A COVID-19 epidemic model with latency period

Z. Liu[(a)*], P. Magal[(b)†‡] Ousmane Seydi [(c)] and Glenn Webb[(d)]

[(a)] *School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People's Republic of China*

[(b)] *Univ. Bordeaux, IMB, UMR 5251, F-33400 Talence, France*
CNRS, IMB, UMR 5251, F-33400 Talence, France

[(c)] *Département Tronc Commun, École Polytechnique de Thiés, Sénégal*

[(d)] *Mathematics Department, Vanderbilt University, Nashville, TN, USA*

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Abstract

At the beginning of a COVID-19 infection, there is a period of time known as the exposed or latency period, before an infected person is capable of transmitting the infection to another person. We develop two differential equations models to account for this period. The first is a model that incorporates infected persons in the exposed class, before transmission is possible. The second is a model that incorporates a time delay in infected persons, before transmission is possible. We apply both models to the COVID-19 epidemic in China. We estimate the epidemiological parameters in the models, such as the transmission rate and the basic reproductive number, using data of reported cases. We thus evaluate the role of the exposed or latency period in the dynamics of a COVID-19 epidemic.

Keywords: corona virus, reported and unreported cases, isolation, quarantine, public closings; epidemic mathematical model

1 Introduction

In [3] it is reported that transmission of COVID-19 infection may occur from an infectious individual, who is not yet symptomatic. In [12] it is reported that COVID-19 infected individuals generally develop symptoms, including mild respiratory symptoms and fever, on an average of 5-6 days after infection (mean 5-6 days, range 1-14 days) . In [13] it is reported that the median time prior to symptom onset is 3 days, the shortest 1 day, and the longest 24 days. It is evident that these time periods play an important role in understanding COVID-19 transmission dynamics.

In this work, we will examine the latency period of COVID-19 infection, that is, the period of time in which newly infected individuals are asymptomatic and noninfectious. We illustrate the latency period in Figure 1 below:

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‡Corresponding authors.

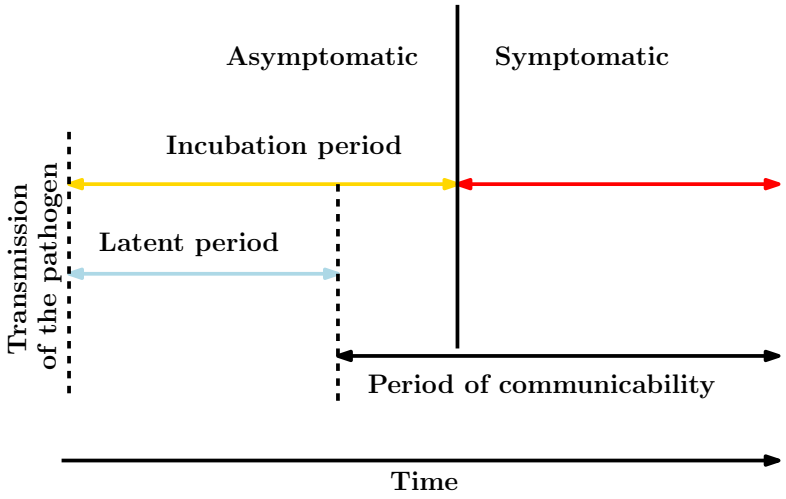

Figure 1: *Key time periods of COVID-19 infection. The latent or exposed period before symptoms and transmissibility, the incubation period before symptoms appear, the symptomatic period, and the transmissibility period, which may overlay the asymptomatic period.*

In the present article we develop two mathematical models to study the impact of the latency period. One is a ODE (ordinary differential equations) model, with an exposed class of infected individuals, who are not yet infectious. The other is a DDE (delay differential equations) model, with a time delay in newly infected individuals, before they become infectious. The DDE model can be derived from a continuous age of infection model, which can be reduced to a system of DDE. The derivation of such models is described in [7]. We refer to [9, 13] for early models with exposure applied to COVID-19.

As mentioned in [5], asymptomatic infectious cases are not usually reported to medical authorities, and reported infectious cases are typically only a fraction of the total number of the symptomatic infectious individuals. In this work, we examine the number of asymptomatic infectious cases and unreported infectious cases, as well as the number of reported infectious cases, for the COVID epidemic in mainland China. We note that public measures in China, beginning January 26, strongly attenuated the epidemic. One of our objectives is to understand how these measures, such as isolation, quarantine, and public closings, reduce the final size of the epidemic. We examine how the latency period, tied contact tracing and to a 14-day medical observation or quarantine period for exposed persons, mitigates the final size of the epidemic.

2 Models

2.1 Model with a compartment E of exposed infected individuals not yet infectious

This model has a compartment in the system of ODE that corresponds to exposed or latent infected individuals. We will designate this model as the **SEIRU** model:

$$\begin{cases} S'(t) = -\tau(t)S(t)[I(t)+U(t)], \\ E'(t) = \tau(t)S(t)[I(t)+U(t)] - \alpha E(t) \\ I'(t) = \alpha E(t) - \nu I(t) \\ R'(t) = \nu_1 I(t) - \eta R(t) \\ U'(t) = \nu_2 I(t) - \eta U(t). \end{cases} \tag{2.1}$$

Here $t \geq t_0$ is time in days, t_0 is the beginning date of the epidemic, $S(t)$ is the number of individuals susceptible to infection at time t, $E(t)$ is the number of asymptomatic noninfectious individuals at time t, $I(t)$ is the number of asymptomatic but infectious individuals at time t, $R(t)$ is the number of reported symptomatic infectious individuals at time t, and $U(t)$ is the number of unreported symptomatic infectious individuals at time t. This system is supplemented by initial data

$$S(t_0) = S_0 > 0,\ E(t_0) = E_0 > 0,\ I(t_0) = I_0 > 0,\ U(t_0) = U_0 > 0, R(t_0) = R_0 = 0. \tag{2.2}$$

The exit flux of the exposed class E is describe by the term $-\alpha E(t)$. The means that the time of exposure follows an exponential law, and the average value of the exposure time is $1/\alpha$, which can be, for

example, 6 hours, 12 hours, 1 day, 2 days, 3 days, etc.... The model contains an asymptomatic infectious class corresponding to the $I(t)$-equation. The dynamics of the symptomatic infectious individuals are decomposed into the $R(t)$-equation, which corresponds to the reported symptomatic infectious individuals (symptomatic infectious with severe symptoms), and the $U(t)$-equation, which corresponds to the unreported symptomatic infectious individuals (symptomatic infectious with mild symptoms). The flux of individuals leaving the class I is $\nu I(t)$. We assume that a fraction f are reported and a fraction $1-f$ are unreported. Thus, $\nu_1 = f\nu$ and $\nu_2 = (1-f)\nu$.

