Hopf bifurcation theorem for second order semi-linear Gurtin-MacCamy equation

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Abstract

In this paper, we prove a Hopf bifurcation theorem for second order semi-linear equations involving non-densely defined operators. Here we use the Crandall and Rabinowitz's approach based on a suitable application of the implicit function theorem. As a special case we obtain the existence of periodic wave trains for the so-called Gurtin-MacCamy problem arising in population dynamics and that couples both spatial diffusion and age structure.

Keywords: Second order semi-linear equations; Integrated semi-groups; Hopf bifurcation; Periodic wave trains; Gurtin-MacCamy equation.

1 Introduction

In this paper, we study periodic wave train solutions for the so-called Gurtin-MacCamy population dynamics. That is an age-structured equation with diffusion in space,

$$\begin{cases}(\partial_t+\partial_a-\Delta_z)u(t,a,z)=-\zeta u(t,a,z), & t\in\mathbb{R}, a>0, z\in\mathbb{R}^N,\\ u(t,0,z)=f\left(\int_0^\infty \beta(a)u(t,a,z)\mathrm{d}a\right), & t\in\mathbb{R}, z\in\mathbb{R}^N.\end{cases} \tag{1.1}$$

Here $\Delta_z=\partial_{z_1}^2+\partial_{z_2}^2+\ldots+\partial_{z_N}^2$ denotes the Laplace operator for the spatial variable $z\in\mathbb{R}^N$ for some integer $N\geq 1$. This equation is called the *Gurtin-MacCamy* equation and was introduced by Gurtin and MacCamy in [10, 11].

In this model, a diffusion process models the spatial displacement. This model describes the simultaneous time evolution of the chronological age and spatial location of the population. We mention that Ducrot and Magal [7] investigated the existence of periodic wave trains for such a problem by developing a center manifold reduction method for second-order semi-linear differential equations on the real line.

We expect that the temporal oscillations generated by the age-structured part would interact with the spatial diffusion to lead to the existence of periodic wave train solutions. Here recall that a couple $(\gamma, U\equiv U(x,a))$ is said to be a periodic wave train profile with speed $\gamma\in\mathbb{R}$ if the solution U is periodic with respect to its variable $x\in\mathbb{R}$, namely there exists a period $T>0$ such that $U(T+\cdot,\cdot)=U(\cdot,\cdot)$, and such that for each direction $e\in\mathbb{S}^{N-1}$ the function $u(t,a,z):=U(z\cdot e+\gamma t,a)$ is an entire solution of (1.1). In other words, the profile (γ,U) is a periodic-in x-solution of the following second-order problem,

$$\begin{cases}\partial_x^2U(x,a)-\gamma\partial_xU(x,a)-\partial_aU(x,a)-\zeta U(x,a)=0, & x\in\mathbb{R}, a>0,\\ U(x,0)=f\left(\int_0^\infty \beta(a)U(x,a)\mathrm{d}a\right).\end{cases} \tag{1.2}$$

Here, we shall prove the existence of wave train solutions of (1.1) by proving the existence of periodic profiles for (1.2). As discussed in Section 2, system (1.2) can be rewritten as a second-order abstract semi-linear problem of the form

$$\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2}-\gamma\frac{\mathrm{d}u(x)}{\mathrm{d}x}+Au(x)+F(\mu,u(x))=0, \quad x\in\mathbb{R}, \tag{1.3}$$

where $\mu \in \mathbb{R}$ is a bifurcation parameter, $\gamma \in \mathbb{R}$ is a given constant, $A : D(A) \subset X \to X$ is a weak Hille-Yosida linear operator (see Assumption 2.1 below) acting on a real Banach space $(X, \|\cdot\|)$ while $F : \overline{D(A)} \to X$ is a given nonlinear map of class C^2. Because of the (weak) Hille-Yosida assumptions for the operator A, (1.3) is not a hyperbolic equation but shares similarities with vector-valued elliptic equations. The equation (1.2) is actually a special case of (1.3) (see Section 2 for more details). In that case, the corresponding operator A turns out to be a non-densely defined Hille-Yosida linear operator. Moreover, (1.3) consists of a more general class of equations, see Ducrot and Magal [7] for more examples.

In this work, we shall develop bifurcation methods to study the existence of periodic solutions of (1.2). We shall more generally focus on the class of second-order equations of the form (1.3). We aim to apply the implicit function theorem inspired by Crandall and Rabinowitz [2] to prove the existence of periodic solutions emanating from Hopf bifurcation before coming back to the special case of (1.2). Here, to apply the implicit function theorem, a significant difficulty is to prove some C^1 regularity property with respect to the period ω of periodic solutions.

More precisely, we make use of the change of variable $v(x) = u(x/\omega)$ to fix the period to 2π, and work in the space of 2π-periodic functions. After this change of variable, the problem (1.3) becomes

$$\omega^2 \frac{\mathrm{d}^2 v(x)}{\mathrm{d}x^2} - \omega\gamma \frac{\mathrm{d}v(x)}{\mathrm{d}x} + Av(x) + F(\mu, v(x)) = 0, \; x \in \mathbb{R}. \tag{1.4}$$

where $\mu \in \mathbb{R}$ is a bifurcation parameter.

In this article, we use the implicit function theorem to derive the existence of map $\mu \to (\omega_\mu, v_\mu)$ where v_μ a $2\pi-$periodic function, solving the above problem (1.4) (whenever $\omega = \omega_\mu$). We use the theory of the sum of commutative operators, and obtain some mild solutions for this abstract vector-valued problem. In order to apply an implicit function theorem, we need to prove the regularity with respect to $\omega > 0$ of the solutions of the following non-homogeneous problem (posed in the space of the continuous and $2\pi-$periodic functions)

$$\left(\omega^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \omega\gamma \frac{\mathrm{d}}{\mathrm{d}x}\right) w(\omega, x) + Aw(\omega, x) = f(x), \; x \in \mathbb{R}. \tag{1.5}$$

In Theorem 3.1, we state a result including especially the C^1 regularity of the map $\omega \to w(\omega, .)$. This property corresponds to the regularity used by Crandall and Rabinowitz in [2]. In their case, they considered first order abstract Cauchy problems of parabolic type, and the regularity with respect to the period ω is inherited from the regularity in time of analytic semigroups. In this work, the operator A is not assumed to be sectorial (nor almost sectorial), and this regularity property roughly follows from the regularity of the resolvent of $\left(\omega^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \omega\gamma \frac{\mathrm{d}}{\mathrm{d}x}\right)$. This regularity property is in sharp contrast with the recent article [6], where the authors studied the Hopf bifurcation for an age-structured equation using this approach based on the implicit function theorem. In [6], some additional regularity assumptions for the birth function was imposed to compensate for this difficulty and to overcome the lack of time differentiability for the semigroup generated by an age-structured equation.

Compared to Ducrot and Magal [7], we obtain the existence of wave solution by using a very different approach. In the present article, we establish the Hopf bifurcation theorem for a second-order abstract semi-linear problem (1.3) directly, instead of reducing it to an ordinary differential equation by the center manifold method developed in [7]. Nevertheless, we believe that the current approach provides an interesting alternative that could be simpler to extend our results, for example, when we replace the local diffusion with the non-local diffusion (see [12] for a recent article on this topic). In addition, in Section 5, we apply the established Hopf bifurcation theorem to prove the existence of periodic wave train for (1.1).

Hopf bifurcation have been previously studied in the context of age-structured models, by Cushing [3, 4], Prüss [20], Swart [22], Bertoni [1], Swart [22] in the 1980s and the 1990s. Hopf bifurcation was reconsidered by using abstract non-densely defined Cauchy problems by Magal and Ruan [17]. An abstract Hopf bifurcation theorem was obtained by Liu, Magal, and Ruan [14]. In [14], a general Hopf bifurcation theorem for systems of age-structured equations was proved. Hopf bifurcation theory is extensively presented in the book of Magal, and Ruan [18]. The method presented in this article, was already employed by Cushing [3,4], and more recently by [6] for first-order Gurtin-MacCamy model. But as far as we know, such a method was not used previously for second-order problems.

This paper is organized as follows. Section 2 is concerned with the statements of the main results obtained in this work and recalls of some results on the integrated semigroups. Section 3 deals with the solvability of (1.3) in the space of periodic continuous functions. We also derive some regularity property of the solutions with respect to the period. Section 4 is devoted to the proof of our main results. Finally Section 5 focuses on an application of these general results to study the existence of periodic wave trains for the Gurtin-MacCamy equation, namely (1.1).

2 Main results

2.1 Hopf bifurcation theorem for abstract elliptic equations

In this section we first give assumptions and then state the main theorems of this paper. Throughout this paper $(X, \|\cdot\|)$ denotes a Banach space. We denote by $\mathcal{L}(X,Y)$ the Banach spaces of bounded linear operators from the Banach space X into the Banach space Y, and we denote for simplicity $\mathcal{L}(X)$ the space $\mathcal{L}(X,X)$. We consider a linear operator $A : D(A) \subset X \to X$ and we denote by $\rho(A)$, the resolvent of A. Throughout the paper, the linear operator A will satisfy the following set of assumptions.

Assumption 2.1. ***(Weak Hille-Yosida Property)*** *Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space $(X, \|\cdot\|)$. We assume that there exist two constants $\omega_A \in \mathbb{R}$ and $M_A \geq 1$ such that the following properties hold true:*

(i) $(\omega_A, +\infty) \subset \rho(A)$;

(ii) $\lim_{\lambda \to +\infty} (\lambda I - A)^{-1} x = 0,\ \forall x \in X$;

(iii) *For each $\lambda > \omega_A$ and each $n \geq 1$ the following resolvent estimate holds*

$$\left\|(\lambda I - A)^{-n}\right\|_{\mathcal{L}(\overline{D(A)})} \leq \frac{M_A}{(\lambda - \omega_A)^n}.$$

Now we will present some important results about integrated semigroups that will be used along this work. We refer the readers to the papers of Ducrot and Magal [7], Ducrot, Magal and Thorel [9], the book of Magal and Ruan [18], and the references cited therein for more details on this topic.

Let us introduce $X_0 := \overline{D(A)}$ which is a Banach space endowed with the norm of X. Let $A_0 : D(A_0) \subset X_0 \to X_0$ (which is a linear operator on X_0) be the part of A in X_0. That is,

$$A_0 x = Ax,\ \forall x \in D(A_0),\ \text{with } D(A_0) := \{x \in D(A) : Ax \in X_0\}.$$

Note that Assumption 2.1 implies that A_0 is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on X_0, denoted by $\{T_{A_0}(t)\}_{t \geq 0}$. Moreover, it satisfies the following estimate

$$\|T_{A_0}(t)\|_{\mathcal{L}(X_0)} \leq M_A e^{\omega_A t},\ \forall t \geq 0.$$

Proposition 2.2. *Let Assumption 2.1 be satisfied. Then $A : D(A) \subset X \to X$ generates a uniquely determined non-degenerate exponentially bounded integrated semigroup $\{S_A(t)\}_{t \geq 0}$. Moreover, for each $x \in X$, each $t \geq 0$ and each $\varrho > \omega_A$, $S_A(t)$ is given by*

$$S_A(t)x = (\varrho I - A_0) \int_0^t T_{A_0}(s)(\varrho I - A)^{-1} x \mathrm{d}s,$$

or equivalently

$$S_A(t)x = \varrho \int_0^t T_{A_0}(s)(\varrho I - A)^{-1} x \mathrm{d}s + [I - T_{A_0}(t)](\varrho I - A)^{-1} x.$$

Furthermore, the map $t \to S_A(t)x$ is continuously differentiable if and only if $x \in X_0$ and

$$\frac{\mathrm{d}S_A(t)x}{\mathrm{d}t} = T_{A_0}(t)x,\ \forall t \geq 0,\ \forall x \in X_0.$$

From now on we define for each $\tau > 0$

$$(S_A * f)(t) = \int_0^t S_A(t-s) f(s) \mathrm{d}s, \quad \forall t \in [0, \tau],$$

whenever $f \in L^1(0,\tau;X)$. Moreover, denote by

$$(S_A \diamond f)(t) = \frac{\mathrm{d}}{\mathrm{d}t}(S_A * f)(t),$$

whenever the convolution map $t \to (S_A * f)(t)$ is continuously differentiable.

We will need further assumptions on the linear operator A that are related to the first order Cauchy problem,

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + f(t), \text{ for } t > 0 \text{ and } u(0) = 0, \tag{2.1}$$

where $f \in L^p_{\text{loc}}([0,\infty);X)$, and $p \geq 1$ is related to the following condition.

Assumption 2.3. *There exists $p \geq 1$ such that for all $f \in L^p_{\text{loc}}([0,\infty);X)$ one has*

$$\|(S_A \diamond f)(t)\| \leq M_A \left(\int_0^t e^{p\omega_A(t-s)} \|f(s)\|^p \, \mathrm{d}s \right)^{\frac{1}{p}}, \ \forall t \geq 0,$$

where M_A and ω_A are from Assumption 2.1.

Proposition 2.4. *Let Assumptions 2.1 and 2.3 be satisfied. For each function $f \in L^p_{\text{loc}}([0,\infty);X)$, the convolution function $S_A * f$ belongs to $C^1([0,\infty);\overline{D(A)})$ and the function $u(t) = \frac{\mathrm{d}}{\mathrm{d}t}(S_A * f)(t)$ is the unique mild (or integrated) solution of the abstract Cauchy problem (2.1). Namely, the function $t \to u(t)$ satisfies*

$$\int_0^t u(s)\mathrm{d}s \in D(A), \ \forall t \geq 0 \text{ and } u(t) = A\int_0^t u(s)\mathrm{d}s + \int_0^t f(s)\mathrm{d}s, \ \forall t \geq 0.$$

Moreover, the following estimate

$$\|(S_A \diamond f)(t)\| \leq M_A \left(\int_0^t e^{p\omega_A(t-s)} \|f(s)\|^p \, \mathrm{d}s \right)^{\frac{1}{p}}, \ \forall t \geq 0,$$

holds true for all $f \in L^p_{\text{loc}}([0,\infty);X)$, and for each $\lambda > \omega_0(A_0)$ one has

$$(\lambda I - A_0)^{-1}(S_A \diamond f)(t) = \int_0^t T_{A_0}(t-s)(\lambda I - A)^{-1} f(s)\mathrm{d}s, \ \forall t \geq 0.$$

Note that when we work on the space of continuous functions, Assumption 2.3 implies the following Assumption 2.5, which is called first order solvability.

Assumption 2.5. *(**First Order Solvability**) Let $\tau > 0$ be fixed. We assume that there exists a map $\delta : [0,\tau] \to [0,+\infty)$ such that*

$$\lim_{t\to 0} \delta(t) = 0,$$

and such that for each continuous $f : [0,\tau] \to X$, there exists $u_f \in C([0,\tau],\overline{D(A)})$ a weak solution of the Cauchy problem

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + f(t), \ t \in [0,\tau] \text{ and } u(0) = 0$$

satisfying the following estimate

$$\|u_f(t)\| \leq \delta(t) \sup_{s\in[0,t]} \|f(s)\|, \ \forall t \in [0,\tau].$$

Here weak solution means that for each $\varrho \in \rho(A)$, one has

$$(\varrho I - A)^{-1} u(\cdot) \in C^1([0,\tau],X),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}[(\varrho I - A)^{-1} u(t)] = -u(t) + (\varrho I - A)^{-1}[f(t) + \varrho u(t)].$$

We now turn to Problem (1.3) and recall a notion of a weak solution for such a second order semi-linear equation, inspired by Ducrot and Magal [7]. To do so let us introduce further notations. For each interval $I \subset \mathbb{R}$, each Banach space $(Y, \|\cdot\|)$ and each weight $\eta \in \mathbb{R}$, the weighted function space $BC^0_\eta(I,Y)$ is defined by

$$BC^0_\eta(I,Y) = \{\varphi \in C(I,Y) : \sup_{x\in I} e^{-\eta|x|} \|\varphi(x)\| < \infty\}. \tag{2.2}$$

It becomes a Banach space when it is endowed with the norm

$$\|\varphi\|_{0,\eta} := \sup_{x\in I} e^{-\eta|x|} \|\varphi(x)\|.$$

We also define for each integer $k \geq 1$ the space $BC^k_\eta(I,Y)$ by

$$BC^k_\eta(I,Y) = \{\varphi \in C^k(I,Y) : \frac{\mathrm{d}^l\varphi}{\mathrm{d}x^l} \in BC^0_\eta(I,Y), l = 0,\dots,k\}. \tag{2.3}$$

These spaces are Banach spaces when they are endowed with the usual weighted uniform norm

$$\|\varphi\|_{k,\eta} = \sum_{m=0}^{k} \left\| \frac{\mathrm{d}^m\varphi}{\mathrm{d}x^m} \right\|_{0,\eta}.$$

We also denote by $C_{2\pi}(\mathbb{R},Y)$ and $C^k_{2\pi}(\mathbb{R},Y)$, the space of the continuous and $2\pi-$periodic functions from $\mathbb{R}$ into Y, and the space of $k-$th differentiable and $2\pi-$periodic functions from $\mathbb{R}$ into Y for $k \in \mathbb{N}\setminus\{0\}$.

