Variation of constants formula and exponential dichotomy for nonautonomous non-densely defined Cauchy problems

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Abstract: In this paper we extend to non-Hille-Yosida case a variation of constants formula for a nonautonomous and nonhomogeneous Cauchy problems obtained first by [22]. By using this variation of constants formula we derive a necessary and sufficient condition for the existence of exponential dichotomy for the evolution family generated by the associated nonautonomous homogeneous problem. We also prove a persistence result of the exponential dichotomy for small perturbations. Finally we illustrate our results by considering two examples. The first example is a parabolic equation with nonlocal and nonautonomous boundary conditions and the second example is an age-structured model which is a hyperbolic equation.

Keywords : Nonautonomous Cauchy problem, non-densely defined Cauchy problem, exponential dichotomy.

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Contents

1 Introduction

In this article we consider the following nonhomogeneous nonautonomous problem

$$\frac{du(t)}{dt} = (A + B(t))u(t) + f(t), \text{ for } t \geq t_0, \text{ and } u(t_0) = x \in \overline{D(A)}, \tag{1.1}$$

where $t_0 \in \mathbb{R}$, $A : D(A) \subset X \to X$ is a linear operator (possibly with non-dense domain that is $\overline{D(A)} \subsetneq X$) on a Banach space $(X, \|\cdot\|)$, $\{B(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(\overline{D(A)}, X)$ is a locally bounded and strongly continuous family of bounded linear operators and $f \in L^1_{loc}(\mathbb{R}, X)$.

Recall that the linear operator A is said to be a Hille-Yosida operator if there exist two constants $\omega \in \mathbb{R}$ and $M \geq 1$, such that the resolvent set of A contains $(\omega, +\infty)$ and the resolvent operator satisfies the usual condition

$$\left\|(\lambda I - A)^{-k}\right\|_{\mathcal{L}(X)} \leq M (\lambda - \omega)^{-k}, \forall \lambda > \omega, \forall k \geq 1.$$

In the following, we will not assume that A is a Hille-Yosida operator since in Assumption 1.1-*i)* the operator norm is taken into $\overline{D(A)}$ instead of X.

Assumption 1.1 *We assume that*

i) There exist two constants $\omega \in \mathbb{R}$ *and* $M \geq 1$, *such that* $(\omega, +\infty) \subset \rho(A)$ *and*

$$\left\|(\lambda I - A)^{-k}\right\|_{\mathcal{L}(\overline{D(A)})} \leq M (\lambda - \omega)^{-k}, \ \forall \lambda > \omega, \ k \geq 1.$$

ii) $\lim_{\lambda\to+\infty} (\lambda I - A)^{-1}x = 0, \ \forall x \in X.$

Set

$$X_0 := \overline{D(A)}$$

and denote by A_0 the part of A on X_0 that is

$$A_0 x = Ax, \forall x \in D(A_0) \text{ and } D(A_0) := \{x \in D(A) : Ax \in X_0\}.$$

Then it is known that Assumption 1.1 is equivalent to $\rho(A) \neq \varnothing$ and A_0 is a densely defined Hille-Yosida linear operator on X_0 (see [33, Lemma 2.1 and Lemma 2.2]). Therefore A_0 generates a strongly continuous semigroup $\{T_{A_0}(t)\}_{t\geq 0} \subset \mathcal{L}(X_0)$.

An important and useful approach to investigate such a non-densely defined Cauchy problem (1.1) is to use the integrated semigroup theory. This notion was introduced first by Arendt [3, 4]. The integrated semigroup generated by A, namely $\{S_A(t)\}_{t\geq 0}$, is a strongly continuous family of bounded linear operators on X that is uniquely defined for each $t \geq 0$ and each $x \in X$ by

$$S_A(t)x = (\mu I - A_0) \int_0^t T_{A_0}(t)(\mu I - A)^{-1} x dt$$

when $\mu > \omega$.

In order to assure the existence of mild solutions of (1.1) we need the following extra assumption.

Assumption 1.2 *For each $\tau > 0$ and each $f \in C([0,\tau], X)$ we assume that there exists $u_f \in C([0,\tau], X_0)$ an integrated (or mild) solution of*

$$\frac{du_f}{dt} = Au_f(t) + f(t), \text{ for } t \geq 0 \text{ and } u_f(0) = 0. \tag{1.2}$$

Moreover we assume that there exists a nondecreasing map $\delta : [0,+\infty) \to [0,+\infty)$ such that

$$\|u_f(t)\| \leq \delta(t) \sup_{s\in[0,t]} \|f(s)\|, \ \forall t \geq 0, \tag{1.3}$$

with

$$\delta(t) \to 0 \text{ as } t \to 0^+.$$

Let $f \in C([0,+\infty), X)$ be fixed. The existence of mild solutions in Assumption 1.2 is equivalent to the continuous time differentiability of the map $t \to (S_A * f)(t)$ from $[0,+\infty)$ into X. Moreover by the uniqueness of the mild solutions of (1.2) (see Thieme [40]) we have

$$u_f(t) = \frac{d}{dt}(S_A * f)(t), \forall t \geq 0,$$

when it exists. Define

$$(S_A \diamond f)(t) := \frac{d}{dt}(S_A * f)(t), \forall t \geq 0.$$

The foregoing Assumption 1.2 needs justification. In fact if A is a Hille-Yosida operator, then Assumption 1.2 holds true as long as $t \to f(t)$ is continuous (see Kellermann and Hieber [26]) and we have the following estimate

$$\|(S_A \diamond f)(t)\| \leq M \int_0^t e^{\omega(t-s)} \|f(s)\| ds$$

and Assumption 1.2 is clearly satisfied. As presented in Magal and Ruan [31], it is possible to obtain some necessary and sufficient conditions on the resolvent operator of A to obtain L^p (for $p \in [1,+\infty)$) estimation on $\|(S_A \diamond f)(t)\|$. Such a result was also investigated by using the notion of bounded semi-p-variation (for $p \in [1,+\infty)$) Thieme [42, Theorem 4.3]. Such a conditions was also investigated by Ducrot, Magal and Prevost [17] in the almost sectorial case.

The following assumption will be required in order to deal with the existence of integrated solutions for the nonhomogeneous equation (1.1).

Assumption 1.3 *Let $\{B(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(X_0, X)$ be a family of bounded linear operators. We assume that $t \to B(t)$ is strongly continuous from $\mathbb{R}$ into $\mathcal{L}(X_0, X)$, that is for each $x \in X_0$ the map $t \to B(t)x$ is continuous from $\mathbb{R}$ into X. We assume that for each integer $n \geq 1$*

$$\sup_{t\in[-n,n]} \|B(t)\|_{\mathcal{L}(X_0,X)} < +\infty.$$

The foregoing assumptions will allow us to obtain the existence of an *evolution family* (see Definition 1.4 below) for the homogeneous Cauchy problem (1.4). Before proceeding let us introduce the notation

$$\Delta := \left\{(t,s) \in \mathbb{R}^2 : t \geq s\right\}$$

and recall the notion of an evolution family.

Definition 1.4 *Let $(Z, \|\cdot\|)$ be a Banach space. A two parameters family of bounded linear operators on Z, $\{U(t,s)\}_{(t,s)\in\Delta}$, is an evolution family if*

i) For each $t, r, s \in \mathbb{R}$ with $t \geq r \geq s$

$$U(t,t) = I_{\mathcal{L}(Z)} \quad \text{and} \quad U(t,r)U(r,s) = U(t,s).$$

ii) For each $x \in Z$, the map $(t,s) \to U(t,s)x$ is continuous from Δ into Z.

If in addition there exist two constants $\widehat{M} \geq 1$ and $\widehat{\omega} \in \mathbb{R}$ such that

$$\|U(t,s)\|_{\mathcal{L}(Z)} \leq \widehat{M} e^{\widehat{\omega}(t-s)}, \ \forall (t,s) \in \Delta,$$

we say that $\{U(t,s)\}_{(t,s)\in\Delta}$ is an exponentially bounded evolution family.

Consider the following homogeneous equation for each $t_0 \in \mathbb{R}$

$$\frac{du(t)}{dt} = (A + B(t))u(t), \ \text{for } t \geq t_0 \text{ and } u(t_0) = x \in X_0. \tag{1.4}$$

By using [31, Theorem 5.2] and [32, Proposition 4.1] we obtain the following Proposition.

Proposition 1.5 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Then the Cauchy problem (1.4) generates a unique evolution family $\{U_B(t,s)\}_{(t,s)\in\Delta} \subset \mathcal{L}(X_0)$. Moreover $U_B(\cdot, t_0)x_0 \in C([t_0, +\infty), X_0)$ is the unique solution of the fixed point problem*

$$U_B(t,t_0)x_0 = T_{A_0}(t - t_0)x_0 + \frac{d}{dt}\int_{t_0}^{t} S_A(t-s)B(s)U_B(s,t_0)x_0 ds, \ \forall t \geq t_0. \tag{1.5}$$

If we assume in addition that

$$\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X_0,X)} < +\infty,$$

then the evolution family $\{U_B(t,s)\}_{(t,s)\in\Delta}$ is exponentially bounded.

The following theorem provides an approximation formula of the solutions of equation (1.1). This is the first main result.

Theorem 1.6 (Variation of constants formula) *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Then for each $t_0 \in \mathbb{R}$, each $x_0 \in X_0$ and each $f \in C([t_0, +\infty], X)$ the unique integrated solution $u_f \in C([t_0, +\infty], X_0)$ of (1.1) is given by*

$$u_f(t) = U_B(t,t_0)x_0 + \lim_{\lambda\to+\infty} \int_{t_0}^{t} U_B(t,s)\lambda(\lambda - A)^{-1} f(s) ds, \ \forall t \geq t_0, \tag{1.6}$$

where the limit exists in X_0. Moreover the convergence in (1.6) is uniform with respect to $t, t_0 \in I$ for each compact interval $I \subset \mathbb{R}$.

This variation of constant formula has been proved by Gühring and Räbiger [22] when A is a Hille-Yosida operator by using extrapolated semigroups. Some extensions of this result have been proved in [10, 11] for nonautonomous Hille-Yosida operators $A(t)$.