The time-dependent parameter $\tau(t)$ is the transmission rate. During the early phase of the epidemic, when the cumulative number of reported cases grows approximately exponential, $\tau(t)$ is a constant value τ_0. After January 23, strong government measures in all of China, such as isolation, quarantine, and public closings, strongly impacted the transmission of new cases. The actual effects of these measures were complex, and we use a time-dependent exponentially decreasing transmission rate $\tau(t)$ to incorporate these effects after the early exponentially increasing phase. The formula for $\tau(t)$ during the exponential decreasing phase is derived by a fitting procedure to the data:

$$\begin{cases} \tau(t) = \tau_0,\ 0 \le t \le N, \\ \tau(t) = \tau_0 \exp\left(-\mu \times (t-N)\right),\ N < t. \end{cases} \tag{2.3}$$

Day N corresponds to the day when the public measures take effect, and μ is the rate at which they take effect. A schematic diagram of the model is given in Figure 2, and the parameters of the model are listed in Table 1 below.

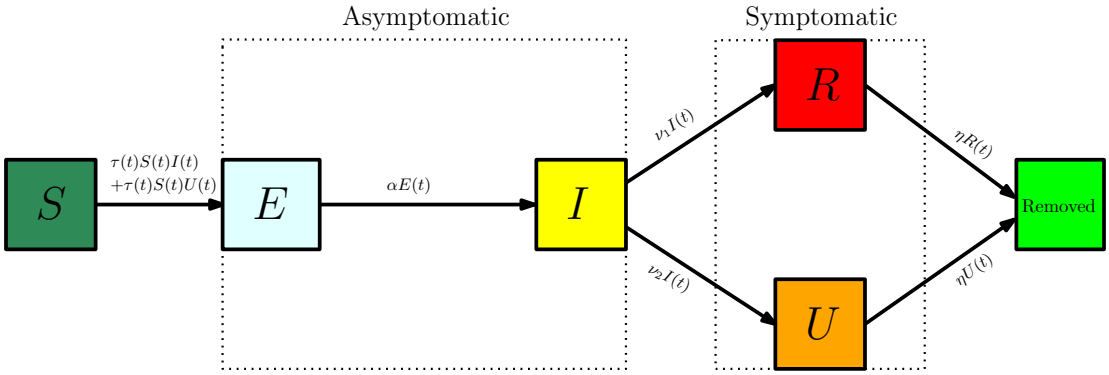

Figure 2: *Flow chart for the model* **SEIRU**.

Symbol	Interpretation	Method
t_0	Time at which the epidemic started	fitted
S_0	Number of susceptible at time t_0	fixed
E_0	Number of asymptomatic and noninfectious at time t_0	fitted
I_0	Number of asymptomatic but infectious at time t_0	fitted
U_0	Number of unreported symptomatic infectious at time t_0	fitted
$\tau(t)$	Transmission rate	fitted
N	First day of the public interventions	fitted
μ	Intensity of the public interventions	fitted
$1/\alpha$	average duration of the exposed noninfectious period	fitted
$1/\nu$	Average time during which asymptomatic infectious are asymptomatic	fixed
f	Fraction of asymptomatic infectious that become reported symptomatic infectious	fixed
$\nu_1 = f\,\nu$	Rate at which asymptomatic infectious become reported symptomatic	fitted
$\nu_2 = (1-f)\,\nu$	Rate at which asymptomatic infectious become unreported symptomatic	fitted
$1/\eta$	Average time symptomatic infectious have symptoms	fixed

Table 1: *Parameters and initial conditions of the model* **SEIRU**.

2.2 Model with a constant time delay δ in the I class equation

This model has a time delay δ in the $I(t)$ equation in the system of DDE that contains the latency period. We will designate this model as the **SEIRU**δ model:

$$\begin{cases} S'(t) = -\tau(t)S(t)[I(t) + U(t)], \\ I'(t) = \tau(t-\delta)S(t-\delta)[I(t-\delta) + U(t-\delta)] - \nu I(t) \\ R'(t) = \nu_1 I(t) - \eta R(t) \\ U'(t) = \nu_2 I(t) - \eta U(t). \end{cases} \tag{2.4}$$

This system is supplemented by initial data

$$S(t_0+\theta) = S_0(\theta) > 0,\ I(t_0+\theta) = I_0(\theta) > 0,\ U(t_0+\theta) = U_0(\theta) > 0, \forall \theta \in [-\delta, 0],\ \text{and}\ R(t_0) = 0. \tag{2.5}$$

In the model **SEIRU**δ, the duration of exposure is constant and equal to δ. The exposed class is given by the integral formula

$$E(t) = \int_{t-\delta}^{t} \tau(\sigma)S(\sigma)[I(\sigma) + U(\sigma)]d\sigma. \tag{2.6}$$

or alternatively, by using the differential equation

$$E'(t) = \tau(t)S(t)[I(t) + U(t)] - \tau(t-\delta)S(t-\delta)[I(t-\delta) + U(t-\delta)]. \tag{2.7}$$

$E(t)$ can be decoupled from the equations in the DDE system **SEIRU**δ, since it can be obtained from $S(t)$, $I(t)$, and $U(t)$. The parameters and initial conditions of the **SEIRU**δ model are given in Table 1, and a schematic diagram of the model is given in Figure 3.

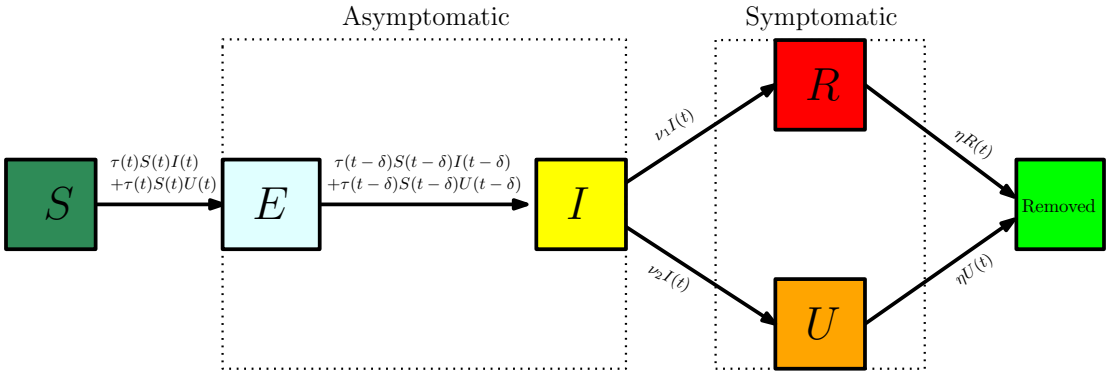

Figure 3: *Flow chart for the model* **SEIRU**δ.