Using these notations, we recall the following notions of solutions for (1.3).

Definition 2.6. *(**Weak and Classical Solutions of** (1.3)) We define different types of solutions for (1.3).*

(i) *We say that $u \in BC^0_0(\mathbb{R}, \overline{D(A)})$ is **a weak solution** of (1.3) if for each $\lambda \in \rho(A)$, we have*

$$(\lambda I - A)^{-1}u \in BC^2_0(\mathbb{R}, \overline{D(A)}),$$

and

$$u = \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \gamma\frac{\mathrm{d}}{\mathrm{d}x}\right)[(\lambda I - A)^{-1}u] + (\lambda I - A)^{-1}[F(u) + \lambda u].$$

(ii) *We say that $u \in BC^0_0(\mathbb{R}, \overline{D(A)})$ is **a classical solution** of (1.3) if*

$$u \in BC^2_0(\mathbb{R}, \overline{D(A)}),$$

$$u(x) \in D(A),\ \forall x \in \mathbb{R} \text{ and } x \to Au(x) \in BC^0_0(\mathbb{R}, X),$$

and

$$0 = \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \gamma\frac{\mathrm{d}}{\mathrm{d}x}\right)u(x) + Au(x) + F(u(x)),\ \forall x \in \mathbb{R}.$$

From the above definition, we can now deal with periodic solutions of (1.3) and make of a change of variable to fix the period to 2π.

Lemma 2.7. *Let $u \in BC^0_0(\mathbb{R}, \overline{D(A)})$ be a $T-$periodic weak solution of (1.3) for some $T > 0$. Then the function $v : \mathbb{R} \to \overline{D(A)}$ given by*

$$x \to v(x) := u\left(\frac{T}{2\pi}x\right) \in C_{2\pi}(\mathbb{R}, \overline{D(A)}),$$

is a weak solution of the equation

$$\omega^2\frac{\mathrm{d}^2v(x)}{\mathrm{d}x^2} - \omega\gamma\frac{\mathrm{d}v(x)}{\mathrm{d}x} + Av(x) + F(v(x)) = 0,\ x \in \mathbb{R} \text{ with } \omega = \frac{2\pi}{T} > 0. \tag{2.4}$$

Similarly if v is a $2\pi-$periodic weak solution of (2.4) for some $\omega > 0$ then the function u given by $u(\cdot) = v(\omega\cdot)$ becomes a $T = \frac{2\pi}{\omega}-$periodic weak solution of (1.3).

In the following we focus on the existence of $2\pi-$periodic (weak) solutions of (2.4) with $\omega > 0$ as a parameter.

Before stating our main Hopf bifurcation theorem, we need to introduce additional assumptions.

Assumption 2.8. *We assume that the essential growth rate of A_0 satisfies*

$$\omega_{0,ess}(A_0) = \lim_{t\to+\infty} \frac{\ln(\|T_{A_0}(t)\|_{ess})}{t} < 0.$$

In the above assumption, $\|L\|_{ess}$ denotes the essential norm of a bounded linear operator L on the Banach space X_0. Recall that it is defined by

$$\|L\|_{ess} = \kappa(L(B_{X_0}(0,1))),$$

wherein $B_{X_0}(0,1) = \{x \in X_0 : \|x\| \leq 1\}$ is the ball of radius 1 in X_0 and $\kappa(B)$ denotes the Kuratowski's measure of non-compactness of B, a bounded subset of X_0, defined by

$$\kappa(B) = \inf\{\epsilon > 0 : B \text{ can be covered by a finite number of balls of radius } \leq \epsilon\}.$$

Next we set

$$\sigma_{cu}(A) := \{\lambda \in \sigma(A) : \text{Re}(\lambda) \geq 0\},$$

and

$$\sigma_s(A) := \{\lambda \in \sigma(A) : \text{Re}(\lambda) < 0\}.$$

Now recalling Assumptions 2.1, 2.5 and Assumption 2.8 and using the results of Magal and Ruan [17, Proposition 3.5], we obtain that there exists a uniquely determined finite rank bounded linear projector $\Pi_{cu} \in \mathcal{L}(X)$ satisfying the following set of properties:

(i) $\Pi_{cu}(\lambda I - A)^{-1} = (\lambda I - A)^{-1}\Pi_{cu}$, $\forall \lambda \in \rho(A)$;

(ii) $A_{cu} \in \mathcal{L}(X_{cu})$ the part of A in $X_{cu} := \Pi_{cu}(X)$ satisfies $\sigma(A_{cu}) = \sigma_{cu}(A)$;

(iii) $A_s \in \mathcal{L}(X_s)$ the part of A in $X_s := (I - \Pi_{cu})(X)$ satisfies $\sigma(A_s) = \sigma_s(A)$.

Therefore this leads us to the following splitting of the state spaces X_0 and X,

$$X_0 = X_{cu} \oplus X_{0s} \text{ and } X = X_{cu} \oplus X_s,$$

wherein we have set $\Pi_s := I - \Pi_{cu}$ and

$$X_{0s} := \Pi_s(X_0) \subset X_0 \text{ and } X_s := \Pi_s(X).$$

We now consider system (1.3) depending on some parameter $\mu \in \mathbb{R}$. To be more precise we consider the following second order equation

$$\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} - \gamma\frac{\mathrm{d}u(x)}{\mathrm{d}x} + Au(x) + F(\mu, u(x)) = 0, \quad x \in \mathbb{R}, \tag{2.5}$$

where $\mu \in \mathbb{R}$ and $u : \mathbb{R} \to \overline{D(A)}$, and $F : \mathbb{R} \times \overline{D(A)} \to X$ is twice continuously differentiable.

Our Hopf bifurcation theorem reads as follows.

Theorem 2.9. ***(Hopf Bifurcation Theorem)*** *Let Assumptions 2.1, 2.3 and 2.8 be satisfied. Assume that $F : \mathbb{R} \times \overline{D(A)} \to X$ is of class C^2 and satisfies*

(i) *$F(\mu, 0) = 0$ for all $\mu \in \mathbb{R}$ and $\partial_u F(0,0) = 0$.*

(ii) *For each μ in some neighborhood of $\mu = 0$, there exists a pair of conjugated simple eigenvalues of $(A + \partial_u F(\mu, 0))_0$, denoted by $\lambda^{(1)}(\mu)$ and $\overline{\lambda^{(1)}(\mu)}$, such that*

$$\lambda^{(1)}(0) = \omega_0^2 + i\gamma\omega_0 \textit{ for some } \omega_0 > 0.$$

We further assume that

$$\sigma(A_0) \cap \mathcal{P}_{\omega_0\mathbb{Z}} = \{\lambda^{(1)}(0), \overline{\lambda^{(1)}(0)}\},$$

where

$$\mathcal{P}_{\omega_0\mathbb{Z}} = \{\xi^2 + i\gamma\xi : \xi \in \omega_0\mathbb{Z}\}.$$

We also assume that the map $\mu \to \lambda^{(1)}(\mu)$ is continuously differentiable and satisfies

$$\text{Re}\left[\frac{1}{\gamma - 2i\omega_0}\frac{\mathrm{d}\lambda^{(1)}(0)}{\mathrm{d}\mu}\right] \neq 0.$$

Then there exist a constant $\eta^* > 0$ and three C^1-maps, $\eta \to \mu(\eta)$ from $(-\eta^*, \eta^*)$ into $\mathbb{R}$, $\eta \to \omega(\eta)$ from $(-\eta^*, \eta^*)$ into $\mathbb{R}$ and $\eta \to u_\eta$ from $(-\eta^*, \eta^*)$ into $BC_0^0(\mathbb{R}, X_0)$, such that for each $\eta \in (-\eta^*, \eta^*) \setminus \{0\}$ the function $u_\eta \in BC_0^0(\mathbb{R}, X_0)$ is a nontrivial and $\omega(\eta)-$periodic weak solution of (2.5) with the parameter value $\mu = \mu(\eta)$ and moreover for $\eta = 0$ one has

$$\omega(0) = \omega_0, \ \mu(0) = 0 \text{ and } u_0 = 0.$$

Remark 2.10. *In the above theorem, if we consider the continuous set*

$$\mathcal{P} = \{\xi^2 + i\gamma\xi : \xi \in \mathbb{R}\},$$

in general, the intersection of the spectrum of A_0 with the continuous set $\mathcal{P}$ may contain extra elements. That is,

$$\sigma(A_0) \cap \mathcal{P} \setminus \{\lambda^{(1)}(0), \overline{\lambda^{(1)}(0)}\} \neq \emptyset.$$

But we assume that the intersection of $\sigma(A_0)$ with the discrete set $\mathcal{P}_{\omega_0\mathbb{Z}}$ only contains the two conjugated eigenvalues $\lambda^{(1)}(0)$ and $\overline{\lambda^{(1)}(0)}$. This condition extends the one which Crandall and Rabinowitz [2] employed for Hopf bifurcation. Moreover, such similar Crandall and Rabinowitz's condition is also used in following Theorems 2.11 and 2.16, see (2.8) and Assumption 2.15-(ii) respectively.

In addition, compared with Ducrot and Magal [7, Theorem 2.11], here we only require that F is of class C^2, which needs less regularity.

We now deal with the persistence of non-degenerate Hopf bifurcation for Problem (2.5) with large speed $\gamma \gg 1$. Observe that if u is a solution of (2.5) for some $\gamma \neq 0$ then the function $v(x) := u(\gamma x)$ satisfies the problem

$$\frac{1}{\gamma^2}\frac{\mathrm{d}^2 v(x)}{\mathrm{d}x^2} - \frac{\mathrm{d}v(x)}{\mathrm{d}x} + Av(x) + F(\mu, v(x)) = 0, \quad x \in \mathbb{R}. \tag{2.6}$$

For $|\gamma| \gg 1$ large enough, the above equation becomes a singular perturbation of the following first order evolution equation

$$\frac{\mathrm{d}v(x)}{\mathrm{d}x} = Av(x) + F(\mu, v(x)), \quad x \in \mathbb{R}. \tag{2.7}$$

Our next result will show that non-degenerate Hopf bifurcation for (2.7) persist for (2.6) when γ is large enough. Our detailed result reads as follows.

Theorem 2.11. *(**Persistence of Hopf bifurcation**) Let Assumptions 2.1, 2.3 and 2.8 be satisfied. Assume that $F : \mathbb{R} \times \overline{D(A)} \to X$ is of the class C^2 and satisfies*

(i) *$F(\mu, 0) = 0$ for all $\mu \in \mathbb{R}$ and $\partial_u F(0, 0) = 0$.*

(ii) *For each μ in some neighborhood of $\mu = 0$, there exists a pair of conjugated simple eigenvalues of $(A + \partial_u F(\mu, 0))_0$, denoted by $\lambda^{(1)}(\mu)$ and $\overline{\lambda^{(1)}(\mu)}$, such that*

$$\lambda^{(1)}(0) = i\omega_0 \text{ for some } \omega_0 > 0,$$

and

$$\sigma(A_0) \cap i\omega_0\mathbb{Z} = \{\lambda^{(1)}(0), \overline{\lambda^{(1)}(0)}\}. \tag{2.8}$$

We furthermore assume that the map $\mu \to \lambda^{(1)}(\mu)$ is continuously differentiable and satisfies

$$\mathrm{Re}\, \frac{\mathrm{d}\lambda^{(1)}(0)}{\mathrm{d}\mu} > 0 \text{ respectively } < 0.$$

Then, there exists a constant $\delta > 0$ and a map $\gamma = \gamma(\mu)$ defined from $(0, \delta)$ (resp. on $(-\delta, 0)$) into $(0, \infty)$, such that each $\mu_0 \in (0, \delta)$ (resp. for each $\mu_0 \in (-\delta, 0)$) is a Hopf bifurcation point for system (2.6) whenever $\gamma = \gamma(\mu_0)$.

2.2 Application to the existence of periodic wave trains for Gurtin-MacCamy model

We now apply Theorem 2.12 to investigate the existence of periodic wave trains with large wave speed γ for system (1.1). Here recall that a wave train profile with speed γ is an entire solution of (1.1) of the form $u(t,a,z) = U(x,a)$ with $x = z \cdot e + \gamma t$ for $e \in \mathbb{S}^{N-1}$ and where the function U is periodic with respect to the variable $x \in \mathbb{R}$. Namely, there exists a period $T > 0$ such that for all $x \in \mathbb{R}$ and $a > 0$ one has $U(x+T,a) = U(x,a)$. This leads us to the following problem of finding an $x-$periodic profile $U \equiv U(x,a)$ and a speed $\gamma \in \mathbb{R}$ solving the problem

$$\begin{cases} \partial_x^2 U(x,a) - \gamma \partial_x U(x,a) - \partial_a U(x,a) - \zeta U(x,a) = 0, & x \in \mathbb{R}, a > 0, \\ U(x,0) = f\left(\nu, \int_0^\infty \beta(a) U(x,a) \mathrm{d}a\right), \end{cases} \tag{2.9}$$

where $\zeta > 0$ is a given and fixed parameter while $\nu \in \mathbb{R}$ is a parameter that will be used as a bifurcation parameter. We also provide the following assumptions.

Assumption 2.12. *We assume that*

(i) *There exists an open interval I such that for each $\nu \in I$ there exists a constant solution $\overline{w}_\nu \in \mathbb{R}$ of the equation*

$$\overline{w}_\nu = f(\nu, \overline{w}_\nu).$$

In the sequel we set

$$\overline{u}_\nu = \begin{pmatrix} 0 \\ \overline{v}_\nu \end{pmatrix} \quad \textit{with} \quad \overline{v}_\nu(a) = \overline{w}_\nu e^{-\zeta a}, \quad \forall \nu \in I.$$

(ii) *Assume that $f \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\chi := \beta(\cdot) e^{-\zeta \cdot} \in L^1(0,\infty)$ with*

$$\int_0^\infty \chi(a) \mathrm{d}a = 1.$$

Under the above assumptions, let us re-write (2.9) as a special case of (1.3). To do so, consider the Banach spaces

$$X = \mathbb{R} \times L^1(0,\infty) \text{ and } X_0 = \{0\} \times L^1(0,\infty),$$

as well as the non-densely defined operator $A : D(A) \subset X \to X$ defined by

$$D(A) = \{0\} \times W^{1,1}(0,\infty) \text{ and } A \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - \zeta \varphi \end{pmatrix}.$$

Observe that $X_0 = \overline{D(A)} \neq X$. Next consider the nonlinear map $G : I \times X_0 \to X$ defined by

$$G\left(\nu, \begin{pmatrix} 0_{\mathbb{R}} \\ \varphi \end{pmatrix}\right) = \begin{pmatrix} f\left(\nu, \int_0^\infty \beta(a) \varphi(a) \mathrm{d}a\right) \\ 0_{L^1} \end{pmatrix}.$$

Next setting $u(x) = \begin{pmatrix} 0 \\ U(x,\cdot) \end{pmatrix}$, system (2.9) re-writes as

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - \gamma \frac{\mathrm{d}u(x)}{\mathrm{d}x} + Au(x) + G(\nu, u(x)) = 0, \quad x \in \mathbb{R}. \tag{2.10}$$

Note that the linear operator $(A, D(A))$ is a Hille-Yosida operator so that Assumptions 2.1 is readily satisfied. Also note that $\omega_{0,ess}(A_0) \leq -\zeta < 0$, so that Assumption 2.8 holds true. We refer to Magal and Ruan [17] for more details. Moreover, A satisfies Assumption 2.3, which can be found in [16,18].