The arguments used in [22] strongly use the fact that A is a Hille-Yosida operator. Actually the estimation used is the following

$$\|\lambda(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq M\frac{\lambda}{\lambda-\omega}, \forall \lambda > \max(0,\omega),$$

as a consequence when $\lambda \to +\infty$ we obtain

$$\limsup_{\lambda\to+\infty} \|\lambda(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} < +\infty.$$

In the context of non-Hille-Yosida operator this last estimation is no longer true. Therefore we need to find another approach to prove our result. In Sections 6 and 7 we will consider some examples of parabolic and hyperbolic equations which lead us to the non-Hille-Yosida case. For example, the linear operator A in Sections 6 which comes from a parabolic equation in L^p space, satisfies the following estimates (see Lemma 6.3)

$$0 < \liminf_{\lambda(\in\mathbb{R})\to+\infty} \lambda^{\frac{2p}{1+p}} \left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(X)} \leq \limsup_{\lambda(\in\mathbb{R})\to+\infty} \lambda^{\frac{2p}{1+p}} \left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(X)} < +\infty$$

where $p \in [1,+\infty)$. It follows that when $p > 1$

$$\lim_{\lambda\to+\infty} \lambda \left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(X)} = +\infty.$$

Therefore proving our results will not consist in adapting arguments from the Hille-Yosida case but require several intermediate technical lemmas in order to obtain the limit

$$\lim_{\lambda\to+\infty} \int_{t_0}^{t} U_B(t,s)\lambda\left(\lambda I - A\right)^{-1} f(s)ds,\ t \geq t_0.$$

Our second main result deals with a necessary and sufficient condition for the evolution family (generated by the homogeneous problem associated to system (1.1)) to have an exponential dichotomy. To be more precise let us first recall some definitions and state our result.

Definition 1.7 *Let $(Z, \|\cdot\|_Z)$ be a Banach space. We say that $\{\Pi(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(Z)$ is a strongly continuous family of projectors on Z if*

$$\Pi(t)\Pi(t) = \Pi(t),\ \forall t \in \mathbb{R},$$

and for each $x \in Z$, $t \to \Pi(t)x$ is continuous from $\mathbb{R}$ into Z.

The following notion of exponential dichotomy will be used in this paper. We refer for instance to [18, 19, 22, 23, 24, 29] and the references therein.

Definition 1.8 *Let $(Z, \|\cdot\|_Z)$ be a Banach space. We say that an evolution family $\{U(t,s)\}_{(t,s)\in\Delta} \subset \mathcal{L}(Z)$ has an exponential dichotomy with constant $\kappa \geq 1$ and exponent $\beta > 0$ if and only if the following properties are satisfied* *i) There exist two strongly continuous families of projectors* $\{\Pi^+(t)\}_{t\in\mathbb{R}}$ *and* $\{\Pi^-(t)\}_{t\in\mathbb{R}}$ *on* Z *such that*

$$\Pi^+(t) + \Pi^-(t) = I_{\mathcal{L}(Z)}, \quad \forall t \in \mathbb{R}.$$

Then we define for all $t \geq s$

$$U^+(t,s) := U(t,s)\Pi^+(s) \quad \text{and} \quad U^-(t,s) := U(t,s)\Pi^-(s).$$

ii) For all $(t,s) \in \Delta$ *we have* $\Pi^+(t)U(t,s) = U(t,s)\Pi^+(s)$ *and then* $\Pi^-(t)U(t,s) = U(t,s)\Pi^-(s)$.

iii) For all $(t,s) \in \Delta$ *the restricted linear operator* $U(t,s)\Pi^-(s)$ *is invertible from* $\Pi^-(s)(Z)$ *into* $\Pi^-(t)(Z)$ *with inverse denoted by* $\bar{U}^-(s,t)$ *and we set*

$$U^-(s,t) := \bar{U}^-(s,t)\Pi^-(t).$$

iv) For all $(t,s) \in \Delta$

$$\|U^+(t,s)\|_{\mathcal{L}(Z)} \leq \kappa e^{-\beta(t-s)} \quad \text{and} \quad \|U^-(s,t)\|_{\mathcal{L}(Z)} \leq \kappa e^{-\beta(t-s)}.$$

In the foregoing Definition 1.8 the notations $+$ and $-$ are used to refer respectively the *forward time* and the *backward time*.

Definition 1.9 *Let* $f \in L^1_{loc}(\mathbb{R}, X)$ *be fixed. A function* $u \in C(\mathbb{R}, X_0)$ *is an integrated solution (or a weak solution) of (1.1) if and only if for each* $t \geq t_0$

$$\int_{t_0}^t u(r)dr \in D(A)$$

and

$$u(t) = u(t_0) + A\int_{t_0}^t u(r)dr + \int_{t_0}^t [B(r)u(r) + f(r)]dr.$$

We will say that u *is a mild solution of (1.1) if*

$$u(t) = T_{A_0}(t-t_0)u(t_0) + \frac{d}{dt}\int_{t_0}^t S_A(t-s)[B(s)u(s) + f(s)]ds, \ \forall t \geq t_0.$$

Actually weak and mild notions of solutions are equivalent (see [31, Corollary 2.12.]). Then our second main result splits into the following two theorems.

Theorem 1.10 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Assume in addition that*

$$\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X_0,X)} < +\infty.$$

Then the following assertions are equivalent

i) The evolution family $\{U_B(t,s)\}_{(t,s)\in\Delta}$ *has an exponential dichotomy.*

ii) For each bounded function $f \in C(\mathbb{R}, X)$, there exists a unique bounded integrated solution $u \in C(\mathbb{R}, X_0)$ of (1.1).

Theorem 1.11 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Assume in addition that*

$$\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X_0,X)} < +\infty.$$

If U_B has an exponential dichotomy with exponent $\beta > 0$, then for each $\eta \in [0, \beta)$ and each $f \in BC^\eta(\mathbb{R}, X)$ with

$$BC^\eta(\mathbb{R}, X) := \left\{ f \in C(\mathbb{R}, X) : \|f\|_\eta := \sup_{t\in\mathbb{R}} e^{-\eta|t|}\|f(t)\| < +\infty \right\}$$

there exists a unique integrated solution $u \in BC^\eta(\mathbb{R}, X_0)$ of (1.1) which is given for each $t \in \mathbb{R}$ by

$$u_f(t) = \lim_{\lambda\to+\infty} \left[\int_{-\infty}^t U_B^+(t, s)\lambda(\lambda - A)^{-1} f(s)ds - \int_t^{+\infty} U_B^-(t, s)\lambda(\lambda - A)^{-1} f(s)ds \right]. \tag{1.7}$$

Moreover the following properties hold true

i) The limit (1.7) exists uniformly on compact subsets of $\mathbb{R}$.

ii) If f is bounded and uniformly continuous with relatively compact range, then the limit (1.7) is uniform on $\mathbb{R}$.

iii) For each $\nu \in (-\beta, 0)$ there exists $C(\nu, \kappa, \beta) > 0$ such that

$$\|u_f\|_\eta \leq C(\nu, \kappa, \beta)\|f\|_\eta,\ \forall \eta \in [0, -\nu].$$

The paper is organized as follows. In section 2 we recall some results concerning integrated semigroups and define the notion of integrated solutions for system (1.1). Section 3 is devoted to a the proof of Theorem 1.6 concerning the variation of constants formula. In Section 4 we prove some uniform convergence results. Theorems 1.10 and 1.11 are proved in Section 5. Finally in Sections 6 and 7 we present two examples to illustrate our results.

2 Preliminaries

In the following lemma we summarize some results proved in Magal and Ruan [33, Lemma 2.1 and Lemma 2.2].

Lemma 2.1 *Let Assumption 1.1 be satisfied. Then we have*

$$\rho(A) = \rho(A_0).$$

Moreover, we have the following properties *i) For each* $\lambda > \omega$

$$D(A_0) = (\lambda I - A)^{-1} X_0 \quad \textit{and} \quad (\lambda I - A)^{-1}_{|X_0} = (\lambda I - A_0)^{-1}.$$

ii) $\overline{D(A_0)} = X_0$.

iii) $\lim\limits_{\lambda \to +\infty} \lambda(\lambda I - A)^{-1} x = x, \ \forall x \in X_0$.

Remark 2.2 *It can be easily proved that* $\lim\limits_{\lambda \to +\infty} \lambda(\lambda I - A)^{-1} x = x$ *uniformly for* x *in a relatively compact subset of* X_0*. This property will be often used in this paper.*

Note that if $(A, D(A))$ satisfies Assumption 1.1, then by Lemma 2.1 we have

$$\left\| (\lambda I - A_0)^{-k} \right\|_{\mathcal{L}(X_0)} \leq M \left(\lambda - \omega\right)^{-k}, \ \forall \lambda > \omega, \ k \geq 1 \text{ and } \overline{D(A_0)} = X_0.$$

Therefore $(A_0, D(A_0))$ generates a strongly continuous semigroup $\{T_{A_0}(t)\}_{t \geq 0} \subset \mathcal{L}(X_0)$ with

$$\|T_{A_0}(t)\|_{\mathcal{L}(X_0)} \leq M e^{\omega t}, \quad \forall t \geq 0.$$

The characterization of an integrated semigroup is summarized in the definition below.

Definition 2.3 *Let* $(X, \| \cdot \|)$ *be a Banach space. A family of bounded linear operators* $\{S(t)\}_{t \geq 0}$ *on* X *is called an integrated semigroup if*

i) $S(0)x = 0, \forall x \in X$.

ii) $t \to S(t)x$ *is continuous on* $[0, +\infty)$ *for each* $x \in X$.

iii) For each $t \geq 0$, $S(t)$ *satisfies*

$$S(s)S(t) = \int_0^s [S(r+t) - S(r)] dr, \ \forall s \geq 0.$$

The integrated semigroup $\{S(t)\}_{t \geq 0}$ *is said to be non-degenerate if*

$$S(t)x = 0, \ \forall t \geq 0 \Rightarrow x = 0.$$

Moreover we will say that $(A, D(A))$ *generates an integrated semigroup* $\{S_A(t)\}_{t \geq 0} \subset \mathcal{L}(X, X_0)$ *that is*

$$x \in D(A) \textit{ and } y = Ax \Leftrightarrow S_A(t)x = tx + \int_0^t S(s) y ds, \ \forall t \geq 0.$$

The following result is well known in the context of integrated semigroups.