2.3 Data for the COVID-19 epidemic in China

In our simulations of models **SEIRU** and **SEIRU**δ for COVID-19 in mainland China, we will use the following data:

January						
19	20	21	22	23	24	25
198	291	440	571	830	1287	1975
26	27	28	29	30	31	
2744	4515	5974	7711	9692	11791	
February						
1	2	3	4	5	6	7
14380	17205	20438	24324	28018	31161	34546
8	9	10	11	12	13	14
37198	40171	42638	44653	46472	48467	49970
15	16	17	18	19	20	21
51091	70548 − 17409	72436 − 17409	74185 − 17409	75002 − 17409	75891 − 17409	76288 − 17409
22	23	24	25	26	27	28
76936 − 17409	77150 − 17409	77658 − 17409	78064 − 17409	78497 − 17409	78824 − 17409	79251 − 17409
29						
79824 − 17409						
March						
1	2	3	4	5	6	7
80026 − 17409	80151 − 17409	80270 − 17409	80409 − 17409	80552 − 17409	80651 − 17409	80695 − 17409
8	9	10	11	12	13	14
80735 − 17409	80754 − 17409	80778 − 17409	80793 − 17409	80813 − 17409	80824 − 17409	80844 − 17409
15	16	17	18			
80860 − 17409	80881 − 17409	80894 − 17409	80928 − 17409			

Table 2: *Cumulative daily reported case data from January 19, 2020 to March 18, 2020, reported for mainland China by the National Health Commission of the People's Republic of China and the Chinese CDC [1]. The data corresponds to cumulative reported cases confirmed by testing.*

3 Estimation of the parameters and initial conditions

The parameters τ, ν, ν_1, ν_2, η, α, δ, as well as the starting time t_0 and the initial conditions $S(t_0)$, $E(t_0)$, $I(t_0)$, $U(t_0)$, are uncertain. Our objective is to identify them from specific time data of reported symptomatic infectious cases. To identify the unreported asymptomatic infectious cases, we assume that the cumulative reported symptomatic infectious cases at time t consist of a constant fraction f of the total number of symptomatic infectious cases at time t. In other words, we assume that the removal rate ν of infectious asymptomatic cases $I(t)$ takes the following form: $\nu = \nu_1 + \nu_2$, where $\nu_1 = f\nu$ is the removal rate of reported symptomatic infectious individuals, and $\nu_2 = (1-f)\nu$ is the removal rate of unreported symptomatic infectious individuals due to all causes. The cumulative number of reported symptomatic infectious cases at time t, denoted by $CR(t)$, is

$$CR(t) = \nu_1 \int_{t_0}^{t} I(s)ds. \tag{3.1}$$

Our method is the following: We assume that $CR(t)$ has the following form when the epidemic is in the early exponentially growing phase:

$$CR(t) = \chi_1 \exp(\chi_2 t) - \chi_3. \tag{3.2}$$

We evaluate χ_1, χ_2, χ_3 using the reported cases data. By using the method in Section 6.1 (Supplementary material), we estimate the starting time t_0 for the models from

$$CR(t_0) = 0 \Leftrightarrow \chi_1 exp(\chi_2 t_0) - \chi_3 = 0 \;\Rightarrow\; t_0 = \frac{1}{\chi_2}\Big(\ln(\chi_3) - \ln(\chi_1)\Big).$$

We fix $S_0 = 1.40005 \times 10^9$, which corresponds to the total population of mainland China. We assume that the variation in $S(t)$ is small during this exponentially growing phase. We fix ν, η, f, α. We assume that

the transmission rate $\tau(t) \equiv \tau_0$ is constant during this exponentially growing phase. We identify τ_0 from χ_1, χ_2, χ_3 for each of the models **SEIRU** and **SEIRU**δ.

3.1 Parameters and initial conditions for model SEIRU

We fix the fraction $f = 0.8$ of symptomatic infectious cases that are reported. Thus, 80% of infectious cases are reported. We assume $1/\nu$, the average time during which the patients are asymptomatic infectious is 5 days or 7 days. We assume that $1/\eta$, the average time during which a patient is symptomatic infectious, is 7 days. Since f is known, we obtain

$$\nu_1 = f\,\nu = 0.8/5\,(\text{or } 0.8/7) \text{ and } \nu_2 = (1-f)\,\nu = 0.2/5\,(\text{or } 0.2/7). \tag{3.3}$$

From Section 6.1 (Supplementary material), we obtain

$$E_0 = \frac{\chi_2+\nu}{\alpha} I_0,\ U_0 = \frac{\nu_2}{\chi_2+\eta} I_0,$$

$$\tau_0 = \frac{(\chi_2+\alpha)E_0}{S_0[I_0+U_0]} = \frac{(\chi_2+\nu)(\chi_2+\alpha)(\chi_2+\eta)}{\alpha S_0(\chi_2+\eta+\nu_2)}. \tag{3.4}$$

From Section 6.2 (Supplementary material), we obtain basic reproductive number $\mathcal{R}_0$ for model **SEIRU**

$$\mathcal{R}_0 = \frac{(\chi_2+\nu)(\chi_2+\alpha)(\chi_2+\eta)}{\alpha\nu(\chi_2+\eta+\nu_2)}\left(1+\frac{(1-f)\,\nu}{\eta}\right).$$

3.2 Parameters and initial conditions for model SEIRUδ

The values of f, ν, and η are the same as for model **SEIRU**. From Section 6.3 (Supplementary material), we obtain

$$S(t_0+\theta) = S_0(\theta) = S_0,\ \theta \in [-\delta, 0], \tag{3.5}$$

$$I(t_0+\theta) = I_0(\theta) = \frac{\chi_3\chi_2}{f\,\nu} e^{\chi_2\theta},\ \theta \in [-\delta, 0], \tag{3.6}$$

$$U_0(t_0+\theta) = U_0(\theta) = \frac{(1-f)\nu}{\eta+\chi_2} I_0(\theta),\ \theta \in [-\delta, 0] \tag{3.7}$$

and

$$\tau_0 = \frac{\chi_2+\nu}{S_0}\frac{\eta+\chi_2}{\nu_2+\eta+\chi_2} e^{\chi_2\delta}. \tag{3.8}$$

From Section 6.4 (Supplementary material), we obtain the basic reproductive number $\mathcal{R}_0$ for model **SEIRU**δ

$$\mathcal{R}_0 = \frac{\tau_0 S_0}{\nu}\left(1+\frac{\nu_2}{\eta}\right) = \frac{\chi_2+\nu}{\nu}\frac{\eta+\chi_2}{\nu_2+\eta+\chi_2} e^{\chi_2\delta}\left(1+\frac{(1-f)\,\nu}{\eta}\right).$$

4 Comparisons of the models with the data

We use the data from Table 2 to numerically simulate models **SEIRU** and **SEIRU**δ.

4.1 Comparison of model SEIRU with data

In Figures 4 and 5, we plot the graphs of $CR(t)$, $CU(t)$, $R(t)$, and $U(t)$ from the numerical simulation of model **SEIRU**. We use $\chi_1 = 0.2254$, $\chi_2 = 0.3762$, $\chi_3 = 1$, $f = 0.8$, $\eta = 1/7$, $t_0 = 3.9607$, and $S_0 = 1400050000$ in both Figures 4 and 5. We take $\nu = 1/5$ in Figure 4 and $\nu = 1/7$ in Figure 5 . We take four different values for α: $1/4,\ 1/2,\ 1,\ 3$ in both Figure 4 and Figure 5. The value of μ is chosen so that the simulations align with the cumulative reported case data. In this way, we are able to predict the future values of the epidemic from early cumulative reported case data. We see from the simulations the following: for Figure 4, with $\nu = 1/5$, the simulations are almost the same; for Figure 5, with $\nu = 1/7$, the simulations are almost the same.

