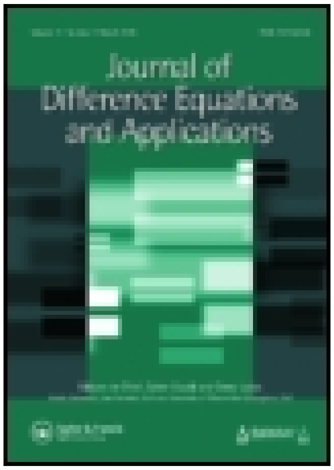

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A Global stabilization result for discrete dynamical systems on a cone

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A Global Stabilization Result for Discrete Dynamical Systems on a Cone

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1 INTRODUCTION AND MAIN RESULTS

In this paper we deal with non-monotone discrete time dynamical systems preserving a cone of a Banach space. Our aim is to obtain a general result concerning the convergence of all solutions of the system to an equilibrium solution, which is what we call a global stabilization result. In Magal [5,6] results are obtained on the existence and uniqueness of non-trivial fixed points, together with a global attractivity result for the case when there is a unique non-trivial fixed point. In this paper we extend the global stabilization result to the case where there are several non-trivial fixed points.

Let K be a cone of a Banach space $(X, \|\cdot\|)$, i.e. K is a closed convex subset of X such that (i) $tK \subset K$ for all $t \geq 0$, and (ii) $-x \notin K$ if $x \in K \setminus \{0\}$. Such cone K induces a partial order on X, denoted by $\leq$ and defined by

$$x \leq y \Leftrightarrow y - x \in K.$$

In the sequel, we denote by X^* the dual space of X (i.e. the space of continuous linear forms on X), and we denote by K^* the dual cone defined by

$$K^* = \{f \in X^*: f(x) \geq 0, \forall x \in K\}.$$

We will say that a cone K of a Banach space $(X, \|\cdot\|)$ is normal if there exists an equivalent norm $\|\cdot\|_1$ on X such that

$$0 \leq x \leq y \Rightarrow \|x\|_1 \leq \|y\|_1, \quad \forall x, y \in K; \tag{1}$$

such norm is said to be monotonic. We will say that $(X, \leq)$ (where $\leq$ is the partial order on X induced by K) is a vector lattice if $\sup(x, y)$ and $\inf(x, y)$ exist for all $(x, y) \in X \times X$. When this is the case we set

$$x^+ = \sup(x, 0), \quad x^- = \sup(-x, 0) \text{ and } |x| = x^+ + x^-$$

for all $x \in X$. One has $x = x^+ - x^-$ (see Schaefer [10, Theorem 1.1, p. 207]). Given a cone K of a Banach space $(X, \|\cdot\|)$ we set $B_K(0, \delta) = \{x \in K: \|x\| \leq \delta\}$ (for $\delta > 0$); for each map $g: A \subset K \to X$ we set

$$\|g\|_{\mathrm{Lip},A} = \sup_{x,y \in A: x \neq y} \|x - y\|^{-1} \|g(x) - g(y)\|.$$

A bounded linear operator $L \in \mathcal{L}(X)$ is said to be positive if $L(K) \subset K$; we denote by $\sigma(L)$ the spectrum, and by $r(L)$ the spectral radius of L. As usual, given a map $T: M \to M$ on a metric space (M, d) we define the iterates $T^m: M \to M$ $(m \in \mathbb{N})$ of T by $T^0 = Id$ and $T^m = T^{m-1} \circ T$ for $m \geq 1$.

In this paper we consider the following type of discrete time dynamical system. Let K be a normal cone of a Banach space $(X, \|\cdot\|)$, such that $(X, \leq)$ is a vector lattice for the partial order $\leq$ induced by K, and let (Λ, d) be a metric space. Let $F: \Lambda \times K \to K$, $(\lambda, x) \mapsto F(\lambda, x) = F_\lambda(x)$ be a continuous map, such that for each $\lambda \in \Lambda$ the mapping $F_\lambda: K \to K$ is asymptotically smooth (as defined by Hale [2, p. 11]) and satisfies $F_\lambda(0) = 0$. We consider then the following initial value problem (for each $\lambda \in \Lambda$):

$$\begin{aligned} x(t+1) &= F_\lambda(x(t)), \quad \forall t \in \mathbb{N}, \\ x(0) &= x_0 \in K. \end{aligned} \tag{2}$$

We will investigate global stabilization results for (2) under the following assumptions:

(H1) There exists some $\lambda_0 \in \Lambda$ such that zero is globally asymptotically stable for F_{λ_0}, and such that F_{λ_0} has a right derivative $D_+F_{\lambda_0}(0) \in \mathcal{L}(X)$ at zero (see Deimling [1, p. 225] for the corresponding definition).

(H2) There exist $v \in K\backslash\{0\}$ and $v^* \in K^*\backslash\{0\}$ such that $v^*(v) = 1$, $D_+F_{\lambda_0}(0)v = v$, $D_+F_{\lambda_0}(0)^*v^* = v^*$, and

$$r((Id - P)D_+F_{\lambda_0}(0)(Id - P)) < 1,$$

where $P \in \mathcal{L}(X)$ is the projection defined by

$$P(x) = v^*(x)v, \quad \forall x \in X.$$

(H3) The map $g : \Lambda \times K \to X$ defined by

$$g(\lambda, x) = F(\lambda, x) - D_+F_{\lambda_0}(0)x, \quad \forall x \in K, \ \forall \lambda \in \Lambda,$$

is such that

$$\lim_{\delta \to 0} \sup_{\lambda \in \Lambda : d(\lambda, \lambda_0) \leq \delta} \|g_\lambda\|_{\mathrm{Lip}, B_K(0,\delta)} = 0.$$

(H4) For each $\lambda \in \Lambda$ there exists $\alpha_\lambda > 0$ such that $B_K(0, \alpha_\lambda)$ is positively invariant for F_λ, and for all $x \in K$ with $\|x\| \geq \alpha_\lambda$ there exists $m = m(x) \in \mathbb{N}$ such that

$$\|F_\lambda^m(x)\| < \alpha_\lambda.$$

(H5) There exists a compact subset $C \subset K$ such that

$$\lim_{R \to 0^+} \delta\left(\cup_{\lambda \in \Lambda : d(\lambda, \lambda_0) \leq R} A_\lambda, C\right) = 0,$$

where $A_\lambda \subset K$ is (for each $\lambda \in \Lambda$) the maximal compact invariant subset of F_λ, and where the distance $\delta(B_1, B_2)$ of $B_1 \subset X$ to $B_2 \subset X$ is defined by $\delta(B_1, B_2) = \sup_{y \in B_1}(\inf_{x \in B_2} \|x - y\|)$.

The following theorem is main result of this paper.

THEOREM 1.1 *Let K be a cone of a Banach space $(X, \|\cdot\|)$ which is normal, and such that $(X, \leq)$ is a vector lattice for the order $\leq$ induced by K. Let (Λ, d) be a metric space. Let $F: \Lambda \times K \to K$ be a continuous map such that for each $\lambda \in \Lambda$ the mapping F_λ is asymptotically smooth and satisfies $F_\lambda(0) = 0$. Assume also that the hypotheses* (H1)–(H5) *are satisfied.*

Then there exists some $\delta > 0$ such that for all $\lambda \in \Lambda$ with $d(\lambda_0, \lambda) \leq \delta$ and for all $x_0 \in K$ the sequence $\{F_\lambda^m(x_0)\}_{m \in \mathbb{N}}$ converges to a fixed point of F_λ in K.