Proposition 2.4 *Let Assumption 1.1 be satisfied. Then* $(A, D(A))$ *generates a uniquely determined non-degenerate exponentially bounded integrated semigroup with*

$$\|S_A(t)\|_{\mathcal{L}(X)} \leq \hat{M} e^{\hat{\omega} t},$$

where $\hat{M} > 0$, $\hat{\omega} > 0$ *and* $(\hat{\omega}, +\infty) \in \rho(A)$*. Moreover the following properties hold* *i) For each $x \in X$, each $t \geq 0$, each $\mu > \omega$, $S_A(t)x$ is given by*

$$S_A(t)x = (\mu I - A_0) \int_0^t T_{A_0}(s) ds (\mu I - A)^{-1},$$

or equivalently

$$S_A(t)x = \mu \int_0^t T_{A_0}(s)(\mu I - A)^{-1} x ds + [I - T_{A_0}(t)](\mu I - A)^{-1} x.$$

ii) The map $t \to S_A(t)x$ is continuously differentiable if and only if $x \in X_0$ and

$$\frac{dS_A(t)x}{dt} = T_{A_0}(t)x, \ \forall t \geq 0, \ \forall x \in X_0.$$

Next we give the notion of integrated solutions for system (1.1).

Definition 2.5 *Let $t_0 \in \mathbb{R}$ and let $f \in L^1_{loc}((t_0, +\infty), X)$ be fixed. A function $u \in C([t_0, +\infty), X_0)$ is an integrated (or mild) solution of (1.1) if and only if for each $t \geq t_0$*

$$\int_{t_0}^t u(r) dr \in D(A)$$

and

$$u(t) = x + A \int_{t_0}^t u(r) dr + \int_{t_0}^t [B(r)u(r) + f(r)] dr.$$

The following result is a direct consequence of Theorem 2.10 in [32].

Theorem 2.6 *Let Assumptions 1.1 and 1.2 be satisfied. Let $t_0 \in \mathbb{R}$ be fixed. Then for all $f \in C([t_0, +\infty), X)$, the map $t \to (S_A * f(t_0 + \cdot))(t - t_0)$ is continuously differentiable from $[t_0, +\infty)$ into X and satisfies the following properties*

i) $(S_A * f(t_0 + \cdot))(t - t_0) \in D(A)$, $\forall t \geq t_0$.

ii) If we set

$$u(t) := (S_A \diamond f(t_0 + \cdot))(t - t_0), \ \forall t \geq t_0,$$

then the following hold

$$u(t) = A \int_{t_0}^t u(s) ds + \int_{t_0}^t f(s) ds, \ \forall t \geq t_0,$$

and

$$\|u(t)\| \leq \delta(t - t_0) \sup_{s \in [t_0, t]} \|f(s)\|, \ \forall t \geq t_0.$$

iii) For all $\lambda \in (\omega, +\infty)$ we have for each $t \geq t_0$

$$(\lambda I - A)^{-1} \frac{d}{dt} (S_A * f(t_0 + \cdot))(t - t_0) = \int_{t_0}^t T_{A_0}(t - s)(\lambda I - A)^{-1} f(s) ds.$$

As a consequence of *iii)* in Theorem 2.6, we obtain the following approximation formula

$$\frac{d}{dt}\int_{t_0}^{t} S_A(t-s)f(s)ds = \lim_{\lambda\to+\infty}\int_{t_0}^{t} T_{A_0}(t-s)\lambda(\lambda I-A)^{-1}f(s)ds,\ \forall t\geq t_0. \tag{2.1}$$

It also follows that for each $t,h\geq 0$

$$(S_A\diamond f)(t+h) = T_{A_0}(h)(S_A\diamond f)(t) + (S_A\diamond f(t+\cdot))(h). \tag{2.2}$$

As an immediate consequence of Theorem 2.6 we obtain the following lemma.

Lemma 2.7 *Let Assumptions 1.1 and 1.2 be satisfied. Let $f\in C(\mathbb{R},X)$. Then the map $(t,t_0)\to(S_A\diamond f(t_0+\cdot))(t-t_0)$ is continuous from Δ into X.*

Proof. Let $(t,t_0),(s,s_0)\in\Delta$. We have

$$\begin{array}{ll} I & := (S_A\diamond f(t_0+\cdot))(t-t_0)-(S_A\diamond f(s_0+\cdot))(s-s_0) \\ & = (S_A\diamond\big[f(t_0+\cdot)-f(s_0+\cdot)\big])(t-t_0) \\ & +(S_A\diamond f(s_0+\cdot))(t-t_0)-(S_A\diamond f(s_0+\cdot))(s-s_0) \end{array}$$

hence by using (2.2)

$$\begin{array}{ll} I & = (S_A\diamond\big[f(t_0+\cdot)-f(s_0+\cdot)\big])(t-t_0) \\ & +\big[T_{A_0}((t-t_0)-(s-s_0))-I\big](S_A\diamond f(s_0+\cdot))(s-s_0) \\ & +(S_A\diamond f(s_0+(s-s_0)+\cdot))((t-t_0)-(s-s_0)) \end{array}$$

whenever $t-t_0\geq s-s_0$. The result follows by using the uniform continuity of f on bounded intervals. ■

By using [32, Proposition 4.1] we obtain the following lemma.

Lemma 2.8 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Let $t_0\in\mathbb{R}$ be fixed. Then for each $x_0\in X_0$ and $f\in C([t_0,+\infty),X)$ there exists a unique integrated solution $u_f\in C([t_0,+\infty),X_0)$ of (1.1) given by*

$$u_f(t) = T_{A_0}(t-t_0)x_0 + \frac{d}{dt}(S_A*((Bu_f)(t_0+\cdot)+f(t_0+\cdot))(t-t_0),\ \forall t\geq t_0,$$

or equivalently

$$u_f(t) = T_{A_0}(t-t_0)x_0 + (S_A\diamond((Bu_f)(t_0+\cdot)+f(t_0+\cdot))(t-t_0),\ \forall t\geq t_0,$$

where we have used the notation $(Bu_f)(t):=B(t)u_f(t)$ for every $t\geq t_0$.

The next result is due to Magal and Ruan [32, Proposition 2.14] and is one of the main tools in studying integrated solutions for non-Hille-Yosida operators.

Proposition 2.9 *Let Assumption 1.1 be satisfied. Let $\varepsilon>0$ be given and fixed. Then, for each $\tau_\varepsilon>0$ satisfying $M\delta(\tau_\varepsilon)\leq\varepsilon$, we have*

$$\|\frac{d}{dt}(S_A*f)(t)\| \leq C(\varepsilon,\gamma)\sup_{s\in[0,t]} e^{\gamma(t-s)}\|f(s)\|,\ \forall t\geq 0,$$

whenever $\gamma\in(\omega,+\infty)$, $f\in C(\mathbb{R}_+,X)$ with

$$C(\varepsilon,\gamma) := \frac{2\varepsilon\max(1,e^{-\gamma\tau_\varepsilon})}{1-e^{(\omega-\gamma)\tau_\varepsilon}}.$$

3 A variation of constants formula

In this section we will prove the first main result of this paper. It deals with the representation of the integrated solution of (1.1) in term of the evolution family $\{U_B(t,s)\}_{(t,s)\in\Delta}$. This result generalizes [22, Theorem 2.2] to the context of non-Hille-Yosida operator. The proof will be given by using several technical lemmas. Note that a direct consequence of Theorem 1.6 is the following

Corollary 3.1 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Then for each $t_0 \in \mathbb{R}$, each $x_0 \in X_0$ and each $f \in C([t_0,+\infty), X_0)$ the unique integrated solution $u_f \in C([t_0,+\infty), X_0)$ of (1.1) is given by*

$$u_f(t) = U_B(t,t_0)x_0 + \int_{t_0}^{t} U_B(t,s)f(s)ds, \ \forall t \geq t_0.$$

Next we prove some technical lemmas that will be crucial for the proof of Theorem 1.6.

Lemma 3.2 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Then for each $h \in C(\Delta, X)$ the following equality holds*

$$\int_{t_0}^{t} \frac{d}{dt}\left[\int_s^t S_A(t-r)h(r,s)dr\right] ds = \frac{d}{dt}\int_{t_0}^{t} S_A(t-r)\left[\int_{t_0}^{r} h(r,s)ds\right] dr$$

for all $(t,t_0) \in \Delta$.

For convenience we will use the following notation

$$R_\lambda(A) := (\lambda I - A)^{-1}, \forall \lambda \in \rho(A).$$

Proof. Let $t_0 \in \mathbb{R}$ be fixed. Let $s \geq t_0$ be given. Then observing that $h(\cdot,s) \in C([s,+\infty), X)$ one can apply Theorem 2.6 to obtain for all $t \geq s$ and $\lambda > \omega$ that

$$\int_s^t T_{A_0}(t-r)\lambda R_\lambda(A)h(r,s)dr = \lambda R_\lambda(A)\frac{d}{dt}\int_s^t S_A(t-r)h(r,s)dr. \tag{3.1}$$

Thus integrating both sides of (3.1) and using Fubini's theorem we obtain for each $t \geq t_0$ and $\lambda > \omega$ that

$$\begin{aligned}\lambda R_\lambda(A)\int_{t_0}^{t}\left[\frac{d}{dt}\int_s^t S_A(t-r)h(r,s)dr\right] ds &= \int_{t_0}^{t}\left[\int_s^t T_{A_0}(t-r)\lambda R_\lambda(A)h(r,s)dr\right] ds \\ &= \int_{t_0}^{t}\left[\int_{t_0}^{r} T_{A_0}(t-r)\lambda R_\lambda(A)h(r,s)ds\right] dr \\ &= \int_{t_0}^{t} T_{A_0}(t-r)\lambda R_\lambda(A)\left[\int_{t_0}^{r} h(r,s)ds\right] dr.\end{aligned}$$

Now observing that

$$\int_{t_0}^{t}\left[\frac{d}{dt}\int_{s}^{t}S_A(t-r)h(r,s)dr\right]ds \in X_0,\ \forall t\geq t_0,$$

the result follows since we have

$$\lim_{\lambda\to+\infty}\lambda R_\lambda(A)\int_{t_0}^{t}\left[\frac{d}{dt}\int_{s}^{t}S_A(t-r)h(r,s)dr\right]ds=\int_{t_0}^{t}\left[\frac{d}{dt}\int_{s}^{t}S_A(t-r)h(r,s)dr\right]ds$$

for all $t\geq t_0$ and (see equality (2.1))

$$\lim_{\lambda\to+\infty}\int_{t_0}^{t}T_{A_0}(t-r)\lambda R_\lambda(A)\left[\int_{t_0}^{r}h(r,s)ds\right]dr=\frac{d}{dt}\int_{t_0}^{t}S_A(t-r)\left[\int_{t_0}^{r}h(r,s)ds\right]dr$$

for all $t\geq t_0$. ■

Using Lemma 3.2 and Proposition 2.9 we can prove the following technical lemma.