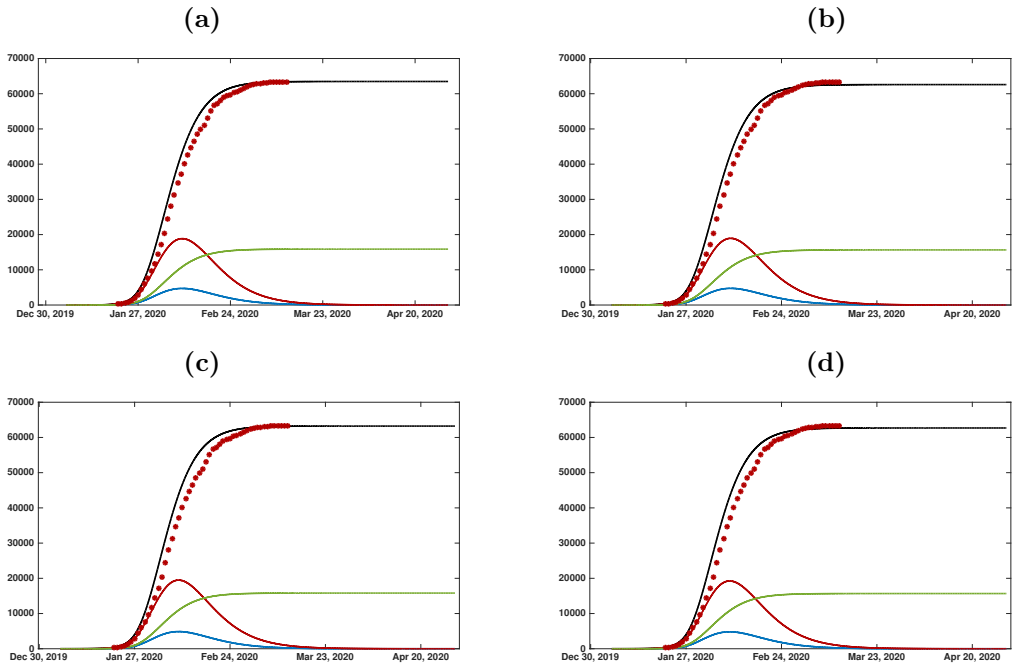

Figure 4: *Graphs of the reported cumulated symptomatic infectious individuals* $t \to CR(t)$ *(black solid line), unreported cumulated symptomatic infectious individuals* $t \to CU(t)$ *(green solid line),* $t \to U(t)$ *(blue solid line), and* $t \to R(t)$ *(red solid line). The red dots are the data of the reported cumulated confirmed cases for mainland China in Table 2. We use* $\chi_1 = 0.2254$, $\chi_2 = 0.3762$, $\chi_3 = 1$, $f = 0.8$, $\eta = 1/7$, $\nu = 1/5$, $t_0 = 3.9607$, *and* $S_0 = 1400050000$. (a) $\mu = 0.1276$, $1/\alpha = 6$ *hours*. (b) $\mu = 0.142$, $1/\alpha = 12$ *hours*. (c) $\mu = 0.166$, $1/\alpha = 1$ *day*. (d) $\mu = 0.25$, $1/\alpha = 3$ *days*.

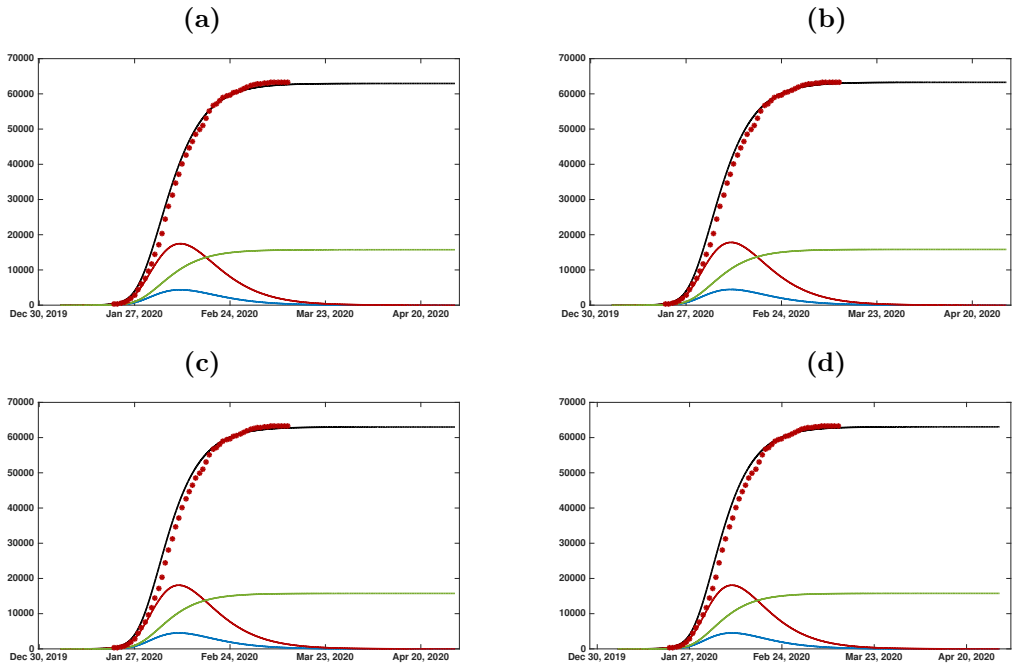

Figure 5: *Graphs of the reported cumulated symptomatic infectious individuals* $t \to CR(t)$ *(black solid line), unreported cumulated symptomatic infectious individuals* $t \to CU(t)$ *(green solid line),* $t \to U(t)$ *(blue solid line), and* $t \to R(t)$ *(red solid line). The red dots are the data of the reported cumulated confirmed cases for mainland China in Table 2. We use* $\chi_1 = 0.2254$, $\chi_2 = 0.3762$, $\chi_3 = 1$, $f = 0.8$, $\eta = 1/7$, $\nu = 1/7$, $t_0 = 3.9607$, *and* $S_0 = 1400050000$. (a) $\mu = 0.1539$, $1/\alpha = 6$ *hours*. (b) $\mu = 0.169$, $1/\alpha = 12$ *hours*. (c) $\mu = 0.198$, $1/\alpha = 1$ *day*. (d) $\mu = 0.3$, $1/\alpha = 3$ *days*.