Remark Since $F_{\lambda_0}(0) = 0$ the right derivative $D_+F_{\lambda_0}(0)$ is a positive operator, and therefore one can use the theory of positive bounded linear operators to verify assumption (H2). To be more precise, (H2) can be replaced by the assumption that $D_+F_{\lambda_0}(0)$ is primitive. (A positive operator $A \in \mathcal{L}(X)$ is primitive if for each $x \in K \setminus \{0\}$ there exists some $m \in \mathbb{N}$ such that $A^m x$ belongs to the quasi-interior of K.) Theorems 19.3 (p. 228) and 19.5 (p. 235) of the book by Deimling [1] give further conditions which imply assumption (H2). We also refer to the books by Schaefer [10,11] for more results on the spectral properties of positive bounded linear operators on a Banach lattice.

In order to distinguish between orbits converging to zero and orbits converging to a non-trivial fixed point we will next assume that we can find a positive operator $Q \in \mathcal{L}(X)$ which is a projection (i.e. $Q^2 = Q$) such that $Q(K) \neq \{0\}$ and for which we have that $\lim_{m\to\infty} F_\lambda^m(x) = 0$ for all $x \in K$ such that $Qx = 0$. Our next result gives additional assumptions on Q which will imply that in fact we have for all $\lambda \in \Lambda \setminus \{\lambda_0\}$ and all $x \in K$ that

$$Qx = 0 \Leftrightarrow \lim_{m\to\infty} F_\lambda^m(x) = 0.$$

For $\lambda \neq \lambda_0$ we call then $M_u = \{x \in K: Qx \neq 0\}$ the zero unstable part, and $K_s = \{x \in K: Qx = 0\}$ the zero stable part.

In order to formulate our hypotheses on Q we need to extend the classical notion of ejectivity. Recall that a subset $A \subset M$ is ejective for a mapping $T: M \to M$ on a metric space (M, d) if there exists a neighborhood V of A in M such that of each $x \in V \setminus A$ we can find some $m \in \mathbb{N}$ such that $T^m(x) \in M \setminus V$. Here we will assume that T has a fixed point $\bar{x} \in M$; given a subset $A \subset M$ such that $\bar{x} \in A \cap \partial A$ we will say that $\bar{x}$ is semi-ejective for T on $M \setminus A$ if there exists a neighborhood V of $\bar{x}$ in M such that for each $x \in V \setminus A$ we can find some $m \in \mathbb{N}$ for which $T^m(x) \in M \setminus V$. In particular, if we take $A = \{\bar{x}\}$ this reduces to the ejectivity of $\{\bar{x}\}$.

We have then the following corollary to Theorem 1.1.

COROLLARY 1.2 *Under the assumptions of Theorem* 1.1 *let $Q \in \mathcal{L}(X)$ be a positive operator which is a projection (i.e. $Q^2 = Q$) such that $Q(K) \neq \{0\}$. Let $K_S = \{x \in K\colon Qx = 0\}$ and $M_u = \{x \in K\colon Qx \neq 0\}$, and assume that for all $\lambda \in \Lambda$ the following holds:*

(H6) *M_u is positively invariant for F_λ.*
(H7) *For $\lambda \neq \lambda_0$ the fixed point 0 of F_λ is semi-ejective on $M_u = K \setminus K_S$.*
(H8) *We have for each $x \in K_S$ that*

$$\lim_{m\to\infty} F_\lambda^m(x) = 0.$$

Then there exists a $\delta > 0$ such that if $0 < d(\lambda_0, \lambda) \leq \delta$ and $x_0 \in M_u$, then the sequence $\{F_\lambda^m(x_0)\}_{m\in\mathbb{N}}$ converges to a fixed point of F_λ in M_u.

As an illustration of Corollary 1.2 we consider a discrete time population dynamics model as introduced by Liu and Cohen [3]. Their model leads to the difference equation

$$\begin{aligned}
x_1(t+1) &= \sum_{i=1}^{n}\left[b_i x_i(t)\exp\left(-\sum_{j=1}^{n}\tilde{\gamma}_{ij}x_j(t)\right)\right]\\
x_2(t+1) &= x_1(t)\exp\left(-\left[M_1 + \sum_{j=1}^{n}\gamma_{1j}x_j(t)\right]\right)\\
&\vdots\\
x_n(t+1) &= x_{n-1}(t)\exp\left(-\left[M_{n-1} + \sum_{j=1}^{n}\gamma_{n-1,j}x_j(t)\right]\right)
\end{aligned} \tag{3}$$

for $t \in \mathbb{N}$, together with the initial condition

$$x_i(0) = x_i \geq 0, \quad \forall_i = 1, \ldots, n;$$

the constants b_i, $\tilde{\gamma}_{ij}$, γ_{kl} and M_k (with $i, j, l = 1, \ldots, n$ and $k = 1, \ldots, n-1$) are all non-negative. In their paper [3] Liu and Cohen obtain an existence and uniqueness result for a non-trivial equilibrium solution of (3). In Magal [6] (see Theorem 1.2) a different method is used to obtain the uniqueness of the non-trivial fixed point, while in Magal [5] (see Theorem 1.4) the global attractivity of this non-trivial fixed point is investigated. We also would like to mention the local stability result of Yicang and Cushing [13]. Their general stability result is applied (see Example 3 in [13]) to some simplified version of the Liu and Cohen [3]. To our knowledge no further results on the asymptotical behavior of (3) are available.

We first introduce some notation; we set

$$\lambda = (b_1, b_2, \dots, b_n)^{\mathrm{T}} \in \mathbb{R}^n_+$$

and

$$R(\lambda) = \sum_{j=1}^{n} b_j l_j, \quad \text{with } l_1 = 1 \text{ and } l_i = \prod_{j=1}^{i-1} \exp(-M_j), \quad \forall i = 2, \dots, n.$$

We also define $p_i = \exp(-M_i)$ $(1 \leq i \leq n-1)$, and denote by $F : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $(\lambda, x) \mapsto F(\lambda, x)$ the mapping defined by the right-hand side of (3).

Next fix some $\lambda_0 = (b_1^0, b_2^0, \dots, b_n^0)^{\mathrm{T}} \in \mathbb{R}^n_+$ such that $R(\lambda_0) = 1$; let $n_0 = \max\{k \mid 1 \leq k \leq n, b_k^0 > 0\}$ and assume the following:

(i) $\tilde{\gamma}_{ii} > 0$ for all $i = 1, \dots, n$ for which $b_i^0 > 0$;
(ii) the positive linear operator

$$L_1 = \begin{pmatrix} b_1^0 & b_2^0 & \dots & \dots & b_{n_0}^0 \\ p_1 & 0 & \dots & \dots & 0 \\ 0 & p_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & 0 & p_{n_0-1} & 0 \end{pmatrix} \in M_{n_0}(\mathbb{R})$$

is primitive.