Now we observe that according to Assumption 2.12 the stationary equation

$$A\overline{u} + G(\nu, \overline{u}) = 0, \quad \overline{u} \in D(A),$$

has a solution that is defined by

$$\overline{u}_\nu = \begin{pmatrix} 0 \\ \overline{v}_\nu \end{pmatrix} \quad \text{with} \quad \overline{v}_\nu(a) = \overline{w}_\nu e^{-\zeta a}, \quad \forall \nu \in I.$$

In order to apply Theorem 2.11 we first recall some known results for the first order differential equation

$$\frac{\mathrm{d}u(x)}{\mathrm{d}x} = Au(x) + G(\nu, u(x)). \tag{2.11}$$

Consider for each $\nu \in I$, the linear operator $B_\nu : D(B_\nu) \subset X \to X$ defined by

$$D(B_\nu) = D(A), \quad B_\nu = A + \partial_u G(\nu, \overline{u}_\nu), \tag{2.12}$$

where ∂_u corresponds to the partial derivative of $G(\nu, u)$ with respect to u. The bounded linear operator $\partial_u G(\nu, \overline{u}_\nu) \in \mathcal{L}(X_0, X)$ is defined by

$$\partial_u G(\nu, \overline{u}_\nu) \begin{pmatrix} 0_{\mathbb{R}} \\ \varphi \end{pmatrix} = \begin{pmatrix} \partial_w f\,(\nu, \overline{w}_\nu) \int_0^\infty \beta(l)\varphi(l)\mathrm{d}l \\ 0_{L^1} \end{pmatrix}, \ \forall \begin{pmatrix} 0_{\mathbb{R}} \\ \varphi \end{pmatrix} \in X_0.$$

Here ∂_w denotes the partial derivative of $f = f(\nu, w)$ with respect to w.

By using the result of Ducrot, Liu and Magal [8], the essential growth rate of the semigroup generated by $(B_\nu)_0$, the part of B_ν in the closure of its domain, satisfies

$$\omega_{0,ess}((B_\nu)_0) \leq -\zeta < 0.$$

The following result follows from [18, Theorem 4.3.27, Lemma 4.4.2, Theorem 4.4.3-(ii)] to which we refer the reader for a proof and more details.

Lemma 2.13. *The spectrum of B_ν in the half plane*

$$\Omega := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -\zeta\}$$

contains only isolated eigenvalues which are poles of the resolvent of B_ν.

Recall that the characteristic function, describing the spectrum of B_ν in Ω, is obtained by computing the resolvent of B_ν as presented in Liu, Magal and Ruan [14]. We define the **characteristic function** for $\nu \in I$ and $\lambda \in \Omega$ as follows

$$\Delta(\nu, \lambda) := 1 - \partial_w f\,(\nu, \overline{w}_\nu) \int_0^\infty \beta(l) e^{-(\zeta+\lambda)l}\mathrm{d}l. \tag{2.13}$$

We now recall a result presented in a more general framework in the Section 5.2 in [14]

Lemma 2.14. *For each $\nu \in I$ the resolvent set $\rho(B_\nu)$ of B_ν satisfies*

$$\lambda \in \rho(B_\nu) \cap \Omega \Leftrightarrow \Delta(\nu, \lambda) \neq 0,$$

or equivalently the spectrum $\sigma(B_\nu) := \mathbb{C} \setminus \rho(B_\nu)$ of B_ν satisfies

$$\lambda \in \sigma(B_\nu) \cap \Omega \Leftrightarrow \Delta(\nu, \lambda) = 0.$$

In addition, we provide the following assumptions on the characteristic equation $\Delta(\nu, \lambda) = 0$ to obtain the existence of Hopf bifurcation of the first order equation (2.11).

Assumption 2.15. *There exists $\nu_0 \in I$ and $\omega_0 > 0$ such that the following properties are satisfied.*

(i)
$$\Delta(\nu_0, \omega_0 i) = 0.$$

(ii) *Crandall and Rabinowitz's condition*
$$\Delta(\nu_0, k\,\omega_0 i) \neq 0, \forall k \in \mathbb{Z} \text{ and } k \neq 1.$$

(iii) *Simplicity of the eigenvalue $\omega_0 i$*
$$\partial_\lambda \Delta(\nu_0, \omega_0 i) \neq 0,$$
which is also equivalent to
$$\partial_w f\,(\nu_0, \overline{w}_{\nu_0}) \int_0^\infty \beta(l)\, l\, e^{-(\zeta+\omega_0 i)l}\mathrm{d}l \neq 0.$$

(iv) *Transversality condition*

$$\text{Re}\left(\partial_\nu \Delta(\nu_0, \omega_0 i) \times \overline{\partial_\lambda \Delta(\nu_0, \omega_0 i)}\right) \neq 0.$$

Recall that Assumption 2.15 combined with Assumption 2.12 implies that $\nu = \nu_0$ is a Hopf bifurcation point for the first order equation (2.11), see Liu, Magal and Ruan [14]. Here we mention that recently we [6] provided a short proof for Hopf bifurcation of the first order equation (2.11) with slightly smoother assumption on β (roughly of bounded variation), which is also motivated by Crandall and Rabinowitz's approach [2].

Now let us re-write system (2.10) for $\nu \in I$ by setting $u = v + \overline{u}_\nu$. This leads to the following parametrized problem

$$\frac{\mathrm{d}^2 v(x)}{\mathrm{d}x^2} - \gamma \frac{\mathrm{d}v(x)}{\mathrm{d}x} + B_\nu v(x) + F(\nu, v(x)) = 0, \tag{2.14}$$

where function $F : I \times X_0 \to X$ is defined by

$$F(\nu, v) = G(\nu, v + \overline{u}_\nu) - G(\nu, \overline{u}_\nu) - \partial_u G(\nu, \overline{u}_\nu)v.$$

Then according to Assumptions 2.12 and 2.15, Theorem 2.11 applies and leads to the following result.

Theorem 2.16. *(**Existence of periodic wave train solutions**) Let Assumptions 2.12 and 2.15 be satisfied. Then there exist $\nu^* \in I, \gamma^* > 0$ large enough and $\eta^* > 0$ such that for each $\gamma \in (\gamma^*, \infty)$, there exists $\nu_\gamma \in (\nu^* - \eta^*, \nu^* + \eta^*)$ such that system (2.9) with $\gamma \in (\gamma^*, \infty)$ undergoes a Hopf bifurcation at $\nu = \nu_\gamma$ around the $x-$independent solution $\overline{u}_{\nu_\gamma}$, and the bifurcation solution is a periodic wave train of (1.1) with speed γ.*

Finally in order to apply Theorem 2.9 and give more general speed γ, we will take an explicit example to provide some estimates on the lower bound of spreading speed γ, which is motivated by Ma and Magal [15], Magal and Ruan [17].

Assumption 2.17. *We assume that $f(u) = ue^{-u}, u \geq 0$ and*

$$\beta(a) = \begin{cases} C_0(a-\tau)^n e^{-\kappa(a-\tau)}, & \text{if } a \geq \tau, \\ 0, & \text{if } a < \tau, \end{cases}$$

with $\tau > 0, \kappa > 0, n \in \mathbb{N}$ or $\tau > 0, \kappa = 0, n = 0$, where C_0 is dependent on the parameters τ, κ and n to ensure that $\int_0^\infty \beta(a) e^{-\zeta a} da = 1$.

We shall consider the following problem depending on some parameter α

$$\begin{cases} \partial_x^2 U(x,a) - \gamma \partial_x U(x,a) - \partial_a U(x,a) - \zeta U(x,a) = 0, & x \in \mathbb{R},\ a > 0, \\ U(x,0) = \alpha f\left(\int_0^\infty \beta(a) U(x,a) \mathrm{d}a\right). \end{cases} \tag{2.15}$$

Here $\alpha > e^2$ is a bifurcation parameter while the nonlinear function f and birth rate function β satisfy Assumption 2.17. With the above assumptions, we have the following Hopf bifurcation result.

Theorem 2.18. *Let Assumption 2.17 be satisfied. If $\gamma^2 > (n+1)/\tau - 2(\zeta + \kappa)$, then for each $k \geq 0$, the number α_k defined in the following*

$$\alpha_k = \exp\left(1 + e^{\omega_k^2 \tau} \left(\left(1 + \frac{\omega_k^2}{\zeta + \kappa}\right)^2 + \frac{\gamma^2 \omega_k^2}{(\zeta+\kappa)^2}\right)^{\frac{n+1}{2}}\right)$$

is a Hopf bifurcation point for the system (2.15) parameterized by α around the positive equilibrium $\overline{U} = \ln \alpha e^{-\zeta a}$, where ω_k is the solution of

$$\gamma \omega \tau + (n+1) \arctan \frac{\gamma \omega}{\zeta + \kappa + \omega^2} = \pi + 2k\pi,\ k \in \mathbb{N}.$$

3 Solvability of a linear second order problem

Throughout this section we consider the solvability of the following second order linear problem of finding $u \in C_{2\pi}(\mathbb{R}, X_0)$, solution of

$$\left(\omega^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \omega\gamma \frac{\mathrm{d}}{\mathrm{d}x}\right) u(x) + Au(x) = f(x),\ x \in \mathbb{R}, \tag{3.1}$$

with $\omega > 0$ and $f \in C_{2\pi}(\mathbb{R}, X)$. Here the solvability of the above linear problem is understood in the weak sense of Definition 2.6. To that aim we extend the methodology developed by Ducrot and Magal in [7] based on the commutative sum of linear operators.

The main result of the section reads as follows.

Theorem 3.1 (Solvability)**.** *Let Assumption 2.1, 2.3 and 2.8 be satisfied. Then for each $\omega > 0$ there exists $K(\omega) \in \mathcal{L}(C_{2\pi}(\mathbb{R}, X_s), C_{2\pi}(\mathbb{R}, X_{0s}))$ such that for all $f \in C_{2\pi}(\mathbb{R}, X)$ and any $\omega > 0$ one has: $u \in C_{2\pi}(\mathbb{R}, X_0)$ is a weak solution of (3.1) if and only if the functions $u_{cu} := \Pi_{cu} u \in C^2_{2\pi}(\mathbb{R}, X_{cu})$ and $u_s := \Pi_s u \in C_{2\pi}(\mathbb{R}, X_s)$ satisfy the following system*

$$\begin{aligned} &u_s = K(\omega)\Pi_s f, \\ &\omega^2 u''_{cu}(x) - \omega\gamma u'_{cu}(x) + A_{cu} u_{cu}(x) = \Pi_{cu} f(x),\ x \in \mathbb{R}. \end{aligned} \tag{3.2}$$

Moreover the map $K : (0, \infty) \times C_{2\pi}(\mathbb{R}, X_s) \to C_{2\pi}(\mathbb{R}, X_{0s})$ given by

$$K(\omega, f) = K(\omega)f,$$

is continuously differentiable with respect to $\omega > 0$.

Remark 3.2. *In Theorem 3.7 we will see that the operator*

$$u(x) \to \left(\omega^2 \partial_x^2 - \gamma\omega\partial_x + A_s\right) u(x)$$

defined on $C^2_{2\pi}(\mathbb{R}, X_s) \cap C_{2\pi}(\mathbb{R}, D(A_s))$ is closable. The closure is invertible, and this inverse is the linear operator $K(\omega)$. The linear operator $K(\omega)$ is nonlocal in x. That is,

$$K(\omega)f(x) = K(\omega)\left[f(\cdot)\right](x)$$

depends on all the values of $x \to f(x)$ (with $x \in \mathbb{R}$).

Remark 3.3. *Recall that A_{cu}, arising in the second equation of (3.2), is a bounded linear operator in the finite dimensional space X_{cu}, so that the second equation in (3.2) is a second order ODE (posed in the finite dimensional space X_{cu}).*

The rest of this section is devoted to the proof of the above theorem. It is split into several parts.

3.1 Strongly continuous semigroups and integrated semigroups on space of $2\pi-$periodic continuous functions

Let $(X, \|\cdot\|)$ be a Banach space. Let $A : D(A) \subset X \to X$ be a linear operator satisfying Assumptions 2.1 and 2.3. Consider the linear operator $\mathcal{A} : D(\mathcal{A}) \subset C_{2\pi}(\mathbb{R}, X) \to C_{2\pi}(\mathbb{R}, X)$ defined by

$$\begin{cases} D(\mathcal{A}) = C_{2\pi}(\mathbb{R}, D(A)), \\ \mathcal{A}(\varphi)(x) = A\varphi(x),\ \forall x \in \mathbb{R}. \end{cases} \tag{3.3}$$

In the above notation $C_{2\pi}(\mathbb{R}, D(A))$, $D(A)$ is endowed with the graph norm.

We now make some precise properties of the linear operator $\mathcal{A}$. First observe that one has

$$\overline{D(\mathcal{A})} = C_{2\pi}(\mathbb{R}, \overline{D(A)}).$$

Next, one has $\rho(\mathcal{A}) = \rho(A)$, and for each $\varphi \in C_{2\pi}(\mathbb{R}, X)$

$$(\lambda I - \mathcal{A})^{-1}(\varphi)(x) = (\lambda I - A)^{-1}\varphi(x),\ \forall \lambda \in \rho(A).$$

It follows that $\mathcal{A}$ satisfies Assumptions 2.1 (i) and (iii). Moreover, for each given $\varphi \in C_{2\pi}(\mathbb{R}, X)$ the subset $\{\varphi(x)\}_{x\in\mathbb{R}}$ is compact. Hence we obtain that

$$\lim_{\lambda\to\infty} \left\|(\lambda - \mathcal{A})^{-1}\varphi\right\|_{C_{2\pi}(\mathbb{R},X)} = 0, \ \forall \varphi \in C_{2\pi}(\mathbb{R}, X),$$

and the operator $\mathcal{A}$ also satisfies Assumption 2.1 (ii) in $C_{2\pi}(\mathbb{R}, X)$.

Furthermore the part of $\mathcal{A}$ in $\overline{D(\mathcal{A})}$, denoted by $\mathcal{A}_0$, is the infinitesimal generator of a C_0-semigroup of bounded linear operators $\{T_{\mathcal{A}_0}(t)\}_{t\geq 0}$ which is defined for each $t \geq 0$, $\varphi \in C_{2\pi}(\mathbb{R}, X)$ by

$$T_{\mathcal{A}_0}(t)(\varphi)(x) = T_{A_0}(t)\varphi(x), \ \forall x \in \mathbb{R}.$$

Here A_0 denotes the part of A in $\overline{D(A)}$. This property implies the growth rate equality

$$\omega_0(\mathcal{A}_0) = \omega_0(A_0).$$

In addition, the linear operator $\mathcal{A}$ is the generator of the integrated semigroup $\{S_{\mathcal{A}}(t)\}_{t\geq 0}$ on $C_{2\pi}(\mathbb{R}, X)$ given for any $t \geq 0$ and each $\varphi \in C_{2\pi}(\mathbb{R}, X)$ by

$$S_{\mathcal{A}}(t)(\varphi)(x) = S_A(t)\varphi(x), \ \forall x \in \mathbb{R}.$$

It follows by Assumption 2.3 that there exists $p \geq 1$ such that for all $f \in L^p_{\text{loc}}([0, \infty); C_{2\pi}(\mathbb{R}, X))$ one has $(S_{\mathcal{A}} \diamond f)(t)(x) = (S_A \diamond f(\cdot, x))(t)$, and

$$\|(S_{\mathcal{A}} \diamond f)(t)\|_{C_{2\pi}(\mathbb{R},X)} \leq M_A \left(\int_0^t e^{p\omega_A(t-s)} \|f(s)\|^p_{C_{2\pi}(\mathbb{R},X)} \mathrm{d}s\right)^{\frac{1}{p}}, \ \forall t \geq 0,$$

where M_A and ω_A are the constants arising in Assumption 2.1 (see Ducrot and Magal [7, Proposition 5.3]).

To sum-up, the linear operator $\mathcal{A} : D(\mathcal{A}) \subset C_{2\pi}(\mathbb{R}, X) \to C_{2\pi}(\mathbb{R}, X)$ satisfies Assumptions 2.1 and 2.3.

3.2 Second order differential operators

Let $(Y, \|\cdot\|)$ be a Banach space. Consider for $\omega > 0$ the linear operator $\mathcal{D}(\omega) : D(\mathcal{D}(\omega)) \subset C_{2\pi}(\mathbb{R}, Y) \to C_{2\pi}(\mathbb{R}, Y)$ defined by

$$\begin{cases} D(\mathcal{D}(\omega)) = C^2_{2\pi}(\mathbb{R}, Y), \\ \mathcal{D}(\omega) = \omega^2\partial^2_x - \gamma\omega\partial_x. \end{cases}$$

First, for each $\omega > 0$, $\mathcal{D}(\omega)$ is the infinitesimal generator of a C_0-semigroup on $C_{2\pi}(\mathbb{R}, Y)$ which is given for $t > 0$ and $\varphi \in C_{2\pi}(\mathbb{R}, Y)$ by

$$T_{\mathcal{D}(\omega)}(t)(\varphi)(x) = \frac{1}{\sqrt{4\pi\omega^2 t}} \int_{\mathbb{R}} e^{-\frac{y^2}{4\omega^2 t}} \varphi(x - \gamma\omega t - y)\mathrm{d}y = \frac{1}{\sqrt{4\pi\omega^2 t}} \int_{\mathbb{R}} e^{-\frac{(x-\gamma\omega t-y)^2}{4\omega^2 t}} \varphi(y)\mathrm{d}y.$$

This explicit formula allows us to obtain the following growth rate estimate.