4.2 Comparison of model SEIRUδ with data

In Figures 6 and 7, we plot the graphs of $CR(t)$, $CU(t)$, $R(t)$, and $U(t)$ from the numerical simulation of model **SEIRU**δ. We use $\chi_1 = 0.2254$, $\chi_2 = 0.3762$, $\chi_3 = 1$, $f = 0.8$, $\eta = 1/7$, $t_0 = 3.9607$, and $S_0 = 1,400,050,000$ in both Figures 6 and 7. We take $\nu = 1/5$ in Figure 6 and $\nu = 1/7$ in Figure 7. We take four different values for δ: $1/4$, $1/2$, 1, 3 in both Figure 6 and Figure 7. The value of μ is chosen so that the simulations align with the cumulative reported case data. In this way, we are able to predict the future values of the epidemic from early cumulative reported case data. We see from the simulations the following: for Figure 6, with $\nu = 1/5$, the simulation for $\delta = 1/4$ is almost the same as $\delta = 1/2$, the simulations for $\delta = 1$ and $\delta = 3$ do not agree with the data, and thus, δ cannot be greater than 5 days; for Figure 7, with $\nu = 1/7$, the simulations for $\delta = 1/4$, $\delta = 1/4$, and $\delta = 1$ are almost the same and for $\delta = 3$, the simulation does not agree with the data, and thus, δ cannot be greater than 7 days.

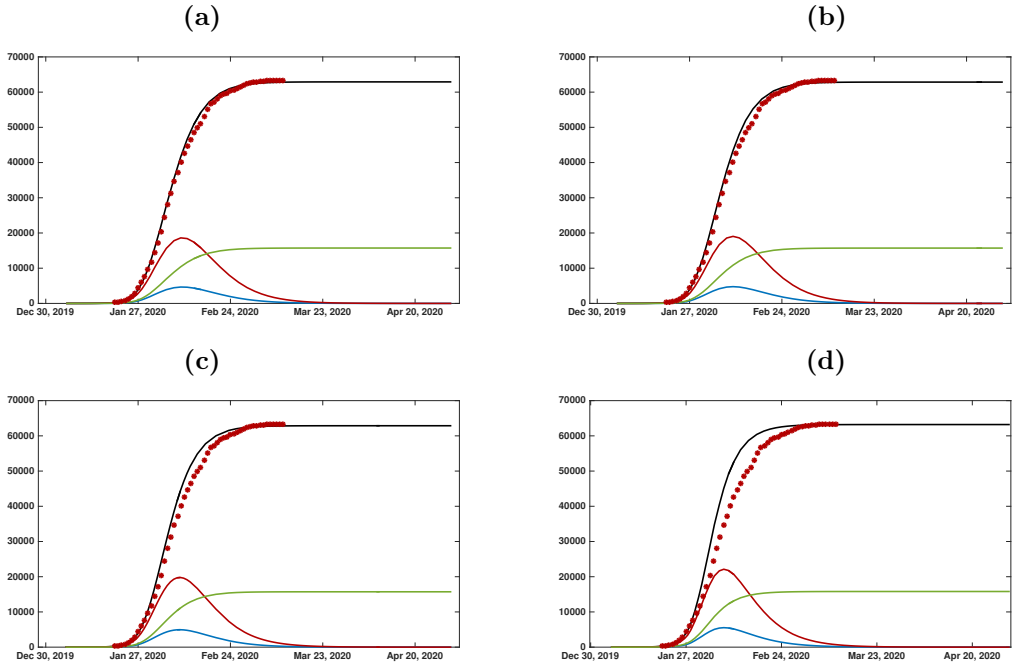

Figure 6: *Graphs of the reported cumulated symptomatic infectious individuals* $t \to CR(t)$ *(black solid line), unreported cumulated symptomatic infectious individuals* $t \to CU(t)$ *(green solid line),* $t \to U(t)$ *(blue solid line), and* $t \to R(t)$ *(red solid line). The red dots are the data of the reported cumulated confirmed cases for mainland China in Table 2. We use* $\chi_1 = 0.2254$, $\chi_2 = 0.3762$, $\chi_3 = 1$, $f = 0.8$, $\eta = 1/7$, $\nu = 1/5$, $t_0 = 3.9607$, *and* $S_0 = 1400050000$. (a) $\mu = 0.1273$, $\delta = 1/4$. (b) $\mu = 0.1432$, $\delta = 1/2$. (c) $\mu = 0.177$, $\delta = 1$. (d) $\mu = 0.373$, $\delta = 3$.

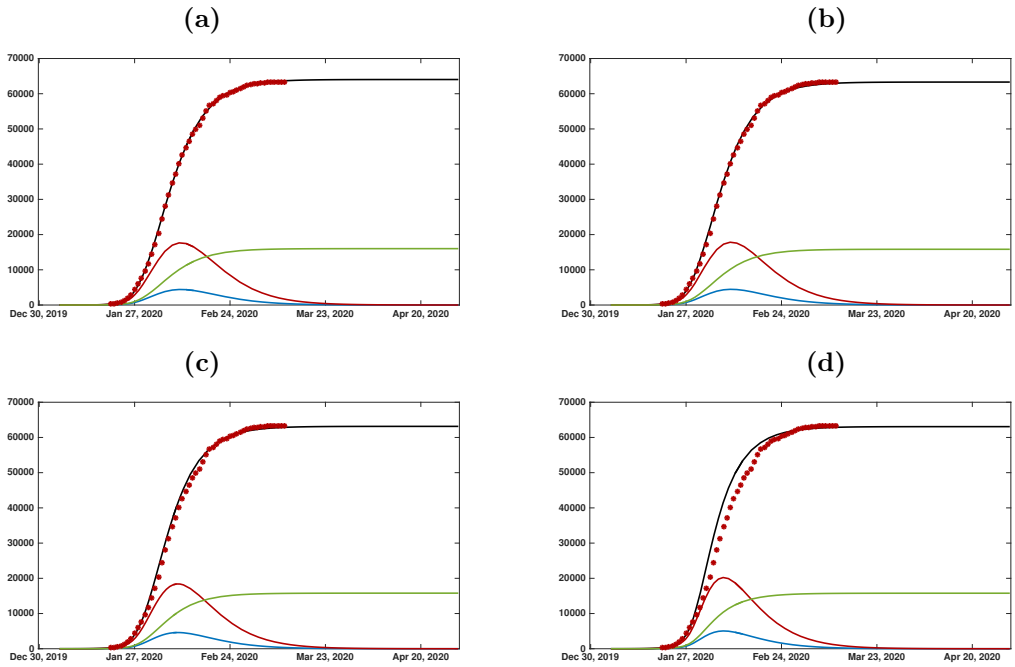

Figure 7: *Graphs of the reported cumulated symptomatic infectious individuals* $t \to CR(t)$ *(black solid line), unreported cumulated symptomatic infectious individuals* $t \to CU(t)$ *(green solid line),* $t \to U(t)$ *(blue solid line), and* $t \to R(t)$ *(red solid line). The red dots are the data of the reported cumulated confirmed cases for mainland China in Table 2. We use* $\chi_1 = 0.2254$, $\chi_2 = 0.3762$, $\chi_3 = 1$, $f = 0.8$, $\eta = 1/7$, $\nu = 1/7$, $t_0 = 3.9607$, *and* $S_0 = 1400050000$. (a) $\mu = 0.1515$, $\delta = 1/4$ day. (b) $\mu = 0.17$, $\delta = 1/2$ day. (c) $\mu = 0.2093$, $\delta = 1$ day. (d) $\mu = 0.454$, $\delta = 3$ days.