Finally, we set

$$Q = \begin{bmatrix} Id_{\mathbb{R}^{n_0}} & 0 \\ 0 & 0 \end{bmatrix} \in M_n(\mathbb{R}).$$

and

$$\Lambda = \left\{\lambda \in \mathbb{R}^n_+ : \lambda_0 \leq \lambda \leq C_1\lambda_0\right\},$$

for some $C_1 > 1$. One has then the following theorem.

THEOREM 1.3 *Under the assumptions* (i) *and* (ii) *there exists some* $\delta > 0$ *such that for all* $\lambda \in \Lambda \backslash \{\lambda_0\}$ *with* $\|\lambda_0 - \lambda\|_{\mathbb{R}^n_+} \leq \delta$ *and for all* $x_0 \in \mathbb{R}^n_+$ *we have*

$$\lim_{m\to\infty} F^m_\lambda(x_0) = 0 \quad \textit{if } Qx_0 = 0,$$

and

$$\lim_{m\to\infty} F^m_\lambda(x_0) = \bar{x} \quad \textit{if } Qx_0 \neq 0;$$

here $\bar{x} = \bar{x}(x_0, \lambda) \in \mathbb{R}^n_+$ *denotes a fixed point of* F_λ *with* $Q\bar{x} \neq 0$.

Remark The main difference between the previous result and Theorem 6.2 of [5] is the simplication of the proof. Indeed, in [5] the uniqueness of the non-trivial equilibrium is necessary. Such a uniqueness result (proved in [6]) is in most cases difficult to prove. Here we do not use such uniqueness result, so Corollary 1.2 is easier to apply.

The foregoing result can be extended to the population dynamics model introduced by Magal and Pelletier [7]. In this model one considers the difference equation

$$\begin{aligned}
x_1(t+1) &= f_1\left(\sum_{i=1}^{n} b_i x_i(t)\right)\\
x_2(t+1) &= x_1(t)\exp(-(M_1 + q_1E(t)))\\
x_3(t+1) &= x_2(t)\exp(-(M_2 + q_2E(t)))\\
&\vdots\\
x_n(t+1) &= x_{n-1}(t)\exp(-(M_{n-1} + q_{n-1}E(t)))\\
E(t+1) &= f_2\left(\sum_{i=1}^{n-1} W_i x_i(t) q_i \tilde{h}(M_i + q_iE(t))\right)
\end{aligned} \tag{4}$$

for $t \in \mathbb{N}$, and with the initial condition

$$x_i(0) = x_i \geq 0 \quad (\forall i = 1, 2, \ldots, n) \text{ and } E(0) = E_0 \geq 0.$$

Again all constants appearing in (4) are non-negative, while the function $\tilde{h} : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$\tilde{h}(x) = \begin{cases} 1 & \text{if } x = 0, \\ \dfrac{1 - \exp(-x)}{x} & \text{if } x > 0. \end{cases}$$

As for the functions $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, 2)$ we assume that they are bounded, of class C^1, and such that $f_i(0) = 0$; more in particular, we assume that f_1 has the form $f_1(s) = sh_1(s)$, with $h_1 : \mathbb{R}_+ \to [0, 1]$ a strictly decreasing C^1-function satisfying $h_1(0) = 1$ and $\lim_{s\to\infty} h_1(s) = 0$. We define λ, $R(\lambda)$, l_i $(1 \leq i \leq n)$ and p_i $(1 \leq i \leq n-1)$ as in the previous example. The state variable $x = (x_1, \ldots, x_n, E)$ in this case belongs to $\mathbb{R}_+^{n+1}$, and the mapping $F = F(\lambda, x)$ defined by the right-hand side of (4) maps $\mathbb{R}_+^n \times \mathbb{R}_+^{n+1}$ into $\mathbb{R}_+^{n+1}$.

Consider again some $\lambda_0 \in \mathbb{R}_+^n$ such that $R(\lambda_0) = 1$; let $n_0 \max\{k \mid 1 \leq k \leq n, b_k^0 > 0\}$, and define a projection Q on $\mathbb{R}^{n+1}$ by

$$Q = \begin{bmatrix} Id_{\mathbb{R}^{n_0}} & 0 \\ 0 & 0 \end{bmatrix} \in M_{n+1}(\mathbb{R}).$$

Taking $\Lambda \subset \mathbb{R}_+^n$ as before we have then a result similar to Theorem 1.3.

THEOREM 1.4 *Under the assumptions* (ii) *there exists some* $\delta > 0$ *such that for all* $\lambda \in \Lambda\backslash\{\lambda_0\}$ *with* $\|\lambda_0 - \lambda\|_{\mathbb{R}_+^n} \leq \delta$ *and for all* $x_0 \in \mathbb{R}_+^{n+1}$ *we have*

$$\lim_{m\to\infty} F_\lambda^m(x_0) = 0, \quad \textit{if } Qx_0 = 0,$$

and

$$\lim_{m\to\infty} F_\lambda^m(x_0) = \bar{x}, \quad \textit{if } Qx_0 \neq 0,$$

where $\bar{x} = \bar{x}(x_0, \lambda) \in \mathbb{R}_+^{n+1}$ *denotes a fixed point of* F_λ *with* $Q\bar{x} \neq 0$.

2 EXISTENCE OF AN ATTRACTOR

In this section we summarize some results proved in Magal [4] using results on discrete time dissipative dynamical systems as found in the book [2] by Hale.

PROPOSITION 2.1 *Let K be a cone of a Banach space $(X, \|\cdot\|)$, and let (Λ, d) be a metric space. Let $F:\Lambda \times K \to K$ be a continuous map such that F_λ is asymptotically smooth for each $\lambda \in \Lambda$. Assume in addition that F satisfies the hypothesis* (H4). *Then there exists for each $\lambda \in \Lambda$ a maximal compact subset $A_\lambda \subset B_K(0, \alpha_\lambda)$ which is invariant under F_λ, stable and attracting for compact subsets of K. Moreover, A_λ is connected.*

PROPOSITION 2.2 *Under the assumptions of Proposition* 2.1 *suppose moreover that $F_\lambda(0) = 0$ for all $\lambda \in \Lambda$, and that F also satisfies* (H1) *and* (H5). *Then for each $\varepsilon > 0$ we can find some $\eta > 0$ such that $A_\lambda \subset B_K(0, \varepsilon)$ for all $\lambda \in \Lambda$ such that $d(\lambda, \lambda_0) \leq \eta$.*

3 SOME REDUCTION RESULTS

In this section we recall some results due to Vanderbauwhede [12] concerning the reduction of discrete time dynamical systems. Let $(X, \|\cdot\|)$ be a Banach space. We denote by $\mathrm{Lip}_b(X)$ the space of bounded Lipschitz continuous mapping from X into itself, and for each $\eta > 0$ we denote by $(Y_\eta^-(X), \|\cdot\|_{Y_\eta^-(X)})$ the Banach space of sequences $y = \{y_{-p} : p \in \mathbb{N}\} \subset X$ such that

$$\|y\|_{Y_\eta^-(X)} = \sup\{\eta^p \|y_{-p}\| : p \in \mathbb{N}\} < +\infty.$$

Next consider a continuous map $T: X \to X$ satisfying the following hypotheses:

(h1) T has the form $T(x) = Ax + g(x)$, with $A \in \mathcal{L}(X)$ and $g \in \mathrm{Lip}_b(X)$;

(h2) X admits a splitting $X = X_1 \oplus X_2$, where X_1 and X_2 are closed subspaces invariant under A and such that

$$a = \sup_{\lambda \in \sigma(A_1)} |\lambda| < b = \inf_{\lambda \in \sigma(A_2)} |\lambda| \leq 1,$$

$$\text{where } A_i = A|_{X_i} \in \mathcal{L}(X_i), \ i = 1, 2.$$

For each $\eta > 0$ we also define

$$M_\eta = \{ y_0 \in X \mid \exists y \in Y_\eta^-(X)\text{: } y \text{ is a negative } T\text{-orbit through } y_0 \}.$$

Observe that $T(M_\eta) \subset M_\eta$.

