\Leftrightarrow \varphi(x) = \tfrac{\alpha}{-\sqrt{\lambda}} e^{-\sqrt{\lambda}x}\alpha + \tfrac{1}{2\sqrt{\lambda}} \int_0^{+\infty} e^{-\sqrt{\lambda}s}\psi(s)ds e^{-\sqrt{\lambda}x} + \tfrac{1}{2\sqrt{\lambda}} \int_0^{+\infty} e^{-\sqrt{\lambda}|x-s|}\psi(s)ds \end{aligned}$$

for $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,(\lambda) > 0$.

So as in the hyperbolic case we obtain

$$0 < \liminf_{\lambda \to +\infty} \lambda^{1/p^*} \left\| (\lambda I - A)^{-1} \right\|_{\mathcal{L}(X)} < \limsup_{\lambda \to +\infty} \lambda^{1/p^*} \left\| (\lambda I - A)^{-1} \right\|_{\mathcal{L}(X)} < +\infty,$$

where

$$p^* = \frac{2p}{1+p}.$$

Note that A is not a Hille-Yosida operator when

$$p \in (1, +\infty).$$

Now by identifying $u(t,.)$ to $v(t) = \begin{pmatrix} 0 \\ u(t,.) \end{pmatrix}$, the PDE (8.6) can be formulated as a non-densely defined Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + F(v(t)) \text{ for } t \geq 0 \text{ with } u(0) = x \in \overline{D(A)}. \tag{8.7}$$

We make the following assumption.

Assumption 8.4 Let $A : D(A) \subset X \to X$ a linear operator on a Banach space X. Assume that there exist two constants, $\omega_A \in \mathbb{R}$ and $M_A > 0$, such that

(a) $\rho(A_0) \supset \{\lambda \in \mathbb{C} :\ \mathrm{Re}\ (\lambda) > \omega_A\}$ and

$$\left\| (\lambda - \omega_A)\left(\lambda I - A_0\right)^{-1} \right\|_{\mathcal{L}(X_0)} \leq M_A,\ \forall \lambda \in \mathbb{C},\ \mathrm{Re}\ (\lambda) > \omega_A;$$

(b) $(\omega_A, +\infty) \subset \rho(A)$ and there exists $p^* \geq 1$ such that

$$\limsup_{\lambda \to +\infty} \lambda^{1/p^*} \left\| (\lambda I - A)^{-1} \right\|_{\mathcal{L}(X)} < +\infty.$$

The above assumption can be reformulated by saying that A_0 is sectorial and A is $\frac{1}{p^*}$-almost sectorial. Moreover, we have the following result (see Ducrot, Magal and Prevost [13, Theorem 3.1]).

Lemma 8.5 *Let Assumption 8.4 be satisfied. Let $\lambda > \omega_A$ and $\hat{p} \in (p^*, +\infty)$ be fixed. Then for each $f \in L^{\hat{p}}((0,\tau), X)$, the map $t \to (S_A * f)(t)$ is continuously differentiable, $(S_A * f)(t) \in D(A)$, $\forall t \in [0,\tau]$, and if we denote by $u(t) = \frac{d}{dt}(S_A * f)(t)$, then*

$$u(t) = A\int_0^t u(s)ds + \int_0^t f(s)ds, \ \forall t \in [0,\tau].$$

Moreover, for each $\beta \in \left(1 - \frac{1}{p^*}, 1 - \frac{1}{\hat{p}}\right)$ and each $t \in [0,\tau]$, the following estimate holds

$$\|(S_A \diamond f)(t)\| \leq M_{\beta,\tau}\int_0^t (t-s)^{-\beta} e^{\omega_A(t-s)}\|f(s)\|ds, \tag{8.8}$$

where $M_{\beta,\tau}$ is some positive constant.

Lemma 8.5 implies that for each $\widehat{p} > p^*$ and each $\tau > 0$, the Cauchy problem (8.7) has a unique integrated solution and there exists a constant $M_{\tau,\widehat{p}} > 0$ such that

$$\|(S_A \diamond f)(t)\| \leq M_{\tau,\widehat{p}}\left(\int_0^t \|f(s)\|^{\widehat{p}}ds\right)^{1/\widehat{p}}, \ \forall t \in [0,\tau].$$

Therefore, the Assumptions 2.1 and 2.8 are satisfied for the parabolic equation (8.8). Thus, the results in sections 3-7 apply to this equation too.

For parabolic problems in dimension n, we refer to Tanabe [34, Section 3.8, p.82], Agranovich [1], and Volpert and Volpert [40] for general estimates for the resolvent of elliptic operators in the n dimensional case.

The goal of this paper was to develop a comprehensive semilinear theory for non-densely defined Cauchy problems when the linear operator is not a Hille-Yosida operator. The above two examples on transport equations with nonlinear boundary conditions and parabolic equations with nonlocal boundary conditions demonstrate that when the linear operator is not Hille-Yosida, we still can formulate the problems into a non-densely defined Cauchy problem and apply the theory of this paper to study these two types of equations. We believe that our results can be applied to discuss various other types of equations, such as retarded and neutral functional differential equations (see Liu, Magal and Ruan [19], Magal and Ruan [22], and Ducrot, Liu and Magal [11]).

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