5 Discussion

We have developed two models **SEIRU** and **SEIRU**δ of the COVID-19 epidemic in China that incorporate key features of this epidemic: (1) the importance of implementation of major government public restrictions designed to mitigate the severity of the epidemic; (2) the importance of both reported and unreported cases in interpreting the number of reported cases; and (3) the importance of asymptomatic infectious cases in the disease transmission. The main difference from our previous papers [5] and [6] is that we consider a latency period in the two models. In model **SEIRU**, an exposed class E is used to model latency. Newly infected individuals enter the class E, where they are neither symptomatic nor infectious. From this exposed class, noninfectious asymptomatic individuals enter an infectious asymptomatic class I. From I, asymptomatic infectious individuals enter a class R or U where they are symptomatic infectious, and later are reported R or unreported U. In model **SEIRU**δ a time delay is used to model latency. Newly infected individuals enter the class I after a fixed time delay δ, and then proceed through classes R and U.

In order to compare the models **SEIRU** and **SEIRU**δ, we use the Median of Absolute Deviation (MAD) as an indicator:

$$MAD = \text{median}(CR - CR_{Data}), \tag{5.1}$$

where CR_{Data} is the vector of the cumulative number of reported cases from Table 2, while CR is the vector of predicted cumulative number of reported cases of the model. The following table summaries the MAD of models **SEIRU** and **SEIRU**δ:

ODE		DDE	
Figure	**MAD$_{ODE}$**	**Figure**	**MAD$_{DDE}$**
Figure 4 (a)	1717	Figure 6 (a)	662
Figure 4 (b)	1251	Figure 6 (b)	796
Figure 4 (c)	1899	Figure 6 (c)	1095
Figure 4 (d)	1485	Figure 6 (d)	1967
Figure 5 (a)	893	Figure 7 (a)	178
Figure 5 (b)	572	Figure 7 (b)	485
Figure 5 (c)	754	Figure 7 (c)	585
Figure 5 (d)	713	Figure 7 (d)	852

Table 3: *Table of the Median Absolute Deviation of model* **SEIRU** *and* **SEIRU**δ.

From Table 3, it is evident that Figure 5(b) is the best fit for model **SEIRU** (which corresponds to $1/\alpha = 12$ hours) and Figure 7(a) is the best fit for model **SEIRU**δ (which corresponds to $\delta = 6$ hours). This mean that for both models, $\nu = 1/7$ is better than $\nu = 1/5$. Furthermore, Table 3 also indicates that model **SEIRU**δ gives a better prediction than model **SEIRU**. We also deduce that the best exposure period varies between 6 hours for model **SEIRU**δ and 12 hours for model **SEIRU**. Our finding is consistent with Zou [14] where the high viral load observed for COVID-19 was used to explain that the transmission can occur at the early stage of the infection. This gives an explanation for the very short exposure period which give the best fit with data here. This also justifies the fact that in our previous articles [5, 6, 8] we neglected the exposure period.

We summarize in the following tables the predicted turning point and final size, respectively, for models **SEIRU** and **SEIRU**δ.

Figure	**Final size**	**Final size Reported**	**Final size Unreported**	**Turning point of** $R(t), U(t)$	**Turning point of** $I(t)$
Figure 4 (a)	79346	63477	15869	day 39.4	day 33.9
Figure 4 (b)	78241	62593	15648	day 39.0	day 33.7
Figure 4 (c)	79036	63229	15807	day 38.9	day 33.6
Figure 4 (d)	78363	62691	15672	day 38.8	day 33.4
Figure 5 (a)	78688	62950	15738	day 39.4	day 33.6
Figure 5 (b)	79097	63278	15819	day 39.2	day 33.5
Figure 5 (c)	78754	63003	15751	day 38.9	day 33.3
Figure 5 (d)	78805	63044	15761	day 38.9	day 33.1

Table 4: *Predicted turning point and final size of the ODE model* **SEIRU**. The turning point for $I(t)$ $U(t)$ and $R(t)$ is the time t at which these functions reach a maximum.

Figure	**Final size**	**Final size Reported**	**Final size Unreported**	**Turning point of** $R(t), U(t)$	**Turning point of** $I(t)$
Figure 6 (a)	78633	62907	15728	day 39.4	day 33.9
Figure 6 (b)	78562	62850	15712	day 39.1	day 33.7
Figure 6 (c)	78590	62872	15718	day 38.6	day 33.4
Figure 6 (d)	79011	63209	15802	day 37.1	day 32.3
Figure 7 (a)	80043	64035	16008	day 39.6	day 33.8
Figure 7 (b)	79139	63312	15827	day 39.3	day 33.5
Figure 7 (c)	78934	63147	15787	day 38.7	day 33.1
Figure 7 (d)	78839	63071	15768	day 37.0	day 33.8

Table 5: *Predicted turning point and final size of the DDE model* **SEIRU**δ. The turning point for $I(t)$ $U(t)$ and $R(t)$ is the time t at which these functions reach a maximum.

For our model without latency in [6], the turning point of the asymptomatic infectious cases $I(t)$ is approximately day 35 = February 4. The turning point of the reported cases $R(t)$ and the unreported cases $U(t)$ is approximately day 41 = February 10, and the final size of cumulative cases is approximately 79,400 with approximately 63,500 reported, 15,900 unreported. For the ODE model **SEIRU**, Figure 5 (b) (the best one according to MAD) predicts a turning point of the asymptomatic infectious cases $I(t)$ at approximately day 34 = February 3. The turning point of the reported cases $R(t)$ and the unreported cases $U(t)$ is approximately day 39 = February 8, and the final size of cumulative cases is approximately 63,278 reported, 15,819 unreported. For the DDE model **SEIRU**δ, Figure 7 (a) (the best one according to MAD) predicts a turning point of the asymptomatic infectious cases $I(t)$ at approximately day 34 = February 3.

The turning point of the reported cases $R(t)$ and the unreported cases $U(t)$ is approximately day 40 = February 9, and the final size of cumulative cases is approximately 64,035 reported 16,008 unreported.

Our analysis of the latency period for the COVID-19 epidemic in mainland China is applicable to COVID-19 epidemics in other regions.

6 Supplementary material

The part is devoted to the parameters estimation's of the models by assuming the reported cases data are exponentially growing. We assume that this exponential phase occurs before any public intervention. Therefore we assume that

$$\tau(t) = \tau_0$$

for both **SEIRU** and **SEIRU**δ models.