The following results were proven by Vanderbauwhede (see [12, Theorems 5, p. 413 and 6, p. 415]).

THEOREM 3.1 *Let $T: X \to X$ be a continuous map satisfying* (h1) *and* (h2), *and let $\eta \in]a, b[$. Then there exists some $C_1 > 0$ (depending only on A and η) such that if $C_1 \|g\|_{\mathrm{Lip}} < 1$ then there exits some $\phi \in \mathrm{Lip}_b(X_2, X_1)$ such that*

$$M_\eta = \{x_2 + \phi(x_2)\text{: } x_2 \in X_2\}.$$

Moreover, for each $x \in M_\eta$ there exists a unique negative T-orbit $y \in Y_\eta^-(X)$ through x.

THEOREM 3.2 *Under the assumptions of Theorem* 3.1 *there exists some $C_2 > 0$ (again depending only on A and η) such that if $C_2 \|g\|_{\mathrm{Lip}} < 1$ we have for each $x \in X$ that*

$$M_\eta \cap \tilde{M}_{1/\eta}(x) = \{H(x)\},$$

where

$$\tilde{M}_{1/\eta}(x) = \left\{ \bar{x} \in X\text{: } \sup_{p \in N} \eta^{-p} \| T^p(x) - T^p(\bar{x}) \| < +\infty \right\},$$

and with $H: X \to M_\eta$ continuous.

Under the hypothesis (h2) we denote by $P \in \mathcal{L}(X)$ the bounded linear projection satisfying

$$\mathrm{Im}(P) = X_2 \text{ and } \mathrm{Ker}(P) = X_1.$$

Also, given a continuous map $T: M \to M$ on a metric space (M, d) and a subset $B \subset M$ we denote by $\omega(B)$ the omega-limit set of B given by (see Hale [2])

$$\omega(B) = \cap_{n \geq 0} Cl\left(\cup_{k \geq n} T^k(B)\right).$$

We now consider some consequences of the Theorems 3.1 and 3.2.

PROPOSITION 3.3 *Under the hypotheses of Theorem 3.1 let M be a compact and connected subset of X and assume the following conditions are satisfied:*

(i) $\dim(X_2) = 1$;
(ii) $C_1\|g\|_{\mathrm{Lip}} < 1$ *and* $C_2\|g\|_{\mathrm{Lip}} < 1$;
(iii) *M is invariant under T, i.e.* $T(M) = M$;
(iv) *there exists some $\bar{x} \in M$ with $T(\bar{x}) = \bar{x}$ and such that $P(\bar{x})$ is an extremal (boundary) point of the interval $P(M)$.*

Then we have for each $z_0 \in X$ for which $\omega(z_0) \subset M$ that there exists some $\bar{x} = \bar{x}(z_0) \in M$ such that $T(\bar{x}) = \bar{x}$ and $\lim_{m\to\infty} T^m(z_0) = \bar{x}$.

The proof of this proposition requires the following lemmas whose proof we omit.

LEMMA 3.4 *Let $J = [c, d] \subset \mathbb{R}$ be a compact interval, and $g : J \to J$ be a continuous map. Assume the following:*

(i) *J is invariant under g, i.e.* $g(J) = J$;
(ii) *for each $x \in J$ there exists exactly one negative g-orbit through x;*
(iii) *either $g(c) = c$ or $g(d) = d$.*

Then g is strictly increasing, $g(c) = c$ and $g(d) = d$.

LEMMA 3.5 *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous map, and let $J = [c, d] \subset \mathbb{R}$ (with $c \leq d$) be a compact interval such that g is strictly increasing on J, $g(c) = c$ and $g(d) = d$. Then we have for each $z_0 \in \mathbb{R}$ for which $\omega(z_0) \subset J$ that there exists some $\bar{x} = \bar{x}(z_0) \in J$ such that $g(\bar{x}) = \bar{x}$ and* $\lim_{m\to+\infty} g^m(z_0) = \bar{x}$.

Proof of Proposition 3.3 It follows from Theorem 3.1 that M_η is the graph of some $\phi \in \mathrm{Lip}_b(X_2, X_1)$. By Theorem 3.2 there exists a continuous map $H : X \to M_\eta$ such that

$$\sup_{m\in\mathbb{N}} \eta^{-m}\|T^m(x) - T^m(H(x))\| < +\infty, \quad \forall x \in X;$$

since $0 < \eta < 1$ this implies that

$$\lim_{m\to+\infty} \|T^m(x) - T^m(H(x))\| = 0, \quad \forall x \in X.$$

Since M is invariant under T each $x \in M$ has a pre-image in M, and hence there exists a negative T-orbit through x which is contained in M.

Since M is bounded such negative T-orbit belongs to $Y_\eta^-(X)$. We conclude that $M \subset M_\eta$ and that

$$x = Px + \phi(Px), \quad \forall x \in M.$$

Moreover, again by Theorem 3.1, there exists a *unique* negative T-orbit through each $x \in M$ which is contained in M. Therefore, taking $J = P(M) \subset X_2$, we can apply Lemmas 3.4 and 3.5 to the mapping $f: X_2 \to X_2$ defined by

$$f(y) = P(T(y + \phi(Py))), \quad \forall y \in X_2.$$

We conclude that if $y_0 \in X_2$ is such that $\omega_f(y_0) \subset P(M)$ then there exists some $\bar{x} = \bar{x}(y_0) \in P(M)$ such that

$$f(\bar{x}) = \bar{x} \text{ and } \lim_{m \to +\infty} f^m(y_0) = \bar{x}.$$

Next fix some $z_0 \in X$ such that $\omega(z_0) \subset M$. Since

$$\lim_{m \to +\infty} \|T^m(z_0) - T^m(H(z_0))\| = 0 \tag{5}$$

it follows that also $\omega(H(z_0)) \subset M$. But $H(z_0) \in M_\eta = \{x_2 + \phi(x_2): x_2 \in X_2\}$ and $T(M_\eta) \subset M_\eta$; therefore we have for each $m \in \mathbb{N}$ that

$$PT^m(H(z_0)) = f^m(PH(z_0))$$

and

$$(Id - P)T^m(H(z_0)) = \phi(f^m(PH(z_0))).$$

Moreover, P is a bounded linear operator, and so

$$\omega(PH(z_0)) \subset P(M).$$

It follows that $\lim_{m \to +\infty} f^m(PH(z_0)) = \bar{x}_P$ for some fixed point $\bar{x}_P = \bar{x}_P(z_0) \in P(M)$ of f; by construction we have then that

$$\lim_{m \to +\infty} T^m(H(z_0)) = \bar{x}_P(z_0) + \phi(\bar{x}_P(z_0)) = \bar{x}(z_0),$$

with $\bar{x}(z_0) \in M$ a fixed point of T. Finally, (5) then implies that also

$$\lim_{m \to +\infty} T^m(z_0) = \bar{x}(z_0),$$

which completes the proof.