Lemma 3.3 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Let $f\in C(\mathbb{R},X)$. Define for each $\lambda>\omega$ and $(t,t_0)\in\Delta$*

$$v_\lambda(t,t_0):=\int_{t_0}^{t}U_B(t,s)\lambda R_\lambda(A)f(s)ds$$

and

$$w(t,t_0):=\frac{d}{dt}\int_{t_0}^{t}S_A(t-s)f(s)ds=(S_A\diamond f(t_0+\cdot))(t-t_0).$$

Then we have the following properties

i) For each $\lambda>\omega$ and $(t,t_0)\in\Delta$

$$v_\lambda(t,t_0)=\frac{d}{dt}\int_{t_0}^{t}S_A(t-r)B(r)v_\lambda(r,t_0)dr+\lambda R_\lambda(A)w(t,t_0),\ \forall t\geq t_0.$$

ii) If in addition $\sup_{t\in\mathbb{R}}\|B(t)\|_{\mathcal{L}(X_0,X)}<+\infty$, then there exists a constant $\gamma>\max(0,\omega)$ such that for each $\lambda>\omega$ and $(t,t_0)\in\Delta$

$$\sup_{s\in[t_0,t]}e^{-\gamma s}\|v_\lambda(s,t_0)\|\leq 2\sup_{s\in[t_0,t]}e^{-\gamma s}\|\lambda R_\lambda(A)w(s,t_0)\|$$

and since $w(s,t_0)\in X_0$ we have

$$\|\lambda R_\lambda(A)w(s,t_0)\|\leq\frac{M|\lambda|}{\lambda-\omega}\|w(s,t_0)\|,\ \forall s\in[t_0,t].$$

Proof. ***(i)*** By using formula (1.5) we obtain for each $\lambda>\omega$ and $t\geq t_0$ that

$$\begin{aligned}v_\lambda(t,t_0)= & \int_{t_0}^{t}T_{A_0}(t-s)\lambda R_\lambda(A)f(s)ds\\ & +\int_{t_0}^{t}\left[\frac{d}{dt}\int_{s}^{t}S_A(t-r)B(r)U_B(r,s)\lambda R_\lambda(A)f(s)dr\right]ds.\end{aligned}$$

Now note that from Theorem 2.6 we have for each $\lambda > \omega$

$$\int_{t_0}^{t} T_{A_0}(t-s)\lambda R_\lambda(A)f(s)ds = \lambda R_\lambda(A)\frac{d}{dt}\int_{t_0}^{t} S_A(t-s)f(s)ds,\ \forall t \geq t_0, \tag{3.2}$$

and from Lemma 3.2 with $h(r,s) = B(r)U_B(r,s)\lambda R_\lambda(A)f(s)$

$$\int_{t_0}^{t}\left[\frac{d}{dt}\int_{s}^{t} S_A(t-r)h(r,s)dr\right]ds = \frac{d}{dt}\int_{t_0}^{t} S_A(t-r)v_\lambda(r,t_0)dr,\ \forall t \geq t_0. \tag{3.3}$$

Then *(i)* follows by combining (3.2) and (3.3)
(ii) To do this we will make use of Proposition 2.9. Let $\varepsilon > 0$ be given such that

$$2\varepsilon \sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X_0,X)} < \frac{1}{4}. \tag{3.4}$$

Let $\tau_\varepsilon > 0$ be given with $M\delta(\tau_\varepsilon) \leq \varepsilon$. By combining Proposition 2.9 together with *i)* we obtain for each $\lambda > \omega$ and $t \geq t_0$ that

$$\|v_\lambda(t,t_0)\| \leq C(\varepsilon,\gamma)\sup_{s\in[t_0,t]}\left[e^{\gamma(t-s)}\ \|B(s)\|_{\mathcal{L}(X_0,X)}\ \|v_\lambda(s,t_0)\|\right] + \|\lambda R_\lambda(A)w(t,t_0)\|,$$

whenever $\gamma \in (\omega,+\infty)$ with

$$C(\varepsilon,\gamma) := \frac{2\varepsilon\max(1,e^{-\gamma\tau_\varepsilon})}{1-e^{(\omega-\gamma)\tau_\varepsilon}}, \tag{3.5}$$

so that

$$\sup_{s\in[t_0,t]} e^{-\gamma s}\|v_\lambda(s,t_0)\| \leq C(\varepsilon,\gamma)\sup_{s\in\mathbb{R}}\|B(s)\|_{\mathcal{L}(X_0,X)}\sup_{s\in[t_0,t]} e^{-\gamma s}\|v_\lambda(s,t_0)\| + \sup_{s\in[t_0,t]}\|\lambda R_\lambda(A)w(s,t_0)\|.$$

By using (3.5) and (3.4) it is easily seen that one can chose $\gamma > \max(0,\omega)$ large enough such that

$$0 \leq C(\varepsilon,\gamma)\sup_{t\in\mathbb{R}}\|B(t)\|_{\mathcal{L}(X_0,X)} < \frac{1}{2},$$

and *(ii)* follows. ■

The next Lemma will be needed in the following.

Lemma 3.4 *Let Assumptions 1.1 and 1.2 be satisfied. Then for each $a, c \in \mathbb{R}$ with $a < c$ and each $x \in X$, the map $t \to (S_A * x\mathbb{1}_{[a,c)}(\cdot))(t)$ is differentiable on $[0,+\infty)$ and*

$$\frac{d}{dt}(S_A * x\mathbb{1}_{[a,c)}(\cdot))(t) = \begin{cases} 0 & \text{if } c \leq 0 \text{ or } t \leq a, \\ S_A(t-a^+)x & \text{if } c > 0 \text{ and } t \in [a,c), \\ T_{A_0}(t-c)S_A(c-a^+)x & \text{if } c > 0 \text{ and } t \geq c \end{cases}$$

with $a^+ := \max(0,a)$.

Proof. The proof is straightforward. ■

Now we have all the materials to prove Theorem 1.6.

Proof of Theorem 1.6. Since the proof is trivial when $f(t) = 0$ it is sufficient to prove our theorem for $x_0 = 0$. Let $t_0 \in \mathbb{R}$ be fixed. Recalling for each $\lambda > \omega$

$$v_\lambda(t, t_0) = \int_{t_0}^{t} U_B(t, s)\lambda R_\lambda(A) f(s) ds, \ \forall t \geq t_0,$$

we will show that the limit

$$\bar{v}(t, t_0) := \lim_{\lambda \to +\infty} v_\lambda(t, t_0), \ \forall t \geq t_0, \tag{3.6}$$

is well defined and is an integrated solution of

$$\frac{dv(t)}{dt} = [A + B(t)]v(t) + f(t), \ t \geq t_0 \text{ and } v(t_0) = 0. \tag{3.7}$$

First of all note that by Lemma 2.8, problem (3.7) admits a unique integrated solution $v(\cdot, t_0) \in C([t_0, +\infty), X_0)$ satisfying

$$v(t, t_0) = (S_A \diamond (Bv(\cdot, t_0))(t_0 + \cdot))(t - t_0) + (S_A \diamond f(t_0 + \cdot))(t - t_0), \ \forall t \geq t_0, \tag{3.8}$$

where we used the notation $(Bv(\cdot, t_0))(t) = B(t)v(t, t_0)$ for every $t \geq t_0$. Furthermore by Lemma 3.3 we also have for each $\lambda > \omega$ and each $t \geq t_0$

$$v_\lambda(t, t_0) = \frac{d}{dt} \int_{t_0}^{t} S_A(t - r)B(r)v_\lambda(r, t_0)dr + \lambda R_\lambda(A)w(t, t_0), \ \forall t \geq t_0, \tag{3.9}$$

with

$$w(t, t_0) = \frac{d}{dt} \int_{t_0}^{t} S_A(t - s)f(s)ds = (S_A \diamond f(t_0 + \cdot))(t - t_0), \ \forall t \geq t_0. \tag{3.10}$$

Then (3.8) and (3.9) can be rewritten, for each $\lambda > \omega$, as the following system

$$\begin{cases} v_\lambda(t, t_0) = (S_A \diamond (Bv_\lambda(\cdot, t_0))(t_0 + \cdot))(t - t_0) + \lambda R_\lambda(A)w(t, t_0), \ t \geq t_0 \\ v(t, t_0) = (S_A \diamond (Bv(\cdot, t_0))(t_0 + \cdot))(t - t_0) + w(t, t_0), \ t \geq t_0 \end{cases} \tag{3.11}$$

where we used the notation $(Bv_\lambda(\cdot, t_0))(t) = B(t)v_\lambda(t, t_0)$ for every $t \geq t_0$.