Lemma 3.4. *The linear operator $\mathcal{D}(\omega)$ for $\omega > 0$ is the infinitesimal generator of strongly continuous semigroup $\{T_{\mathcal{D}(\omega)}(t)\}_{t\geq 0}$ of bounded linear operators on $C_{2\pi}(\mathbb{R}, Y)$. Moreover the following estimate holds true*

$$\left\|T_{\mathcal{D}(\omega)}(t)\right\|_{\mathcal{L}(C_{2\pi}(\mathbb{R},Y))} \leq 1, \ \forall t \geq 0, \omega > 0,$$

and therefore, $\omega_0(\mathcal{D}(\omega)) \leq 0$ for $\omega > 0$.

We continue this section by deriving some regularity estimates that will be used in the following.

Lemma 3.5. *For any $\omega > 0$, $\varphi \in C_{2\pi}(\mathbb{R}, Y)$ and $t > 0$ the map $x \to T_{\mathcal{D}(\omega)}(t)\varphi(x)$ belongs to $C^\infty(\mathbb{R}, Y)$. Moreover there exist positive constants $C > 0$ and $C'' > 0$ such that for all $n \geq 0$, $t > 0$ and $\omega > 0$ one has:*

$$\left\|\partial^n_x T_{\mathcal{D}(\omega)}(t)\varphi\right\|_{C_{2\pi}(\mathbb{R},Y)} \leq C'' a_n \left(\frac{Ce}{\omega^2 t}\right)^{\frac{n}{2}} \|\varphi\|_{C_{2\pi}(\mathbb{R},Y)}, \ \forall \varphi \in C_{2\pi}(\mathbb{R}, Y),$$

where we have set

$$a_n = \begin{cases} j! \text{ if } n = 2j, \\ \sqrt{j!(j-1)!} \text{ if } n = 2j - 1. \end{cases}$$

Proof. Let us note that one has for all $t > 0$ and $\omega > 0$

$$T_{\mathcal{D}(\omega)}(t) = T_{\omega^2\partial_x^2}(t)T_{-\omega\gamma\partial_x}(t) = T_{\partial_x^2}(\omega^2 t)T_{-\gamma\partial_x}(\omega t) = T_{-\gamma\partial_x}(\omega t)T_{\partial_x^2}(\omega^2 t).$$

Observe also that if $f \in C^n_{2\pi}(\mathbb{R}, Y)$ for some $n \geq 1$, then for all $0 \leq k \leq n$ one has

$$\partial_x^k T_{-\omega\gamma\partial_x}(t)f = T_{-\omega\gamma\partial_x}(t)\partial_x^k f.$$

Now when $f \in C_{2\pi}(\mathbb{R}, Y)$, then $T_{\partial_x^2}(\omega^2 t)f \in C^n_{2\pi}(\mathbb{R}, Y)$ for any $n \geq 0$, so that for any $f \in C_{2\pi}(\mathbb{R}, Y)$ and $t > 0$ one has

$$\partial_x^n T_{\mathcal{D}(\omega)}(t)f = T_{-\gamma\partial_x}(\omega t)\partial_x^n T_{\partial_x^2}(\omega^2 t)f.$$

Now according to the proof of Pazy [19, Theorem 5.2], one knows that there exists a positive constant $C > 0$ such that for all $n \geq 0$ and $t > 0$ one has

$$\frac{1}{n!}\left\|(\partial_x^2)^n T_{\partial_x^2}(t)\right\|_{\mathcal{L}(C_{2\pi}(\mathbb{R},Y))} = \frac{1}{n!}\left\|T^{(n)}_{\partial_x^2}(t)\right\|_{\mathcal{L}(C_{2\pi}(\mathbb{R},Y))} \leq \left(\frac{Ce}{t}\right)^n.$$

As a consequence we get for any $n \in \mathbb{N}$ and $t > 0$:

$$\left\|\partial_x^{2n} T_{\mathcal{D}(\omega)}(t)\right\|_{\mathcal{L}(C_{2\pi}(\mathbb{R},Y))} \leq n!\left(\frac{Ce}{\omega^2 t}\right)^n.$$

On the other hand, by the Gagliardo-Nirenberg interpolation inequality, there exists some constant $C' > 0$ such that for all $n \geq 1$, $\omega > 0$ and $t > 0$ one has

$$\begin{aligned}\left\|\partial_x^{2n-1} T_{\mathcal{D}(\omega)}(t)\right\|_{\mathcal{L}(C_{2\pi}(\mathbb{R},Y))} &\leq C'\left\|\partial_x^{2n} T_{\mathcal{D}(\omega)}(t)\right\|^{\frac{1}{2}}_{\mathcal{L}(C_{2\pi}(\mathbb{R},Y))}\left\|\partial_x^{2(n-1)} T_{\mathcal{D}(\omega)}(t)\right\|^{\frac{1}{2}}_{\mathcal{L}(C_{2\pi}(\mathbb{R},Y))} \\ &\leq C'\sqrt{n!(n-1)!}\left(\tfrac{Ce}{\omega^2 t}\right)^{n-\frac{1}{2}}.\end{aligned}$$

Then it follows that there exists some constant $C'' > 0$ such that for all $n \geq 0$, $\omega > 0$ and $t > 0$ the following estimate holds:

$$\left\|\partial_x^n T_{\mathcal{D}(\omega)}(t)\right\|_{\mathcal{L}(C_{2\pi}(\mathbb{R},Y))} \leq C'' a_n \left(\frac{Ce}{\omega^2 t}\right)^{\frac{n}{2}},$$

which completes the proof of the lemma. $\square$

To investigate further regularity, let us introduce for $\eta \in \mathbb{R}$ and $q \geq 1$ the weighted space $L^q_\eta(0, \infty; Y)$ defined by

$$\varphi \in L^q_\eta(0, \infty; Y) \Leftrightarrow \begin{cases}\varphi \in L^q_{\text{loc}}([0, \infty); Y), \\ x \to e^{\eta x}\varphi(x) \in L^q(0, \infty; Y).\end{cases}$$

This weighted space is endowed with the weighted L^q-norm

$$\|\varphi\|_{L^q_\eta(0,\infty;Y)} = \|e^{\eta\cdot}\varphi(\cdot)\|_{L^q(0,\infty;Y)},\ \forall \varphi \in L^q_\eta(0, \infty; Y).$$

Our next result reads as follows.

Proposition 3.6. *Let $\eta > 0$ be given and $q \geq 1$ be given. The map $\mathcal{Q} : (0, \infty) \times C_{2\pi}(\mathbb{R}, Y) \to L^q_{-\eta}(0, \infty; C_{2\pi}(\mathbb{R}, Y))$ given by*

$$\mathcal{Q}(\omega, f)(t) = T_{\mathcal{D}(\omega)}(t)f = T_{-\gamma\partial_x}(\omega t)T_{\partial_x^2}(\omega^2 t)f,$$

is continuously differentiable with respect to $\omega > 0$. Its derivative with respect to ω is given by

$$\partial_\omega \mathcal{Q}(\omega, f)(t) = (2\omega t\partial_x^2 - t\gamma\partial_x)T_{\mathcal{D}(\omega)}(t)f,\ \forall \omega > 0,\ f \in C_{2\pi}(\mathbb{R}, Y),$$

and we have the following key estimate

$$\|\partial_\omega \mathcal{Q}(\omega, f)(t)\|_{C_{2\pi}(\mathbb{R},Y)} \leq M\left[\omega^{-1} + \omega^{-1}t^{1/2}\right]\|f\|_{C_{2\pi}(\mathbb{R},Y)},$$

for some constant $M > 0$ independent of ω, t and f.

Proof. Note that $\mathcal{Q}$ is continuous and a linear bounded operator with respect to $f \in C_{2\pi}(\mathbb{R}, Y)$. To prove the proposition, it remains to investigate some properties of the derivatives with respect to ω. Fix $f \in C_{2\pi}(\mathbb{R}, Y)$ and set

$$g(\omega, t, \cdot) = T_{\omega^2\partial_x^2 - \omega\gamma\partial_x}(t) f(\cdot) = T_{-\gamma\partial_x}(\omega t) T_{\partial_x^2}(\omega^2 t) f(\cdot).$$

Recall that for any $f \in C^1_{2\pi}(\mathbb{R}; Y)$ and $t > 0$ one has

$$h^{-1}\left[T_{-\gamma\partial_x}(t+h)f - T_{-\gamma\partial_x}(t)f\right] \to -\gamma\partial_x f(\cdot - \gamma t) = -\gamma\partial_x T_{-\gamma\partial_x}(t)f \text{ as } h \to 0 \text{ in } C_{2\pi}(\mathbb{R}, Y).$$

Next, since for any $t > 0$ and $f \in C_{2\pi}(\mathbb{R}, Y)$ one has $T_{\partial_x^2}(t)f \in C^\infty_{2\pi}(\mathbb{R}, Y)$, it follows that one has for any $\omega > 0$, $t > 0$

$$h^{-1}\left[g(\omega + h, t, \cdot) - g(\omega, t, \cdot)\right] \to \left(2\omega t\partial_x^2 - t\gamma\partial_x\right) T_{\omega^2\partial_x^2 - \omega\gamma\partial_x}(t) f \text{ as } h \to 0 \text{ in } C_{2\pi}(\mathbb{R}, Y).$$

Now we make use of Lemma 3.5 and the Lebesgue convergence theorem to show that the limit above holds in $L^q_{-\eta}(0, \infty; C_{2\pi}(\mathbb{R}, Y))$. Indeed, due to Lemma 3.5 there exists some constant $M > 0$ such that for all $t > 0$, $\omega > 0$ and $f \in C_{2\pi}(\mathbb{R}, Y)$ we have

$$\left\|\left(2\omega t\partial_x^2 - t\gamma\partial_x\right) T_{\omega^2\partial_x^2 - \omega\gamma\partial_x}(t) f\right\|_{C_{2\pi}(\mathbb{R}, Y)} \leq M\left[\omega^{-1} + \omega^{-1}t^{1/2}\right]\|f\|_{C_{2\pi}(\mathbb{R}, Y)}. \tag{3.4}$$

As a consequence, from the mean value theorem, we obtain that all h small enough, there exists some constant $\tilde{M} = \tilde{M}(\omega) > 0$ such that

$$\left\|h^{-1}\left[g(\omega + h, t, \cdot) - g(\omega, t, \cdot)\right]\right\|_{C_{2\pi}(\mathbb{R}, Y)} \leq \tilde{M}\left[1 + t^{1/2}\right]\|f\|_{C_{2\pi}(\mathbb{R}, Y)},$$

and $t \to 1 + t^{1/2} \in L^q_{-\eta}(0, \infty; \mathbb{R})$ since $q \geq 1$ and $\eta > 0$. To conclude, the Lebesgue convergence theorem ensures that for any $\omega > 0$ and $f \in C_{2\pi}(\mathbb{R}, Y)$ one has

$$h^{-1}\left[\mathcal{Q}(\omega + h, f) - \mathcal{Q}(\omega, f)\right] \to \partial_\omega\mathcal{Q}(\omega, f) := \left(2\omega t\partial_x^2 - t\gamma\partial_x\right) T_{\omega^2\partial_x^2 - \omega\gamma\partial_x}(t) f \text{ as } h \to 0 \text{ in } L^q_{-\eta}(0, \infty; C_{2\pi}(\mathbb{R}, Y)).$$

It remains to check that the map $\partial_\omega\mathcal{Q}(\omega, f)$ is continuous from $(0, \infty) \times C_{2\pi}(\mathbb{R}, Y)$ into $L^q_{-\eta}(0, \infty; C_{2\pi}(\mathbb{R}, Y))$. To do so, fix $\omega_1 > 0$ and $f_1 \in C_{2\pi}(\mathbb{R}, Y)$. Then for any $\omega > 0$ and $f \in C_{2\pi}(\mathbb{R}, Y)$ we have

$$\begin{aligned}
&\left(2\omega_1 t\partial_x^2 - t\gamma\partial_x\right) T_{\omega_1^2\partial_x^2 - \omega_1\gamma\partial_x}(t) f_1 - \left(2\omega t\partial_x^2 - t\gamma\partial_x\right) T_{\omega^2\partial_x^2 - \omega\gamma\partial_x}(t) f \\
={}& \left(2\omega_1 t\partial_x^2 - t\gamma\partial_x\right) T_{\omega_1^2\partial_x^2 - \omega_1\gamma\partial_x}(t) f_1 - \left(2\omega t\partial_x^2 - t\gamma\partial_x\right) T_{\omega^2\partial_x^2 - \omega\gamma\partial_x}(t) f_1 \\
&+ \left(2\omega t\partial_x^2 - t\gamma\partial_x\right) T_{\omega^2\partial_x^2 - \omega\gamma\partial_x}(t)(f_1 - f) \\
={}& \left(2\omega_1 t\partial_x^2\right)\left[T_{\omega_1^2\partial_x^2 - \omega_1\gamma\partial_x}(t) - T_{\omega^2\partial_x^2 - \omega\gamma\partial_x}(t)\right] f_1 + 2(\omega_1 - \omega)t\partial_x^2 T_{\omega^2\partial_x^2 - \omega\gamma\partial_x}(t) f_1 \\
&- t\gamma\partial_x\left[T_{\omega_1^2\partial_x^2 - \omega_1\gamma\partial_x}(t) - T_{\omega^2\partial_x^2 - \omega\gamma\partial_x}(t)\right] f_1 + \left(2\omega t\partial_x^2 - t\gamma\partial_x\right) T_{\omega^2\partial_x^2 - \omega\gamma\partial_x}(t)(f_1 - f).
\end{aligned}$$

Now observe that for any $k \geq 0$ and any $t > 0$ one has

$$\partial_x^k\left[T_{\omega_1^2\partial_x^2}(t) - T_{\omega^2\partial_x^2}(t)\right] f_1 \to 0 \text{ as } \omega \to \omega_1 \text{ in } C_{2\pi}(\mathbb{R}, Y).$$

Hence one obtains for any $k \geq 0$ and all $t > 0$

$$\partial_x^k\left[T_{\omega_1^2\partial_x^2 - \omega_1\gamma\partial_x}(t) - T_{\omega^2\partial_x^2 - \omega\gamma\partial_x}(t)\right] f_1 \to 0 \text{ as } \omega \to \omega_1 \text{ in } C_{2\pi}(\mathbb{R}, Y).$$

Next one has for some constant $M > 0$ (independent of ω, t and f_1, see Lemma 3.5)

$$\left\|2(\omega_1 - \omega)t\partial_x^2 T_{\omega^2\partial_x^2 - \omega\gamma\partial_x}(t) f_1\right\|_{C_{2\pi}(\mathbb{R}, Y)} \leq 2|\omega_1 - \omega|\frac{M}{\omega^2}\|f_1\|_{C_{2\pi}(\mathbb{R}, Y)} \to 0 \text{ as } \omega \to \omega_1 \text{ for any } t > 0,$$

while due to Lemma 3.5 we have

$$\left\|\left(2\omega t\partial_x^2 - t\gamma\partial_x\right) T_{\omega^2\partial_x^2 - \omega\gamma\partial_x}(t)(f_1 - f)\right\|_{C_{2\pi}(\mathbb{R}, Y)} \leq M\left[\omega^{-1} + \omega^{-1}t^{1/2}\right]\|f - f_1\|_{C_{2\pi}(\mathbb{R}, Y)}.$$

As a consequence of the above estimates, we have reached that for any $t > 0$

$$\left(2\omega_1 t\partial_x^2 - t\gamma\partial_x\right) T_{\omega_1^2\partial_x^2 - \omega_1\gamma\partial_x}(t) f_1 - \left(2\omega t\partial_x^2 - t\gamma\partial_x\right) T_{\omega^2\partial_x^2 - \omega\gamma\partial_x}(t) f \to 0$$

in $C_{2\pi}(\mathbb{R}, Y)$ as $(\omega, f) \to (\omega_1, f_1) \in (0, \infty) \times C_{2\pi}(\mathbb{R}, Y)$. Finally the convergence for the topology $L^q_{-\eta}(0, \infty; C_{2\pi}(\mathbb{R}, Y))$ follows from the application of the Lebesgue convergence theorem using (3.4), and the result follows. ∎

3.3 Proof of Theorem 3.1

This section is devoted to the proof of Theorem 3.1. Recall that $A : D(A) \subset X \to X$ is a linear operator satisfying Assumptions 2.1, 2.3 and 2.8. We define $\mathcal{A}_s : D(\mathcal{A}) \subset C_{2\pi}(\mathbb{R}, X_s) \to C_{2\pi}(\mathbb{R}, X_s)$ as in (3.3) with the part A_s of A in its stable space X_s. Note again that Assumption 2.3 implies Assumption 2.5. We are now concerned with the linear operator $\mathcal{D}(\omega) + \mathcal{A}_s$ for some parameter $\omega > 0$, and we apply Thieme [21, Theorem 4.7] to have the following result.