4 PROOF OF THEOREM 1.1 AND COROLLARY 1.2

In this section we prove Theorem 1.1, starting with some auxiliary results. First we mention the following lemma from Nussbaum [8, Lemma 2.4, p. 56].

LEMMA 4.1 *Let K be a cone in a Banach space $(X, \|\cdot\|)$. Then $X_K = K - K$ is a Banach space when endowed with the norm $\|\cdot\|_K$ defined by*

$$\|x\|_K = \inf\{\|u\| + \|v\|: u, v \in K, u - v = x\}, \quad \forall x \in X_K.$$

It is not difficult to see that the canonical injection of $(X_K, \|\cdot\|_K)$ into $(X, \|\cdot\|)$ is continuous; indeed we have

$$\|x\| \leq \|x\|_K \text{ if } x \in X_K \quad \text{and} \quad \|x\| = \|x\|_K \text{ if } x \in K.$$

Therefore on K the two induced topologies coincide.

LEMMA 4.2 *Let K be a cone in a Banach space $(X, \|\cdot\|)$. Assume that*

(i) *K is a normal cone of $(X, \|\cdot\|)$;*
(ii) *$(X, \leq)$ is a vector lattice for the partial order $\leq$ induced by K.*

Then the mappings $x \mapsto x^+$ and $x \mapsto x^-$ are Lipschitz continuous.

Proof Since K is normal cone we can without loss of generality assume that the norm $\|\cdot\|$ on X is monotonic, i.e. such that $\|x\| \leq \|y\|$ for all $x, y \in X$ satisfying $0 \leq x \leq y$. Since for each $x \in X$ we have that $x^+ \leq |x|$ and $x^- \leq |x|$ this monotonicity implies that

$$\|x^+\| \leq \| |x| \| \text{ and } \|x^-\| \leq \| |x| \|, \quad \forall x \in X. \tag{6}$$

Define a norm $\|\cdot\|_K$ on $X = K - K$ as in Lemma 4.1; from that lemma we know that $(X, \|\cdot\|_K)$ is a Banach space and that $\|x\| \leq \|x\|_K$ for all $x \in X$. By the bounded inverse theorem it follows that there exists some $C > 0$ such that $\|x\|_K \leq C\|x\|$ for all $x \in X$, and the two norms $\|\cdot\|$ and $\|\cdot\|_K$ on X are equivalent. Therefore, to prove the lemma it is sufficient to prove that the mappings $x \mapsto x^+$ and $x \mapsto x^-$ are Lipschitz continuous in the Banach space $(X, \|\cdot\|_K)$.

It follows from the monotonicity of the norm $\|\cdot\|$ that

$$\|x\|_K = \|x^+\| + \|x^-\|, \quad \forall x \in X. \tag{7}$$

Moreover, we know from Theorem 1.1, p. 207 in Schaefer [10] that

$$|x^+ - y^+| \le |x - y|, \quad \forall x, y \in X. \tag{8}$$

Using (6)–(8) and the triangular inequality we find then for all $x, y \in X$ that

$$\begin{aligned}
\|x^+ - y^+\|_K &= \|(x^+ - y^+)^+\| + \|(x^+ - y^+)^-\| \\
&\le 2\| \, |x^+ - y^+| \, \| \\
&\le 2\| \, |x - y| \, \| \\
&\le 2\big[\|(x - y)^+\| + \|(x - y)^-\|\big] \\
&= 2\|x - y\|_K.
\end{aligned}$$

This shows that the mapping $x \mapsto x^+$ is continuous; the same holds for the mapping $x \mapsto x^- = (-x)^+$, and the proof is complete.

PROPOSITION 4.3 *Under the assumptions of Lemma* 4.2 *let* (Λ, d) *be a metric space,* $\lambda_0 \in \Lambda$ *and* $g : \Lambda \times K \to X$ *a continuous map. Assume in addition that*

$$\lim_{R \to 0^-} \sup_{\lambda \in \Lambda : d(\lambda_0, \lambda) \le R} \|g_\lambda\|_{\mathrm{Lip}, B_K(0,R)} = 0.$$

Then the map $\tilde{g} : \Lambda \times X \to X$ *defined by*

$$\tilde{g}_\lambda(x) = g_\lambda(x^+) - g_\lambda(x^-), \quad \forall x \in X, \ \forall \lambda \in \Lambda,$$

is continuous, and such that

$$\lim_{R \to 0^+} \sup_{\lambda \in \Lambda : d(\lambda_0, \lambda) \le R} \|\tilde{g}_\lambda\|_{\mathrm{Lip}, B_X(0,R)} = 0.$$

Proof It follows from Lemma 4.2 that there exists some $C > 0$ such that

$$\|x^+ - y^+\| \le C|x - y| \text{ and } \|x^- - y^-\| \le C\|x - y\|, \quad \forall x, y \in X.$$

So, assuming that $\|g_\lambda\|_{\mathrm{Lip},B_K(0,R)} < +\infty$, we have for all $x, y \in B_X(0, R/C)$ that

$$\begin{aligned}\|\tilde{g}_\lambda(x) - \tilde{g}_\lambda(y)\| &\leq \|g_\lambda(x^+) - g_\lambda(y^+)\| + \|g_\lambda(x^-) - g_\lambda(y^-)\| \\ &\leq \|g_\lambda\|_{\mathrm{Lip},B_K(0,R)}\big(\|x^+ - y^+\| + \|x^- - y^-\|\big) \\ &\leq 2C\|g_\lambda\|_{\mathrm{Lip},B_K(0,R)}\|x - y\|.\end{aligned}$$

From this the conclusion follows immediately.