6.1 Method to estimate the parameters and initial conditions of SEIRU from the number of reported cases

In the following we fix f, ν, η, α.

Step 1: Since f, α, η and ν are fixed, we know that

$$\nu_1 = f\nu \text{ and } \nu_2 = (1-f)\,\nu.$$

Step 2: By using equation (3.1) and (3.2) we obtain

$$CR'(t) = \nu_1 I(t) \Leftrightarrow \chi_1\chi_2 exp\,(\chi_2 t) = \nu_1 I(t) \tag{6.1}$$

and

$$\frac{exp\,(\chi_2 t)}{exp\,(\chi_2 t_0)} = \frac{I(t)}{I(t_0)},$$

and therefore

$$I(t) = I_0 exp\,(\chi_2\,(t-t_0))\,. \tag{6.2}$$

Moreover by using (6.2) at $t = t_0$

$$I_0 = \frac{\chi_1\chi_2 exp\,(\chi_2 t_0)}{f\,\nu} = \frac{\chi_3\chi_2}{f\,\nu}. \tag{6.3}$$

Step 3: In order to evaluated the parameters of the model we replace $S(t)$ by $S_0 = 1.40005 \times 10^9$ in the right-hand side of (2.1) (which is equivalent to neglecting the variation of susceptibles due to the epidemic, which is consistent with the fact that $t \to CR(t)$ grows exponentially). Therefore, it remains to estimate τ_0, E_0, and U_0 in the following system:

$$\begin{cases} E'(t) = \tau_0 S_0[I(t) + U(t)] - \alpha E(t) \\ I'(t) = \alpha E(t) - \nu I(t) \\ U'(t) = \nu_2 I(t) - \eta U(t). \end{cases} \tag{6.4}$$

By using the second equation we obtain

$$E(t) = \frac{1}{\alpha}\left[I'(t) + \nu I(t)\right],$$

and therefore by using (6.2) we must have

$$I(t) = I_0 \exp\left(\chi_2\,(t-t_0)\right) \text{ and } E(t) = E_0 \exp\left(\chi_2\,(t-t_0)\right).$$

Then, by using the first equation we obtain

$$U(t) = \frac{1}{\tau_0 S_0}\left[E'(t) + \alpha E(t)\right] - I(t)$$

and then

$$U(t) = U_0 \exp\left(\chi_2\,(t-t_0)\right).$$

By substituting these expressions into (6.4), we obtain

$$\begin{cases} \chi_2 E_0 = \tau_0 S_0[I_0 + U_0] - \alpha E_0 \\ \chi_2 I_0 = \alpha E_0 - \nu I_0 \\ \chi_2 U_0 = \nu_2 I_0 - \eta U_0. \end{cases} \tag{6.5}$$

Remark 6.1 *Here we fix τ_0 in such a way that the value χ_2 becomes the dominant eigenvalue of the linearized equation (6.5), and (E_0, I_0, U_0) is the positve eigenvector associated to this dominant eigenvalue χ_2. Thus, we apply implicitly the Perron-Frobenius theorem. Moreover the exponentially growing solution $(E(t), I(t), U(t))$ that we consider (which is starting very close to $(0,0,0)$) follows the direction of the positive eigenvector associated with the dominant eigenvalue χ_2.*

From the second and third equations of (6.5) we obtain

$$E_0 = \frac{\chi_2 + \nu}{\alpha} I_0,\ U_0 = \frac{\nu_2}{\chi_2 + \eta} I_0,$$

and by substituting these expressions into the first equation of (6.5) we obtain

$$\tau_0 = \frac{(\chi_2 + \alpha)E_0}{S_0[I_0 + U_0]} = \frac{(\chi_2 + \nu)(\chi_2 + \alpha)(\chi_2 + \eta)}{\alpha S_0(\chi_2 + \eta + \nu_2)}. \tag{6.6}$$

6.2 Computation of the basic reproductive number $\mathcal{R}_0$ of model SEIRU

In this section we apply results in Diekmann, Heesterbeek and Metz [2] and Van den Driessche and Watmough [11]. The linearized equation of the infectious part of the system is given by

$$\begin{cases} E'(t) = \tau S_0[I(t) + U(t)] - \alpha E(t) \\ U'(t) = \nu_2 I(t) - \eta U(t). \\ I'(t) = \alpha E(t) - \nu I(t) \end{cases} \tag{6.7}$$

The corresponding matrix is

$$A = \begin{bmatrix} -\alpha & \tau S_0 & \tau S_0 \\ 0 & -\eta & \nu_2 \\ \alpha & 0 & -\nu \end{bmatrix}$$

and the matrix A can be rewritten as

$$A = V - S$$

where

$$V = \begin{bmatrix} 0 & \tau S_0 & \tau S_0 \\ 0 & 0 & \nu_2 \\ \alpha & 0 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \nu \end{bmatrix}.$$

Therefore, the next generation matrix is

$$VS^{-1} = \begin{bmatrix} 0 & \frac{\tau S_0}{\eta} & \frac{\tau S_0}{\nu} \\ 0 & 0 & \frac{\nu_2}{\nu} \\ 1 & 0 & 0 \end{bmatrix}$$

and we obtain that

$$\mathcal{R}_0 = \frac{\tau S_0}{\nu}\left(1 + \frac{\nu_2}{\eta}\right). \tag{6.8}$$

By using (6.6) we obtain

$$\mathcal{R}_0 = \frac{(\chi_2 + \nu)(\chi_2 + \alpha)(\chi_2 + \eta)}{\alpha S_0(\chi_2 + \eta + \nu_2)} \frac{S_0}{\nu}\left(1 + \frac{\nu_2}{\eta}\right)$$

and by using $\nu_2 = (1 - f)\,\nu$ we obtain

$$\mathcal{R}_0 = \frac{(\chi_2 + \nu)(\chi_2 + \alpha)(\chi_2 + \eta)}{\alpha\nu(\chi_2 + \eta + \nu_2)}\left(1 + \frac{(1 - f)\,\nu}{\eta}\right). \tag{6.9}$$

6.3 Method to estimate the parameters of model SEIRUδ from the number of reported cases

Step 1: We have

$$\nu_1 = f\nu \text{ and } \nu_2 = (1 - f)\,\nu.$$

Step 2: By using equation (3.2) we obtain

$$CR'(t) = \nu_1 I(t) \Leftrightarrow \chi_1\chi_2 exp\,(\chi_2 t) = \nu_1 I(t) \tag{6.10}$$

and

$$\frac{exp\,(\chi_2 t)}{exp\,(\chi_2 t_0)} = \frac{I(t)}{I(t_0)},$$

and therefore

$$I(t) = I(t_0) exp\,(\chi_2\,(t - t_0))\,. \tag{6.11}$$

Moreover, by using (6.10) at $t = t_0$,

$$I(t_0) = \frac{\chi_1\chi_2 exp\,(\chi_2 t_0)}{f\,\nu} = \frac{\chi_3\chi_2}{f\,\nu},\quad U(t_0) = \frac{\nu_2}{\chi_2 + \eta}\,I_0. \tag{6.12}$$