Theorem 3.7. *Let Assumptions 2.1 and 2.5 be satisfied and let $\omega > 0$ be given. Then the linear operator $\mathcal{D}(\omega)+\mathcal{A}_s : D(\mathcal{D}(\omega))\cap D(\mathcal{A}_s) \subset C_{2\pi}(\mathbb{R}, X_s) \to C_{2\pi}(\mathbb{R}, X_s)$ for $\omega > 0$ is closable, and its closure, denoted by $\overline{\mathcal{D}(\omega) + \mathcal{A}_s} : D(\overline{\mathcal{D}(\omega) + \mathcal{A}_s}) \subset C_{2\pi}(\mathbb{R}, X_s) \to C_{2\pi}(\mathbb{R}, X_s)$ satisfies Assumptions 2.1 and 2.5. More precisely, it satisfies the following properties:*

(i) *The following inclusion holds true*

$$C^2_{2\pi}(\mathbb{R}, X_s) \cap C_{2\pi}(\mathbb{R}, D(A_s)) \subset \overline{D(\overline{\mathcal{D}(\omega) + \mathcal{A}_s})} = C_{2\pi}(\mathbb{R}, X_{0s}).$$

(ii) *The part of $\overline{\mathcal{D}(\omega) + \mathcal{A}_s}$ in $C_{2\pi}(\mathbb{R}, \overline{D(A_s)})$, denoted by $(\overline{\mathcal{D}(\omega) + \mathcal{A}_s})_0$ is the infinitesimal generator of a C_0-semigroup $\left\{T_{(\overline{\mathcal{D}(\omega)+\mathcal{A}_s})_0}(t)\right\}_{t\geq 0}$ on $C_{2\pi}(\mathbb{R}, \overline{D(A_s)})$, and*

$$T_{(\overline{\mathcal{D}(\omega)+\mathcal{A}_s})_0}(t) = T_{\mathcal{D}(\omega)}(t)T_{\mathcal{A}_{0s}}(t) = T_{\mathcal{A}_{0s}}(t)T_{\mathcal{D}(\omega)}(t),\ \forall t \geq 0,$$

so that the following growth rate estimate holds true

$$\omega_0((\overline{\mathcal{D}(\omega) + \mathcal{A}_s})_0) \leq \omega_0(A_{0s}) < 0.$$

(iii) *The linear operator $\overline{\mathcal{D}(\omega) + \mathcal{A}_s}$ generates an exponential bounded (non degenerate) integrated semigroup $\left\{S_{\overline{\mathcal{D}(\omega)+\mathcal{A}_s}}(t)\right\}_{t\geq 0}$ of bounded linear operators on $C_{2\pi}(\mathbb{R}, X_s)$ given by*

$$S_{\overline{\mathcal{D}(\omega)+\mathcal{A}_s}}(t) = (S_{\mathcal{A}_s} \diamond T_{\mathcal{D}(\omega)}(t - \cdot))(t),\ \forall t \geq 0.$$

(iv) *The equality*

$$-(\overline{\mathcal{D}(\omega) + \mathcal{A}_s})u = v \text{ and } u \in D(\overline{\mathcal{D}(\omega) + \mathcal{A}_s})$$

holds if and only if

$$\left[(\lambda I - \mathcal{D}(\omega))^{-1} + (\varrho I - \mathcal{A}_s)^{-1}\right] u = (\lambda I - \mathcal{D}(\omega))^{-1}(\varrho I - \mathcal{A}_s)^{-1} [v + (\lambda + \varrho)u]$$

for some $\lambda \in \rho(\mathcal{D}(\omega))$ and $\varrho \in \rho(\mathcal{A}_s)$. This last formula is also equivalent to

$$(\varrho I - A_s)^{-1}u(\cdot) \in C^2_{2\pi}(\mathbb{R}, X_{0s}),$$

and for some $\varrho \in \rho(A_s)$ and any $x \in \mathbb{R}$,

$$-\left(\omega^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \gamma\omega\frac{\mathrm{d}}{\mathrm{d}x}\right)(\varrho I - A_s)^{-1}u(x) = -u(x) + (\varrho I - A_s)^{-1}[v(x) + \varrho u(x)].$$

Note that due to Theorem 3.7-(ii), one has $0 \in \rho\left(\overline{\mathcal{D}(\omega) + \mathcal{A}_s}\right)$, and for each $\omega > 0$ we define $K(\omega) \in \mathcal{L}\left(C_{2\pi}(\mathbb{R}, X_s), C_{2\pi}(\mathbb{R}, X_{0s})\right)$ by $K(\omega) = \left(\overline{\mathcal{D}(\omega) + \mathcal{A}_s}\right)^{-1}$. With this notation and recalling that X_{cu} is finite dimensional, the first part of Theorem 3.1 holds.

To complete the proof of the result, it remains to prove the regularity of the map $(\omega, f) \to K(\omega)f$. Recall from Ducrot and Magal [7] that $K(\omega)$ is given by

$$K(\omega)f = \lim_{t\to+\infty}(S_{\mathcal{A}_s} \diamond T_{\mathcal{D}(\omega)}(t - \cdot)f)(t).$$

Next the smoothness with respect to ω follows from Proposition 3.6 by applying some results of Ducrot, Magal and Thorel [9].

Recall that the weighted space $L^q_\eta(0, \infty; Y)$ is endowed with the weighted L^q-norm

$$\|\varphi\|_{L^q_\eta(0,\infty;Y)} = \|e^{\eta\cdot}\varphi(\cdot)\|_{L^q(0,\infty;Y)},\ \forall\varphi \in L^q_\eta(0, \infty; Y).$$

Recall also that $\omega_0(A_{0s}) < 0$ by Assumption 2.8, so if $\delta > 0$ with

$$\omega_0(A_{0s}) + \delta < 0,$$

then given $p \geq 1$ in Assumption 2.3, for each $\eta > 0$, the operator $\mathbb{K}_{\mathcal{A}_s+\delta} : L^p_\eta(0,\infty; C_{2\pi}(\mathbb{R}, X_s)) \to C_{2\pi}(\mathbb{R}, X_{0s})$ defined by

$$\mathbb{K}_{\mathcal{A}_s+\delta}(g) := \lim_{t\to+\infty}(S_{\mathcal{A}_s+\delta} \diamond g(t-\cdot))(t), \ \forall g \in L^p_\eta(0,\infty; C_{2\pi}(\mathbb{R}, X_s))$$

is a bounded linear operator and satisfies the following estimate (see Ducrot, Magal and Thorel [9, Lemma 3.1]): for each $\delta > 0$ and $\widehat{\omega}_A > \omega_0(A_{0s})$ with $\delta + \widehat{\omega}_A < 0$ there exists some constant $\widehat{M}_A > 0$ such that

$$\|\mathbb{K}_{\mathcal{A}_s+\delta}\|_{\mathcal{L}(L^p_\eta(0,\infty;C_{2\pi}(\mathbb{R},X_s)),C_{2\pi}(\mathbb{R},X_{0s}))} \leq \frac{\widehat{M}_A^{1+\frac{1}{p}}}{1-e^{(\widehat{\omega}_A+\delta-\eta)}} := \widetilde{M}_A(\eta,\delta).$$

Now fix $\delta > 0$ with $\omega_0(A_{0s}) + \delta < 0$ and observe that for all $f \in C_{2\pi}(\mathbb{R}, X_s)$ and $\omega > 0$, one has

$$t \to e^{-\delta t} T_{\mathcal{D}(\omega)}(t) f \in L^q_\eta(0,\infty; C_{2\pi}(\mathbb{R}, X_s)),$$

for all $q \geq 1$ and $\eta \in (0,\delta)$. Fix such $\eta \in (0,\delta)$ and recall that $p \geq 1$ is given in Assumption 2.3. Note that for all $(\omega, f) \in (0,\infty) \times C_{2\pi}(\mathbb{R}, X_s)$ one has,

$$K(\omega) f = \mathbb{K}_{\mathcal{A}_s+\delta}\left(e^{-\delta\cdot} T_{\mathcal{D}(\omega)}(\cdot) f\right).$$

As announced above, the smoothness follows from Proposition 3.6 together with the boundedness of the linear operator $\mathbb{K}_{\mathcal{A}_s+\delta}$ from $L^p_\eta(0,\infty; C_{2\pi}(\mathbb{R}, X_s))$ into $C_{2\pi}(\mathbb{R}, X_{0s})$. This completes the proof of Theorem 3.1.

4 Proof of Theorem 2.9

This section is devoted to the proof of Theorem 2.9. Note that Theorem 2.11 can be established directly by combing Theorem 2.9 and the arguments in Ducrot and Magal [7, Theorem 7.7]. Thus we only prove Theorems 2.9.

Up to time rescaling we assume with loss of generality that

$$\omega_0 = 1. \tag{4.1}$$

We now investigate the existence of a $2\pi/\omega$-periodic $x \to u(x)$ solution of the elliptic equation

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - \gamma \frac{\mathrm{d}u(x)}{\mathrm{d}x} + A u(x) + F(\mu, u(x)) = 0, \quad x \in \mathbb{R}, \tag{4.2}$$

with ω close to 1 and (μ, u) close to $(0,0)$. We use the change of variables

$$v(x) = u(x/\omega).$$

Thus v becomes a 2π-periodic solution of the problem

$$\omega^2 \frac{\mathrm{d}^2 v(x)}{\mathrm{d}x^2} - \gamma\omega \frac{\mathrm{d}v(x)}{\mathrm{d}x} + A v(x) + F(\mu, v(x)) = 0, \quad x \in \mathbb{R}. \tag{4.3}$$

Now the existence of nontrivial $2\pi/\omega-$periodic solution of (4.2) becomes equivalent to the one of nontrivial $2\pi-$ periodic solution of (4.3). We shall apply the implicit function theorem to investigate the existence of nontrivial $2\pi-$periodic solution of (4.3) for μ close to 0. Projecting the above equation (4.3) on X_s and X_{cu} formally yields for $x \in \mathbb{R}$

$$\begin{cases} \omega^2 \dfrac{\mathrm{d}^2 v_{cu}(x)}{\mathrm{d}x} - \omega\gamma \dfrac{\mathrm{d}v_{cu}(x)}{\mathrm{d}x} + A_{cu} v_{cu}(x) = -\Pi_{cu} F(\mu, v_{cu}(x) + v_s(x)), \\ \omega^2 \dfrac{\mathrm{d}^2 v_s(x)}{\mathrm{d}x} - \omega\gamma \dfrac{\mathrm{d}v_s(x)}{\mathrm{d}x} + A_s v_s(x) = -\Pi_s F(\mu, v_{cu}(x) + v_s(x)), \end{cases} \tag{4.4}$$

where $v_h(x) = \Pi_h v(x)$ for $h \in \{s, cu\}$. Recall that $\dim X_{cu} < \infty$ so that the first equation is an ODE in X_{cu}. A rigorous splitting is described in the following lemma whose proof is based on Theorem 3.1.

Lemma 4.1. *Let Assumptions 2.1, 2.3 and 2.8 be satisfied. Consider the second order differential operator for $h \in \{s, cu\}$ defined on $C_{2\pi}(\mathbb{R}, X_h)$ by $\mathcal{D}(\omega) = \omega^2\partial^2 - \gamma\omega\partial$ with $\omega > 0$. Then $v \in C_{2\pi}(\mathbb{R}, \overline{D(A)})$ is a weak solution of (2.5) if and only if the maps*

$$\begin{pmatrix} v'_{cu} \\ v_{cu} \end{pmatrix} \in C^1_{2\pi}(\mathbb{R}, X_{cu} \times X_{cu}) \text{ and } v_s = \Pi_s v \in C_{2\pi}(\mathbb{R}, X_s),$$

satisfy:

(i) *The function v_s is a weak solution of*

$$v_s(x) = -K(\omega)\left[\Pi_s F(\mu, (v_{cu} + v_s)(\cdot))\right](x) \quad \left(\Leftrightarrow -\left(\overline{\mathcal{D}(\omega) + \mathcal{A}_s}\right) v_s = \Pi_s F(\mu, v_{cu} + v_s)\right);$$

(ii) *The function (v'_{cu}, v_{cu}) satisfies, for all $x \in \mathbb{R}$, the problem*

$$\omega^2 \frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} v'_{cu} \\ v_{cu} \end{pmatrix} + \begin{pmatrix} -\gamma\omega I & A_{cu} \\ -\omega^2 I & 0 \end{pmatrix}\begin{pmatrix} v'_{cu} \\ v_{cu} \end{pmatrix} = \begin{pmatrix} -\Pi_{cu}F(\mu, v_{cu}(x) + v_s(x)) \\ 0 \end{pmatrix}.$$

Here and in the sequel, $\mathcal{A}$ denotes the linear operator associated to A as in (3.3), while for each $h \in \{s, cu, 0\}, \mathcal{A}_h$ denotes the linear operator associated with A_h as above.

We define

$$(A + \partial_u F(\mu, 0))_{cu}\, x_{cu} = A_{cu}\, x_{cu} + \Pi_{cu}\partial_u F(\mu, 0)\, x_{cu}, \forall x_{cu} \in X_{cu},$$

which is a linear operator from X_{cu} to itself. The above notation also clarifies the notation used in our previous paper Ducrot and Magal [7, Theorem 2.11]. Moreover one has

$$\sigma\left((A + \partial_u F(\mu, 0))_{cu}\right) = \sigma\left((A + \partial_u F(\mu, 0))_0\right) \cap \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}.$$

According to the above Lemma 4.1, we set

$$\mathcal{B}(\omega, \mu) = \begin{pmatrix} \frac{\gamma}{\omega} I & -\frac{1}{\omega^2}(A + \partial_u F(\mu, 0))_{cu} \\ I & 0 \end{pmatrix} = \begin{pmatrix} \frac{\gamma}{\omega} I & -\frac{1}{\omega^2}(A_{cu} + \Pi_{cu}\partial_u F(\mu, 0)) \\ I & 0 \end{pmatrix} \in \mathcal{L}(X_{cu} \times X_{cu}),$$

and

$$u(x) := \begin{pmatrix} v'_{cu}(x) \\ v_{cu}(x) \end{pmatrix} \in X_{cu} \times X_{cu}.$$

We also set $G : \mathbb{R}^2 \times (X_{cu} \times X_{cu}) \times X_{0s} \to X_{cu} \times X_{cu}$ and $H : \mathbb{R} \times (X_{cu} \times X_{cu}) \times X_{0s} \to X_s$ defined by

$$G\left(\omega, \mu, u, v_s\right) = \begin{pmatrix} -\frac{1}{\omega^2}\Pi_{cu}F(\mu, v_{cu} + v_s) + \Pi_{cu}\partial_u F(\mu, 0)v_{cu} \\ 0 \end{pmatrix}$$

and

$$H\left(\mu, u, v_s\right) = -\Pi_s F(\mu, v_{cu} + v_s).$$

Together with these notations (4.4) becomes equivalent to the following system posed for $x \in \mathbb{R}$

$$\begin{cases} \dfrac{\mathrm{d}u(x)}{\mathrm{d}x} = \mathcal{B}(\omega, \mu)u(x) + G(\omega, \mu, u(x), v_s(x)), \\ v_s(x) = K(\omega)\left[H(\mu, u(\cdot), v_s(\cdot))\right](x). \end{cases} \tag{4.5}$$

Note that the linear operator $\mathcal{B}$ satisfies the following spectrum properties (see Ducrot and Magal [7, Lemma 7.1])

$$\sigma(\mathcal{B}(1, 0)) = \{0\} \cup \{z \in \mathbb{C} : \gamma z - z^2 \in \sigma(A_{cu})\}.$$

