Positively Invariant Subset for Non-Densely Defined Cauchy Problems

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Abstract

This study develops a generalized notion of sub tangential condition to establish the positive invariance of a closed subset under the semiflow generated by a semi-linear non densely defined Cauchy problem. We also remark that the sufficient condition for the positivity of the semiflow implies our sub tangentiality condition. In other words, our sub tangential condition is more powerful since it can be used to show the positive invariance of a much larger class of closed subset. As an illustration we apply our results to an age-structured equation in L^p space which is only defined on a closed subset of L^p.

Key words. Semilinear differential equations, non-dense domain, integrated semigroup, positively invariant subset, age structured models.

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1 Introduction

The current paper is a continuation of [11] in which we have developed the monotonicity and comparison principle for the following abstract semi-linear Cauchy problem

$$\frac{du(t)}{dt} = Au(t) + F(t, u(t)), \text{ for } t \geq 0, \text{ with } u(0) = u_0 \in \overline{D(A)}, \tag{1.1}$$

where $A : D(A) \subset X \to X$ is a linear operator on a Banach space X, and $F : [0, \infty) \times \overline{D(A)} \to X$ is continuous. We assume that the map $x \to F(t, x)$ is Lipschitz on the bounded sets of $\overline{D(A)}$ uniformly with respect to t in a bounded interval of $[0, \infty)$. Here $D(A)$ is not necessarily dense in X and A is not necessarily a Hille-Yosida operator. The main purpose of this paper is to further study the invariance of subset for system (1.1).

The invariance of subset for differential equation has a long history which starts with the seminal paper of the Japanese mathematician Nagumo [13] in 1942. The result for ordinary differential equations was subsequently rediscovered by Brezis [4] and Hartman [6], and it was further extended to ordinary differential equation in ordered Banach spaces by Walter [19] and Redheffer and Walter [15]. Several extensions to partial differential equations were proposed later on by Redheffer and Walter [16] and Martin [10] for parabolic equations, etc. Martin and Smith [12] further investigated comparison/differential inequalities and invariant sets for abstract functional differential equations and reaction-diffusion systems that have time delays in the nonlinear reaction terms, and their developed results have had many applications. We refer to the book of Pavel and Motreanu [14] for an extensive study of densely defined semi-linear Cauchy problem. In [14] the authors studied the positive invariance for general closed subset subjected to tangency condition. They also conidered positive invariance of time dependent closed subset and extended their results to semilinear differential inclusion problems. The case of closed convex subset for non-densely defined Cauchy problems with a Hille-Yosida linear operator perturbed by Lipschitz continuous non linear map has been studied by Thieme [17]. The goal of this article is to extend the results of Thieme [17] from the Hille-Yosida case to the non Hille-Yosida case. It is worth noting that the non Hille-Yosida case induces several difficulties due to the problem of non uniform boundedness of $\lambda(\lambda - A)^{-1}$ whenever λ becomes. To overcome these difficulties, we adopt an approach which is somewhat different from Thieme [17]. Our key step is to establish the estimates on the integrated semigroup with regulated functions rather than continuous functions (see Lemma 3.3 and Remark 3.4). Combining these new estimates with our generalized sub tangential condition (see Assumption 3.5), we successfully establish the invariance of subset for system (1.1) (see Theorem 3.7).

In section 4, we apply our results to the two species age structured model with local competition in age (i.e. the competition for resources occurs in between individuals with the same age only)

$$\begin{cases} \dfrac{\partial u_i(t,a)}{\partial t} + \dfrac{\partial u_i(t,a)}{\partial a} = -u_i(t,a)\left(\mu_i(a) + u_1(t,a) + u_2(t,a)\right), \\ u_i(t,0) = \displaystyle\int_0^{+\infty} \beta_i(a) u_i(t,a) \mathrm{d}a \\ u_i(0,.) = u_{i0} \in \mathrm{L}^p_+((0,\infty),\mathbb{R}),\ p \in [1,+\infty),\ i = 1,2. \end{cases}$$

The paper is organized as follows. In section 2, we recall some basic results for non densely defined Cauchy problems. We develop the sub-tangential condition to establish the positive invariance of a closed subset for non densely defined non Hille-Yosida semilinear Cauchy problems in section 3, which is the main part in this paper. In section 4, we apply our developed result to age-structured models.

2 Preliminary results

For the convenience in the subsequent presentation, we first collect some existing results of integrated semigroups and non densely defined Cauchy problems in this section.

2.1 Existing results of Integrated Semigroups

Let $A : D(A) \subset X \to X$ be a linear operator. Define

$$X_0 := \overline{D(A)}$$

and $A_0 : D(A_0) \subset X_0 \to X_0$ the part of A in X_0 that is

$$A_0 x = Ax, \quad \forall x \in D(A_0),$$

and

$$D(A_0) = \{x \in D(A) : Ax \in X_0\}.$$

We impose the following assumption:

Assumption 2.1 *Suppose that*

(i) *There exist two constants $\omega_A \in \mathbb{R}$ and $M_A \geq 1$, such that $(\omega_A, +\infty) \subset \rho(A)$ and*

$$\left\|(\lambda I - A)^{-n}\right\|_{\mathcal{L}(X_0)} \leq M_A \left(\lambda - \omega_A\right)^{-n}, \ \forall \lambda > \omega_A, \ \forall n \geq 1.$$

(ii) $\lim_{\lambda \to +\infty} (\lambda I - A)^{-1} x = 0, \ \forall x \in X.$

In view of Assumption 2.1, we emphasize that A is not necessarily a Hille-Yosida linear operator since the operator norm in Assumption 2.1-(i) is taken into $X_0 \subseteq X$ (where the inclusion can be strict) instead of X. From Lemma 2.2.10 in [9], we see that $\rho(A) = \rho(A_0)$ if $\rho(A) \neq \emptyset$. Combining this fact with Assumption 2.1, we conclude that $(A_0, D(A_0))$ is a Hille-Yosida linear operator of type (ω_A, M_A) and generates a strongly continuous semigroup $\{T_{A_0}(t)\}_{t \geq 0} \subset \mathcal{L}(X_0)$ with

$$\|T_{A_0}(t)\|_{\mathcal{L}(X_0)} \leq M_A e^{\omega_A t}, \quad \forall t \geq 0.$$

As a consequence

$$\lim_{\lambda \to +\infty} \lambda \left(\lambda I - A\right)^{-1} x = x$$

only for $x \in X_0$. We note that the above limit does not necessarily exist whenever x belongs to X (see also Lemma 2.2.11 and Lemma 2.4.4 in [9]).

We summarize the above discussions as follows.

Lemma 2.2 *Assumption 2.1 is satisfied if and only if there exist two constants, $M_A \geq 1$ and $\omega_A \in \mathbb{R}$, such that $(\omega_A, +\infty) \subset \rho(A)$ and A_0 is the infinitesimal generator of a C_0-semigroup $\{T_{A_0}(t)\}_{t\geq 0}$ on X_0 which satisfies $\|T_{A_0}(t)\|_{\mathcal{L}(X_0)} \leq M_A e^{\omega_A t}, \forall t \geq 0$.*

Next, we consider the non homogeneous Cauchy problem

$$v'(t) = \ Av(t) + f(t), \quad t \geq 0 \quad \text{and} \quad v(0) = v_0 \in X_0, \tag{2.1}$$

with $f \in L^1_{\text{loc}}(\mathbb{R}, X)$.

The integrated semi-group plays central roles in the study of non-homogeneous Cauchy problems. This notion was first introduced by Ardent [1, 2]. For the results of Hille-Yosida operator, we refer to the books Arendt et al. [3]. The theory of integrated semi-group for Non-Hille-Yosida operator can be found in [18, 7, 9]. We also refer to the book of Magal and Ruan [9] for more references and results on this topic.

Definition 2.3 *Let Assumption 2.1 be satisfied. Then $\{S_A(t)\}_{t\geq 0} \in \mathcal{L}(X)$ the* ***integrated semigroup generated by*** *A is a strongly continuous family of bounded linear operators on X, which is defined by*

$$S_A(t)x = (\lambda I - A_0) \int_0^t T_{A_0}(l)(\lambda I - A)^{-1} x dl, \forall t \geq 0, \ x \in X,$$

where $\lambda \in \rho(A)$.

In order to obtain existence and uniqueness of solutions for (2.1) whenever f is a continuous map, we will require the following assumption.

Assumption 2.4 *Assume that for any $\tau > 0$ and $f \in C\left([0, \tau], X\right)$ there exists $v_f \in C\left([0, \tau], X_0\right)$ an integrated (mild) solution of*

$$\frac{dv_f(t)}{dt} = Av_f(t) + f(t), \ \textit{for } t \geq 0 \textit{ and } v_f(0) = 0,$$

that is to say that

$$\int_0^t v_f(r)dr \in D(A), \ \forall t \geq 0$$

and

$$v_f(t) = A \int_0^t v_f(r)dr + \int_0^t f(r)dr, \ \forall t \geq 0.$$

Moreover we assume that there exists a non decreasing map $\delta : [0, +\infty) \to [0, +\infty)$ such that

$$\|v_f(t)\| \leq \delta(t) \sup_{s \in [0,t]} \|f(s)\|, \ \forall t \geq 0,$$

with $\delta(t) \to 0$ as $t \to 0^+$.

Remark 2.5 *Note that Assumption 2.4 is equivalent (see [8]) to the assumption that there exists a non-decreasing map* $\delta : [0, +\infty) \to [0, +\infty)$ *such that for each* $\tau > 0$ *and each* $f \in C([0, \tau], X)$ *the map* $t \to (S_A * f)(t)$ *is differentiable in* $[0, \tau]$ *with*

$$\|(S_A \diamond f)(t)\| \leq \delta(t) \sup_{s\in[0,t]} \|f(s)\|, \ \forall t \in [0, \tau],$$

where $(S_A * f)(t)$ *and* $(S_A \diamond f)(t)$ *will be defined below in Theorem 2.7 and equation (2.3).*

Remark 2.6 *It is important to point out the fact Assumption 2.4 is also equivalent to saying that* $\{S_A(t)\}_{t\geq 0} \subset \mathcal{L}(X, X_0)$ *is of bounded semi-variation on* $[0, t]$ *for any* $t > 0$ *that is to say that*

$$V^{\infty}(S_A, 0, t) := \sup \left\{ \left\| \sum_{i=0}^{n-1} [S_A(t_{j+1}) - S_A(t_j)]x_j \right\| \right\} < +\infty$$

where the supremum is taken over all partitions $0 = t_0 < \cdots < t_n = t$ *of* $[0, t]$ *and all elements* $x_1, \ldots, x_n \in X$ *with* $\|x_j\| \leq 1$*, for* $j = 1, 2, \ldots, n$*. Moreover the non-decreasing map* $\delta : [0, +\infty) \to [0, +\infty)$ *in Assumption 2.4 is defined by*

$$\delta(t) := \sup_{s\in[0,t]} V^{\infty}(S_A, 0, s), \ \forall t \geq 0.$$

The following result is proved in [8, Theorem 2.9].

Theorem 2.7 *Let Assumptions 2.1 and 2.4 be satisfied. Then for each* $\tau > 0$ *and each* $f \in C([0, \tau], X)$ *the map*

$$t \to (S_A * f)(t) := \int_0^t S_A(t - s)f(s)ds$$

is continuously differentiable, $(S_A * f)(t) \in D(A), \forall t \in [0, \tau]$*, and if we set* $u(t) = \frac{d}{dt}(S_A * f)(t)$*, then*

$$u(t) = A \int_0^t u(s)ds + \int_0^t f(s)ds, \ \forall t \in [0, \tau].$$

Moreover we have

$$\|u(t)\| \leq \delta(t) \sup_{s\in[0,t]} \|f(s)\|, \ \forall t \in [0, \tau].$$

Furthermore, for each $\lambda \in (\omega, +\infty)$ *we have for each* $t \in [0, \tau]$ *that*

$$(\lambda I - A)^{-1} \frac{d}{dt}(S_A * f)(t) = \int_0^t T_{A_0}(t - s)(\lambda I - A)^{-1} f(s)ds. \tag{2.2}$$

From now on we will use the following notation

$$(S_A \diamond f)(t) := \frac{d}{dt}(S_A * f)(t). \tag{2.3}$$

From (2.2) and using the fact that $(S_A \diamond f)(t) \in X_0$, we deduce the approximation formula

$$(S_A \diamond f)(t) = \lim_{\lambda \to +\infty} \int_0^t T_{A_0}(t-s)\lambda (\lambda I - A)^{-1} f(s) ds.$$

A consequence of the approximation formula is the following

$$(S_A \diamond f)(t+s) = T_{A_0}(s)(S_A \diamond f)(t) + (S_A \diamond f(t+.))(s), \forall t, s \geq 0.$$

The following result is proved by Magal and Ruan [7, Theorem 3.1], which will be used and applied to the operator $A - \gamma B$ in the subsequent section.