Let $I \subset \mathbb{R}$ be a compact subset of $\mathbb{R}$. To show that (3.6) exists uniformly for $t \geq t_0$ in I, we will make use of Proposition 2.9. We have from (3.11) that for each $\lambda > \omega$ and each $t \geq t_0$

$$v_\lambda(t, t_0) - v(t, t_0) = (S_A \diamond (B(v_\lambda(\cdot, t_0) - v(\cdot, t_0)))(t_0 + \cdot))(t - t_0) + [\lambda R_\lambda(A) - I]w(t, t_0), \tag{3.12}$$

with the notation

$$(B(v_\lambda(\cdot,t_0)-v(\cdot,t_0)))(t) := B(t)(v_\lambda(t,t_0)-v(t,t_0)),\ \forall t \geq t_0.$$

Let $\varepsilon > 0$ be given such that

$$2\varepsilon \sup_{t\in I}\|B(t)\|_{\mathcal{L}(X_0,X)} < \frac{1}{4}. \tag{3.13}$$

Let $\tau_\varepsilon > 0$ be given with $M\delta(\tau_\varepsilon) \leq \varepsilon$. Then by using (3.12) and Proposition 2.9 we obtain for each $\lambda > \omega$ and each $t \geq t_0$ with $t, t_0 \in I$ that

$$\begin{aligned}\|v_\lambda(t,t_0)-v(t,t_0)\| \leq\ & C(\varepsilon,\gamma) \sup_{\substack{s\geq t_0\\ s,t_0\in I}} \left[e^{\gamma(t-s)}\ \|B(s)\|_{\mathcal{L}(X_0,X)}\ \|v_\lambda(s,t_0)-v(s,t_0)\|\right] \\ & +\|[\lambda R_\lambda(A)-I]w(t,t_0)\|,\end{aligned}$$

whenever $\gamma \in (\omega,+\infty)$ with

$$C(\varepsilon,\gamma) := \frac{2\varepsilon \max(1,e^{-\gamma\tau_\varepsilon})}{1-e^{(\omega-\gamma)\tau_\varepsilon}},$$

so that

$$\begin{aligned}\sup_{\substack{s\geq t_0\\ s,t_0\in I}} e^{-\gamma s}\|v_\lambda(s,t_0)-v(s,t_0)\| \leq\ & C(\varepsilon,\gamma) \sup_{s\in I}\|B(s)\|_{\mathcal{L}(X_0,X)} \sup_{\substack{s\geq t_0\\ s,t_0\in I}} e^{-\gamma s}\|v_\lambda(s,t_0)-v(s,t_0)\| \\ & + \sup_{\substack{s\geq t_0\\ s,t_0\in I}} \|[\lambda R_\lambda(A)-I]w(s,t_0)\|.\end{aligned}$$

By using (3.13) one can chose $\gamma > 0$ large enough such that

$$0 \leq C(\varepsilon,\gamma) \sup_{t\in I}\|B(t)\|_{\mathcal{L}(X_0,X)} < \frac{1}{2},$$

providing for all $\lambda > \omega$ that

$$\sup_{\substack{s\geq t_0\\ s,t_0\in I}} e^{-\gamma s}\|v_\lambda(s,t_0)-v(s,t_0)\| \leq 2 \sup_{\substack{s\geq t_0\\ s,t_0\in I}} e^{-\gamma s}\|[\lambda R_\lambda(A)-I]w(s,t_0)\|.$$

Hence recalling that the limit $\lim_{\lambda\to+\infty} \lambda R_\lambda(A)y = y$ is uniform on relatively compact sets of X_0 and by observing that $w(\cdot,\cdot)$ maps $I \times I$ into a relatively compact set of X_0, we obtain

$$\lim_{\lambda\to+\infty} \sup_{\substack{s\geq t_0\\ s,t_0\in I}} e^{-\gamma s}\|[\lambda R_\lambda(A)-I]w(s,t_0)\| = 0,$$

that is

$$\lim_{\lambda\to+\infty} \sup_{\substack{s\geq t_0\\ s,t_0\in I}} e^{-\gamma s}\|v_\lambda(s,t_0)-v(s,t_0)\| = 0.$$

Since I is bounded, this implies

$$\lim_{\lambda\to+\infty} \sup_{\substack{s\geq t_0\\ s,t_0\in I}} \|v_\lambda(s,t_0)-v(s,t_0)\| = 0.$$

The proof is complete. ∎

4 A uniform convergence result

Let $BUC(\mathbb{R}, X)$ be the space of all bounded and uniformly continuous functions on $\mathbb{R}$. The next proposition gives a uniform convergence result subject to f belonging to an appropriate subspace of $BUC(\mathbb{R}, X)$.

Proposition 4.1 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Assume in addition that*

$$\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X_0,X)} < +\infty.$$

Let $f \in BUC(\mathbb{R}, X)$ with relatively compact range. Then, for any fixed $t_0 > 0$ the limit

$$\lim_{\lambda\to+\infty} \int_{t-t_0}^{t} U_B(t,s)\lambda R_\lambda(A)f(s)ds,$$

exists uniformly for $t \in \mathbb{R}$.

Proof. Let $t_0 > 0$ be fixed. Recall that for each $\lambda > \omega$ we have

$$v_\lambda(t, t-t_0) = \int_{t-t_0}^{t} U_B(t,s)\lambda R_\lambda(A)f(s)ds, \ \forall t \in \mathbb{R}.$$

Thus by using similar arguments in the proof of *(ii)* in Lemma 3.3 we have for each $t \in \mathbb{R}$, each $\lambda > \omega$ and $\mu > \omega$ that

$$\sup_{s\in[t-t_0,t]} e^{-\gamma s}\|v_\lambda(s,s-t_0)-v_\mu(s,s-t_0)\| \le 2 \sup_{s\in[t-t_0,t]} e^{-\gamma s}\|[\lambda R_\lambda(A)-\mu R_\mu(A)]w(s,s-t_0)\|,$$

with $\gamma > \max(0, \omega)$ (large enough) and

$$w(t_1,t_2) = (S_A \diamond f(t_2+\cdot))(t_1-t_2), \ \forall (t_1,t_2) \in \Delta.$$

Hence for each $t \in \mathbb{R}$, each $\lambda > \omega$ and $\mu > \omega$

$$\begin{aligned}
\|v_\lambda(t,t-t_0) - v_\mu(t,t-t_0)\| &\le 2 \sup_{s\in[t-t_0,t]} e^{\gamma(t-s)}\|[\lambda R_\lambda(A) - \mu R_\mu(A)]w(s,s-t_0)\| \\
&\le 2\, e^{\gamma t_0} \sup_{s\in[t-t_0,t]} \|[\lambda R_\lambda(A) - \mu R_\mu(A)]w(s,s-t_0)\|.
\end{aligned}$$

Then to prove our proposition, it is sufficient to show that

$$\lim_{\lambda,\mu\to+\infty} \sup_{s\in\mathbb{R}} \|[\lambda R_\lambda(A) - \mu R_\mu(A)]w(s,s-t_0)\| = 0.$$

This can be achieved by proving that $w(\cdot, \cdot - t_0)$ maps $\mathbb{R}$ into a relatively compact subset of X_0. To do so we will prove that for any $\varepsilon > 0$, there exists a relatively compact set K such that

$$w(t,t-t_0) \in K + c\varepsilon B_{X_0}(0,1), \ \forall t \in \mathbb{R},$$

for some constant $c > 0$ and $B_{X_0}(0,1)$ the closed unit ball of X_0.

Let $\varepsilon > 0$ be given and fixed. Then since f has its range in a relatively compact subset of X, there exist $\eta = \frac{t_0}{n} > 0$, with $n \in \mathbb{N} \setminus \{0\}$ and a function $g : \mathbb{R} \to X$ such that g is constant on each interval $[k\eta, (k+1)\eta)$, $k \in \mathbb{Z}$. Moreover the range of g is contained in a finite set $K_0 \subset X$ and

$$\sup_{t\in\mathbb{R}} \|f(t) - g(t)\| \leq \varepsilon.$$

Note that g can be written as

$$g(t) = \sum_{k\in\mathbb{Z}} x_k \mathbb{1}_{[k\eta, k\eta+\eta)}(t), \ \forall t \in \mathbb{R},$$

with $x_k \in K_0$ for all $k \in \mathbb{Z}$. Then by Lemma 3.4 it is easy to see that $t \to (S_A * g)(t)$ is differentiable on $[0, +\infty)$ and we can write

$$w(t, t-t_0) = (S_A \diamond g(t - t_0 + \cdot))(t_0) + (S_A \diamond (f-g)(t - t_0 + \cdot))(t_0), \ \forall t \in \mathbb{R}.$$

Let $t \in \mathbb{R}$ be fixed. Note that one can write

$$t = k_0\eta + r, \quad \text{with } r \in [0, \eta) \text{ and } k_0 \in \mathbb{Z},$$

providing that (recalling $t_0 = n\eta$)

$$\begin{aligned}
(S_A \diamond g(t - t_0 + \cdot))(t_0) \quad &= \frac{d}{dt}\int_0^{t_0} S_A(t_0 - s) g(t - t_0 + s) ds \\
&= \frac{d}{dt}\int_{t-t_0}^{t} S_A(t - s) g(s) ds \\
&= \frac{d}{dt}\int_{(k_0-n)\eta+r}^{k_0\eta+r} S_A(t - s) g(s) ds \\
&= \sum_{i=0}^{n-1} \frac{d}{dt}\int_{(k_0-i-1)\eta+r}^{(k_0-i)\eta+r} S_A(k_0\eta + r - s) g(s) ds \\
&= \sum_{i=0}^{n-1} \frac{d}{dt}\left[\int_{(k_0-i-1)\eta+r}^{(k_0-i)\eta} S_A(k_0\eta + r - s) x_{k_0-i-1} ds \right. \\
&\qquad \left. + \int_{(k_0-i)\eta}^{(k_0-i)\eta+r} S_A(k_0\eta + r - s) x_{k_0-i} ds \right],
\end{aligned}$$

therefore we obtain

$$\begin{aligned}
(S_A \diamond g(t-t_0+\cdot))(t_0) &= \sum_{i=0}^{n-1} \left[S_A(i\eta+\eta) - S_A(i\eta+r)\right] x_{k_0-i-1} \\
&\qquad + \sum_{i=0}^{n-1} \left[S_A(i\eta+r) - S_A(i\eta)\right] x_{k_0-i} \\
&= \sum_{i=1}^{n} \left[S_A(i\eta) - S_A(i\eta-\eta+r)\right] x_{k_0-i} \\
&\qquad + \sum_{i=0}^{n-1} \left[S_A(i\eta+r) - S_A(i\eta)\right] x_{k_0-i} \\
&= [S_A(n\eta) - S_A(n\eta-\eta+r)] x_{k_0-n} \\
&\qquad + \sum_{i=1}^{n-1} \left[S_A(i\eta+r) - S_A(i\eta-\eta+r)\right] x_{k_0-i} + S_A(r)x_{k_0} \\
&= T_{A_0}(n\eta-\eta+r)S_A(\eta-r)x_{k_0-n} \\
&\qquad + \sum_{i=1}^{n-1} T_{A_0}(i\eta-\eta+r)S_A(\eta)x_{k_0-i} + S_A(r)x_{k_0},
\end{aligned}$$

so that we can claim that $t \to (S_A \diamond g(t-t_0+\cdot))(t_0)$ has its range in

$$K = \left\{ \sum_{k=0}^{n} T_{A_0}(s_k)S_A(l_k)x_k : 0 \le s_k, l_k \le t_0 \text{ and } x_k \in K_0,\ k=0,\ldots,n \right\}.$$

Then recalling that

$$(t,x) \in [0,+\infty) \times X \to S(t)x \quad \text{and} \quad (t,x) \in [0,+\infty) \times X_0 \to T(t)x$$

are continuous, K is clearly compact.