Recalling that $\omega_0 = 1$ (see (4.1)), Assumption-(ii) in Theorem 2.9 implies that there exist two continuously differentiable maps $\mu \to \lambda^{(1)}(\mu)$ and $\mu \to \overline{\lambda^{(1)}(\mu)}$ in the spectrum of $(A + \partial_u F(\mu, 0))_0$, such that

$$\lambda^{(1)}(0) = 1 + i\gamma.$$

We further assume that

$$\sigma(A_0) \cap \mathcal{P}_{\mathbb{Z}} = \{\lambda^{(1)}(0), \overline{\lambda^{(1)}(0)}\},$$

where

$$\mathcal{P}_{\mathbb{Z}} = \{\xi^2 + i\gamma\xi : \xi \in \mathbb{Z}\},$$

and that the map $\mu \to \lambda^{(1)}(\mu)$ is continuously differentiable and satisfies

$$\mathrm{Re}\left[\frac{1}{\gamma - 2i}\frac{\mathrm{d}\lambda^{(1)}(0)}{\mathrm{d}\mu}\right] \neq 0.$$

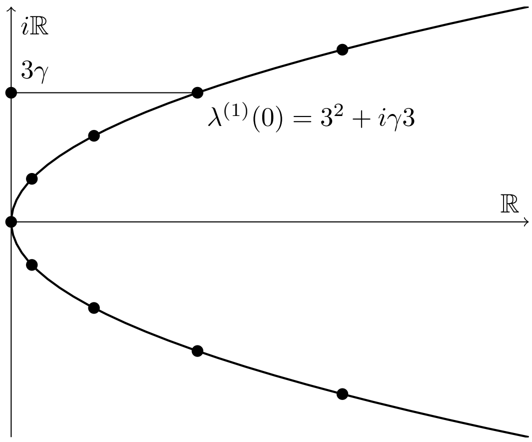

Figure 1: *In this figure we plot both the continuous parabola* $\mathcal{P} = \{\xi^2 + i\gamma\xi : \xi \in \mathbb{R}\}$ *(black solid curve) and discrete parabola* $\mathcal{P}_{\mathbb{Z}} = \{\xi^2 + i\gamma\xi : \xi \in \mathbb{Z}\}$ *(black dots).*

Claim 4.2. *There exists* $\eta > 0$ *small enough and a map* $\lambda^{(2)} : (-\eta, \eta) \to \mathbb{C}$ *of class* C^1 *such that*

$$\lambda^{(2)}(0) = i, \tag{4.6}$$

and

$$\gamma\lambda^{(2)}(\mu) - \left(\lambda^{(2)}(\mu)\right)^2 = \lambda^{(1)}(\mu),\ \forall \mu \in (-\eta, \eta)\,. \tag{4.7}$$

Note that the conjugated equation reads as

$$\gamma\overline{\lambda^{(2)}(\mu)} - \left(\overline{\lambda^{(2)}(\mu)}\right)^2 = \overline{\lambda^{(1)}(\mu)},\ \forall \mu \in (-\eta, \eta)\,.$$

In order to prove the claim we apply the implicit function theorem to the map

$$V(\mu, z) := \gamma z - z^2 - \lambda^{(1)}(\mu),\ z \in \mathbb{C},\ \mu \in \mathbb{R}.$$

Observe that V is of class C^1 with respect to μ and z due to to the C^1 regularity of $\mu \to \lambda^{(1)}(\mu)$. Moreover, we have

$$V(0, i) = 0 \text{ and } \partial_z V(0, i) = \gamma - 2i \neq 0.$$

This proves the claim. Further, up to reduce η if necessary, we have $\lambda^{(2)}(\mu) \neq 0$ for all $\mu \in (-\eta, \eta)$. Fix such an η and denote

$$J := (-\eta, \eta). \tag{4.8}$$

Now, by differentiating both sides of (4.7) we obtain

$$\frac{\mathrm{d}\lambda^{(2)}(0)}{\mathrm{d}\mu}(\gamma - 2i) = \frac{\mathrm{d}\lambda^{(1)}(0)}{\mathrm{d}\mu},$$

so that

$$\mathrm{Re}\frac{\mathrm{d}\lambda^{(2)}(0)}{\mathrm{d}\mu} \neq 0. \tag{4.9}$$

Now set

$$\lambda^{(2)}(\omega, \mu) := \frac{\lambda^{(2)}(\mu)}{\omega},\ \omega > 0 \text{ and } \mu \in J.$$

Then one can verify that $\lambda^{(2)}(\omega,\mu) \in \sigma(\mathcal{B}(\omega,\mu))$. Indeed, if $\lambda^{(2)}(\omega,\mu)$ is an eigenvalue of $\mathcal{B}(\omega,\mu)$ associated with $(r_1,r_2) \in X_{cu} \times X_{cu} \setminus \{(0,0)\}$, then it follows that

$$\begin{cases} \frac{\gamma}{\omega} r_1 - \frac{1}{\omega^2}(A + \partial_u F(\mu,0))_{cu} r_2 = \lambda^{(2)}(\omega,\mu) r_1, \\ r_1 = \lambda^{(2)}(\omega,\mu) r_2, \end{cases}$$

which implies that

$$(A + \partial_u F(\mu,0))_{cu} r_2 = \left[\gamma\left(\lambda^{(2)}(\omega,\mu)\omega\right) - \left(\lambda^{(2)}(\omega,\mu)\omega\right)^2\right] r_2.$$

Since $\lambda^{(2)}(\mu) \neq 0$ for all $\mu \in J$, it follows that $r_2 \neq 0$. Otherwise $(r_1,r_2) = (0,0)$. Hence

$$\gamma\left(\lambda^{(2)}(\omega,\mu)\omega\right) - \left(\lambda^{(2)}(\omega,\mu)\omega\right)^2 \in \sigma\left((A + \partial_u F(\mu,0))_{cu}\right).$$

Together with the above claim (see (4.7)) we can also conclude that $\lambda^{(2)}(\omega,\mu) = \frac{\lambda^{(2)}(\mu)}{\omega}$. Moreover, we have

$$\sigma(\mathcal{B}(1,0)) \cap i\mathbb{Z} = \{\lambda^{(2)}(0), \overline{\lambda^{(2)}(0)}\} = \{\pm i\}. \tag{4.10}$$

Since $m := \dim X_{cu} < \infty$ and thanks to the isomorphism, we now use the notation $\mathbb{R}^{2m}$ to represent $X_{cu} \times X_{cu}$ in the following context. First using a change of basis, for any fixed v_s we rewrite the first equation of (4.5) under the following form

$$\begin{cases} \dfrac{\mathrm{d}u^1(x)}{\mathrm{d}x} = \mathcal{B}^1(\omega,\mu)u^1(x) + G^1(\omega,\mu,u^1(x),u^2(x),v_s(x)) \in \mathbb{R}^2, \\ \dfrac{\mathrm{d}u^2(x)}{\mathrm{d}x} = \mathcal{B}^2(\omega,\mu)u^2(x) + G^2(\omega,\mu,u^1(x),u^2(x),v_s(x)) \in \mathbb{R}^{2m-2}. \end{cases} \tag{4.11}$$

The equations of system (4.11) are obtained by using the projector Π_1 on the eigenspace of $\mathcal{B}(\omega,\mu)$ associated with $\lambda^{(2)}(\omega,\mu)$ and $\overline{\lambda^{(2)}(\omega,\mu)}$ parallel to the direct sum of the generalized eigenspaces associated to the remaining eigenvalues in the spectrum of $\mathcal{B}(\omega,\mu)$. The first equation of (4.11) is obtained by applying Π_1 to the u-equation of (4.5), and the second equation is obtained by applying $I - \Pi_1$ to the u-equation of (4.5).

We now observe that

$$\begin{pmatrix} \mathcal{B}^1(\omega,\mu) & 0 \\ 0 & \mathcal{B}^2(\omega,\mu) \end{pmatrix}$$

has the same eigenvalues as $\mathcal{B}(\omega,\mu)$ because they are similar matrices. Further, according to assumptions in Theorem 2.9, the following properties are satisfied:

(i) For $\omega > 0$ and $\mu \in J$, where $J = (-\eta,\eta)$ defined in (4.8), the functions $(\omega,\mu) \to \mathcal{B}^1(\omega,\mu)$ and $(\omega,\mu) \to \mathcal{B}^2(\omega,\mu)$ are of class C^1 by the assumptions on F;

(ii) The matrix $\mathcal{B}^1$ has the following properties by (4.6) and (4.9) (and recalling that we assumed at the beginning of the section that $\omega_0 = 1$ in (4.1))

$$\mathcal{B}^1(\omega,\mu) = \frac{1}{\omega}\begin{pmatrix} \alpha(\mu) & \chi(\mu) \\ -\chi(\mu) & \alpha(\mu) \end{pmatrix} \tag{4.12}$$

with $\alpha(\mu) := \operatorname{Re}\lambda^{(2)}(\mu)$ and $\chi(\mu) := \operatorname{Im}\lambda^{(2)}(\mu)$. Furthermore, we have $\alpha(0) = 0$ and $\chi(0) = 1$ as well as the tranversality condition $\alpha'(0) \neq 0$;

(iii) Observe that $\mathcal{B}^1(\omega,\mu)$ also satisfies

$$\partial_\omega \mathcal{B}^1(1,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \tag{4.13}$$

(iv) The matrix $I-e^{\mathcal{B}^2(1,0)2\pi}$ is invertible from $\mathbb{R}^{2m-2}$ into itself, which is a consequence of Assumption 2.9-(ii) by (4.10). Indeed, the kernel $N(I-e^{\mathcal{B}^2(1,0)2\pi})=\{0\} \Leftrightarrow N(I-e^{2\pi J_{\lambda_i}})=\{0\}, \forall i=1,\ldots,q \Leftrightarrow e^{2\pi\lambda_i}\neq 1, \forall i=1,\ldots,q$, where $\lambda_i, i=1,\ldots,q$ are the eigenvalues of $\mathcal{B}^2(1,0)$ and J_{λ_i} are the Jordan blocks of $\mathcal{B}^2(1,0)$. That is there exists an invertible matrix P such that

$$P^{-1}\mathcal{B}^2(1,0)P=\begin{pmatrix} J_{\lambda_1} & 0 & \ldots & 0\\ 0 & J_{\lambda_2} & \ldots & 0\\ \vdots & \ddots & \ddots & 0\\ 0 & \ldots & 0 & J_{\lambda_q}\end{pmatrix}.$$

(v) The functions G^1 and G^2 are of class C^2;

(vi) $G^i(1,\mu,0,0,0)=0$ for $i=1,2$ and for all μ in a small neighborhood of 0;

(vii) $\partial_z G^i(1,\mu,0,0,0)=0$ for $i=1,2$ and for all μ in a small neighborhood of 0 with $z\in\{u^1,u^2,v_s\}$.

Then system (4.11) is equivalent to

$$\frac{\mathrm{d}u(x)}{\mathrm{d}x}=C(\omega,\mu)u+P(\omega,\mu,u,v_s),$$

and

$$C(\omega,\mu)=\begin{pmatrix}\mathcal{B}^1(\omega,\mu) & 0\\ 0 & \mathcal{B}^2(\omega,\mu)\end{pmatrix} \text{ and } P(\omega,\mu,u,v_s)=\begin{pmatrix}G^1(\omega,\mu,u^1,u^2,v_s)\\ G^2(\omega,\mu,u^1,u^2,v_s)\end{pmatrix}.$$

Note that when $x\to(u(x),v_s(x))$ is a solution of (4.5), it satisfies

$$\begin{cases} u(x)=e^{C(\omega,\mu)x}u(0)+\int_0^x e^{C(\omega,\mu)(x-s)}P(\omega,\mu,u(s),v_s(s))\mathrm{d}s,\\ v_s(x)=K(\omega)[H(\mu,u(\cdot),v_s(\cdot))](x).\end{cases}$$

In order to investigate the existence of a $2\pi-$periodic solution of the above integral equation coupled with an operator equation, in addition to the space $C_{2\pi}([0,2\pi],\mathbb{R}^{2m})$ of the $2\pi-$periodic continuous functions with values in $\mathbb{R}^{2m}$, we also introduce $C_0([0,2\pi],\mathbb{R}^{2m})$, the Banach space of continuous functions g such that $g(0)=0$ with values in $\mathbb{R}^{2m}$.

Next recall that $J=(-\eta,\eta)$ given in (4.8), and define the map $\mathcal{F}:(0,\infty)\times J\times C_{2\pi}([0,2\pi],\mathbb{R}^{2m})\times C_{2\pi}([0,2\pi],X_{0s})\to C_0([0,2\pi],\mathbb{R}^{2m})\times C_{2\pi}([0,2\pi],X_{0s})$ by

$$\mathcal{F}\left(\omega,\mu,\begin{pmatrix}u\\ v_s\end{pmatrix}\right)(x)=\begin{pmatrix}u(x)\\ v_s(x)\end{pmatrix}-\begin{pmatrix}e^{C(\omega,\mu)x}u(0)\\ 0\end{pmatrix}-\begin{pmatrix}\int_0^x e^{C(\omega,\mu)(x-s)}P(\omega,\mu,u(s),v_s(s))\mathrm{d}s\\ K(\omega)[H(\mu,u(\cdot),v_s(\cdot))](x)\end{pmatrix}.$$

We next aim at investigating the zeros of the equation

$$\mathcal{F}(\omega,\mu,z)=0,\quad \text{with } z=\begin{pmatrix}u\\ v_s\end{pmatrix}$$

for (ω,μ,z) close to $(1,0,0)$ using the implicit function theorem, where the last 0 indicates the zero element in $C_{2\pi}([0,2\pi],\mathbb{R}^{2m})\times C_{2\pi}([0,2\pi],X_{0s})$.

Computation of the derivatives of $\mathcal{F}$: We can calculate the derivatives directly. To do so we use the following notations for $z,\widehat{z}\in C_{2\pi}([0,2\pi],\mathbb{R}^{2m})\times C_{2\pi}([0,2\pi],X_{0s})$:

$$z=\begin{pmatrix}u\\ v_s\end{pmatrix} \text{ and } \widehat{z}=\begin{pmatrix}\widehat{u}\\ \widehat{v}_s\end{pmatrix}.$$

Then we get for any $(\omega,\mu,z)\in(0,\infty)\times J\times C_{2\pi}([0,2\pi],\mathbb{R}^{2m})\times C_{2\pi}([0,2\pi],X_{0s})$ and $\hat{z}\in C_{2\pi}([0,2\pi],\mathbb{R}^{2m})\times C_{2\pi}([0,2\pi],X_{0s})$:

$$\partial_z\mathcal{F}(\omega,\mu,z)(\hat{z})(x)=\begin{pmatrix}\hat{u}(x)\\ \hat{v}_s(x)\end{pmatrix}-\begin{pmatrix}e^{C(\omega,\mu)x}\hat{u}(0)\\ 0\end{pmatrix}-\begin{pmatrix}\int_0^x e^{C(\omega,\mu)(x-s)}\partial_z P(\omega,\mu,u(s),v_s(s))\hat{z}(s)\mathrm{d}s\\ K(\omega)[\partial_z H(\mu,u,v_s)\hat{z}(\cdot)](x)\end{pmatrix},$$

$$\partial_\mu\partial_z\mathcal{F}(\omega,\mu,z)(\hat{z})(x)=\begin{pmatrix}-\partial_\mu C(\omega,\mu)xe^{C(\omega,\mu)x}\hat{u}(0)\\ 0\end{pmatrix}-\begin{pmatrix}\int_0^x \Xi_1(\omega,\mu,u,v_s,x,s)\hat{z}(s)\mathrm{d}s\\ K(\omega)[\partial_\mu\partial_z H(\mu,u,v_s)\hat{z}(\cdot)](x)\end{pmatrix},$$

and by Theorem 3.1

$$\partial_\omega \partial_z \mathcal{F}(\omega,\mu,z)(\hat{z})(x) = \begin{pmatrix} -\partial_\omega C(\omega,\mu) x e^{C(\omega,\mu)x} \hat{u}(0) \\ 0 \end{pmatrix} - \begin{pmatrix} \int_0^x \Xi_2(\omega,\mu,u,v_s,x,s)\hat{z}(s)\mathrm{d}s \\ \partial_\omega K(\omega)[\partial_z H(\mu,u,v_s)\hat{z}(\cdot)](x) \end{pmatrix},$$

where

$$\Xi_1(\omega,\mu,u,v_s,x,s) = \partial_\mu C(\omega,\mu)(x-s)e^{C(\omega,\mu)(x-s)}\partial_z P(\omega,\mu,u(s),v_s(s)) + e^{C(\omega,\mu)(x-s)}\partial_\mu\partial_z P(\omega,\mu,u(s),v_s(s)),$$
$$\Xi_2(\omega,\mu,u,v_s,x,s) = \partial_\omega C(\omega,\mu)(x-s)e^{C(\omega,\mu)(x-s)}\partial_z P(\omega,\mu,u(s),v_s(s)) + e^{C(\omega,\mu)(x-s)}\partial_\omega\partial_z P(\omega,\mu,u(s),v_s(s)).$$