Theorem 2.8 (Bounded Linear Perturbation)
Let Assumptions 2.1 and 2.4 be satisfied. Assume $L \in \mathcal{L}(X_0, X)$ is a bounded linear operator. Then $A + L : D(A) \subset X \to X$ satisfies the conditions in Assumptions 2.1 and 2.4. More precisely, if we fix $\tau_L > 0$ such that

$$\delta(\tau_L) \|L\|_{\mathcal{L}(X_0,X)} < 1,$$

and if we denote by $\{S_{A+L}(t)\}_{t\geq 0}$ the integrated semigroup generated by $A + L$, then for any $f \in C([0, \tau_L], X)$, we have

$$\left\| \frac{d}{dt}(S_{A+L} * f) \right\| \leq \frac{\delta(t)}{1 - \delta(\tau_L)\|L\|_{\mathcal{L}(X_0,X)}} \sup_{s\in[0,t]} \|f(s)\|, \ \forall t \in [0, \tau_L].$$

The following result is proved in [8, Lemma 2.13].

Lemma 2.9 *Let Assumptions 2.1 and 2.4 be satisfied. Then*

$$\lim_{\lambda \to +\infty} \|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} = 0.$$

It follows that if $B \in \mathcal{L}(X_0, X)$, then for all $\lambda > 0$ large enough the linear operator $\lambda I - A - B$ is invertible and its inverse can be written as follows

$$(\lambda I - A - B)^{-1} = (\lambda I - A)^{-1}\left[I - B(\lambda I - A)^{-1}\right]^{-1}.$$

2.2 Existence and Uniqueness of a Maximal Semiflow

Consider now the non-autonomous semi-linear Cauchy problem

$$U(t,s)x = x + A\int_s^t U(l,s)x dl + \int_s^t F(l, U(l,s)x)dl, \quad t \geq s \geq 0, \tag{2.4}$$

and the following problem

$$U(t,s)x = T_{A_0}(t-s)x + \frac{d}{dt}(S_A * F(.+s, U(.+s,s)x)(t-s), \ t \geq s \geq 0. \tag{2.5}$$

We will make the following assumption.

Assumption 2.10 *Assume that $F : [0, +\infty) \times \overline{D(A)} \to X$ is a continuous map such that for each $\tau_0 > 0$ and each $\xi > 0$, there exists $K(\tau_0, \xi) > 0$ such that*

$$\|F(t,x) - F(t,y)\| \leq K(\tau_0, \xi) \|x - y\|$$

whenever $t \in [0, \tau_0]$, $y, x \in X_0$, and $\|x\| \leq \xi, \|y\| \leq \xi$.

In the following definition τ is the blow-up time of maximal solutions of (2.4).

Definition 2.11 (Non autonomous maximal semiflow)
Consider two maps $\tau : [0, +\infty) \times X_0 \to (0, +\infty]$ and $U : D_\tau \to X_0$, where

$$D_\tau = \left\{ (t, s, x) \in [0, +\infty)^2 \times X_0 : s \leq t < s + \tau(s, x) \right\}.$$

*We say that U is **a maximal non-autonomous semiflow on** X_0 if U satisfies the following properties*

(i) $\tau(r, U(r,s)x) + r = \tau(s,x) + s, \forall s \geq 0, \forall x \in X_0, \forall r \in [s, s + \tau(s,x))$.

(ii) $U(s,s)x = x, \forall s \geq 0, \forall x \in X_0$.

(iii) $U(t,r)U(r,s)x = U(t,s)x, \forall s \geq 0, \forall x \in X_0, \forall t, r \in [s, s + \tau(s,x))$ *with* $t \geq r$.

(iv) *If $\tau(s,x) < +\infty$, then*

$$\lim_{t \to (s + \tau(s,x))^-} \|U(t,s)x\| = +\infty.$$

Set

$$D = \left\{ (t, s, x) \in [0, +\infty)^2 \times X_0 : t \geq s \right\}.$$

The following theorem is the main result in this section, which was proved in [7, Theorem 5.2].

Theorem 2.12 *Let Assumptions 2.1, 2.4 and 2.10 be satisfied. Then there exists a map $\tau : [0, +\infty) \times X_0 \to (0, +\infty]$ and a maximal non-autonomous semiflow $U : D_\tau \to X_0$, such that for each $x \in X_0$ and each $s \geq 0$, $U(.,s)x \in C([s, s + \tau(s,x)), X_0)$ is a unique maximal solution of (2.4) (or equivalently a unique maximal solution of (2.5)). Moreover, D_τ is open in D and the map $(t,s,x) \to U(t,s)x$ is continuous from D_τ into X_0.*

3 Positive invariance of a closed subset

In this section we will study the positive invariance of a closed subset by imposing the so called sub-tangential condition. Our results extend those in [14, 17] since we focus on the study of non densely defined non Hille-Yosida semilinear Cauchy problems.

3.1 Statements of main results

We start with some lemmas that will be used in the subsequent discussions.

Lemma 3.1 *Suppose that Assumptions 2.1 and 2.4 are satisfied. Let* $0 \leq a < b$ *and* $x \in X$ *be given and define*

$$f(t) := x\mathbb{1}_{[a,b]}(t), \ \forall t \geq 0.$$

Then $t \to (S_A * f)(t)$ *is differentiable in* $[0, +\infty)$ *and*

$$(S_A \diamond f)(t) = \frac{d}{dt}(S_A * f)(t) = S_A((t-a)^+)x - S_A((t-b)^+)x, \ \forall t \geq 0,$$

where $\sigma^+ := \max(0, \sigma), \forall \sigma \in \mathbb{R}$.

Proof. We observe that

$$(S_A * f)(t) = \begin{cases} \int_a^t S_A(t-s)xds & \text{if} \quad t \in [a,b], \\ \int_a^b S_A(t-s)xds & \text{if} \quad t \geq b, \\ 0 & \text{if} \quad 0 \leq t \leq a, \end{cases}$$

which is equivalent to

$$(S_A * f)(t) = \begin{cases} \int_0^{t-a} S_A(s)xds & \text{if} \quad t \in [a,b], \\ \int_{t-b}^{t-a} S_A(s)xds & \text{if} \quad t \geq b, \\ 0 & \text{if} \quad 0 \leq t \leq a. \end{cases}$$

Then the formula follows by computing the time derivative. ■

By using similar arguments in the proof of Lemma 3.1 one can easily obtain the following results.

Lemma 3.2 *Suppose that Assumptions 2.1 and 2.4 are satisfied, and* $0 \leq a < b$ *is given. Let* $a = t_0 < \cdots < t_n = b$ *be a partition of* $[a,b]$*, and* $f : [a,b] \to X$ *be the step function defined by*

$$f(t) := \sum_{i=0}^{n-1} x_i \mathbb{1}_{[t_i, t_{i+1})}(t), \ \forall t \in [a,b) \quad \textit{and} \quad f(b) = f(t_{n-1}) = x_{n-1}.$$

Then $t \to (S_A * f(a+\cdot))(t-a)$ *is differentiable in* $[a,b]$ *and for any* $t \in [t_k, t_{k+1}]$, $k = 0, \ldots, n-1$ *one has*

$$(S_A \diamond f(a+\cdot))(t-a) = \sum_{i=0}^{k-1} [S_A(t-t_i) - S_A(t-t_{i+1})]x_i + S_A(t-t_k)x_k.$$

Recall that $f : [a,b] \to X$ is a regulated function if the limit from the right side $\lim_{s \to t^+} f(s)$ exists for each $t \in [a,b)$, and the limit from the left side $\lim_{s \to t^-} f(s)$ exists for each $t \in (a,b]$. For each $0 \leq a < b$, we assume $\text{Reg}([a,b], X)$ denotes the space of regulated functions from $[a,b]$ to X, and we also denote by $\text{Step}([a,b], X)$ the space of step functions from $[a,b]$ to X.

The following lemma extends the property described in Assumption 2.4 for the space of continuous functions to the space of regulated functions.

Lemma 3.3 *Suppose that Assumptions 2.1 and 2.4 are satisfied, and* $0 \leq a < b$ *is given. Then for any* $f \in \mathrm{Reg}([a,b],X)$ *we have*

$$\|(S_A \diamond f(a+\cdot))(t-a)\| \leq \delta(t-a) \sup_{s\in[a,t]} \|f(s)\|, \ \forall t \in [a,b].$$

Proof. Since $\mathrm{Step}([a,b],X)$ is dense in $\mathrm{Reg}([a,b],X)$ for the topology of uniform convergence (see Dieudonne [5, p.139]), it is sufficient to prove the result for $f \in \mathrm{Step}([a,b],X)$ and apply the linear extension theorem to the bounded linear operator

$$f \in \mathrm{Step}([a,b],X) \mapsto (S_A \diamond f)(\cdot).$$

Let $f \in \mathrm{Step}([a,b],X)$ be a non zero step function given by

$$f(t) := \sum_{i=0}^{n-1} x_i \mathbb{1}_{[t_i,t_{i+1})}(t), \ \forall t \in [a,b), \ \text{ and } f(b) = f(t_{n-1}) = x_{n-1}$$

with $a = t_0 < t_1 < \cdots < t_n = b$. Let $t \in [a,b]$ be given and fixed. Then there exists $k \in \{0,\ldots,n-1\}$ such that $t \in [t_k,t_{k+1}]$. Hence by Lemma 3.2 we have

$$\begin{aligned}(S_A \diamond f(a+\cdot))(t-a) &= \sum_{i=0}^{k-1}[S_A(t-t_i) - S_A(t-t_{i+1})]x_i + S_A(t-t_k)x_k \\ &= \sum_{i=0}^{k} S_A(t-t_{k-i})x_{k-i} - \sum_{i=1}^{k} S_A(t-t_{k-i+1})x_{k-i}.\end{aligned}$$

Setting

$$\bar{t}_i = t - t_{k-i+1}, \ i = 1,\ldots,k \ \text{ and } \ \bar{t}_0 := 0$$

and

$$\bar{x}_i := \frac{x_{k-i}}{\alpha}, \ i = 0,\ldots,k$$

with $\alpha := \max_{i=1,\ldots,k} \|x_i\| > 0$. Then we obtain

$$(S_A \diamond f(a+\cdot))(t-a) = \alpha \sum_{i=0}^{k}[S_A(\bar{t}_{i+1}) - S_A(\bar{t}_i)]\bar{x}_i.$$

Since $0 = \bar{t}_0 < \cdots < \bar{t}_{k+1} = t-a$ and $\|\bar{x}_i\| \leq 1$ for all $i = 1,\ldots,k$, it follows from Remark 2.6 that

$$\|(S_A \diamond f(a+\cdot))(t-a)\| \leq \alpha V^\infty(S_A,0,t-a) \leq \alpha\delta(t-a)$$

and the result follows by observing that

$$\alpha := \max_{i=1,\ldots,k} \|x_i\| = \sup_{s\in[a,t]} \|f(s)\|.$$

■

Remark 3.4 *By Lemma 3.3, we remark that the results in Remark 2.5, Theorem 2.7, and Theorem 2.8 are still valid when f is a regulated function rather than a continuous function.*

In order to prove the invariance property of a closed subset $C_0 \subset X_0$ we need to make the following assumption.