To complete the proof it remains to give an estimate of $z(\cdot, \cdot - t_0)$ with

$$z(t_1,t_2) := (S_A \diamond (f-g)(t_2+\cdot))(t_1-t_2),\ \forall (t_1,t_2) \in \Delta.$$

By using Proposition 2.9 one obtains

$$\|z(t_1,t_2)\| \le C(1,\gamma_0) \sup_{t \in [0,t_1-t_2]} e^{\gamma_0(t_1-t_2-t)} \|f(t_2+t) - g(t_2+t)\|,\ \forall (t_1,t_2) \in \Delta.$$

with $\gamma_0 > \max(0,\omega)$, $M\delta(\tau_1) \le 1$ and

$$C(1,\gamma_0) := \frac{2\max(1, e^{-\gamma_0\tau_1})}{1 - e^{(\omega-\gamma_0)\tau_1}}.$$

Therefore

$$\begin{aligned}
\sup_{(t_1,t_2)\in\Delta} \|z(t_1,t_2)\| &\le C(1,\gamma_0)e^{\gamma_0(t_1-t_2)} \sup_{t\in\mathbb{R}} \|f(t) - g(t)\| \\
&\le C(1,\gamma_0)e^{\gamma_0(t_1-t_2)}\varepsilon,
\end{aligned}$$

that is

$$\sup_{t\in\mathbb{R}} \|z(t,t-t_0)\| \le C(1,\gamma_0)e^{\gamma_0 t_0}\varepsilon,$$

and the result follows. ■

5 Exponential dichotomy

In this section we consider the complete orbits of the Cauchy problem (1.1). Namely we consider a continuous map $u : (-\infty, +\infty) \to X_0$ as a mild solution of

$$\frac{du(t)}{dt} = (A + B(t))u(t) + f(t), \text{ for } t \in \mathbb{R}. \tag{5.1}$$

This part is devoted to the proof of Theorems 1.10 and 1.11. We will give necessary and sufficient conditions for the evolution family $\{U_B(t,s)\}_{(t,s)\in\Delta} \subset \mathcal{L}(X_0)$ to have an exponential dichotomy. More precisely we will prove that the existence of an exponential dichotomy for $\{U_B(t,s)\}_{(t,s)\in\Delta}$ is equivalent to the existence of an integrated solution $u \in C(\mathbb{R}, X_0)$ for all f in an appropriate subspace of $C(\mathbb{R}, X)$.

In what follows when $\{U(t,s)\}_{(t,s)\in\Delta} \subset \mathcal{L}(Z)$ has an exponential dichotomy we define its associate Green's operator function by

$$\Gamma(t,s) := \begin{cases} U^+(t,s), & \text{if } t \geq s, \\ -U^-(s,t), & \text{if } t < s. \end{cases}$$

Remark 5.1 *It is well known that when $\{U(t,s)\}_{(t,s)\in\Delta} \subset \mathcal{L}(Z)$ has an exponential dichotomy, then for each $x \in Z$, the map $(t,s) \in \mathbb{R}^2 \to U^-(t,s)x$ is continuous from $\mathbb{R}^2$ into Z (see [37, Lemma VI.9.15] or [20, Lemma 9.17]).*

Remark 5.2 *It is easy to obtain from condition i) in Definition 1.8 that*

$$\Pi^+(t)\Pi^-(t) = \Pi^-(t)\Pi^+(t) = 0_{\mathcal{L}(Z)}.$$

We also trivially have

$$U^+(t,t) = \Pi^+(t) \quad \text{and} \quad U^+(t,r)U^+(r,l) = U^+(t,l), \quad \forall t \geq r \geq l,$$

while

$$U^-(t,t) = \Pi^-(t) \quad \text{and} \quad U^-(t,r)U^-(r,l) = U^-(t,l). \quad \forall t, r, l \in \mathbb{R},$$

It follows that U^+ (respectively U^-) is a strongly continuous semiflow (respectively flow). One may also observe that

$$U^-(t,r)U(r,l) = U^-(t,l), \quad \forall (r,t), (r,l) \in \Delta$$

and

$$U^+(t,r)U(r,l) = U^+(t,l), \quad \forall (t,r), (r,l) \in \Delta.$$

Notation 5.3 *Let $(Z, \|\cdot\|)$ be a Banach space. The following weighted Banach space will be used in the following*

$$BC^\eta(\mathbb{R}, Z) := \left\{ f \in C(\mathbb{R}, Z) : \|f\|_\eta := \sup_{t\in\mathbb{R}} e^{-\eta|t|}\|f(t)\|_Z < +\infty \right\}, \quad \eta \geq 0.$$

Note that we have the following continuous embedding

$$BC^{\eta_1}(\mathbb{R}, Z) \subseteq BC^{\eta_2}(\mathbb{R}, Z) \quad \textit{if} \quad \eta_1 \leq \eta_2.$$

If $\eta = 0$ we set

$$BC(\mathbb{R}, Z) := \left\{ f \in C(\mathbb{R}, Z) : \|f\|_\infty := \sup_{t\in\mathbb{R}} \|f(t)\|_Z < +\infty \right\}$$

and define

$$C_0(\mathbb{R}, Z) := \left\{ f \in BC(\mathbb{R}, Z) : \lim_{t\to\pm\infty} f(t) = 0 \right\}.$$

The following result is well known in the context of exponential dichotomy. We refer for instance to [6, 27, 28].

Theorem 5.4 *Let Z be a Banach space. Let $\{U(t,s)\}_{(t,s)\in\Delta} \subset \mathcal{L}(Z)$ be an exponentially bounded evolution family. Consider the following integral equation*

$$u(t) = U(t,t_0)u(t_0) + \int_{t_0}^{t} U(t,s)f(s)ds, \ (t,t_0) \in \Delta. \tag{5.2}$$

Then the following properties are equivalent

- *i)* $\{U(t,s)\}_{(t,s)\in\Delta} \subset \mathcal{L}(Z)$ *has an exponential dichotomy.*
- *ii) Let $\mathcal{F}(\mathbb{R}, Z)$ be the space $BC(\mathbb{R}, Z)$ or $C_0(\mathbb{R}, Z)$. Then for any $f \in \mathcal{F}(\mathbb{R}, Z)$ there exists a unique solution $u \in \mathcal{F}(\mathbb{R}, Z)$ of (5.2).*

Moreover, if $\{U(t,s)\}_{(t,s)\in\Delta}$ has an exponential dichotomy, then for each $f \in \mathcal{F}(\mathbb{R}, Z)$ the unique solution $u \in \mathcal{F}(\mathbb{R}, Z)$ of (5.2) is given by

$$u(t) = \int_{-\infty}^{+\infty} \Gamma(t,s)f(s)ds, \ \forall t \in \mathbb{R},$$

where $\{\Gamma(t,s)\}_{(t,s)\in\mathbb{R}^2} \subset \mathcal{L}(Z)$ is the Green's operator function associated to $\{U(t,s)\}_{(t,s)\in\Delta}$.

In what follows we will give an analogue of Theorem 5.4 for the evolution family $\{U_B(t,s)\}_{(t,s)\in\Delta} \subset \mathcal{L}(X_0)$. To do so we will first prove some estimates.

Proposition 5.5 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Assume in addition that*

$$\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X_0,X)} < +\infty.$$

Then there exists a non-decreasing function $\delta^* : [0,+\infty) \to [0,+\infty)$ with $\delta^*(t) \to 0$ as $t \to 0^+$ such that for each $f \in C(\mathbb{R}, X)$ and $\lambda > w+1$ the map

$$v_\lambda(t,t_0) = \int_{t_0}^{t} U_B(t,s)\lambda R_\lambda(A)f(s)ds, \ (t,t_0) \in \Delta,$$

satisfies

$$\|v_\lambda(t,t_0)\| \le \delta^*(t-t_0) \sup_{s\in[t_0,t]} \|f(s)\|, \ \forall (t,t_0) \in \Delta.$$

Proof. Let $\lambda > \omega$ be given. Thus by Lemma 3.3 there exists $\gamma > \max(0, w)$ large enough (independent of t_0) such that for each $t \geq t_0$

$$\sup_{s \in [t_0, t]} e^{-\gamma s} \|v_\lambda(s, t_0)\| \leq 2 \sup_{s \in [t_0, t]} e^{-\gamma s} \|\lambda R_\lambda(A) w(s, t_0)\|$$

with

$$w(t_1, t_2) = (S_A \diamond f(t_2 + \cdot))(t_1 - t_2), \ \forall (t_1, t_2) \in \Delta.$$

Since $w(t_1, t_2) \in X_0$ for all $(t_1, t_2) \in \Delta$ and by Assumption 1.3

$$\|w(t_1, t_2)\| \leq \delta(t_2 - t_1) \sup_{s \in [t_1, t_2]} \|f(s)\|, \ \forall (t_1, t_2) \in \Delta,$$

it follows that for each $\lambda > \omega$ and $t \geq t_0$ that

$$\begin{aligned} \sup_{s \in [t_0, t]} e^{-\gamma s} \|v_\lambda(s, t_0)\| & \leq 2 \sup_{s \in [t_0, t]} e^{-\gamma s} \|\lambda R_\lambda(A_0) w(s, t_0)\| \\ & \leq 2 \frac{M|\lambda|}{\lambda - \omega} \sup_{s \in [t_0, t]} e^{-\gamma s} \|w(s, t_0)\| \\ & \leq 2 \frac{M|\lambda|}{\lambda - \omega} \sup_{s \in [t_0, t]} \left[e^{-\gamma s} \delta(s - t_0) \sup_{l \in [t_0, s]} \|f(l)\| \right]. \end{aligned}$$

Then by using the fact that δ is non-decreasing and $\gamma > 0$ we obtain for each $\lambda > \omega$ and $t \geq t_0$ that

$$\sup_{s \in [t_0, t]} e^{-\gamma s} \|v_\lambda(s, t_0)\| \leq 2 \frac{M|\lambda|}{\lambda - \omega} e^{-\gamma t_0} \delta(t - t_0) \sup_{s \in [t_0, t]} \|f(s)\|, \ \forall t \geq t_0,$$

providing that

$$\|v_\lambda(t, t_0)\| \leq 2 \frac{M|\lambda|}{\lambda - \omega} e^{\gamma(t - t_0)} \delta(t - t_0) \sup_{s \in [t_0, t]} \|f(s)\|, \ \forall t \geq t_0.$$

This conclusion follows by using the fact that

$$\lambda > \omega + 1 \Rightarrow \frac{|\lambda|}{\lambda - w} < 1 + |\omega|.$$

■

In the rest of this paper, the following assumption will be used.