After plugging $(\omega,\mu,z) = (1,0,0)$ into above equalities, we have by (vi) and (vii)

$$\partial_z \mathcal{F}(1,0,0)(z)(x) = \begin{pmatrix} u(x) \\ v_s(x) \end{pmatrix} - \begin{pmatrix} e^{C(1,0)x}u(0) \\ 0 \end{pmatrix}, \tag{4.14}$$

$$\partial_\mu \partial_z \mathcal{F}(1,0,0)(z)(x) = -\begin{pmatrix} \partial_\mu C(1,0) x e^{C(1,0)x} u(0) \\ 0 \end{pmatrix}, \tag{4.15}$$

$$\partial_\omega \partial_z \mathcal{F}(1,0,0)(z)(x) = -\begin{pmatrix} \partial_\omega C(1,0) x e^{C(1,0)x} u(0) \\ 0 \end{pmatrix}. \tag{4.16}$$

State space decomposition: Note that by (4.12), the kernel $N(\partial_z \mathcal{F}(1,0,0))$ can be given as follows,

$$N(\partial_z \mathcal{F}(1,0,0)) = \operatorname{span}\{e_1,e_2\} \times \left\{0_{C_{2\pi}([0,2\pi],X_{0s})}\right\},$$

where the functions e_1 and e_2 in $C_{2\pi}([0,2\pi],\mathbb{R}^{2m})$ are given by

$$e_1 = e_1(x) := \begin{pmatrix} \sin x \\ \cos x \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ and } e_2 = e_2(x) := \begin{pmatrix} \cos x \\ -\sin x \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.17}$$

Now define the closed space $\mathcal{X}_1 := \operatorname{span}\{e_1,e_2\} \subset C_{2\pi}([0,2\pi],\mathbb{R}^{2m})$ and

$$\mathcal{X}_2 := \left\{ z \in C_{2\pi}([0,2\pi],\mathbb{R}^{2m}) : \int_0^{2\pi} z(x)e_i(x)\mathrm{d}x = 0_{\mathbb{R}^{2m}} \text{ for } i = 1,2 \right\},$$

where the multiplication under the above integral is understood componentwise. This space turns out to be a complement of $\mathcal{X}_1$ as stated in the next lemma.

Lemma 4.3. *We have the following state space decomposition*

$$C_{2\pi}([0,2\pi],\mathbb{R}^{2m}) = \mathcal{X}_1 \oplus \mathcal{X}_2.$$

Proof. This property is directly inherited from the decomposition of $L^2\big((0,2\pi),\mathbb{R}^{2m}\big)$ as

$$L^2\big((0,2\pi),\mathbb{R}^{2m}\big) = \mathcal{X}_1 \oplus \mathcal{X}_1^\perp,$$

with

$$\mathcal{X}_1^\perp = \left\{ z \in L^2\big((0,2\pi),\mathbb{R}^{2m}\big) : \int_0^{2\pi} z(x)e_i(x)\mathrm{d}x = 0 \text{ for } i = 1,2 \right\}.$$

Now if $z \in C_{2\pi}([0,2\pi],\mathbb{R}^{2m})$ then $z \in L^2\big((0,2\pi),\mathbb{R}^{2m}\big)$ and the above $L^2\big((0,2\pi),\mathbb{R}^{2m}\big)-$decomposition ensures that there exist unique $z_1 \in \mathcal{X}_1$ and $z_2 \in \mathcal{X}_1^\perp$ such that

$$z = z_1 + z_2.$$

Now since $z_1 = c_1e_1 + c_2e_2$ (for some constants c_1 and c_2) this ensures that $z_2 = z - z_1$ is also continuous and $z_2 \in \mathcal{X}_2$. The state space decomposition follows. □

Now define the map $h : \mathbb{R} \times (0, \infty) \times J \times \mathcal{X}_2 \times C_{2\pi}([0, 2\pi], X_{0s}) \to C_0([0, 2\pi], \mathbb{R}^{2m}) \times C_{2\pi}([0, 2\pi], X_{0s})$ by

$$h(t, \omega, \mu, y) = \begin{cases} t^{-1}\mathcal{F}\left(\omega, \mu, t\begin{pmatrix} e_1 + y_{cu} \\ y_s \end{pmatrix}\right), & \text{if } t \neq 0, \\ \partial_z \mathcal{F}(\omega, \mu, 0)\begin{pmatrix} e_1 + y_{cu} \\ y_s \end{pmatrix}, & \text{if } t = 0, \end{cases} \quad \text{with } y = \begin{pmatrix} y_{cu} \\ y_s \end{pmatrix},$$

where e_1 is defined in (4.17). Observe that since $F = F(\mu, u)$ is of class C^2, h is of class C^1. One has $h(0, 1, 0, 0) = 0$ while the partial derivative with respect to (ω, μ, y) is given, for all $(\tilde{\omega}, \tilde{\mu}, \tilde{y}) \in \mathbb{R}^2 \times \mathcal{X}_2 \times C_{2\pi}([0, 2\pi], X_{0s})$ with $\tilde{y} = (\tilde{y}_{cu}, \tilde{y}_s)^T$, by

$$D_{(\omega,\mu,y)}h(0, 1, 0, 0)(\tilde{\omega}, \tilde{\mu}, \tilde{y}) = \partial_z \mathcal{F}(1, 0, 0)\begin{pmatrix} \tilde{y}_{cu} \\ \tilde{y}_s \end{pmatrix} + \tilde{\mu}\partial_\mu \partial_z \mathcal{F}(1, 0, 0)\begin{pmatrix} e_1 \\ 0 \end{pmatrix} + \tilde{\omega}\partial_\omega \partial_z \mathcal{F}(1, 0, 0)\begin{pmatrix} e_1 \\ 0 \end{pmatrix}.$$

Hence, by using (4.14)-(4.16) we obtain

$$\begin{aligned} &D_{(\omega,\mu,y)}h(0, 1, 0, 0)(\tilde{\omega}, \tilde{\mu}, \tilde{y}) \\ &= \begin{pmatrix} \tilde{y}_{cu}(x) \\ \tilde{y}_s(x) \end{pmatrix} - \begin{pmatrix} e^{C(1,0)x}\tilde{y}_{cu}(0) \\ 0 \end{pmatrix} - \tilde{\mu}\begin{pmatrix} \partial_\mu C(1, 0)xe^{C(1,0)x}e_1(0) \\ 0 \end{pmatrix} - \tilde{\omega}\begin{pmatrix} \partial_\omega C(1, 0)xe^{C(1,0)x}e_1(0) \\ 0 \end{pmatrix}. \end{aligned}$$

The second main result of the proof is the following lemma.

Lemma 4.4. *The bounded linear operator*

$$D_{(\omega,\mu,y)}h(0, 1, 0, 0) \in \mathcal{L}\left(\mathbb{R}^2 \times \mathcal{X}_2 \times C_{2\pi}([0, 2\pi], X_{0s}), C_0([0, 2\pi], \mathbb{R}^{2m}) \times C_{2\pi}([0, 2\pi], X_{0s})\right)$$

is invertible.

Proof. To prove the above lemma, let $w \in C_0([0, 2\pi], \mathbb{R}^{2m}) \times C_{2\pi}([0, 2\pi], X_{0s})$ be given. Set $W := D_{(\omega,\mu,y)}h(0, 1, 0, 0)$ and investigate the equation

$$W(\tilde{\omega}, \tilde{\mu}, \tilde{y}) = w = (w_{cu}, w_s)^T. \tag{4.18}$$

First observe that the stable part $\tilde{y}_s$ can be solved easily as follows,

$$\tilde{y}_s = w_s.$$

Thus in the following we only focus on the unstable and center parts $\tilde{y}_{cu}$. To do so, for $l = (l_1, l_2) \in \mathbb{R}^{2m} = \mathbb{R}^2 \times \mathbb{R}^{2m-2}$ we define projections maps $P_1 : \mathbb{R}^{2m} \to \mathbb{R}^2$ and $P_2 : \mathbb{R}^{2m} \to \mathbb{R}^{2m-2}$ by $P_1 l = l_1$ and $P_2 l = l_2$.

Setting $\tilde{y}_i = P_i \tilde{y}_{cu}$ and $w_i = P_i w_{cu}$ for $i = 1, 2$ and projecting the unstable and center parts of (4.18) with P_1 and P_2, the system becomes for all $x \in [0, 2\pi]$

$$\begin{aligned} w_1(x) &= \tilde{y}_1(x) - e^{\mathcal{B}^1(1,0)x}\tilde{y}_1(0) - \tilde{\mu}\partial_\mu \mathcal{B}^1(1, 0)xe^{\mathcal{B}^1(1,0)x}P_1 e_1(0) - \tilde{\omega}\partial_\omega \mathcal{B}^1(1, 0)xe^{\mathcal{B}^1(1,0)x}P_1 e_1(0), \\ w_2(x) &= \tilde{y}_2(x) - e^{\mathcal{B}^2(1,0)x}\tilde{y}_2(0). \end{aligned} \tag{4.19}$$

Since $\tilde{y}_2$ is $2\pi-$periodic and $I - e^{\mathcal{B}^2(1,0)2\pi}$ is invertible, we obtain

$$\tilde{y}_2(x) = e^{\mathcal{B}^2(1,0)x}\left(I - e^{\mathcal{B}^2(1,0)2\pi}\right)^{-1} w_2(2\pi) + w_2(x),$$

so that $\tilde{y}_2 \in C_{2\pi}([0, 2\pi], \mathbb{R}^{2m-2})$.

We now turn to the solution of the first equation of (4.19). To do so, note from (4.12) and (4.13) that we have

$$x\partial_\mu \mathcal{B}^1(1, 0)e^{\mathcal{B}^1(1,0)x}P_1 e_1(0) = x\alpha'(0)\begin{pmatrix} \sin x \\ \cos x \end{pmatrix} + x\chi'(0)\begin{pmatrix} \cos x \\ -\sin x \end{pmatrix},$$

while

$$x\partial_\omega \mathcal{B}^1(1, 0)e^{\mathcal{B}(1,0)x}P_1 e_1(0) = -x\begin{pmatrix} \cos x \\ -\sin x \end{pmatrix}.$$

As a consequence, the first equation in (4.19) rewrites as finding $\tilde{y}_1 \in P_1\mathcal{X}_2$ and $(\tilde{\mu}, \tilde{\omega}) \in \mathbb{R}^2$ such that

$$\tilde{y}_1(x) - e^{\mathcal{B}^1(1,0)x}\tilde{y}_1(0) + c_1 x \begin{pmatrix} \sin x \\ \cos x \end{pmatrix} + c_2 x \begin{pmatrix} \cos x \\ -\sin x \end{pmatrix} = w_1(x), \tag{4.20}$$

with

$$c_1 = -\alpha'(0)\tilde{\mu}, \quad c_2 = \tilde{\omega} - \chi'(0)\tilde{\mu}.$$

Since $\tilde{y}_1$ is $2\pi-$periodic, taking $t = 2\pi$ in the above equation yields, since $e^{\mathcal{B}^1(1,0)2\pi} = I$,

$$\tilde{y}_1(2\pi) - e^{\mathcal{B}^1(1,0)2\pi}\tilde{y}_1(0) = \tilde{y}_1(0) - e^{\mathcal{B}^1(1,0)2\pi}\tilde{y}_1(0) = 0.$$

Now (4.20) becomes

$$c_1 2\pi \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 2\pi \begin{pmatrix} 1 \\ 0 \end{pmatrix} = w_1(2\pi) := \begin{pmatrix} w_1^1(2\pi) \\ w_1^2(2\pi) \end{pmatrix}.$$

Therefore we obtain $c_1 = w_1^2(2\pi)/2\pi$ and $c_2 = w_1^1(2\pi)/2\pi$. Since $\alpha'(0) \neq 0$, this allows us to recover $\tilde{\mu}$ and $\tilde{\omega}$ as,

$$\tilde{\mu} = -\frac{w_1^2(2\pi)}{2\pi\alpha'(0)} \text{ and } \tilde{\omega} = -\frac{\chi'(0)w_1^2(2\pi)}{2\pi\alpha'(0)} + \frac{w_1^1(2\pi)}{2\pi}.$$

Next we focus on $\tilde{y}_1$. To do so, we set

$$Y_1(x) := w_1(x) - c_1 x \begin{pmatrix} \sin x \\ \cos x \end{pmatrix} - c_2 x \begin{pmatrix} \cos x \\ -\sin x \end{pmatrix},$$

so that $\tilde{y}_1 \in P_1\mathcal{X}_2$ satisfies

$$\tilde{y}_1(x) - e^{\mathcal{B}^1(1,0)x}\tilde{y}_1(0) = Y_1(x), \ \forall x \in [0, 2\pi].$$

Here note that $Y_1 \in C_{2\pi}([0, 2\pi], \mathbb{R}^2) \cap C_0([0, 2\pi], \mathbb{R}^2)$. Consider the linear operator $Q : C_{2\pi}([0, 2\pi], \mathbb{R}^2) \to C_{2\pi}([0, 2\pi], \mathbb{R}^2)$ defined by

$$Q(z)(x) = e^{\mathcal{B}^1(1,0)x}z(0), \ \forall x \in [0, 2\pi],$$

so that the above equation rewrites as

$$\tilde{y}_1 \in P_1\mathcal{X}_2 \text{ and } (I - Q)\tilde{y}_1 = Y_1 \in C_{2\pi}([0, 2\pi], \mathbb{R}^2) \cap C_0([0, 2\pi], \mathbb{R}^2).$$

Now note that Q is a projector (i.e $Q^2 = Q$) and since $x \to e^{\mathcal{B}^1(1,0)x}$ is $2\pi-$periodic, one has

$$R(I - Q) = C_{2\pi}([0, 2\pi], \mathbb{R}^2) \cap C_0([0, 2\pi], \mathbb{R}^2) \text{ and } N(I - Q) = R(Q) = \text{span}\{P_1 e_1, P_1 e_2\}.$$

Since $C_{2\pi}([0, 2\pi], \mathbb{R}^2) = P_1\mathcal{X}_2 \oplus \text{span}\{P_1 e_1, P_1 e_2\}$, we obtain that the linear bounded operator $(I-Q)_{|P_1\mathcal{X}_2}$ is bijective from $P_1\mathcal{X}_2$ onto $C_{2\pi}([0, 2\pi], \mathbb{R}^2) \cap C_0([0, 2\pi], \mathbb{R}^2)$. Hence by the bounded inverse theorem we end up with

$$\tilde{y}_1 = (I - Q)^{-1}_{|P_1\mathcal{X}_2} Y_1.$$

To sum up the above analysis, we have obtained for each $w \in C_0([0, 2\pi], \mathbb{R}^{2m}) \times C_{2\pi}([0, 2\pi], X_{0s})$, there exists a unique $(\tilde{\omega}, \tilde{\mu}, \tilde{y}) \in \mathbb{R}^2 \times \mathcal{X}_2 \times C_{2\pi}([0, 2\pi], X_{0s})$ satisfying (4.18), which completes the proof that W is invertible. □

Last part of the proof of Theorem 2.9: To conclude the proof of the Hopf bifurcation Theorem 2.9, we apply the implicit function theorem (see Deimling [5, Theorem 15.2]) to the function $h : \mathbb{R} \times (0, \infty) \times J \times \mathcal{X}_2 \times C_{2\pi}([0, 2\pi], X_{0s}) \to C_0([0, 2\pi], \mathbb{R}^{2m}) \times C_{2\pi}([0, 2\pi], X_{0s})$ and we deduce that there exists a C^1-mapping $(\omega, \mu, y) : (-\delta, \delta) \to \mathbb{R}^2 \times \mathcal{X}_2 \times C_{2\pi}([0, 2\pi], X_{0s})$, for some $\delta > 0$ small enough, such that

$$h(t, \omega(t), \mu(t), y(t)) = 0, \ \forall t \in (-\delta, \delta).$$

By the definition of h, this is equivalent to

$$\mathcal{F}\left(\omega(t), \mu(t), \begin{pmatrix} t(e_1 + y_{cu}(t)) \\ y_s(t) \end{pmatrix}\right) = 0,$$

when $t \neq 0$ with $(\omega(0), \mu(0), z(0)) = (1, 0, 0)$. We see that $\left(\omega(t), \mu(t), \begin{pmatrix} t(e_1 + y_{cu}(t)) \\ y_s(t) \end{pmatrix}\right)$ is the desired curve of solution of $\mathcal{F} = 0$. Thus the proof is complete.