Assumption 3.5 (Sub-Tangential Condition) *Let C_0 be a closed subset of X_0. We assume that there exists a bounded linear operator $B : X_0 \to X$ such that for each $\xi > 0$ and each $\sigma > 0$ there exists $\gamma = \gamma(\xi, \sigma) > 0$ such that*

$$\lim_{h \to 0^+} \frac{1}{h} d\left(T_{(A-\gamma B)_0}(h)x + S_{A-\gamma B}(h)\left[F(t,x) + \gamma Bx\right], C_0\right) = 0,$$

whenever $x \in C_0$ with $\|x\| \leq \xi$ and $t \in [0, \sigma]$. Here the map $x \to d(x, C_0)$ is the Hausdorff semi-distance which is defined as

$$d(x, C_0) := \inf_{y \in C_0} \|x - y\|.$$

Remark 3.6 *Recall that the usual assumption for the non negativity of the mild solutions of (1.1) is covered by Assumption 3.5. In fact X_{0+} is positively invariant with respect to semiflow generated by (1.1) if for each $\xi > 0$ and each $\sigma > 0$ there exists $\gamma = \gamma(\xi, \sigma) > 0$ such that*

$$T_{(A-\gamma B)_0}(h)x + S_{A-\gamma B}(h)[F(t,x) + \gamma Bx] \in X_{0+}$$

whenever $x \in X_{0+}$ with $\|x\| \leq \xi$ and $t \in [0, \sigma]$.

The main result of this article is the following theorem.

Theorem 3.7 (Positive invariant Subset) *Let Assumptions 2.1, 2.4, 2.10 and 3.5 be satisfied. Then for each $x \in C_0$ and each $s \geq 0$, we have*

$$U(t,s)x \in C_0, \forall t \in [s, s + \tau(s,x)).$$

3.2 Proof of Theorem 3.7

This subsection is devoted to the proof of Theorem 3.7. We fix the initial condition $x_0 \in C_0$ and $s = 0$. Set $\rho := 2(\|x_0\| + 1)$ and define

$$F_\gamma(t,x) := F(t,x) + \gamma Bx, \ \forall (t,x) \in [0, +\infty) \times X_0.$$

Let $\Lambda := \Lambda(\rho) > 0$ be the constant such that

$$\|F_\gamma(t,x) - F_\gamma(t,y)\| \leq \Lambda \|x - y\|, \forall t \in [0, \rho], \ \forall x, y \in B(0, \rho). \tag{3.1}$$

Therefore by setting

$$\Gamma := 2\Lambda\rho + \sup_{t \in [0,\rho]} \|F_\gamma(t, x_0)\|,$$

we obtain

$$\|F_\gamma(t,x)\| \le \Gamma,\ \forall t\in[0,\rho],\ \forall x\in B(0,\rho). \tag{3.2}$$

Let $\gamma := \gamma(\rho) > 0$ be a constant such that

$$\lim_{h\to 0^+} \frac{1}{h} d\left(T_{(A-\gamma B)_0}(h)x + S_{A-\gamma B}(h)F_\gamma(t,x), C_0\right) = 0,$$

whenever $x\in C_0$, $\|x\|\le\rho$ and $t\in[0,\rho]$.

Then by Theorem 2.8, $A-\gamma B : D(A)\subset X\to X$ satisfies Assumptions 2.1 and 2.4. Hence combining Theorem 2.8 and Lemma 3.3 (see Remark 3.4) we know that if we fix $\tau_\gamma>0$ such that

$$\gamma\delta\left(\tau_\gamma\right)\|B\|_{\mathcal{L}(X_0,X)} < 1,$$

then there exists a non decreasing map $\delta_\gamma : [0,+\infty)\to[0,+\infty)$ with

$$\lim_{t\to 0^+}\delta_\gamma(t) = 0$$

such that for each $f\in \operatorname{Reg}([a,b],X)$, $0\le a<b\le\tau_\gamma$

$$\left\|\left(S_{A-\gamma B}\diamond f(a+\cdot)\right)(t-a)\right\| \le \delta_\gamma(t-a)\sup_{s\in[a,t]}\|f(s)\|,\ \forall t\in[a,b]. \tag{3.3}$$

To shorten the notations we set

$$\omega_\gamma := \omega_{A-\gamma B} \ \text{ and } \ M_\gamma := M_{A-\gamma B}.$$

Let $\tau\in(0,\min(\tau(0,x),\tau_\gamma,\rho))$ be small enough to satisfy

$$\Gamma\delta_\gamma(h) + M_\gamma e^{\omega_\gamma^+ h}h + \|T_{(A-\gamma B)_0}(h)x_0\| \le \rho,\ \forall h\in[0,\tau] \tag{3.4}$$

with

$$\omega_\gamma^+ = \max(0,\omega_\gamma),$$

and

$$0 < \Lambda\delta_\gamma(\tau) < 1, \tag{3.5}$$

where Λ has been defined as an upper bound for the Lipschitz norm of F_γ on $B(0,\rho)\cap C_0$ in (3.1).

Construction of the knots : Let $\varepsilon\in(0,1)$ be fixed. We define by induction a sequence $(l_k,y_k)\in[0,\tau]\times C_0$ where the index k is a non-negative integer possibly unbounded. For $k=0$ we start with

$$l_0 = 0 \ \text{ and } y_0 = x_0\in C_0.$$

In order to compute the next increment, we define for each integer $k\ge 0$

$$\begin{aligned} I_k = \{\eta\in(0,\varepsilon^*):\quad & \|F_\gamma(l,y)-F_\gamma(l_k,y_k)\|\le\varepsilon,\ \forall |l-l_k|\le\eta,\ \forall y\in B(y_k,\eta)\cap C_0,\\ & \frac{1}{\eta}d\left(T_{(A-\gamma B)_0}(\eta)y_k + S_{A-\gamma B}(\eta)F_\gamma(l_k,y_k), C_0\right) < \frac{\varepsilon}{2}\\ & \text{and } \|T_{(A-\gamma B)_0}(\eta)y_k - y_k\| \le \varepsilon\} \end{aligned} \tag{3.6}$$

where $\varepsilon^* := \min(\varepsilon, \rho)$.

Set

$$r_k := \sup(I_k) > 0 \text{ and } l_{k+1} := \min\left(l_k + \frac{r_k}{2}, \tau\right). \tag{3.7}$$

We define

$$y_{k+1} = y_k \in C_0 \text{ if } l_{k+1} = \tau.$$

Otherwise if $l_{k+1} = l_k + \frac{r_k}{2} < \tau$, then

$$0 < l_{k+1} - l_k = \frac{r_k}{2} < r_k$$

hence

$$l_{k+1} - l_k \in I_k.$$

Thus, it follows that

$$\frac{1}{l_{k+1} - l_k} d\left(T_{(A-\gamma B)_0}(l_{k+1} - l_k)y_k + S_{A-\gamma B}(l_{k+1} - l_k)F_\gamma(l_k, y_k), C_0\right) < \frac{\varepsilon}{2}.$$

Therefore, we can find $y_{k+1} \in C_0$ satisfying

$$\frac{1}{l_{k+1} - l_k}\|T_{(A-\gamma B)_0}(l_{k+1} - l_k)y_k + S_{A-\gamma B}(l_{k+1} - l_k)F_\gamma(l_k, y_k) - y_{k+1}\| \leq \frac{\varepsilon}{2}.$$

Setting

$$H_k := \frac{1}{l_{k+1} - l_k}\left[y_{k+1} - T_{(A-\gamma B)_0}(l_{k+1} - l_k)y_k - S_{A-\gamma B}(l_{k+1} - l_k)F_\gamma(l_k, y_k)\right] \in X_0.$$

Then it follows that

$$H_k \in X_0 \text{ and } \|H_k\| \leq \frac{\varepsilon}{2} \tag{3.8}$$

and

$$y_{k+1} = T_{(A-\gamma B)_0}(l_{k+1} - l_k)y_k + S_{A-\gamma B}(l_{k+1} - l_k)F_\gamma(l_k, y_k) + (l_{k+1} - l_k)H_k \in C_0.$$

Lemma 3.8 *Let Assumptions 2.1, 2.4, 2.10 and 3.5 be satisfied. Then the knots (l_k, y_k), $k \geq 0$ satisfy the following properties*

(i) *For all $k > m \geq 0$ we have*

$$\begin{aligned} y_k &= T_{(A-\gamma B)_0}(l_k - l_m)y_m + \sum_{i=m}^{k-1}(l_{i+1} - l_i)T_{(A-\gamma B)_0}(l_k - l_{i+1})H_i \\ &\quad + \sum_{i=m}^{k-1} T_{(A-\gamma B)_0}(l_k - l_{i+1})S_{A-\gamma B}(l_{i+1} - l_i)F_\gamma(l_i, y_i) \end{aligned} \tag{3.9}$$

(ii) $y_k \in B(0, \rho) \cap C_0$ *for any* $k \geq 0$.

(iii) *For all* $k > m \geq 0$ *we have*

$$\|y_k - T_{(A-\gamma B)_0}(l_k - l_m)y_m\| \leq \Gamma\delta_\gamma(l_k - l_m) + \frac{\varepsilon}{2}M_\gamma e^{\omega_\gamma^+(l_k-l_m)}(l_k - l_m).$$

Proof. Proof of (i): Let $k > m \geq 0$ be given. Recall that for all $i = 0, \ldots, k-1$ we have

$$y_{i+1} = T_{(A-\gamma B)_0}(l_{i+1} - l_i)y_i + S_{A-\gamma B}(l_{i+1} - l_i)F_\gamma(l_i, y_i) + (l_{i+1} - l_i)H_i.$$

Define the linear operator $L_i : X_0 \to X_0$ by

$$L_i := T_{(A-\gamma B)_0}(l_{i+1} - l_i), \ i = 0, \ldots, k-1.$$

Hence

$$y_{i+1} = L_i y_i + S_{A-\gamma B}(l_{i+1} - l_i)F_\gamma(l_i, y_i) + (l_{i+1} - l_i)H_i, \ i = 1, \ldots, k-1.$$

In order to use a variation of constants formula, we introduce the evolution family

$$U(i,j) = L_{i-1} \cdots L_j \ \text{ if } i > j \ \text{ and } U(i,i) = I_{X_0}.$$

Then it follows from the semigroup property that

$$U(i,j) = T_{(A-\gamma B)_0}(l_i - l_j), \ \text{ if } i \geq j.$$

By using a discrete variation of constants formula, we have for integers $k \geq m \geq 0$

$$\begin{aligned} y_k &= U(k,m)y_m + \sum_{i=m}^{k-1} U(k,i+1)[S_{A-\gamma B}(l_{i+1} - l_i)F_\gamma(l_i, y_i) + (l_{i+1} - l_i)H_i] \\ &= T_{(A-\gamma B)_0}(l_k - l_m)y_m + \sum_{i=m}^{k-1}(l_{i+1} - l_i)T_{(A-\gamma B)_0}(l_k - l_{i+1})H_i \\ &\quad + \sum_{i=m}^{k-1} T_{(A-\gamma B)_0}(l_k - l_{i+1})S_{A-\gamma B}(l_{i+1} - l_i)F_\gamma(l_i, y_i). \end{aligned}$$

Proof of (ii): We will argue by recurrence. The property is true for $k = 0$ since $y_0 = x_0 \in B(0,\rho) \cap C_0$. Assume that for $k \geq 1$

$$y_0, \ldots, y_{k-1} \in B(0,\rho) \cap C_0.$$

We are in a position to show that $y_k \in B(0,\rho) \cap C_0$. In view of (3.9), for any $m = 0, \ldots, k-1$, we have

$$y_k - T_{(A-\gamma B)_0}(l_k - l_m)y_m = \sum_{i=m}^{k-1} T_{(A-\gamma B)_0}(l_k - l_{i+1})[S_{A-\gamma B}(l_{i+1} - l_i)F_\gamma(l_i, y_i) + (l_{i+1} - l_i)H_i].$$