PROPOSITION 4.4 *Let $(X, \|\cdot\|)$ be a Banach space, (Λ, d) a metric space, $\lambda_0 \in \Lambda$, and $\tilde{g} : \Lambda \times X \to X$ a continuous map such that*

$$\lim_{R\to 0^+} \sup_{\lambda\in\Lambda: d(\lambda_0,\lambda)\leq R} \|\tilde{g}_\lambda\|_{\mathrm{Lip},B_X(0,R)} = 0.$$

Let $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ be a Lipschitz continuous map such that

$$\begin{aligned}&\chi(s) = 0 && \text{if } s \geq 2,\\ &\chi(s) \in [0,1] && \text{if } 1 < s < 2,\\ &\chi(s) = 1 && \text{if } 0 \leq s \leq 1.\end{aligned}$$

Then the continuous map $\tilde{\tilde{g}} : \mathbb{R}_+^* \times \Lambda \times X \to X$ is defined by

$$\tilde{\tilde{g}}_{\rho,\lambda}(x) = \chi\big(\rho^{-1}\|x\|\big)\tilde{g}_\lambda(x), \quad \forall x \in X,\ \forall\lambda \in \Lambda,\ \forall\rho > 0,$$

has the following properties:

(a) *$\tilde{\tilde{g}}_{\rho,\lambda}(x) = g_\lambda(x)$ for all $x \in B_K(0, \rho)$, all $\lambda \in \Lambda$ and all $\rho > 0$;*
(b) *there exists some $R > 0$ such that $\tilde{\tilde{g}}_{R,\lambda} \in \mathrm{Lip}_b(X)$ for all $\lambda \in \Lambda$ such that $d(\lambda_0, \lambda) \leq R$;*
(c) $\lim_{R\to 0^+} \sup_{\lambda\in\Lambda: d(\lambda_0,\lambda)\leq R} \big\|\tilde{\tilde{g}}_{R,\lambda}\big\|_{\mathrm{Lip}} = 0$.

Proof Let $R_1 > 0$ be sufficiently small such that

$$\|\tilde{g}_\lambda\|_{\mathrm{Lip},B_X(0,R_1)} < +\infty, \quad \forall\lambda \in \Lambda\colon d(\lambda_0, \lambda) \leq R_1.$$

Let

$$\mathrm{Lip}\big(\tilde{\tilde{g}}_{R,\lambda}\big)(x_0) = \limsup_{x\to x_0: x\neq x_0} \|x - x_0\|^{-1}\big\|\tilde{\tilde{g}}_{R,\lambda}(x) - \tilde{\tilde{g}}_{R,\lambda}(x_0)\big\|, \quad \forall x_0 \in X.$$

and $\rho = R_1/2$. It follows immediately from the definition of $\tilde{\tilde{g}}_{\rho,\lambda}$ that

$$\begin{aligned}\left\|\tilde{\tilde{g}}_{\rho,\lambda}(x) - \tilde{\tilde{g}}_{\rho,\lambda}(x_0)\right\| \leq \|\tilde{g}_\lambda(x)\| \left|\chi\left(\rho^{-1}\|x\|\right) - \chi\left(\rho^{-1}\|x_0\|\right)\right| \\ + \chi\left(\rho^{-1}\|x_0\|\right)\|\tilde{g}_\lambda(x) - \tilde{g}_\lambda(x_0)\|;\end{aligned}$$

from the fact that χ is null outside the ball of radius 2 one gets for all $\lambda \in \Lambda$ with $d(\lambda_0, \lambda) \leq R_1$ that

$$\begin{aligned}&\mathrm{Lip}\left(\tilde{\tilde{g}}_{\rho,\lambda}\right)(x_0) = 0, \quad \text{if } \|x_0\| > R_1, \text{ and} \\ &\mathrm{Lip}\left(\tilde{\tilde{g}}_{\rho,\lambda}\right)(x_0) \leq (2\|\chi\|_{\mathrm{Lip}} + 1)\|\tilde{g}_\lambda\|_{\mathrm{Lip},B(0,R_1)}, \quad \text{if } \|x_0\| \leq R_1.\end{aligned}$$

Let $x, y \in X$, let $\lambda \in \Lambda$ be such that $d(\lambda_0, \lambda) \leq R_1$, and consider the Lipschitzian map $\varphi : [0,1] \to \mathbb{R}$ defined by

$$\varphi(t) = \left\|\tilde{\tilde{g}}_{\rho,\lambda}(x + t(y - x)) - \tilde{\tilde{g}}_{\rho,\lambda}(x)\right\|, \quad \forall t \in [0,1].$$

From $\varphi(0) = 0$ and the Lipschitz continuity of φ it follows (see for example Rudin [9]) that there exists an integrable map $h \in L^1(0,1)$ such that

$$\varphi(t) = \varphi(t) - \varphi(0) = \int_0^t h(s)\,\mathrm{d}s \text{ and } \varphi'(t) = h(t), \quad \forall t \in [0,1]\backslash N,$$

where N is a set of zero Lebesque measure. Using the properties of the norm and the definition of φ we also have for each $t, t_0 \in [0,1]$ that

$$|\varphi(t) - \varphi(t_0)| \leq \left\|\tilde{\tilde{g}}_{\rho,\lambda}(x + t(y - x)) - \tilde{\tilde{g}}_{\rho,\lambda}(x + t_0(y - x))\right\|.$$

It follows that

$$|h(t)| \leq \mathrm{Lip}\left(\tilde{\tilde{g}}_{\rho,\lambda}\right)(x + t(y - x))\|y - x\|, \quad \forall t \in [0,1]\backslash N,$$

and by integrating this inequality we get

$$\begin{aligned}\left\|\tilde{\tilde{g}}_{\rho,\lambda}(y) - \tilde{\tilde{g}}_{\rho,\lambda}(x)\right\| = \varphi(1) &\leq \int_0^1 |h(s)|\,\mathrm{d}s \\ &\leq \left[\sup_{s\in[0,1]} \mathrm{Lip}\left(\tilde{\tilde{g}}_{\rho,\lambda}\right)(x + s(y - x))\right]\|y - x\|.\end{aligned}$$

This further implies

$$\|\tilde{\tilde{g}}_{\rho,\lambda}(y) - \tilde{\tilde{g}}_{\rho,\lambda}(x)\| \leq [2\|\chi\|_{\mathrm{Lip}} + 1]\|\tilde{g}_\lambda\|_{\mathrm{Lip},B(0,R_1)}\|y - x\|, \quad \forall x, y \in X, \tag{9}$$

and therefore

$$\tilde{\tilde{g}}_{R_1/2,\lambda} \in \mathrm{Lip}(X), \quad \forall \lambda \in \Lambda\colon d(\lambda, \lambda_0) \leq R_1.$$

Moreover,

$$\tilde{\tilde{g}}_{R_1/2,\lambda}(x) = 0 \quad \text{if } \|x\| > R_1, \text{ and } \tilde{\tilde{g}}_{\rho,\lambda} \in \mathrm{Lip}(X), \quad \forall x \in X,$$

and therefore

$$\tilde{\tilde{g}}_{R_1/2,\lambda} \in \mathrm{Lip}_b(X), \quad \forall \lambda \in \Lambda\colon d(\lambda, \lambda_0) \leq R_1;$$

by using Eq. (9) one also gets

$$\lim_{R\to 0^+} \sup_{\lambda\in\Lambda:d(\lambda_0,\lambda)\leq R} \|\tilde{\tilde{g}}_{R,\lambda}\|_{\mathrm{Lip}} = 0.$$

This completes the proof of Proposition 4.4.