Assumption 5.6 *Assume that* $\{U_B(t, s)\}_{(t,s) \in \Delta} \subset \mathcal{L}(X_0)$ *has an exponential dichotomy with exponent* $\beta > 0$*, constant* $\kappa \geq 1$ *and strongly continuous projectors* $\{\Pi_B^+(t)\}_{t \in \mathbb{R}} \subset \mathcal{L}(X_0)$ *and* $\{\Pi_B^-(t)\}_{t \in \mathbb{R}} \subset \mathcal{L}(X_0)$.

Note that if $\{U_B(t, s)\}_{(t,s) \in \Delta}$ has an exponential dichotomy, then combining Remark 5.2 and condition *iv)* in Definition 1.8 we have

$$\sup_{t \in \mathbb{R}} \|\Pi_B^+(t)\|_{\mathcal{L}(Z)} \leq \kappa \text{ and } \sup_{t \in \mathbb{R}} \|\Pi_B^-(t)\|_{\mathcal{L}(Z)} \leq \kappa. \tag{5.3}$$

Proposition 5.7 *Let Assumption 1.1 be satisfied. Let* $\{U(t,s)\}_{(t,s)\in\Delta} \subset \mathcal{L}(X_0)$ *be a given evolution family such that there exist* $\widehat{M} \geq 1$, $\widehat{\omega} \in \mathbb{R}$ *and*

$$\|U(t,s)\|_{\mathcal{L}(X_0)} \leq \widehat{M} e^{\widehat{\omega}(t-s)},\ \forall (t,s) \in \Delta.$$

Assume that for each $f \in C(\mathbb{R}, X)$ *the map*

$$v_\lambda(t,t_0) = \int_{t_0}^{t} U(t,s)\lambda R_\lambda(A) f(s) ds,\ (t,t_0) \in \Delta,$$

satisfies

$$\|v_\lambda(t,t_0)\| \leq \delta^{**}(t-t_0) \sup_{s\in[t_0,t]} \|f(s)\|,\ \forall (t,t_0) \in \Delta,$$

with $\delta^{**} : [0,+\infty) \to [0,+\infty)$ *a non-decreasing function such that* $\delta^{**}(t) \to 0$ *as* $t \to 0^+$. *Let* $\varepsilon > 0$ *be given and fixed. Then, for each* $\tau_\varepsilon > 0$ *satisfying* $\widehat{M}\delta^{**}(\tau_\varepsilon) \leq \varepsilon$ *and each* $\lambda > \omega + 1$ *we have*

$$\|v_\lambda(t,t_0)\| \leq \widetilde{C}(\varepsilon,\gamma,\widehat{\omega},\widehat{M}) \sup_{s\in[t_0,t]} e^{\gamma(t-s)} \|f(s)\|,\ \forall (t,t_0) \in \Delta,$$

whenever $\gamma > \widehat{\omega}$ *and* $f \in C(\mathbb{R}, X)$ *with*

$$\widetilde{C}(\varepsilon,\gamma,\widehat{\omega},\widehat{M}) := \widehat{M} e^{\max(0,\widehat{\omega})\tau_\varepsilon} \frac{2\varepsilon \max(1, e^{-\gamma\tau_\varepsilon})}{1 - e^{(\widehat{\omega}-\gamma)\tau_\varepsilon}}.$$

Proof. Without loss of generality we can assume that $t_0 = 0$. Let $\tau_\varepsilon > 0$ be given such that

$$\widehat{M}\delta^{**}(s) \leq \varepsilon,\ \forall s \in [0,\tau_\varepsilon].$$

Let $t \geq 0$ be fixed. Then since we can write $t = n\tau_\varepsilon + \theta$ with $\theta \in [0,\tau_\varepsilon)$ and $n \in \mathbb{N}$ we obtain

$$\begin{aligned}
v_\lambda(t,0) &= \int_0^t U(t,s)\lambda R_\lambda(A) f(s) ds \\
&= \sum_{k=0}^{n-1} \int_{k\tau_\varepsilon}^{(k+1)\tau_\varepsilon} U(t,s)\lambda R_\lambda(A) f(s) ds + \int_{n\tau_\varepsilon}^{t} U(t,s)\lambda R_\lambda(A) f(s) ds \\
&= \sum_{k=0}^{n-1} U(t,(k+1)\tau_\varepsilon) \int_{k\tau_\varepsilon}^{(k+1)\tau_\varepsilon} U((k+1)\tau_\varepsilon,s)\lambda R_\lambda(A) f(s) ds \\
&\qquad + U(t,n\tau_\varepsilon) \int_{n\tau_\varepsilon}^{t} U(n\tau_\varepsilon,s)\lambda R_\lambda(A) f(s) ds \\
&= \sum_{k=0}^{n-1} U(t,(k+1)\tau_\varepsilon) v_\lambda((k+1)\tau_\varepsilon,k\tau_\varepsilon) + U(t,n\tau_\varepsilon) v_\lambda(t,n\tau_\varepsilon),
\end{aligned}$$

so that

$$v_\lambda(t,0) = U(t,n\tau_\varepsilon) \sum_{k=0}^{n-1} U(n\tau_\varepsilon,(k+1)\tau_\varepsilon) v_\lambda((k+1)\tau_\varepsilon,k\tau_\varepsilon) + U(t,n\tau_\varepsilon) v_\lambda(t,n\tau_\varepsilon). \quad (5.4)$$

Next observe that for all $(r_0, r_1) \in \Delta$ and $r \geq r_0$ with $0 \leq r_0 - r_1 \leq \tau_\varepsilon$ we have

$$\begin{aligned}
\|U(r, r_0)v_\lambda(r_0, r_1)\| &\leq \widehat{M} e^{\widehat{\omega}(r-r_0)} \|v_\lambda(r_0, r_1)\| \\
&\leq e^{\widehat{\omega}(r-r_0)} \widehat{M} \delta^*(r_0 - r_1) \sup_{s\in[r_1,r_0]} \|f(s)\| \\
&\leq e^{\widehat{\omega}(r-r_0)} \varepsilon \sup_{s\in[r_1,r_0]} \|f(s)\|.
\end{aligned} \tag{5.5}$$

Let $\gamma > \widehat{\omega}$ be fixed. Set $\varepsilon_1 := \max(1, e^{-\gamma\tau_\varepsilon})$. Let $k \in \mathbb{N}$ and $r \in [k\tau_\varepsilon, (k+1)\tau_\varepsilon]$ be given and fixed. Then if $\gamma \geq 0$ we have

$$\varepsilon \sup_{s\in[k\tau_\varepsilon,r]} \|f(s)\| = \varepsilon \sup_{s\in[k\tau_\varepsilon,r]} e^{-\gamma s} e^{\gamma s} \|f(s)\| \leq \varepsilon_1 e^{\gamma r} \sup_{s\in[k\tau_\varepsilon,r]} e^{-\gamma s} \|f(s)\| \tag{5.6}$$

while if $\gamma < 0$

$$\begin{aligned}
\varepsilon \sup_{s\in[k\tau_\varepsilon,r]} \|f(s)\| &= \varepsilon \sup_{s\in[k\tau_\varepsilon,r]} e^{-\gamma s} e^{\gamma s} \|f(s)\| \\
&\leq \varepsilon e^{\gamma k\tau_\varepsilon} \sup_{s\in[k\tau_\varepsilon,r]} e^{-\gamma s} \|f(s)\| \\
&\leq \varepsilon e^{\gamma r} e^{-\gamma(r-k\tau_\varepsilon)} \sup_{s\in[k\tau_\varepsilon,r]} e^{-\gamma s} \|f(s)\| \\
&\leq \varepsilon e^{\gamma r} e^{-\gamma\tau_\varepsilon} \sup_{s\in[k\tau_\varepsilon,r]} e^{-\gamma s} \|f(s)\| \\
&\leq \varepsilon_1 e^{\gamma r} \sup_{s\in[k\tau_\varepsilon,r]} e^{-\gamma s} \|f(s)\|.
\end{aligned}$$

Therefore for each $k \in \mathbb{N}$, each $r \in [k\tau_\varepsilon, (k+1)\tau_\varepsilon]$ and $\gamma > \widehat{\omega}$ we obtain

$$\varepsilon \sup_{s\in[k\tau_\varepsilon,r]} \|f(s)\| \leq \varepsilon_1 e^{\gamma r} \sup_{s\in[k\tau_\varepsilon,r]} e^{-\gamma s} \|f(s)\|. \tag{5.7}$$

By (5.5) and (5.7) we obtain for each $k \in \mathbb{N}$, each $r \geq (k+1)\tau_\varepsilon$ and $\gamma > \widehat{\omega}$ that

$$\|U(r, (k+1)\tau_\varepsilon)v_\lambda((k+1)\tau_\varepsilon, k\tau_\varepsilon)\| \leq e^{(\widehat{\omega}-\gamma)(r-(k+1)\tau_\varepsilon)} \varepsilon_1 e^{\gamma r} \sup_{s\in[k\tau_\varepsilon,(k+1)\tau_\varepsilon]} e^{-\gamma s} \|f(s)\|. \tag{5.8}$$