Then it follows that

$$\|y_k - T_{(A-\gamma B)_0}(l_k - l_m)y_m\| \leq \|W_{k,m}\| + \|Z_{k,m}\|,$$

where

$$W_{k,m} := \sum_{i=m}^{k-1} T_{(A-\gamma B)_0}(l_k - l_{i+1}) S_{A-\gamma B}(l_{i+1} - l_i) F_\gamma(l_i, y_i)$$

and

$$Z_{k,m} := \sum_{i=m}^{k-1} (l_{i+1} - l_i) T_{(A-\gamma B)_0}(l_k - l_{i+1}) H_i.$$

Next, we do estimates of $W_{k,m}$ and $Z_{k,m}$. Since $H_i \in X_0$ and $\|H_i\| \leq \frac{\varepsilon}{2}$, for any $i = m, \ldots, k-1$, it is easy to obtain from (3.8) that

$$\begin{aligned} \|Z_{k,m}\| &\leq \sum_{i=m}^{k-1} (l_{i+1} - l_i) \frac{\varepsilon}{2} M_\gamma e^{\omega_\gamma (l_{i+1} - l_i)} \\ &\leq \frac{\varepsilon}{2} M_\gamma e^{\omega_\gamma^+ (l_k - l_m)} (l_k - l_m), \end{aligned} \tag{3.10}$$

where

$$\omega_\gamma^+ = \max(0, \omega_\gamma).$$

In order to estimate $W_{k,m}$, we will rewrite it in a more convenient form. Using the following relationship

$$T_{(A-\gamma B)_0}(\sigma) S_{A-\gamma B}(h) = S_{A-\gamma B}(\sigma + h) - S_{A-\gamma B}(\sigma), \ \forall \sigma \geq 0, \forall h \geq 0,$$

we see that

$$W_{k,m} = \sum_{i=m}^{k-1} [S_{A-\gamma B}(l_k - l_i) - S_{A-\gamma B}(l_k - l_{i+1})] F_\gamma(l_i, y_i).$$

By Lemma 3.2 we have

$$W_{k,m} = (S_{A-\gamma B} \diamond f_\gamma(l_m + \cdot))(l_k - l_m)$$

with step function

$$f_\gamma(t) = F_\gamma(l_i, y_i), \ \forall t \in [l_i, l_{i+1}), \ i = m, \ldots, k-1 \ \text{ and } \ f_\gamma(l_k) = F_\gamma(l_{k-1}, y_{k-1}).$$

Therefore by using the inequality (3.3) with $a = l_m$ and $b = l_k$ it follows that

$$\begin{aligned} \|W_{k,m}\| &= \|(S_{A-\gamma B} \diamond f_\gamma(l_m + \cdot))(l_k - l_m)\| . \\ &\leq \delta_\gamma(l_k - l_m) \sup_{s \in [l_m, l_k]} \|f_\gamma(s)\| \\ &= \delta_\gamma(l_k - l_m) \max_{i=m,\ldots,k-1} \|F_\gamma(l_i, y_i)\|. \end{aligned} \tag{3.11}$$

By using (3.2) and the induction assumption, we deduce that

$$\max_{i=m,\ldots,k-1} \|F_\gamma(l_i, y_i)\| \leq \Gamma.$$

Then it follows from (3.10) and (3.11) that

$$\|y_k - T_{(A-\gamma B)_0}(l_k - l_m)y_m\| \leq \Gamma\delta_\gamma(l_k - l_m) + \frac{\varepsilon}{2}M_\gamma e^{\omega_\gamma^+(l_k-l_m)}(l_k - l_m)$$

for $m = 0, \ldots, k-1$. To conclude the proof of (ii) we note that

$$\|y_k - T_{(A-\gamma B)_0}(l_k - l_0)x_0\| = \|y_k - T_{(A-\gamma B)_0}(l_k)x_0\| \leq \Gamma\delta_\gamma(l_k) + M_\gamma e^{\omega_\gamma^+ l_k} l_k$$

and

$$\begin{aligned}\|y_k\| &\leq \|y_k - T_{(A-\gamma B)_0}(l_k)x_0\| + \|T_{(A-\gamma B)_0}(l_k)x_0\| \\ &\leq \Gamma\delta_\gamma(l_k) + M_\gamma e^{\omega_\gamma^+ l_k} l_k + \|T_{(A-\gamma B)_0}(l_k)x_0\|.\end{aligned}$$

Since $l_k \in [0, \tau]$, the inequality (3.4) implies that $y_k \in B(0, \rho) \cap C_0$.

Proof of (iii): The proof follows the same lines in (ii). ∎

Lemma 3.9 *Let Assumptions 2.1, 2.4, 2.10 and 3.5 be satisfied. Then there exists an integer $n_\varepsilon \geq 1$ such that $l_{n_\varepsilon} = \tau$. That is to say that we have a finite number of knots (l_k, y_k), $k = 0, \ldots, n_\varepsilon$ with*

$$0 = l_0 < l_1 < \cdots < l_{n_\varepsilon - 1} < l_{n_\varepsilon} = \tau \quad \text{and} \quad y_0, y_1, \ldots, y_{n_\varepsilon} \in C_0,\ y_0 = x_0.$$

Proof. We will use proof by contradiction. Assume that $l_k < \tau$ for all $k \geq 0$. That is to say that

$$l_{k+1} = l_k + \frac{r_k}{2},\ \forall k \geq 0.$$

Since the sequence is strictly increasing, there exists $l^* \leq \tau$ such that $l_k \to l^*$ as $k \to +\infty$ and $l_k < l^*$ for each $k \geq 0$. This also implies that

$$\lim_{k \to +\infty} r_k = 0. \tag{3.12}$$

In order to contradict (3.12), we will prove that there exists k_0 large enough and $\eta^* > 0$ such that $\eta^* \in I_k$ for all $k \geq k_0$. This will mean that $r_k = \sup I_k \geq \eta^* > 0$ for all $k \geq k_0$.

Let us show that $\{y_k\}_{k \geq 0}$ is a Cauchy sequence. To this end, we let $m \geq 0$ be arbitrary and $k \geq j > m$ be given. Then from Lemma 3.8, for all $k \geq j > m$, we have

$$\begin{aligned}\|y_k - y_j\| &\leq \|y_k - T_{(A-\gamma B)_0}(l_k - l_m)y_m\| \\ &+ \|T_{(A-\gamma B)_0}(l_k - l_m)y_m - T_{(A-\gamma B)_0}(l_j - l_m)y_m\| \\ &+ \|T_{(A-\gamma B)_0}(l_j - l_m)y_m - y_j\| \\ &\leq \Gamma\delta_\gamma(l_k - l_m) + M_\gamma e^{\omega_\gamma^+(l_k - l_m)}(l_k - l_m) \\ &+ \|T_{(A-\gamma B)_0}(l_k - l_m)y_m - T_{(A-\gamma B)_0}(l_j - l_m)y_m\| \\ &+ \Gamma\delta_\gamma(l_j - l_m) + M_\gamma e^{\omega_\gamma^+(l_j - l_m)}(l_j - l_m).\end{aligned}$$

Then

$$\limsup_{k,j\to+\infty} \|y_k - y_j\| \le 2\Gamma\delta_\gamma(l^* - l_m) + 2M_\gamma e^{\omega_\gamma^+(l^*-l_m)}(l^* - l_m).$$

Since m is arbitrary and

$$\lim_{m\to+\infty}[2\Gamma\delta_\gamma(l^* - l_m) + 2M_\gamma e^{\omega_\gamma^+(l^*-l_m)}(l^* - l_m)] = 0,$$

we deduce that $\{y_k\}_{k\ge 0}$ is a Cauchy sequence in $B(0,\rho)\cap C_0$. Therefore there exists $y^* \in B(0,\rho)\cap C_0$ such that

$$\lim_{k\to+\infty} y_k = y^* \in C_0.$$

Since $y^* \in C_0$ we have

$$\lim_{h\to 0^+} \frac{1}{h} d\left(T_{(A-\gamma B)_0}(h)y^* + S_{A-\gamma B}(h)F_\gamma(l^*, y^*), C_0\right) = 0.$$

By using the above limit, we can find $\eta^* \in (0, \frac{\varepsilon}{4})$ small enough such that

$$0 < \eta^* < \frac{\varepsilon}{4} < \varepsilon^* \tag{3.13}$$

and

$$\frac{1}{\eta^*} d\left(T_{(A-\gamma B)_0}(\eta^*)y^* + S_{A-\gamma B}(\eta^*)F_\gamma(l^*, y^*), C_0\right) \le \frac{\varepsilon}{4} \tag{3.14}$$

and (by using the continuity of $(l,y) \to T_{(A-\gamma B)_0}(l)y$)

$$\|T_{(A-\gamma B)_0}(\eta^*)y^* - y^*\| \le \frac{\varepsilon}{2}$$

and (by using the continuity of $(l,y) \to F_\gamma(l,y)$)

$$|l^* - l| \le 2\eta^* \text{ and } \|y - y^*\| \le 2\eta^* \Rightarrow \|F_\gamma(l,y) - F_\gamma(l^*, y^*)\| \le \frac{\varepsilon}{2}. \tag{3.15}$$

To obtain a contradiction, we will use the 1-Lipschitz continuity of $x \in X \to d(x, C_0)$ combined with the continuity of $(l,y) \to F_\gamma(l,y)$ and $(l,y) \to T_{(A-\gamma B)_0}(l)y$ at (l^*, y^*). Thus there exists $k_0 \ge 0$ large enough such that for all $k \ge k_0$ one has

$$\begin{cases} \|F_\gamma(l_k, y_k) - F_\gamma(l^*, y^*)\| \le \frac{\varepsilon}{2} \\ \|T_{(A-\gamma B)_0}(\eta^*)y_k - T_{(A-\gamma B)_0}(\eta^*)y^*\| \le \frac{\varepsilon}{4} \\ \|y_k - y^*\| \le \eta^* \text{ and } 0 < |l^* - l_k| \le \eta^* \end{cases} \tag{3.16}$$

since η^* is fixed and $y_k \to y^*$ and $l_k \to l^*$.

By using (3.14) and $y_k \to y^*$ and $l_k \to l^*$, we obtain for each $k \ge k_0$ (taking possibly k_0 larger)

$$\frac{1}{\eta^*} d\left(T_{(A-\gamma B)_0}(\eta^*)y_k + S_{A-\gamma B}(\eta^*)F_\gamma(l_k, y_k), C_0\right) < \frac{\varepsilon}{2}, \ \forall k \ge k_0. \tag{3.17}$$

Next we note that for any $k \geq k_0$

$$0 \leq l - l_k \leq \eta^* \Rightarrow |l - l^*| \leq |l - l_k| + |l^* - l_k| \leq 2\eta^*$$

and

$$\|y - y_k\| \leq \eta^* \Rightarrow \|y - y^*\| \leq \|y - y_k\| + \|y^* - y_k\| \leq 2\eta^*.$$

Combining (3.14)-(3.15) with (3.16), it follows that for any $k \geq k_0$

$$\|F_\gamma(l, y) - F_\gamma(l_k, y_k)\| \leq \|F_\gamma(l, y) - F_\gamma(l^*, y^*)\| + \|F_\gamma(l^*, y^*) - F_\gamma(l_k, y_k)\| \leq \varepsilon$$

whenever

$$|l - l_k| \leq \eta^* \text{ and } \|y - y_k\| \leq \eta^*.$$

In view of (3.13), (3.14) and (3.16), we further have

$$\begin{aligned} \|T_{(A-\gamma B)_0}(\eta^*)y_k - y_k\| \quad \leq \quad & \|T_{(A-\gamma B)_0}(\eta^*)y_k - T_{(A-\gamma B)_0}(\eta^*)y^*\| \\ & + \|T_{(A-\gamma B)_0}(\eta^*)y^* - y^*\| + \|y^* - y_k\| \leq \varepsilon. \end{aligned} \tag{3.18}$$