Step 3: In order to evaluate the parameters of the model **SEIRU**δ, we replace $S(t)$ by $S_0 = 1.40005 \times 10^9$ in the right-hand side of (2.4) (which is equivalent to neglecting the variation of susceptibles due to the epidemic, and is consistent with the fact that $t \to CR(t)$ grows exponentially). Therefore, it remains to estimate τ_0 and η in the following system:

$$\begin{cases} I'(t) = \tau S_0[I(t-\delta) + U(t-\delta)] - \nu I(t) \\ U'(t) = \nu_2 I(t) - \eta U(t). \end{cases} \tag{6.13}$$

By using the first equation we obtain

$$U(t) = \frac{1}{\tau S_0}\,[I'(t) + \nu I(t)] - I(t),$$

and therefore by using (6.11) we must have

$$I(t) = I(t_0)\exp\left(\chi_2\,(t - t_0)\right) \text{ and } U(t) = U(t_0)\exp\left(\chi_2\,(t - t_0)\right),$$

so by substituting these expressions into (6.13) we obtain

$$\begin{cases} \chi_2 I(t_0) = \tau S_0[I(t_0) + U_(0)]e^{-\chi_2\delta} - \nu I(t_0) \\ \chi_2 U(t_0) = \nu_2 I(t_0) - \eta U(t_0). \end{cases} \tag{6.14}$$

Remark 6.2 *Here we fix τ_0 in such a way that the value χ_2 becomes the dominant eigenvalue of the linearized equation (6.14) and $(I(t_0), U(t_0))$ is the positve eigenvector associated to this dominant eigenvalue χ_2. Thus, we apply implicitly the Perron-Frobenius theorem. Moreover the exponentially growing solution $(I(t), U(t))$ that we consider (which is starting very close to $(0,0)$) follows the direction of the positive eigenvector associated with the dominant eigenvalue χ_2.*

By dividing the first equation of (6.14) by $I(t_0)$ we obtain

$$\chi_2 = \tau S_0\left[1 + \frac{U(t_0)}{I(t_0)}\right] e^{-\chi_2\delta} - \nu$$

and hence

$$\frac{U(t_0)}{I(t_0)} = \frac{(\chi_2 + \nu)}{\tau S_0} e^{\chi_2\delta} - 1. \tag{6.15}$$

By using the second equation of (6.14) we obtain

$$\frac{U(t_0)}{I(t_0)} = \frac{\nu_2}{\eta + \chi_2}. \tag{6.16}$$

By using (6.15) and (6.16) we obtain

$$\tau = \frac{(\chi_2 + \nu)}{S_0} e^{\chi_2\delta} \frac{\eta + \chi_2}{\nu_2 + \eta + \chi_2}. \tag{6.17}$$

By using (6.12) we compute

$$U(t_0) = \frac{\nu_2}{\eta + \chi_2} I(t_0) = \frac{(1-f)\nu}{\eta + \chi_2} I(t_0). \tag{6.18}$$

6.4 Computation of the basic reproductive number $\mathcal{R}_0$ of model SEIRUδ

The linearized equation of the infectious part of the system is given by

$$\begin{cases} I'(t) = \tau S_0[I(t-\delta) + U(t-\delta)] - \nu I(t), \\ U'(t) = \nu_2 I(t) - \eta U(t). \end{cases} \tag{6.19}$$

We apply the results in Thieme [10] to the linear operator $A : D(A) \subset X \to X$ where

$$X = \mathbb{R}^2 \times C([-\delta, 0], \mathbb{R}^2)$$

$$A \begin{pmatrix} 0_{\mathbb{R}} \\ 0_{\mathbb{R}} \\ I \\ U \end{pmatrix} = \begin{pmatrix} -I'(0) + \tau S_0[I(-\delta) + U(-\delta)] - \nu I(0) \\ -U'(0) + \nu_2 I(0) - \eta U(0) \\ I' \\ U' \end{pmatrix}$$

with

$$D(A) = \{0_{\mathbb{R}}\}^2 \times C^1([-\delta, 0], \mathbb{R}^2).$$

We split A into

$$C \begin{pmatrix} 0_{\mathbb{R}} \\ 0_{\mathbb{R}} \\ I \\ U \end{pmatrix} = \begin{pmatrix} \tau S_0[I(-\delta) + U(-\delta)] \\ \nu_2 I(0) \\ 0_C \\ 0_C \end{pmatrix}$$

$$B \begin{pmatrix} 0_{\mathbb{R}} \\ 0_{\mathbb{R}} \\ I \\ U \end{pmatrix} = \begin{pmatrix} -I'(0) - \nu I(0) \\ -U'(0) - \eta U(0) \\ I' \\ U' \end{pmatrix}$$

By using Theorem 3.5 in [4] we obtain that $-B$ is invertible and

$$(-B)^{-1} \begin{pmatrix} \alpha \\ \beta \\ I \\ U \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \tilde{I} \\ \tilde{U} \end{pmatrix}$$

where

$$\tilde{I}(\theta) = \nu^{-1}[\alpha + I(0)] + \int_\theta^0 I(\sigma) d\sigma$$
$$\tilde{U}(\theta) = \eta^{-1}[\beta + U(0)] + \int_\theta^0 U(\sigma) d\sigma$$

Thus we can compute

$$C(-B)^{-1} \begin{pmatrix} \alpha \\ \beta \\ I \\ U \end{pmatrix}$$

and since the range of C is contained into $\mathbb{R}^2 \times \{0_C\}^2$ it is sufficient to compute

$$C(-B)^{-1} \begin{pmatrix} \alpha \\ \beta \\ 0_C \\ 0_C \end{pmatrix} = \begin{pmatrix} \tau S_0[\frac{\alpha}{\nu} + \frac{\beta}{\eta}] \\ \frac{\nu_2}{\nu}\alpha \\ 0_C \\ 0_C \end{pmatrix}.$$

Therefore, the next generation matrix is

$$VS^{-1} = \begin{bmatrix} \frac{\tau S_0}{\nu} & \frac{\tau S_0}{\eta} \\ \frac{\nu_2}{\nu} & 0 \end{bmatrix}$$

which is a Leslie matrix, and the basic reproductive number $\mathcal{R}_0$ is

$$\mathcal{R}_0 = \frac{\tau S_0}{\nu}\left(1 + \frac{\nu_2}{\eta}\right). \tag{6.20}$$

By using (6.17) and $\nu_2 = (1-f)\,\nu$, we obtain

$$\mathcal{R}_0 = \frac{\chi_2+\nu}{\nu}\frac{\eta+\chi_2}{\nu_2+\eta+\chi_2}e^{\chi_2\delta}\left(1+\frac{(1-f)\,\nu}{\eta}\right). \tag{6.21}$$

Conflicts of Interest: Declare conflicts of interest or state "The authors declare no conflict of interest."

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