Proof of Theorem 1.1 Let us first consider the map $g : \Lambda \times K \to X$ defined by

$$g(\lambda, x) = F(\lambda, x) - D_+F_{\lambda_0}(0)x, \quad \forall x \in K, \ \forall \lambda \in \Lambda.$$

Under assumption (H3) of Theorem 1.1 one can apply Proposition 4.3 to g, and then Proposition 4.4 to the map $\tilde{g}$. Let $\tilde{\tilde{g}} : \mathbb{R}_+^* \times \Lambda \times X \to X$ be defined as in the Proposition 4.4. Then one knows that this map satisfies the following properties:

(a) $\tilde{\tilde{g}}_{\rho,\lambda}(x) = g_\lambda(x), \forall x \in B_K(0, \rho), \forall \lambda \in \Lambda, \forall \rho > 0$;
(b) $\exists R > 0$: $\tilde{\tilde{g}}_{R,\lambda} \in \mathrm{Lip}_b(X), \forall \lambda \in \Lambda\colon d(\lambda_0, \lambda) \leq R$; and
(c) $\lim_{R\to 0^-} \sup_{\lambda\in\Lambda:d(\lambda_0,\lambda)\leq R} \|\tilde{\tilde{g}}_{R,\lambda}\|_{\mathrm{Lip}} = 0$.

Let P be defined as in assumption (H2), let $a = r((Id - P) \times D_+F_{\lambda_0}(0)(Id - P))$, and fix some $\eta \in]a, 1[$. Let $R_0 > 0$ be fixed such that for all $\lambda \in \Lambda$ with $d(\lambda_0, \lambda) \leq R_0$ we have

$$\tilde{\tilde{g}}_{R_0,\lambda} \in \mathrm{Lip}_b(X), \quad C_1 \left\|\tilde{\tilde{g}}_{R_0,\lambda}\right\|_{\mathrm{Lip}} < 1 \quad \text{and} \quad C_2 \left\|\tilde{\tilde{g}}_{R_0,\lambda}\right\|_{\mathrm{Lip}} < 1,$$

where $C_1 = C_1(D_+F_{\lambda_0}(0), \eta)$ and $C_2 = C_2(D_+F_{\lambda_0}(0), \eta)$ are the constants introduced in Theorems 3.1 and 3.2.

Under the hypothesis (H4) the Proposition 2.1 applies. Then one knows that for each $\lambda \in \Lambda$ there exists a connected subset $A_\lambda \subset B_K(0, \alpha_\lambda)$ which is compact maximal invariant under F_λ, stable for F_λ, and attracts all the compacts subsets of K by F_λ. Under assumptions (H1), (H4), and (H5) Proposition 2.2 applies; it follows that one can find a $\delta_0 > 0$ such that

$$A_\lambda \subset B_K\left(0, \frac{R_0}{2}\right), \quad \forall \lambda \in B_\Lambda(\lambda_0, \delta_0).$$

Next consider for each $\lambda \in \Lambda$ the mapping $\tilde{\tilde{F}}_{R_0,\lambda} : X \to X$ defined by

$$\tilde{\tilde{F}}_{R_0,\lambda}(x) = D_+F_{\lambda_0}(0)x + \tilde{\tilde{g}}_{R_0,\lambda}(x), \quad \forall x \in X.$$

For each $\lambda \in \Lambda \backslash \{\lambda_0\}$ such that $d(\lambda_0, \lambda) \leq \delta_1 = \min(\delta_0, R_0)$ we may then apply Proposition 3.3 to $\tilde{\tilde{F}}_{R_0,\lambda}$, with $M = A_\lambda$ and $\bar{x}_0 = 0$ ($A_\lambda \subset K$ and P is a positive linear operator). It follows that for all such λ and for all $z_0 \in X$ such that

$$\lim_{m \to +\infty} \delta\left(\tilde{\tilde{F}}^m_{R_0,\lambda}(z_0), A_\lambda\right) = \lim_{m \to +\infty} \inf_{y \in A_\lambda} \left\|\tilde{\tilde{F}}^m_{R_0,\lambda}(z_0) - y\right\| = 0,$$

there exists a fixed point $\bar{x}_\lambda(z_0) \in A_\lambda$ of $\tilde{\tilde{F}}_{R_0,\lambda}$ such that

$$\lim_{m \to +\infty} \tilde{\tilde{F}}^m_{R_0,\lambda}(z_0) = \bar{x}_\lambda(z_0).$$

By construction, $\tilde{\tilde{F}}_{R_0,\lambda}$ and F_λ coincide on $B_K(0, R_0)$, and since $A_\lambda \subset B_K(0, R_0/2)$ we conclude that $\bar{x}_\lambda$ is also a fixed point of F_λ.

Fix some $\lambda \in \Lambda \backslash \{\lambda_0\}$ with $d(\lambda_0, \lambda) \leq \delta_1$ and some $z_0 \in K$. Since A_λ attracts all the compact subsets of K one has

$$\lim_{m \to +\infty} \delta(F^m_\lambda(z_0), A_\lambda) = 0.$$

Moreover, since $A_\lambda \subset B_K(0, R_0/2)$ there exists $m_0 \in \mathbb{N}$ such that

$$F_\lambda^m(z_0) \in B_K\left(0, \frac{2R_0}{3}\right), \quad \forall m \geq m_0.$$

But $\tilde{\tilde{F}}_{R_0,\lambda}$ and F_λ coincide on $B_K(0, R_0)$, so

$$F_\lambda^{p+m_0}(z_0) = F_\lambda^p(F_\lambda^{m_0}(z_0)) = \tilde{\tilde{F}}_{R_0,\lambda}^p(F_\lambda^{m_0}(z_0)), \quad \forall p \in \mathbb{N},$$

and

$$\lim_{p \to +\infty} \delta\left(\tilde{\tilde{F}}_{R_0,\lambda}^p(F_\lambda^{m_0}(z_0)), A_\lambda\right) = 0.$$

From this we deduce the there exists a fixed point $\bar{x}_\lambda(z_0) \in A_\lambda$ of F_λ such that

$$\lim_{m \to +\infty} F_\lambda^m(z_0) = \lim_{p \to +\infty} \tilde{\tilde{F}}_{R_0,\lambda}^p(F_\lambda^{m_0}(z_0))(z_0) = \bar{x}_\lambda(z_0).$$

This completes the proof of Theorem 1.1.

Proof of Corollary 1.2 As hypotheses (H1)–(H5) are verified we can apply Theorem 1.1; it follows that there exists some $\delta > 0$ such that for all $\lambda \in \Lambda \backslash \{\lambda_0\}$ such that $d(\lambda_0, \lambda) \leq \delta$, and for all $z_0 \in K$ the sequence $\{F_\lambda^m(z_0)\}_{m \in \mathbb{N}}$ converges to a fixed point $\bar{z}$ of F_λ in K. If $z_0 \in M_u = \{x \in K\colon Qx \neq 0\}$ this fixed point must also belong to M_u; indeed, if $\bar{z} \in K \backslash M_u$ then the hypothesis (H8) implies that $\bar{z} = 0$, which contradicts the hypotheses (H5) and (H6).