Finally it follows from (3.17)-(3.18) that $0 < \eta^* \in I_k$ for all $k \geq k_0$ which contradicts (3.12). ■

Construction of the approximate solution: Recall that from property (i) of Lemma 3.8 we have for each $m = 0, \dots, k-1$ and each $k \geq 1$

$$\begin{aligned} y_k \quad = \quad & T_{(A-\gamma B)_0}(l_k - l_m)y_m + \sum_{i=m}^{k-1}(l_{i+1} - l_i)T_{(A-\gamma B)_0}(l_k - l_{i+1})H_i \\ & + \sum_{i=m}^{k-1} T_{(A-\gamma B)_0}(l_k - l_{i+1})S_{A-\gamma B}(l_{i+1} - l_i)F_\gamma(l_i, y_i). \end{aligned} \tag{3.19}$$

For each $t \in [l_k, l_{k+1}]$ and each $k = 0, \dots, n_\varepsilon - 1$, we set

$$\begin{aligned} u_\varepsilon(t) \quad := \quad & T_{(A-\gamma B)_0}(t - l_0)y_0 + S_{A-\gamma B}(t - l_k)F_\gamma(l_k, y_k) + (t - l_k)H_k \\ & + \sum_{i=0}^{k-1}(l_{i+1} - l_i)T_{(A-\gamma B)_0}(t - l_{i+1})H_i \\ & + \sum_{i=0}^{k-1} T_{(A-\gamma B)_0}(t - l_{i+1})S_{A-\gamma B}(l_{i+1} - l_i)F_\gamma(l_i, y_i) \end{aligned} \tag{3.20}$$

with the convention

$$\sum_{i=m}^{p} = 0 \text{ if } p < m.$$

By using the semigroup property for $t \to T_{(A-\gamma B)_0}(t)$, we deduce from (3.19) and (3.20) that

$$u_\varepsilon(t) = T_{(A-\gamma B)_0}(t - l_k)y_k + S_{A-\gamma B}(t - l_k)F_\gamma(l_k, y_k) + (t - l_k)H_k, \ \forall t \in [l_k, l_{k+1}].$$

Then it is clear that $u_\varepsilon(t)$ is well defined and continuous from $[0,\tau]$ into X_0 and

$$u_\varepsilon(l_k) = y_k,\ \forall k = 0,\ldots,n_\varepsilon.$$

Next we rewrite $u_\varepsilon(t)$ into a form that will be convenient for our subsequent discussions. By using the relationship

$$S_{A-\gamma B}(h+\sigma) - S_{A-\gamma B}(\sigma) = T_{(A-\gamma B)_0}(\sigma)S_{A-\gamma B}(h),\ \forall h \geq 0,\ \forall \sigma \geq 0$$

one can rewrite from (3.20) the formula of u_ε as

$$\begin{aligned} u_\varepsilon(t) &= T_{(A-\gamma B)_0}(t-l_0)y_0 + S_{A-\gamma B}(t-l_k)F_\gamma(l_k,y_k) + (t-l_k)H_k \\ &\quad + \sum_{i=0}^{k-1}(l_{i+1}-l_i)T_{(A-\gamma B)_0}(t-l_{i+1})H_i \\ &\quad + \sum_{i=0}^{k-1}[S_{A-\gamma B}(t-l_{i+1}) - S_{A-\gamma B}(t-l_i)]F_\gamma(l_i,y_i), \forall t \in [l_k,l_{k+1}]. \end{aligned}$$

Setting

$$f_\gamma(t) = F_\gamma(l_i,y_i),\ \forall t \in [l_i,l_{i+1}),\ i=0,\ldots,n_\varepsilon-1,\ f_\gamma(l_{n_\varepsilon}) = F_\gamma(l_{n_\varepsilon-1},y_{n_\varepsilon-1}) \tag{3.21}$$

and remembering that $y_0 = x_0$, by Lemma 3.2 we obtain for each $t \in [l_k,l_{k+1}]$,

$$\begin{aligned} u_\varepsilon(t) &= T_{(A-\gamma B)_0}(t-l_0)x_0 + (S_{A-\gamma B} \diamond f_\gamma(l_0+\cdot))(t-l_0) \\ &\quad + (t-l_k)H_k + \sum_{i=0}^{k-1}(l_{i+1}-l_i)T_{(A-\gamma B)_0}(t-l_{i+1})H_i. \end{aligned} \tag{3.22}$$

Similar arguments also give for any $t \in [l_k,l_{k+1}]$ and each integer $m \in [0,k]$

$$\begin{aligned} u_\varepsilon(t) &= T_{(A-\gamma B)_0}(t-l_m)y_m + (S_{A-\gamma B} \diamond f_\gamma(l_m+\cdot))(t-l_m) \\ &\quad + (t-l_k)H_k + \sum_{i=m}^{k-1}(l_{i+1}-l_i)T_{(A-\gamma B)_0}(t-l_{i+1})H_i. \end{aligned} \tag{3.23}$$

By using again (3.2), we also have the following estimate that for any $t \in [l_m,l_k]$ with $k \geq m$,

$$\|(S_{A-\gamma B} \diamond f_\gamma(l_m+\cdot))(t-l_m)\| \leq \Gamma\delta_\gamma(t-l_m). \tag{3.24}$$

Lemma 3.10 *Let Assumptions 2.1, 2.4, 2.10 and 3.5 be satisfied. Then the approximate solution $u_\varepsilon(t)$ in (3.22) satisfies the following properties*

(i) *There exits a constant $\hat{M}_0 > 0$ such that*

$$\|u_\varepsilon(t) - y_k\| \leq \hat{M}_0(\varepsilon + \delta_\gamma(\varepsilon)),\ \forall t \in [l_k,l_{k+1}]$$

with $k = 0,\ldots,n_\varepsilon - 1$.

(ii) $u_\varepsilon(t) \in B(0,\rho),\ \forall t \in [0,\tau]$.

(iii) *There exists a constant* $\hat{M}_1 > 0$ *such that for all* $t \in [0,\tau]$

$$\left\| u_\varepsilon(t) - T_{(A-\gamma B)_0}(t)x_0 - (S_{A-\gamma B} \diamond F_\gamma(\cdot, u_\varepsilon(\cdot))(t) \right\| \le \hat{M}_1(\varepsilon + \delta_\gamma(\varepsilon)). \tag{3.25}$$

Proof. We first prove that, for each $t \in [l_m, l_p]$ with $p \ge m \ge 0$ and each $\bar{y} \in X_0$, we have

$$\|u_\varepsilon(t) - \bar{y}\| \le \|T_{(A-\gamma B)_0}(t-l_m)y_m - \bar{y}\| + \Gamma\delta_\gamma(t-l_m) + \frac{\varepsilon}{2}M_\gamma(t-l_m)e^{\omega_\gamma^+(t-l_m)}. \tag{3.26}$$

Let $p > m \ge 0$ be given. From (3.23) we have

$$\begin{aligned} u_\varepsilon(t) - \bar{y} &= T_{(A-\gamma B)_0}(t-l_m)y_m - \bar{y} + (S_{A-\gamma B} \diamond f_\gamma(l_m + \cdot))(t-l_m) \\ &\quad + (t-l_k)H_k + \sum_{i=m}^{k-1}(l_{i+1}-l_i)T_{(A-\gamma B)_0}(t-l_{i+1})H_i,\ \forall t \in [l_k, l_{k+1}] \end{aligned}$$

with $m \le k \le p-1$. Hence

$$\begin{aligned} \|u_\varepsilon(t) - \bar{y}\| &\le \|T_{(A-\gamma B)_0}(t-l_m)y_m - \bar{y}\| + \|(S_{A-\gamma B} \diamond f_\gamma(l_m + \cdot))(t-l_m)\| \\ &\quad + (t-l_k)\|H_k\| + \sum_{i=m}^{k-1}(l_{i+1}-l_i)\|T_{(A-\gamma B)_0}(t-l_{i+1})H_i\|,\ \forall t \in [l_k, l_{k+1}] \end{aligned}$$

with $m \le k \le p-1$. In view of (3.24), and

$$H_i \in X_0 \ \text{ and } \|H_i\| \le \frac{\varepsilon}{2},\ i = 0, \ldots, n_\varepsilon,$$

we see that, for any $t \in [l_k, l_{k+1}]$ with $m \le k \le p-1$,

$$\begin{aligned} \|u_\varepsilon(t) - \bar{y}\| &\le \|T_{(A-\gamma B)_0}(t-l_m)y_m - \bar{y}\| + \Gamma\delta_\gamma(t-l_m) + (t-l_k)\frac{\varepsilon}{2} \\ &\quad + \sum_{i=m}^{k-1} M_\gamma e^{\omega_\gamma(t-l_{i+1})}\frac{\varepsilon}{2}(l_{i+1}-l_i) \\ &\le \|T_{(A-\gamma B)_0}(t-l_m)y_m - \bar{y}\| + \Gamma\delta_\gamma(t-l_m) + \frac{\varepsilon}{2}M_\gamma(t-l_m)e^{\omega_\gamma^+(t-l_m)}, \end{aligned}$$

which proves (3.26).
Proof of (i): By using (3.26) with $m = k$, $p = k+1$ and $\bar{y} = y_k$, for each $t \in [l_k, l_{k+1}]$, it follows that

$$\|u_\varepsilon(t) - y_k\| \le \|T_{(A-\gamma B)_0}(t-l_k)y_k - y_k\| + \Gamma\delta_\gamma(t-l_k) + \frac{\varepsilon}{2}M_\gamma(t-l_k)e^{\omega_\gamma^+(t-l_k)}.$$

Observing that

$$t \in [l_k, l_{k+1}] \Rightarrow t - l_k \le l_{k+1} - l_k \le \frac{r_k}{2} < r_k \le \varepsilon \Rightarrow t - l_k \in I_k$$

where I_k and r_k are defined respectively in (3.6) and (3.7). Then we deduce that

$$\|T_{(A-\gamma B)_0}(t-l_k)y_k - y_k\| \leq \varepsilon, \ \forall t \in [l_k, l_{k+1}]$$

and

$$\|u_\varepsilon(t) - y_k\| \leq \varepsilon + \Gamma\delta_\gamma(\varepsilon) + \frac{\varepsilon}{2}M_\gamma \varepsilon e^{\omega_\gamma^+ \varepsilon}, \ \forall t \in [l_k, l_{k+1}]. \tag{3.27}$$

This proves (i).

Proof of (ii): In view of (3.26) with $m = 0$, $p = n_\varepsilon$ and $\bar{y} = 0$, and using the fact $l_0 = 0$ and $y_0 = x_0$, we deduce that

$$\|u_\varepsilon(t)\| \leq \|T_{(A-\gamma B)_0}(t)x_0\| + \Gamma\delta_\gamma(t) + M_\gamma e^{\omega_\gamma^+ t} t, \ \forall t \in [0, \tau].$$

Then the fact $0 \leq t \leq \tau$ together with the inequality (3.4) imply that

$$\|u_\varepsilon(t)\| \leq \rho, \ \forall t \in [0, \tau].$$

Proof of (iii): Let

$$v_\varepsilon(t) = u_\varepsilon(t) - T_{(A-\gamma B)_0}(t)x_0 - (S_{A-\gamma B} \diamond F_\gamma(\cdot, u_\varepsilon(\cdot))(t), \ \forall t \in [0, \tau].$$

We further define

$$g_\gamma(t) := f_\gamma(t) - F_\gamma(t, u_\varepsilon(t)), \ \forall t \in [0, \tau]$$

or equivalently

$$g_\gamma(t) = \begin{cases} F_\gamma(l_k, y_k) - F_\gamma(t, u_\varepsilon(t)) & \text{if} \quad t \in [l_k, l_{k+1}), \ k = 0, \ldots, n_\varepsilon - 1 \\ F_\gamma(l_{n_\varepsilon - 1}, y_{n_\varepsilon - 1}) - F_\gamma(l_{n_\varepsilon}, y_{n_\varepsilon}) & \text{if} \quad t = l_{n_\varepsilon}, \end{cases} \tag{3.28}$$

where f_γ is defined in (3.21) and n_ε has been defined in Lemma 3.9.