5 Periodic wave train

In this section we prove Theorem 2.18. We consider the following problem depending on some parameter α

$$\begin{cases} \partial_x^2 U(x,a) - \gamma \partial_x U(x,a) - \partial_a U(x,a) - \zeta U(x,a) = 0, \quad x \in \mathbb{R},\, a > 0, \\ U(x,0) = \alpha f\left(\int_0^\infty \beta(a) U(x,a) \mathrm{d}a\right). \end{cases} \tag{5.1}$$

Here we use $\nu = \alpha > e^2$ as a bifurcation parameter while the nonlinear function f and birth rate function β satisfy Assumption 2.17. As explained in Section 2, this problem has a positive $x-$independent equilibrium $\overline{U}(a) = \ln \alpha \, e^{-\zeta a}$. The aim of this section is to apply Theorem 2.9 to this example and show that this positive equilibrium may undergo a Hopf bifurcation leading to the existence of periodic wave train. To do so let us recall that (5.1) rewrites as (2.14) (with $\nu = \alpha$). Hence to apply Theorem 2.9 we are looking at $\sigma(B_\alpha) \cap \mathcal{P}_{\omega_0 \mathbb{Z}}$, where B_α is given in (2.12). Using Lemma 2.14 and (2.13) one knows that for any $\alpha > e^2$ one has

$$\lambda \in \sigma(B_\alpha) \cap \{z \in \mathbb{C} : \operatorname{Re} \lambda > -\zeta\} \Leftrightarrow \Delta(\alpha, \lambda) = 0,$$

where the function Δ is given by

$$\Delta(\alpha, \lambda) = 1 - (1 - \ln \alpha) \int_0^\infty \beta(a) e^{-(\lambda+\zeta)a} \mathrm{d}a,$$

which more explicitly rewrites as (see Magal and Ruan [17, Chapter 5])

$$\Delta(\alpha, \lambda) = 1 - (1 - \ln \alpha) e^{-\lambda \tau} \left(1 + \frac{\lambda}{\zeta + \kappa}\right)^{-n-1}.$$

Now, as mentioned above, we are concerned with finding $\omega_0 > 0$, $\alpha_0 > e^2$ and a smooth function $\lambda = \lambda(\alpha)$ defined in a neighborhood I of α_0 such that

$$\Delta(\alpha, \lambda(\alpha)) = 0, \ \forall \alpha \in I \text{ with } \lambda(\alpha_0) = \omega_0^2 + i\gamma\omega_0,$$

together with the transversality condition

$$\operatorname{Re}\left(\frac{1}{\gamma - 2i\omega_0} \frac{\mathrm{d}\lambda(\alpha_0)}{\mathrm{d}\alpha}\right) \neq 0.$$

To study the above properties, define the function $\widetilde{\Delta}$ by

$$\widetilde{\Delta}(\alpha, \lambda) = \Delta(\alpha, \gamma\lambda - \lambda^2) = 1 - (1 - \ln \alpha) e^{(\lambda^2 - \gamma\lambda)\tau} \left(1 + \frac{\gamma\lambda - \lambda^2}{\zeta + \kappa}\right)^{-n-1}.$$

We are now in the position to look for purely imaginary roots $\lambda = \pm i\omega$ with $\omega > 0$ of the characteristic equation

$$\widetilde{\Delta}(\alpha, \lambda) = 0. \tag{5.2}$$

Proposition 5.1. *Let Assumption 2.17 be satisfied and the parameters $\zeta > 0, \tau > 0, \kappa > 0, n \in \mathbb{N}$ be fixed. Then the characteristic equation (5.2) has a pair of purely imaginary solutions $\pm i\omega$ if and only if $\omega > 0$ is a solution of*

$$\gamma\omega\tau + (n+1)\arctan \frac{\gamma\omega}{\zeta + \kappa + \omega^2} = \pi + 2k\pi, \ k \in \mathbb{N}, \tag{5.3}$$

and

$$\alpha = \exp\left(1 + e^{\omega^2 \tau}\left(\left(1 + \frac{\omega^2}{\zeta + \kappa}\right)^2 + \frac{\gamma^2\omega^2}{(\zeta + \kappa)^2}\right)^{\frac{n+1}{2}}\right). \tag{5.4}$$

Furthermore, if

$$\gamma^2 > \frac{n+1}{\tau} - 2(\zeta + \kappa), \tag{5.5}$$

then for each $k \in \mathbb{N}$, there exists exactly one solution ω_k for equation (5.3), i.e. the characteristic equation (5.2) has exactly one pair of purely imaginary eigenvalues $\pm i\omega_k$ with $\omega_k > 0$ for each

$$\alpha_k = \exp\left(1 + e^{\omega_k^2\tau}\left(\left(1 + \frac{\omega_k^2}{\zeta+\kappa}\right)^2 + \frac{\gamma^2\omega_k^2}{(\zeta+\kappa)^2}\right)^{\frac{n+1}{2}}\right). \tag{5.6}$$

Moreover, $\omega_k \to \infty$ and $\alpha_k \to \infty$, as $k \to \infty$.

If $\omega_k(n) > 0$ is the solution of equation (5.3) with fixed $k \in \mathbb{N}$ for any $n \in \mathbb{N}$, then

$$\omega_k(n) \to 0 \ \text{and}\ \alpha_k(n) \to e^2, \ \text{as}\ n \to \infty. \tag{5.7}$$

Proof. Under Assumption 2.17, if the characteristic equation (5.2) admits a pair of purely imaginary solutions $\lambda = \pm i\omega$ with $\omega > 0$, then (5.2) can be expressed

$$(1 - \ln\alpha)e^{-\omega^2\tau}(r(\omega))^{-n-1}e^{-i[\gamma\omega\tau+(n+1)\theta(\omega)]} = 1, \tag{5.8}$$

where

$$r(\omega) = \sqrt{\left(1+\frac{\omega^2}{\zeta+\kappa}\right)^2 + \frac{\gamma^2\omega^2}{(\zeta+\kappa)^2}}, \quad \theta(\omega) = \arctan\frac{\gamma\omega}{\zeta+\kappa+\omega^2}.$$

Therefore, by separating the real and imaginary parts of (5.8), we can obtain that ω satisfies (5.3)-(5.4). Thus the characteristic equation (5.2) has a pair of purely imaginary solutions $\lambda = \pm i\omega$ with $\omega > 0$ if and only if ω is a solution of (5.3) and (5.4).

Let

$$h(\omega) = \gamma\omega\tau + (n+1)\arctan\frac{\gamma\omega}{\zeta+\kappa+\omega^2},\ \omega \geq 0.$$

Then we have

$$h'(\omega) = \gamma\tau + \frac{\gamma(n+1)(\zeta+\kappa-\omega^2)}{(\zeta+\kappa+\omega^2)^2+\gamma^2\omega^2}.$$

Set $y = \omega^2$ and define

$$g(y) = \gamma\tau\Big((\zeta+\kappa+y)^2 + \gamma^2 y\Big) + \gamma(n+1)(\zeta+\kappa-y).$$

It can be checked that if (5.5) is satisfied, then $g(y)$ is positive for all $y \geq 0$. This implies that $h'(\omega) > 0$ for all $\omega > 0$ and that h is strictly increasing. Noting $h(0) = 0$ and $h(\omega) \to \infty$ as $\omega \to \infty$, it follows from the continuity of $h(\omega)$ that the equation (5.3) has exactly one solution $\omega_k > 0$ for each $k \in \mathbb{N}$ with parameter α_k satisfying (5.6).

Moreover, let $\omega_k > 0$ be such that

$$\gamma\tau\omega_k + (n+1)\arctan\frac{\gamma\omega_k}{\zeta+\kappa+\omega_k^2} = \pi + 2k\pi,\ \forall k \in \mathbb{N}.$$

It follows that $\omega_k \to \infty$ by letting $k \to \infty$ on both sides of the equality. For such $\{\omega_k\}_{k\in\mathbb{N}}$ given above, define $\{\alpha_k\}_{k\in\mathbb{N}}$ by (5.6). Then the limit $\alpha_k \to \infty$ as $k \to \infty$ can be obtained from (5.6) and $\omega_k \to \infty$ as $k \to \infty$.

Fix $k \in \mathbb{N}$ and let $\omega_k(n) > 0$ be the solution of (5.3) for any $n \in \mathbb{N}$, i.e

$$\gamma\tau\omega_k(n) + (n+1)\arctan\frac{\gamma\omega_k(n)}{\zeta+\kappa+\omega_k^2(n)} = \pi + 2k\pi. \tag{5.9}$$

Then (5.9) yields

$$\gamma\tau\omega_k(n) + (n+1)\arctan\frac{\gamma\omega_k(n)}{\zeta+\kappa+\omega_k^2(n)} = \gamma\tau\omega_k(n+1) + (n+2)\arctan\frac{\gamma\omega_k(n+1)}{\zeta+\kappa+\omega_k^2(n+1)}. \tag{5.10}$$

It follows from the mean value theorem that there exists some ξ between $\frac{\omega_k(n)}{\zeta+\kappa+\omega_k^2(n)}$ and $\frac{\omega_k(n+1)}{\zeta+\kappa+\omega_k^2(n+1)}$ such that

$$\arctan\frac{\omega_k(n)}{\zeta+\kappa+\omega_k^2(n)} - \arctan\frac{\omega_k(n+1)}{\zeta+\kappa+\omega_k^2(n+1)} = \frac{\gamma(\zeta+\kappa-\xi^2)}{(\zeta+\kappa+\xi^2)^2+\gamma^2\xi^2}(\omega_k(n)-\omega_k(n+1)). \tag{5.11}$$

Therefore, by replacing (5.10) into (5.11), we can rewrite (5.10) as

$$h'(\xi)\Big(\omega_k(n)-\omega_k(n+1)\Big)=\arctan\frac{\omega_k(n+1)}{\zeta+\kappa+\omega_k^2(n+1)}. \tag{5.12}$$

Thanks to the positivity of $\omega_k(n)$ and h', we obtain from (5.12) that

$$\omega_k(n)>\omega_k(n+1)>0, \text{ for all } n\in\mathbb{N},$$

which implies that $\omega_k(n)$ is strictly decreasing for $n\in\mathbb{N}$. Thus, the monotone bounded convergence theorem for the sequence $\{\omega_k(n)\}$ implies that the limit $\lim_{n\to\infty}\omega_k(n)$ exists. Let $\omega_k^*=\lim_{n\to\infty}\omega_k(n)$. Then we have $\omega_k^*\geq 0$ by the positivity of the sequence $\{\omega_k(n)\}$. We claim that $\omega_k^*=0$. In fact, if $\omega_k^*>0$, we obtain a contradiction by letting $n\to\infty$ on both sides of (5.9). Therefore $\omega_k^*=0$.

By letting $n\to\infty$ on both sides of (5.9), we have

$$\pi+2k\pi=\lim_{n\to\infty}\left(\gamma\tau\omega_k(n)+(n+1)\arctan\frac{\gamma\omega_k(n)}{\zeta+\kappa+\omega_k^2(n)}\right)=\lim_{n\to\infty}(n+1)\arctan\frac{\gamma\omega_k(n)}{\zeta+\kappa+\omega_k^2(n)},$$

which implies that

$$\frac{\gamma\omega_k(n)}{\zeta+\kappa}\sim\frac{\pi+2k\pi}{n+1} \text{ as } n\to\infty.$$

Finally, the limit $\alpha_k(n)\to e^2$ as $n\to\infty$ in (5.7) follows from (5.6) and the above asymptotic for $\omega_k(n)$. □

The following theorem provides the tranversality condition for (5.1).

Theorem 5.2. *Let Assumption 2.17 be satisfied and the parameters $\zeta>0,\tau>0,\kappa>0,n\in\mathbb{N}$ be fixed. If (5.5) is satisfied, for each $k\geq 0$, if $\pm i\omega_k$ with $\omega_k>0$ is the purely imaginary roots of the characteristic equation associated to α_k defined in Proposition 5.1, then there exists $\rho_k>0$ and a C^1-map $\hat{\lambda}_k:(\alpha_k-\rho_k,\alpha_k+\rho_k)\to\mathbb{C}$ such that*

$$\hat{\lambda}_k(\alpha_k)=i\omega_k,\quad \widetilde{\Delta}(\alpha,\hat{\lambda}_k(\alpha))=0,\quad \forall\alpha\in(\alpha_k-\rho_k,\alpha_k+\rho_k)$$

satisfying the transversality condition

$$\operatorname{Re}\frac{\mathrm{d}\hat{\lambda}_k(\alpha_k)}{\mathrm{d}\alpha}>0.$$

Proof. By differentiating the characteristic function $\widetilde{\Delta}=\widetilde{\Delta}(\alpha,\lambda)$ with λ and α and recalling (5.2), we have

$$\frac{\partial\widetilde{\Delta}(\alpha,\lambda)}{\partial\lambda}=(\gamma-2\lambda)\left(\tau+\frac{n+1}{\zeta+\kappa+\gamma\lambda-\lambda^2}\right),\qquad \frac{\partial\widetilde{\Delta}(\alpha,\lambda)}{\partial\alpha}=\frac{1}{\alpha(1-\ln\alpha)}.$$

Therefore, by setting $(\alpha,\lambda)=(\alpha_k,i\omega_k)$ in the above equations, we have

$$\operatorname{Re}\frac{\partial\widetilde{\Delta}(\alpha_k,i\omega_k)}{\partial\lambda}=\gamma\tau+\frac{\gamma(n+1)(\zeta+\kappa-\omega_k^2)}{(\zeta+\kappa+\omega_k^2)^2+\gamma^2\omega_k^2}=h'(\omega_k),\qquad \frac{\partial\widetilde{\Delta}(\alpha_k,i\omega_k)}{\partial\alpha}=\frac{1}{\alpha_k(1-\ln\alpha_k)}<0. \tag{5.13}$$

It follows from Proposition 5.1 that when γ satisfies (5.5), we have

$$\operatorname{Re}\frac{\partial\widetilde{\Delta}(\alpha_k,i\omega_k)}{\partial\lambda}>0.$$

Then the implicit function theorem around each $(\alpha_k,i\omega_k)$ ensures that for each $k\geq 0$, there exists $\rho_k>0$ and a C^1-map $\hat{\lambda}_k:(\alpha_k-\rho_k,\alpha_k+\rho_k)\to\mathbb{C}$ such that

$$\hat{\lambda}_k(\alpha_k)=i\omega_k,\quad \widetilde{\Delta}(\alpha,\hat{\lambda}_k(\alpha))=0,\quad \forall\alpha\in(\alpha_k-\rho_k,\alpha_k+\rho_k).$$

Furthermore, by differentiating $\widetilde{\Delta}(\alpha,\hat{\lambda}(\alpha))=0$ with respect to α, we obtain

$$\frac{\partial\widetilde{\Delta}(\alpha,\hat{\lambda}_k(\alpha))}{\partial\alpha}+\frac{\partial\widetilde{\Delta}(\alpha,\hat{\lambda}_k(\alpha))}{\partial\lambda}\frac{\mathrm{d}\hat{\lambda}_k(\alpha)}{\mathrm{d}\alpha}=0,\quad \forall\alpha\in(\alpha_k-\rho_k,\alpha_k+\rho_k). \tag{5.14}$$

As a consequence, $\operatorname{Re}\frac{\mathrm{d}\hat{\lambda}_k(\alpha_k)}{\mathrm{d}\alpha}>0$ follows by substituting (5.13) into (5.14) with $\alpha=\alpha_k$. □

In this example (5.1), we set $\lambda_k(\alpha) = -\hat{\lambda}_k(\alpha)^2 + \gamma\hat{\lambda}_k(\alpha)$ so that

$$\lambda_k(\alpha_k) = \omega_k^2 + i\gamma\omega_k,$$

while

$$\frac{1}{\gamma - 2i\omega_k}\frac{\mathrm{d}\lambda_k(\alpha_k)}{\mathrm{d}\alpha} = \frac{\mathrm{d}\hat{\lambda}_k(\alpha_k)}{\mathrm{d}\alpha}.$$

Thus the assumptions in Theorem 2.9 are satisfied. Finally, Proposition 5.1 and Theorem 5.2 allow us to use the Hopf bifurcation theorem established in Theorem 2.9, and this completes the proof of Theorem 2.18.

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