5 PROOF OF THEOREMS 1.3 AND 1.4

Proof of Theorem 1.3 We give the principal arguments of the proof. For this particular application of our general results we work in the space $X = \mathbb{R}^n$ with the norm $\|x\|_\infty = \max\{|x_i|\colon i = 1, \ldots, n\}$; we take $K = \mathbb{R}_+^n$, and we recall that $\Lambda = \{\lambda = (b_1, \ldots, b_n)^{\mathrm{T}} \in \mathbb{R}_+^n\colon \lambda_0 \leq \lambda \leq C_1\lambda_0\}$, for some $C_1 > 1$. The mapping F associated to the Eq. (3)

can be written as

$$F(\lambda, x) = L(\lambda, x)x, \quad \forall x \in \mathbb{R}_+^n, \ \forall \lambda \in \Lambda,$$

where $L : \Lambda \times \mathbb{R}_+^n \to M_n(\mathbb{R}_+)$ is a continuous map from $\Lambda \times \mathbb{R}_+^n$ into the set of non-negative matrices which has the form

$$L(\lambda, x) = \begin{bmatrix} L_1(\lambda, x) & 0 \\ L_2(\lambda, x) & L_3(\lambda, x) \end{bmatrix}, \quad \text{with } L_1(\lambda, x) \in M_{n_0}(\mathbb{R}).$$

The choice of $\lambda_0 \in \mathbb{R}_+^n$ and the definition of n_0 imply that $r(L_1(\lambda_0, 0)) = 1$ (since $R(\lambda_0) = \sum_{i=1}^n b_i^0 l_i = 1$) and $r(L_3(\lambda_0, 0)) = 0$. For $\lambda \in \Lambda \backslash \{\lambda_0\}$ we have $\lambda > \lambda_0$, and hence $R(\lambda) > 1$ and $r(L_1(\lambda, 0)) > 1$; also $r(L_3(\lambda, 0)) = 0$ for all such λ. The assumption (ii) implies that $L_1(\lambda_0, 0)$ is primitive; since for all $x \in \mathbb{R}_+^n$ and all $\lambda \in \Lambda$ we can find some sufficiently small constant $C = C(\lambda, x) > 0$ such that $L_1(\lambda, x) \geq CL_1(\lambda_0, 0)$ it follows that $L_1(\lambda, x)$ is primitive for all $(\lambda, x) \in \Lambda \times \mathbb{R}_+^n$.

The hypothesis (H1) can be verified using the assumption (i) in combination with the Liapunov function

$$V(x) = \max\left\{\frac{x_i}{l_i} : i = 1, \ldots, n\right\}.$$

To verify (H2) we observe that $F(\lambda_0, \cdot)$ is clearly right differentiable at zero, with

$$D_+F(\lambda_0, 0) = \begin{bmatrix} L_1(\lambda_0, 0) & 0 \\ L_2(\lambda_0, 0) & L_3(\lambda_0, 0) \end{bmatrix}.$$

It follows then from our preceding remarks that 1 is a simple eigenvalue of $D_+F(\lambda_0, 0)$, and that there are no other eigenvalues in the peripheral spectrum of $D_+F(\lambda_0, 0)$. The theory of non-negative matrices then implies (H2). Assumption (H3) is an immediate consequences of the regularity of the exponential map. It follows from the assumption (i) that

$$\alpha_1 = \min\left\{\frac{\tilde{\gamma}_{ii}}{b_i^0} : i = 1, \ldots, n_0\right\} > 0;$$

setting

$$\alpha_2(\lambda) = \max\left\{\frac{b_i}{b_i^0} : i = 1, \ldots, n_0\right\}, \quad \forall \lambda = (b_1, \ldots, b_n)^{\mathrm{T}} \in \Lambda,$$

we have then that

$$\alpha_2(\lambda) x \exp(-\alpha_1 x) \leq \alpha_2(\lambda) \frac{\exp(-1)}{\alpha_1}, \quad \forall x \geq 0.$$

It then follows that

$$F_\lambda^n(x) \in B_{\mathbb{R}_+^n}\left(0, \alpha_2(\lambda) \frac{\exp(-1)}{\alpha_1}\right) \subset \mathrm{Int}_{\mathbb{R}_+^n}\left[B_{\mathbb{R}_+^n}\left(0, \tilde{M}(\lambda)\right)\right], \quad \forall x \in \mathbb{R}_+^n,$$

where we have put

$$\tilde{M}(\lambda) = \alpha_2(\lambda) \frac{\exp(-1)}{\alpha_1} + 1.$$

Since the ball $B_{\mathbb{R}_+^n}(0, \tilde{M}(\lambda)) = \{x \in \mathbb{R}_+^n : \|x\|_\infty \leq \tilde{M}(\lambda)\}$ is positively invariant under $F(\lambda, \cdot)$ it follows that (H4) is satisfied by taking $\alpha_\lambda = \tilde{M}(\lambda)$. Proposition 2.1 then implies that $A_\lambda \subset B_{\mathbb{R}_+^n}(0, \tilde{M}(\lambda))$ if A_λ is a compact and maximal invariant subset under $F(\lambda, \cdot)$; it follows that (H5) is satisfied by taking

$$C = \cup_{\lambda \in \Lambda : \|\lambda - \lambda_0\|_\infty \leq \delta} B_{\mathbb{R}_+^n}\left(0, \tilde{M}(\lambda)\right).$$

Assumption (H6) is a direct consequence of the fact that the block matrix $L_1(\lambda, x) \in M_{n_0}(\mathbb{R})$ is primitive for all $\lambda \in \Lambda$ and for all $x \in \mathbb{R}_+^n$.

Next we prove assumption (H7). Fix some $\lambda \in \Lambda / \{\lambda_0\}$; since $r(L_1(\lambda, 0)) > 1$ we can also fix some $\gamma \in]0, 1[$ such that $r(L_1(\lambda, 0))\gamma > 1$. The continuity of the map L implies that there exists some $\eta > 0$ such that

$$QF(\lambda, x) \geq \gamma Q L(\lambda, 0) x = \gamma \begin{bmatrix} L_1(\lambda, 0) & 0 \\ 0 & 0 \end{bmatrix} Qx, \quad \forall x \in B_{\mathbb{R}_+^n}(0, \eta).$$

Suppose that $x \in B_{\mathbb{R}^n_-}(0, \eta)$ is such that $Qx \neq 0$ and $F^m(\lambda, x) \in B_{\mathbb{R}^n_-}(0, \eta)$ for all $m \in \mathbb{N}$; then we must have

$$QF^m(\lambda, x) \geq \gamma^m \begin{bmatrix} L_1^m(\lambda, 0) & 0 \\ 0 & 0 \end{bmatrix} Qx,$$

and from $r(L_1(\lambda, 0))\gamma > 1$ and the fact that $L_1(\lambda, 0)$ is primitive we conclude that

$$\lim_{m \to \infty} \|F^m(\lambda, x)\|_\infty = +\infty.$$

This contradiction shows that 0 a semi-ejective fixed point of $F(\lambda, \cdot)$ on M_u.

It remains to verify the assumption (H8). We have

$$F^m(\lambda, x) \leq \begin{bmatrix} 0 & 0 \\ 0 & L_3^m(\lambda, 0) \end{bmatrix} x, \quad \forall x \in \mathbb{R}^n_+ \colon Qx = 0,$$

and since $r(L_3) = 0$ we conclude that

$$\lim_{m \to \infty} F^m(\lambda, x) = 0.$$

We conclude that under the hypotheses (i) and (ii) the Corollary 1.2 applies, and the proof of Theorem 1.3 is complete.

Proof of Theorem 1.4 The proof of this theorem uses the same kind of arguments as in the proof of Theorem 1.3; we leave the detail to the reader.

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