Then using (3.23) we get

$$v_\varepsilon(t) = (S_{A-\gamma B} \diamond g_\gamma(\cdot))(t) + (t - l_k)H_k + \sum_{i=0}^{k-1}(l_{i+1} - l_i)T_{(A-\gamma B)_0}(t - l_{i+1})H_i, \ \forall t \in [l_k, l_{k+1}].$$

Since $g_\gamma \in \text{Reg}([0, \tau], X)$, it follows that

$$\begin{aligned} \|v_\varepsilon(t)\| &\leq \delta_\gamma(t) \sup_{s \in [0,t]} \|g_\gamma(s)\| + \frac{\varepsilon}{2} M_\gamma (t - l_0) e^{\omega_\gamma^+ t} \\ &\leq \delta_\gamma(\tau) \sup_{s \in [0,t]} \|g_\gamma(s)\| + \frac{\varepsilon}{2} M_\gamma \tau e^{\omega_\gamma^+ \tau}. \end{aligned}$$

Therefore one can obtain (3.25) by estimating

$$\sup_{s \in [0,t]} \|g_\gamma(s)\|, \ \forall t \in [0, \tau].$$

In view of (3.28), it follows that

$$\|g_\gamma(t)\| \leq \|F_\gamma(l_k, y_k) - F_\gamma(t, y_k)\| + \|F_\gamma(t, y_k) - F_\gamma(t, u_\varepsilon(t))\|, \ t \in [l_k, l_{k+1}]$$

with $k = 0, \ldots, n_\varepsilon$. Observing that if $t \in [l_k, l_{k+1}]$, then

$$t - l_k \leq l_{k+1} - l_k \leq \frac{r_k}{2} < r_k \leq \rho \Rightarrow t - l_k \in I_k \text{ and } t \in [0, \rho]$$

where I_k and r_k are defined respectively in (3.6) and (3.7). This observation together with the fact

$$u_\varepsilon(t) \in B(0, \rho), \ \forall t \in [0, \tau],$$

imply that

$$\|g_\gamma(t)\| \leq \varepsilon + \Lambda \|y_k - u_\varepsilon(t)\|, \ \forall t \in [l_k, l_{k+1}].$$

Finally we infer from (3.27) that

$$\|g_\gamma(t)\| \leq \varepsilon + \Lambda[\varepsilon + \Gamma \delta_\gamma(\varepsilon) + \frac{\varepsilon}{2} M_\gamma \varepsilon e^{\omega_\gamma^+ \varepsilon}], \ \forall t \in [l_k, l_{k+1}].$$

The result follows. ■

Existence of solution in C_0: At this stage, the approximated solution $t \to u_\varepsilon(t)$ only belongs to C_0 for $t = l_k$ (since $u(l_k) = y_k \in C_0$). In this last part of the proof, we take the limit when $\varepsilon \to 0$ and after proving that the limit exists (by using Cauchy sequences), we will prove that the limit solution takes its value in C_0.

We first prove that the approximated solution $(u_\varepsilon)_{\varepsilon \in (0, \varepsilon^*)}$ forms a Cauchy sequence in $C([0, \tau], X_0)$ and its limit is a solution of system (1.1). Indeed, by using property (iii) of Lemma 3.10, we have

$$\|u_\varepsilon(t) - u_\sigma(t)\| \leq \hat{M}_1[\varepsilon + \delta_\gamma(\varepsilon) + \sigma + \delta_\gamma(\sigma)] + \delta_\gamma(t) \sup_{s \in [0,t]} \|F_\gamma(s, u_\varepsilon(s)) - F_\gamma(s, u_\sigma(s))\|.$$

Since

$$u_\varepsilon(t), u_\sigma(t) \in B(0, \rho), \ \forall t \in [0, \tau], \ 0 < \tau \leq \rho,$$

we obtain

$$\|u_\varepsilon(t) - u_\sigma(t)\| \leq \hat{M}_1[\varepsilon + \delta_\gamma(\varepsilon) + \sigma + \delta_\gamma(\sigma)] + \delta_\gamma(\tau)\Lambda \sup_{s \in [0,\tau]} \|u_\varepsilon(s) - u_\sigma(s)\|, \ \forall t \in [0, \tau].$$

In view of (3.5), we have $0 < \delta_\gamma(\tau)\Lambda < 1$, and hence,

$$\sup_{t \in [0,\tau]} \|u_\varepsilon(t) - u_\sigma(t)\| \leq \frac{\hat{M}_1}{1 - \delta_\gamma(\tau)\Lambda}[\varepsilon + \delta_\gamma(\varepsilon) + \sigma + \delta_\gamma(\sigma)].$$

Therefore $(u_\varepsilon)_{\varepsilon \in (0, \varepsilon^*)} \in C([0, \tau], X_0)$ is a Cauchy sequence in $C([0, \tau], X_0)$ endowed with the supremum norm. Then there exists $u \in C([0, \tau], X_0)$ such that

$$\lim_{\varepsilon \to 0^+} \sup_{t \in [0,\tau]} \|u_\varepsilon(t) - u(t)\| = 0.$$

Letting ε tend to zero in (3.25), it is straightforward that

$$\begin{array}{rcl} u(t) & = & T_{(A-\gamma B)_0}(t)x_0 + (S_{A-\gamma B} \diamond F_\gamma(\cdot, u(\cdot))(t) \\ & = & T_{A_0}(t)x_0 + (S_A \diamond F(\cdot, u(\cdot))(t),\ \forall t \in [0,\tau],\ \forall t \in [0,\tau]. \end{array}$$

That is to say that $u \in C([0,\tau], X_0)$ is a mild solution of (1.1) in $[0,\tau]$. Finally using property (i) of Lemma 3.10, we see that

$$d(u_\varepsilon(t), C_0) \leq \hat{M}_0(\varepsilon + \delta_\gamma(\varepsilon)),\ \forall t \in [0,\tau] \Rightarrow \lim_{\varepsilon \to 0^+} d(u_\varepsilon(t), C_0) = 0,\ \forall t \in [0,\tau].$$

By the continuity of $x \in X_0 \mapsto d(x, C_0)$, we further see that

$$d(u(t), C_0) = \lim_{\varepsilon \to 0^+} d(u_\varepsilon(t), C_0),\ \forall t \in [0,\tau] \Rightarrow u(t) \in C_0,\ \forall t \in [0,\tau].$$

4 Applications

4.1 Single species model

We consider the following age structured model with local competition in age

$$\begin{cases} \dfrac{\partial u(t,a)}{\partial t} + \dfrac{\partial u(t,a)}{\partial a} = -u(t,a)\left(\mu(a) + u(t,a)\right) \\ u(t,0) = \displaystyle\int_0^{+\infty} \beta(a) u(t,a) \mathrm{d}a \\ u(0,.) = u_0 \in \mathrm{L}^p_+((0,\infty),\mathbb{R}),\ p \in [1,+\infty). \end{cases} \tag{4.1}$$

It is important to note that due to the intra-species competition term $-u(t,a)^2$, the right hand side of the model (4.1) is not well defined in $\mathrm{L}^p_+((0,\infty),\mathbb{R})$, $p \in [1,+\infty)$. However it is well defined in any subset of $\mathrm{L}^p_+((0,\infty),\mathbb{R}) \cap \mathrm{L}^\infty_+((0,\infty),\mathbb{R})$. Let us denote by

$$\Pi(l,a) := \exp\left(-\int_l^a \mu(s)\mathrm{d}s\right),\ a \geq 0$$

the probability that an individual of age l survives to age a if there is no additional mortality. Therefore we define the basic reproductive number of (4.1) as

$$\mathcal{R}_0 := \int_0^{+\infty} \beta(a)\Pi(0,a)\mathrm{d}a,$$

which represents the average number of offprints produced by one individual during its life span.

Assumption 4.1 *We assume that β and μ belong to $\mathrm{L}^\infty_+((0,\infty),\mathbb{R})$ and there exist two constants $a_0 \geq 0$ and $\mu_0 > 0$ such that*

$$\mu(a) \geq \mu_0,\ \textit{for } a \geq a_0.$$

Equilibria: An equilibrium of (4.1) will have the following form

$$\bar{u}(a) = \frac{\Pi(0,a)\bar{b}}{1 + \int_0^a \Pi(0,\theta)\mathrm{d}\theta\, \bar{b}} \tag{4.2}$$

with $\bar{b} \geq 0$ satisfying

$$\bar{b} = \int_0^\infty \beta(a) \frac{\Pi(0,a)\bar{b}}{1 + \int_0^a \Pi(0,\theta)\mathrm{d}\theta\, \bar{b}} \mathrm{d}a.$$

Therefore there exists a unique positive equilibrium if and only if

$$1 < \mathcal{R}_0 = \int_0^{+\infty} \beta(a)\Pi(0,a)\mathrm{d}a.$$

Moreover the positive equilibrium $\bar{u}(a)$ is defined by (4.2) where $\bar{b} > 0$ is the unique solution of the scalar equation

$$\Delta(b) = 1 \;\text{ with } \Delta(b) := \int_0^\infty \beta(a) \frac{\Pi(0,a)}{1 + \int_0^a \Pi(0,\theta)\mathrm{d}\theta\, b} \mathrm{d}a. \tag{4.3}$$

Let us note that since the map $b \mapsto \Delta(b)$ is decreasing in $[0, +\infty)$ we have

$$\Delta(b) \leq 1, \;\forall b \geq \bar{b}. \tag{4.4}$$

This latter inequality (4.4) will be used in the proof of the main result of this section.

Main result : For each $b > 0$ we define the sub domain of $\mathrm{L}_+^p((0,\infty),\mathbb{R}) \cap \mathrm{L}_+^\infty((0,\infty),\mathbb{R})$

$$\widehat{C}_b = \left\{ u_0 \in \mathrm{L}_+^p((0,\infty),\mathbb{R}) : 0 \leq u_0(a) \leq \frac{\Pi(0,a)b}{1 + \int_0^a \Pi(0,\theta)\mathrm{d}\theta\, b} \right\}.$$

Theorem 4.2 *Let Assumption 4.1 be satisfied, and $\mathcal{R}_0 > 1$. Assume that $\bar{b} > 0$ is the unique solution of (4.3). Then for each $b \geq \bar{b}$ the sub domain $\widehat{C}_b$ is positively invariant by the semiflow generated by (4.1).*

The proof of the main result will be decomposed into several steps. Before proceeding let us define for each $b \geq \bar{b}$ the map $a \mapsto \bar{u}_b(a)$ by

$$\bar{u}_b(a) := \frac{\Pi(0,a)b}{1 + \int_0^a \Pi(0,\theta)\mathrm{d}\theta\, b}, \;\forall a \geq 0$$

and observe that due to Assumption 4.1 we have

$$\bar{u}_b \in \mathrm{L}_+^q((0,\infty),\mathbb{R}), \; q \in [1, +\infty].$$

Furthermore $a \mapsto \bar{u}_b(a)$ satisfies the following ordinary differential equation

$$\begin{cases} \dfrac{\mathrm{d}\bar{u}_b(a)}{\mathrm{d}a} = -\bar{u}_b(a)\left(\mu(a) + \bar{u}_b(a)\right), \ a > 0 \\ \bar{u}_b(0) = b. \end{cases} \tag{4.5}$$

Step 1: (Truncated system) We introduce the truncation function $\chi : \mathbb{R} \to \mathrm{L}^\infty_+((0,\infty),\mathbb{R})$ defined by

$$\chi(s,\cdot) = \min(\bar{u}_b(\cdot), s^+), \ \forall s \in \mathbb{R}$$

and observe that for each $\varphi \in \mathrm{L}^p_+((0,\infty),\mathbb{R})$ we have

$$0 \leq \varphi(a) \leq \bar{u}_b(a), \quad \text{a.e } a \geq 0 \Rightarrow \chi(\varphi(a),a) = \varphi(a), \quad \text{a.e } a \geq 0.$$

Next we consider the following truncated system

$$\begin{cases} \dfrac{\partial u(t,a)}{\partial t} + \dfrac{\partial u(t,a)}{\partial a} = -u(t,a)[\mu(a) + \chi(u(t,a),a)] \\ u(t,0) = \displaystyle\int_0^{+\infty} \beta(a)u(t,a)\mathrm{d}a \\ u(0,.) = u_0 \in \mathrm{L}^p_+((0,+\infty),\mathbb{R}), \ p \in [1,+\infty) \end{cases} \tag{4.6}$$

which is well defined in $\mathrm{L}^p_+((0,+\infty),\mathbb{R})$. The strategy is to prove that for each $u_0 \in \widehat{C}_b$ there exists a unique mild solution of (4.6) lying in $\widehat{C}_b$ and since the two systems (4.1) and (4.6) coincide in $\widehat{C}_b$ the result follows.
Step 2 : (Abstract reformulation) Set

$$X = \mathbb{R} \times \mathrm{L}^p((0,+\infty),\mathbb{R})$$

endowed with the usual product norm. Consider the linear operator $A : D(A) \subset X \to X$

$$A\begin{pmatrix} 0_\mathbb{R} \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' \end{pmatrix}$$

and

$$D(A) = \{0_\mathbb{R}\} \times \mathrm{W}^{1,p}((0,+\infty),\mathbb{R})$$

and note that the closure of the domain of A is

$$X_0 := \overline{D(A)} = \{0_\mathbb{R}\} \times \mathrm{L}^p((0,+\infty),\mathbb{R}).$$

Consider the non linear maps $F_0 : \mathrm{L}^p((0,+\infty),\mathbb{R}) \to \mathbb{R}$ and $F_1 : \mathrm{L}^p((0,+\infty),\mathbb{R}) \to \mathrm{L}^p((0,+\infty),\mathbb{R})$ defined respectively by

$$F_0(\varphi) = \int_0^{+\infty} \beta(a)\varphi(a)\mathrm{d}a$$

and

$$F_1(\varphi)(a) = -\varphi(a)[\mu(a) + \chi(\varphi(a),a)], \quad \text{for a.e. } a \geq 0.$$

Next we consider $F : X_0 \to X$ defined by

$$F\begin{pmatrix} 0_{\mathbb{R}} \\ \varphi \end{pmatrix} = \begin{pmatrix} F_0(\varphi) \\ F_1(\varphi) \end{pmatrix}.$$

By identifying $u(t,.)$ with $v(t) := \begin{pmatrix} 0_{\mathbb{R}} \\ u(t,.) \end{pmatrix}$ we can rewrite the partial differential equation (4.6) as the following abstract Cauchy problem

$$v'(t) = Av(t) + F(v(t)), \text{ for } t \geq 0, \; v(0) = \begin{pmatrix} 0_{\mathbb{R}} \\ u_0 \end{pmatrix} \in X_0.$$

It is well known that the linear operator $A : D(A) \subset X_0 \to X_0$ is not Hille-Yosida for $p > 1$ but fulfills the conditions of Assumption 2.1 (see [7, Section 6]). By using similar arguments in [7] one can also show that Assumption 2.4 is satisfied. It can be easily checked that F is Lipschitz on bounded sets of X_0. Therefore in what follows we will only verify that Assumption 3.5 is satisfied.

Step 3 : (Verification of Assumption 3.5) We consider the following closed subset as a candidate for the application of our results

$$C_b = \{0_{\mathbb{R}}\} \times \widehat{C}_b.$$

Since C_b is a bounded set, by letting $B = I$, it is enough to show that there exists $\gamma > 0$ such that

$$\lim_{h \to 0^+} \frac{1}{h} d\left(T_{(A-\gamma I)_0}(h)x + S_{A-\gamma I}(h)\left[F(x) + \gamma x\right], C_b\right) = 0, \; \forall x \in C_b.$$

Next we set

$$\bar{x}_b := \begin{pmatrix} 0_{\mathbb{R}} \\ \bar{u}_b \end{pmatrix} \in C_b \cap D(A).$$

Let $x = \begin{pmatrix} 0_{\mathbb{R}} \\ \varphi \end{pmatrix} \in C_b$ be given. Then we have

$$[F(x) + \gamma x] - [F(\bar{x}_b) + \gamma x_b] = \begin{pmatrix} \int_0^{+\infty} \beta(a)[\varphi(a) - \bar{u}_b(a)]\mathrm{d}a \\ (\varphi - \bar{u}_b)(\gamma - \mu - \varphi - \bar{u}_b) \end{pmatrix}.$$

Thanks to the boundedness of $\widehat{C}_b$ and Assumption 4.1 one can chose $\gamma > 0$ large enough depending only on $\bar{u}_b$ and μ such that for each $x \in C_b$ we have

$$F(\bar{x}_b) + \gamma x_b \geq F(x) + \gamma x \geq 0, \text{ in } X_+ := \mathbb{R}_+ \times \mathrm{L}^p_+((0, +\infty), \mathbb{R}).$$

Next we define

$$v_b(h) = T_{(A-\gamma I)_0}(h)\bar{x}_b + S_{A-\gamma I}(h)\left[F(\bar{x}_b) + \gamma \bar{x}_b\right], \; \forall h > 0$$

and

$$v(h) = T_{(A-\gamma I)_0}(h)x + S_{A-\gamma I}(h)\left[F(x) + \gamma x\right], \; \forall h > 0.$$

Then we see that the continuous maps $h \mapsto v(h)$ and $h \mapsto v_b(h)$ are mild solutions respectively of

$$\frac{dv_b(h)}{dh} = Av_b(h) + F(\bar{x}_b), \ h > 0, \ v_b(0) = \bar{x}_b$$

and

$$\frac{dv(h)}{dh} = Av(h) + F(x), \ h > 0, \ v(0) = x.$$

It is fairly standard to prove that A is resolvent positive that is $(\lambda I - A)^{-1} X_+ \subset X_+$ for large λ. Therefore we infer from [11, Theorem 4.5] that

$$0 \leq v(h) \leq v_b(h), \ \forall h \geq 0.$$

Observing that $\bar{x}_b \in D(A) \cap X_+$ we have

$$A\bar{x}_b + F(\bar{x}_b) = \begin{pmatrix} -\bar{u}_b(0) + F_0(\bar{u}_b) \\ -\bar{u}_b' - \bar{u}_b(\mu + \bar{u}_b) \end{pmatrix}$$

hence using (4.5) we obtain

$$A\bar{x}_b + F(\bar{x}_b) = \begin{pmatrix} -b + F_0(\bar{u}_b) \\ 0_{\mathrm{L}^p} \end{pmatrix} \leq 0 \Leftrightarrow F_0(\bar{u}_b) \leq b$$

but by (4.4) one knows that $F_0(\bar{u}_b) \leq b$ if and only if $b \geq \bar{b}$. Therefore using [11, Theorem 5.5] we obtain that the map $h \mapsto v_b(h)$ is decreasing for each $b \geq \bar{b}$ so that

$$0 \leq v(h) \leq v_b(h) \leq v_b(0) = \bar{x}_b, \ \forall h \geq 0.$$

Finally we deduce that for each $b \geq \bar{b}$

$$v(h) \in C_b, \ \forall h \geq 0 \Rightarrow \frac{1}{h} d(v(h), C_b) = 0, \ \forall h > 0.$$

4.2 Two species model

We generalize the single species system (4.1) to the following two species system

$$\begin{cases} \dfrac{\partial u_i(t,a)}{\partial t} + \dfrac{\partial u_i(t,a)}{\partial a} = -u_i(t,a) \left(\mu_i(a) + u_1(t,a) + u_2(t,a)\right) \\ u_i(t,0) = \displaystyle\int_0^{+\infty} \beta_i(a) u_i(t,a) \mathrm{d}a \\ u_i(0,.) = u_{i0} \in \mathrm{L}^p_+((0,\infty),\mathbb{R}), \ p \in [1,+\infty), \ i = 1,2. \end{cases} \tag{4.7}$$

As in the single species model system, we note that system (4.7) is not well defined in $\mathrm{L}^p_+((0,\infty),\mathbb{R})^2$, due to the inter-species competition terms $-u_1(t,a)u_2(t,a)$ and the intra-species competition terms $-u_i(t,a)^2$, $i = 1,2$. Fortunately, we will be able to study the existence of solutions and invariant subsets by using a general abstract framework. For convenience, we define

$$\Pi_i(l,a) := \exp\left(-\int_l^a \mu_i(s)\mathrm{d}s\right),\ a \geq 0, i = 1,2.$$

We note that $\Pi_i(l,a)$ stands for the probability that an individual of species i with age l survives to age a. Therefore we define the basic reproductive number of species type i as

$$\mathcal{R}_{0i} := \int_0^{+\infty} \beta_i(a)\Pi_i(0,a)\mathrm{d}a,\ i = 1,2.$$

Assumption 4.3 *For $i = 1,2$, we assume that β_i and μ_i belongs to $\mathrm{L}^\infty_+((0,\infty),\mathbb{R})$ and there exist two constants $a_0 \geq 0$ and $\mu_0 > 0$ such that*

$$\mu_i(a) \geq \mu_0,\ \textit{for } a \geq a_0.$$

Equilibria: Assume that for $i = 1,2$

$$\mathcal{R}_{0i} > 1.$$

Then we have the following trivial and semi-trivial equilibria of (4.7)

$$(0_{L^1}, 0_{L^1}),\ (\bar{u}_{\bar{b}_1}, 0_{L^1}),\ (0_{L^1}, \bar{u}_{\bar{b}_2})$$

where for each $i = 1,2$

$$\bar{u}_{\bar{b}_i}(a) = \frac{\Pi_i(0,a)\bar{b}_i}{1 + \int_0^a \Pi_i(0,\theta)\mathrm{d}\theta\,\bar{b}_i},\ \forall a \geq 0$$

with $\bar{b}_i \geq 0$ the unique solution of

$$1 = \int_0^\infty \beta_i(a)\frac{\Pi_i(0,a)}{1 + \int_0^a \Pi_i(0,\theta)\mathrm{d}\theta\,\bar{b}_i}\mathrm{d}a.$$

Define for $i = 1,2$ and each $b > 0$ the characteristic equations

$$\Delta_i(b) = 1 \ \text{ with } \Delta_i(b) := \int_0^\infty \beta_i(a)\frac{\Pi_i(0,a)}{1 + \int_0^a \Pi_i(0,\theta)\mathrm{d}\theta\, b}\mathrm{d}a. \tag{4.8}$$

Let us note that for $i = 1,2$, the map $b \mapsto \Delta_i(b)$ is decreasing in $[0,+\infty)$ and we have

$$\Delta_i(b) \leq 1,\ \forall b \geq \bar{b}_i. \tag{4.9}$$

This latter inequality (4.9) will be used in the proof of the main result of this section.

Main result : For each $b_1 > 0$ and $b_2 > 0$ we define the sub domains of $\mathrm{L}^p_+((0,\infty),\mathbb{R}) \cap \mathrm{L}^\infty_+((0,\infty),\mathbb{R})$ as

$$\widehat{C}_{b_i} = \left\{ u_{i0} \in \mathrm{L}^p_+((0,\infty),\mathbb{R}) : 0 \leq u_{i0}(a) \leq \frac{\Pi_i(0,a)b_i}{1+\int_0^a \Pi_i(0,\theta)\mathrm{d}\theta\, b_i} \right\}.$$

Theorem 4.4 *Let Assumption 4.1 be satisfied. For $i=1,2$, we assume that $\mathcal{R}_{0i} > 1$, and $\bar{b}_i > 0$ is the unique solution of (4.8). If $b_1 \geq \bar{b}_1$ and $b_2 \geq \bar{b}_2$ then the sub domain $\widehat{C}_{b_1} \times \widehat{C}_{b_2}$ is positively invariant by the semiflow generated by (4.7).*

The proof of the main result will be decomposed into several steps. Before proceeding let us define for $i = 1,2$ and each $b_i \geq \bar{b}_i$ the map $a \mapsto \bar{u}_{b_i}(a)$ by

$$\bar{u}_{b_i}(a) := \frac{\Pi_i(0,a)b_i}{1+\int_0^a \Pi_i(0,\theta)\mathrm{d}\theta\, b_i},\ \forall a \geq 0$$

and observe that due to Assumption 4.1 we have

$$\bar{u}_{b_i} \in \mathrm{L}^q_+((0,\infty),\mathbb{R}),\ q \in [1,+\infty].$$

Furthermore for $i = 1,2$ the map $a \mapsto \bar{u}_{b_i}(a)$ satisfies the following ordinary differential equation

$$\begin{cases} \dfrac{\mathrm{d}\bar{u}_{b_i}(a)}{\mathrm{d}a} = -\bar{u}_{b_i}(a)\left(\mu_i(a) + \bar{u}_{b_i}(a)\right),\ a > 0 \\ \bar{u}_{b_i}(0) = b_i. \end{cases} \tag{4.10}$$

Step 1: (Truncated system) For $i = 1,2$, we introduce the truncation functions $\chi_i : \mathbb{R} \to \mathrm{L}^\infty_+((0,\infty),\mathbb{R})$ defined by

$$\chi_i(s,\cdot) = \min(\bar{u}_{b_i}(\cdot), s^+),\ \forall s \in \mathbb{R}$$

and observe that for each $\varphi \in \mathrm{L}^p_+((0,\infty),\mathbb{R})$ we have

$$0 \leq \varphi(a) \leq \bar{u}_{b_i}(a),\quad \text{a.e } a \geq 0 \Rightarrow \chi_i(\varphi(a),a) = \varphi(a),\quad \text{a.e } a \geq 0.$$

Next we consider the following truncated system for $i = 1,2$

$$\begin{cases} \dfrac{\partial u_i(t,a)}{\partial t} + \dfrac{\partial u_i(t,a)}{\partial a} = -u_i(t,a)[\mu_i(a) + \chi_1(u_1(t,a),a) + \chi_2(u(_2t,a),a)] \\ u_i(t,0) = \displaystyle\int_0^{+\infty} \beta_i(a)u_i(t,a)\mathrm{d}a \\ u_i(0,.) = u_{i0} \in \mathrm{L}^p_+((0,+\infty),\mathbb{R}),\ p \in [1,+\infty) \end{cases} \tag{4.11}$$

which is well defined in $\mathrm{L}^p_+((0,+\infty),\mathbb{R})^2$. Next we will prove that for each $(u_{01},u_{02}) \in \widehat{C}_{b_1} \times \widehat{C}_{b_2}$ there exists a unique mild solution of (4.11) lying in $\widehat{C}_{b_1} \times \widehat{C}_{b_2}$. Then using the fact that the two systems (4.11) and (4.7) coincide

in $\widehat{C}_{b_1} \times \widehat{C}_{b_2}$ we obtain the desired results.
Step 2 : (Abstract reformulation) We first recall that A, X, X_0, and X_+ are defined in the previous subsection. Setting $\mathbb{X} := X \times X$, and we define the diagonal matrix of linear operators $\mathbb{A} : D(\mathbb{A}) \subset \mathbb{X} \to \mathbb{X}$ as

$$\mathbb{A} := \begin{pmatrix} A & 0_X \\ 0_X & A \end{pmatrix}.$$

Thus we have

$$D(\mathbb{A}) = D(A) \times D(A) \text{ and } \overline{D(\mathbb{A})} = X_0 \times X_0$$

and we set

$$\mathbb{X}_0 := \overline{D(\mathbb{A})}.$$

Consider for $i = 1, 2$ the non linear maps $\mathbb{F}_{0i} : \mathrm{L}^p((0, +\infty), \mathbb{R}) \to \mathbb{R}$ and $\mathbb{F}_{1i} : \mathrm{L}^p((0, +\infty), \mathbb{R})^2 \to \mathrm{L}^p((0, +\infty), \mathbb{R})$ respectively by

$$\mathbb{F}_{0i}(\varphi) = \int_0^{+\infty} \beta_i(a)\varphi(a)\mathrm{d}a$$

and

$$\mathbb{F}_{1i}(\varphi_1, \varphi_2)(a) = -\varphi_i(a)[\mu_i(a) + \chi_1(\varphi_1(a), a) + \chi_2(\varphi_2(a), a)], \quad \text{for a.e } a \geq 0.$$

Next we consider $\mathbb{F} : \mathbb{X}_0 \to \mathbb{X}$ defined by

$$\mathbb{F}\begin{pmatrix} \begin{pmatrix} 0_{\mathbb{R}} \\ \varphi_1 \end{pmatrix} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ \varphi_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \mathbb{F}_{01}(\varphi_1) \\ \mathbb{F}_{11}(\varphi_1, \varphi_2) \end{pmatrix} \\ \begin{pmatrix} \mathbb{F}_{02}(\varphi_2) \\ \mathbb{F}_{12}(\varphi_1, \varphi_2) \end{pmatrix} \end{pmatrix}.$$

By setting

$$\mathbf{v}(t) := \begin{pmatrix} \begin{pmatrix} 0_{\mathbb{R}} \\ u_1(t, .) \end{pmatrix} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ u_2(t, .) \end{pmatrix} \end{pmatrix}, \; t \geq 0 \text{ and } \mathbf{v}_0 := \begin{pmatrix} \begin{pmatrix} 0_{\mathbb{R}} \\ u_{10} \end{pmatrix} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ u_{20} \end{pmatrix} \end{pmatrix}$$

we can rewrite the partial differential equation (4.11) as the following abstract Cauchy problem

$$\mathbf{v}'(t) = \mathbb{A}\mathbf{v}(t) + \mathbb{F}(\mathbf{v}(t)), \text{ for } t \geq 0, \; \mathbf{v}(0) = \mathbf{v}_0 \in \mathbb{X}_0.$$

Verification of Assumptions 2.1 and 2.4 can be done by using similar arguments in [7]. Furthermore one can easily prove that $\mathbb{F}$ is Lipschitz on bounded sets of $\mathbb{X}_0$. Therefore in what follows we will only verify that Assumption 3.5 is satisfied.

Step 3 : (Verification of Assumption 3.5) We consider the following closed subset as a candidate for the application of our results

$$C_{b_1,b_2} = \{0_{\mathbb{R}}\} \times \widehat{C}_{b_1} \times \{0_{\mathbb{R}}\} \times \widehat{C}_{b_2}.$$

Since C_{b_1,b_2} is a bounded set, by letting $B = \mathbb{I}$, it is enough to show that there exists $\gamma > 0$ such that

$$\lim_{h\to 0^+} \frac{1}{h} d\left(T_{(\mathbb{A}-\gamma\mathbb{I})_0}(h)x + S_{\mathbb{A}-\gamma\mathbb{I}}(h)\left[\mathbb{F}(x) + \gamma x\right], C_{b_1,b_2}\right) = 0, \ \forall x \in C_{b_1,b_2}.$$

Next we set

$$\bar{x}_{b_1,b_2} := \begin{pmatrix} \begin{pmatrix} 0_{\mathbb{R}} \\ \bar{u}_{b_1} \end{pmatrix} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ \bar{u}_{b_2} \end{pmatrix} \end{pmatrix} \in C_{b_1,b_2} \cap D(\mathbb{A}).$$

Denote by $\mathbb{X}_+ := X_+ \times X_+$ the positive cone of $\mathbb{X}$, and we also define the induced cone in $\mathbb{X}_0$ as $\mathbb{X}_{0+} := \mathbb{X}_0 \cap \mathbb{X}_+$. Using the same arguments for the proof in the single species model one obtains that there exists $\gamma > 0$ large enough such that for each $x \in C_{b_1,b_2}$ we have

$$[\mathbb{F}(x) + \gamma x] \leq [\mathbb{F}(\bar{x}_{b_1,b_2}) + \gamma \bar{x}_{b_1,b_2}] \ \text{ in } \mathbb{X}_+.$$

Since $\mathbb{A}$ is a diagonal matrix of operators with diagonal entries that are resolvent positive it follows that $\mathbb{A}$ is also resolvent positive. Set

$$v_{b_1,b_2}(h) = T_{(\mathbb{A}-\gamma\mathbb{I})_0}(h)\bar{x}_{b_1,b_2} + S_{\mathbb{A}-\gamma\mathbb{I}}(h)\left[\mathbb{F}(\bar{x}_{b_1,b_2}) + \gamma \bar{x}_{b_1,b_2}\right], \ \forall h > 0$$

and

$$v(h) = T_{(\mathbb{A}-\gamma\mathbb{I})_0}(h)x + S_{\mathbb{A}-\gamma\mathbb{I}}(h)\left[\mathbb{F}(x) + \gamma x\right], \ \forall h > 0.$$

Then we can observe that the continuous maps $h \mapsto v(h)$ and $h \mapsto v_{b_1,b_2}(h)$ are mild solutions respectively of

$$\frac{dv_{b_1,b_2}(h)}{dh} = \mathbb{A}v_{b_1,b_2}(h) + \mathbb{F}(\bar{x}_{b_1,b_2}), \ h > 0, \ v_{b_1,b_2}(0) = \bar{x}_{b_1,b_2}$$

and

$$\frac{dv(h)}{dh} = \mathbb{A}v(h) + \mathbb{F}(x), \ h > 0, \ v(0) = x \in C_{b_1,b_2}$$

Using [11, Theorem 4.5] it follows that

$$0 \leq v(h) \leq v_{b_1,b_2}(h), \ \forall h \geq 0.$$

Next recalling that $\bar{x}_{b_1,b_2} \in D(\mathbb{A}) \cap \mathbb{X}_+$ we have

$$\mathbb{A}\bar{x}_{b_1,b_2} + \mathbb{F}(\bar{x}_{b_1,b_2}) = \begin{pmatrix} \begin{pmatrix} -\bar{u}_{b_1}(0) + \mathbb{F}_{01}(\bar{u}_{b_1}) \\ -\bar{u}_{b_1}' - \bar{u}_{b_1}(\mu_1 + \bar{u}_{b_1} + \bar{u}_{b_2}) \end{pmatrix} \\ \begin{pmatrix} -\bar{u}_{b_2}(0) + \mathbb{F}_{02}(\bar{u}_{b_2}) \\ -\bar{u}_{b_2}' - \bar{u}_{b_2}(\mu_2 + \bar{u}_{b_1} + \bar{u}_{b_2}) \end{pmatrix} \end{pmatrix}$$

hence using (4.10) we obtain

$$\mathbb{A}\bar{x}_{b_1,b_2} + \mathbb{F}(\bar{x}_{b_1,b_2}) = \begin{pmatrix} \begin{pmatrix} -b_1 + \mathbb{F}_{01}(\bar{u}_b) \\ -\bar{u}_{b_1}\bar{u}_{b_2} \end{pmatrix} \\ \begin{pmatrix} -b_2 + \mathbb{F}_{02}(\bar{u}_{b_2}) \\ -\bar{u}_{b_2}\bar{u}_{b_1} \end{pmatrix} \end{pmatrix} \leq 0 \text{ if } \mathbb{F}_{0i}(\bar{u}_{b_i}) \leq b_i, \; i = 1,2$$

but by (4.9) one knows that $\mathbb{F}_{0i}(\bar{u}_{b_i}) \leq b_i$ if $b_i \geq \bar{b}_i$. Therefore using [11, Theorem 5.5] we obtain that the map $h \mapsto v_{b_1,b_2}(h)$ is decreasing if $b_i \geq \bar{b}_i$ for $i = 1,2$ so that

$$0 \leq v(h) \leq v_{b_1,b_2}(h) \leq v_{b_1,b_2}(0) = \bar{x}_{b_1,b_2}, \; \forall h \geq 0.$$

Finally we deduce that for each $b \geq \bar{b}$

$$v(h) \in C_{b_1,b_2}, \; \forall h \geq 0 \Rightarrow \frac{1}{h} d(v(h), C_{b_1,b_2}) = 0, \; \forall h > 0.$$

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