Since $t - n\tau_\varepsilon \in [0, \tau_\varepsilon)$ we have from (5.5) and (5.7) that

$$\begin{aligned}
\|U(t, n\tau_\varepsilon)v_\lambda(t, n\tau_\varepsilon)\| &\leq e^{\widehat{\omega}(t-n\tau_\varepsilon)} \varepsilon \sup_{s\in[n\tau_\varepsilon,t]} \|f(s)\| \\
&\leq e^{\widehat{\omega}(t-n\tau_\varepsilon)} \varepsilon_1 e^{\gamma t} \sup_{s\in[n\tau_\varepsilon,t]} e^{-\gamma s} \|f(s)\| \\
&\leq e^{\max(0,\widehat{\omega})\tau_\varepsilon} \varepsilon_1 e^{\gamma t} \sup_{s\in[n\tau_\varepsilon,t]} e^{-\gamma s} \|f(s)\|,
\end{aligned}$$

and by using (5.4) and (5.8) we obtain

$$
\begin{aligned}
\|v_\lambda(t,0)\| \;&\le \widehat{M}e^{\widehat{\omega}(t-n\tau_\varepsilon)}\sum_{k=0}^{n-1}\|U(n\tau_\varepsilon,(k+1)\tau_\varepsilon)v_\lambda((k+1)\tau_\varepsilon,k\tau_\varepsilon)\| + \|U(t,n\tau_\varepsilon)v_\lambda(t,n\tau_\varepsilon)\| \\
&\le \widehat{M}e^{\widehat{\omega}(t-n\tau_\varepsilon)}\sum_{k=0}^{n-1}e^{(\widehat{\omega}-\gamma)(n\tau_\varepsilon-(k+1)\tau_\varepsilon)}\varepsilon_1 e^{\gamma n\tau_\varepsilon}\sup_{s\in[k\tau_\varepsilon,(k+1)\tau_\varepsilon]}e^{-\gamma s}\|f(s)\| \\
&\qquad +e^{\max(0,\widehat{\omega})\tau_\varepsilon}\varepsilon_1 e^{\gamma t}\sup_{s\in[n\tau_\varepsilon,t]}e^{-\gamma s}\|f(s)\| \\
&\le \widehat{M}e^{\widehat{\omega}(t-n\tau_\varepsilon)}e^{\gamma n\tau_\varepsilon}\left[\sum_{k=0}^{n-1}e^{(\widehat{\omega}-\gamma)(n-1-k))\tau_\varepsilon}\right]\varepsilon_1\sup_{s\in[0,t]}e^{-\gamma s}\|f(s)\| \\
&\qquad +e^{\max(0,\widehat{\omega})\tau_\varepsilon}\varepsilon_1 e^{\gamma t}\sup_{s\in[n\tau_\varepsilon,t]}e^{-\gamma s}\|f(s)\| \\
&\le \widehat{M}e^{(\widehat{\omega}-\gamma)(t-n\tau_\varepsilon)}e^{\gamma t}\left[\sum_{k=0}^{n-1}e^{(\widehat{\omega}-\gamma)k\tau_\varepsilon}\right]\varepsilon_1\sup_{s\in[0,t]}e^{-\gamma s}\|f(s)\| \\
&\qquad +e^{\max(0,\widehat{\omega})\tau_\varepsilon}\varepsilon_1 e^{\gamma t}\sup_{s\in[0,t]}e^{-\gamma s}\|f(s)\|.
\end{aligned}
$$

Then since $\widehat{\omega}-\gamma<0$ we obtain

$$
\begin{aligned}
\|v_\lambda(t,0)\| &\le \widehat{M}e^{\max(0,\widehat{\omega})\tau_\varepsilon}e^{\gamma t}\left[1+\sum_{k=0}^{+\infty}(e^{(\widehat{\omega}-\gamma)\tau_\varepsilon})^k\right]\varepsilon_1\sup_{s\in[0,t]}e^{-\gamma s}\|f(s)\| \\
&\le \widehat{M}e^{\max(0,\widehat{\omega})\tau_\varepsilon}e^{\gamma t}\left[\frac{2}{1-e^{(\widehat{\omega}-\gamma)\tau_\varepsilon}}\right]\varepsilon_1\sup_{s\in[0,t]}e^{-\gamma s}\|f(s)\|.
\end{aligned}
$$

The proof is complete. ■

As a direct consequence of Propositions 5.7 and 5.5 we obtain the following result.

Proposition 5.8 *Let Assumptions 1.1, 1.2, 1.3 and 5.6 be satisfied. Assume in addition that*

$$\sup_{t\in\mathbb{R}}\|B(t)\|_{\mathcal{L}(X_0,X)}<+\infty.$$

Let $\varepsilon>0$ be given and fixed. Then, for each $\tau_\varepsilon>0$ satisfying $\kappa\delta^*(\tau_\varepsilon)\le\varepsilon$ and each $\lambda>\omega+1$, the map

$$v_\lambda(t,t_0)=\int_{t_0}^t U_B(t,s)\lambda R_\lambda(A)f(s)ds,\;(t,t_0)\in\Delta,$$

satisfies

$$\|\Pi^+(t)v_\lambda(t,t_0)\|\le\widehat{C}(\varepsilon,\gamma)\sup_{s\in[t_0,t]}e^{\gamma(t-s)}\|f(s)\|,\;\forall(t,t_0)\in\Delta,$$

whenever $\gamma>-\beta$ and $f\in C(\mathbb{R},X)$ with

$$\widehat{C}(\varepsilon,\gamma):=\kappa\frac{2\varepsilon\max(1,e^{-\gamma\tau_\varepsilon})}{1-e^{-(\beta+\gamma)\tau_\varepsilon}}. \tag{5.9}$$

Proposition 5.9 *Let Assumptions 1.1, 1.2, 1.3 and 5.6 be satisfied. Assume in addition that*

$$\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X_0,X)} < +\infty.$$

Let $\varepsilon > 0$ be given and fixed. Then, for each $\tau_\varepsilon > 0$ satisfying $\kappa\delta^*(\tau_\varepsilon) \leq \varepsilon$ and each $\lambda > \omega + 1$ the map

$$v_\lambda(t,t_0) = \int_{t_0}^{t} U_B(t,s)\lambda R_\lambda(A) f(s)ds, \ (t,t_0) \in \Delta,$$

satisfies

$$\|U_B^-(t_0,t)v_\lambda(t,t_0)\| \leq \widehat{C}(\varepsilon,\gamma) \sup_{s\in[t_0,t]} e^{\gamma(s-t_0)}\|f(s)\|, \ \forall (t,t_0) \in \Delta,$$

whenever $\gamma > -\beta$ and $f \in C(\mathbb{R},X)$ with $\widehat{C}(\varepsilon,\gamma)$ defined in (5.9).

Proof. Let $(t,t_0) \in \Delta$ be given. Without loss of generality one can assume that $t = 0$. From now on fix $t_0 \leq 0$. Let $\tau_\varepsilon > 0$ be given such that

$$\kappa\delta^*(s) \leq \varepsilon, \ \forall s \in [0,\tau_\varepsilon].$$

Then since we can write $t_0 = -n\tau_\varepsilon - \theta$ with $\theta \in [0,\tau_\varepsilon)$ and $n \in \mathbb{N}$ we obtain

$$\begin{aligned}
U_B^-(0,t_0)v_\lambda(0,t_0) &= \int_{t_0}^{0} U_B^-(0,s)\lambda R_\lambda(A)f(s)ds \\
&= \sum_{k=0}^{n-1} \int_{-(k+1)\tau_\varepsilon}^{-k\tau_\varepsilon} U_B^-(0,s)\lambda R_\lambda(A)f(s)ds \\
&\qquad + \int_{t_0}^{-n\tau_\varepsilon} U_B^-(0,s)\lambda R_\lambda(A)f(s)ds \\
&= \sum_{k=0}^{n-1} U_B^-(0,-k\tau_\varepsilon) \int_{-(k+1)\tau_\varepsilon}^{-k\tau_\varepsilon} U_B^-(-k\tau_\varepsilon,s)\lambda R_\lambda(A)f(s)ds \\
&\qquad + U_B^-(0,-n\tau_\varepsilon) \int_{t_0}^{-n\tau_\varepsilon} U_B^-(-n\tau_\varepsilon,s)\lambda R_\lambda(A)f(s)ds,
\end{aligned}$$

so that

$$\Pi^-(0)v_\lambda(0,t_0) = \sum_{k=0}^{n-1} U_B^-(0,-k\tau_\varepsilon)v_\lambda(-k\tau_\varepsilon,-(k+1)\tau_\varepsilon) + U_B^-(0,-n\tau_\varepsilon)v_\lambda(-n\tau_\varepsilon,t_0). \tag{5.10}$$

Since $U_B^-(0,t_0)$ is invertible from $\Pi^-(t_0)(X_0)$ into $\Pi^-(0)(X_0)$ with inverse $U_B^-(t_0,0)$, by applying $U_B^-(t_0,0)$ to (5.10) we obtain

$$U_B^-(t_0,0)v_\lambda(0,t_0) = \sum_{k=0}^{n-1} U_B^-(t_0,-k\tau_\varepsilon)v_\lambda(-k\tau_\varepsilon,-(k+1)\tau_\varepsilon) + U_B^-(t_0,-n\tau_\varepsilon)v_\lambda(-n\tau_\varepsilon,t_0).$$

and by using the evolution property of U_B^- it follows that

$$\begin{aligned} U_B^-(t_0,0)v_\lambda(0,t_0) &= U_B^-(t_0,-n\tau_\varepsilon)\sum_{k=0}^{n-1}U_B^-(-n\tau_\varepsilon,-k\tau_\varepsilon)v_\lambda(-k\tau_\varepsilon,-(k+1)\tau_\varepsilon) \\ &\qquad\qquad\qquad\qquad +U_B^-(t_0,-n\tau_\varepsilon)v_\lambda(-n\tau_\varepsilon,t_0). \end{aligned} \tag{5.11}$$

Next observe that for all $(r_0,r_1)\in\Delta$ and $r\le r_1$ with $0\le r_0-r_1\le\tau_\varepsilon$, we have

$$\begin{aligned} \|U_B^-(r,r_0)v_\lambda(r_0,r_1)\| &\le \kappa e^{-\beta(r_0-r)}\|v_\lambda(r_0,r_1)\| \\ &\le e^{-\beta(r_0-r)}\kappa\delta^*(r_0-r_1)\sup_{s\in[r_1,r_0]}\|f(s)\| \\ &\le e^{-\beta(r_0-r)}\varepsilon\sup_{s\in[r_1,r_0]}\|f(s)\|. \end{aligned} \tag{5.12}$$

Let $\gamma>-\beta$ be fixed. Set $\varepsilon_1:=\max(1,e^{-\gamma\tau_\varepsilon})$. Let $k\in\mathbb{N}$ and $r\in[-(k+1)\tau_\varepsilon,-k\tau_\varepsilon]$ be given and fixed. Then if $\gamma\ge 0$ we have

$$\varepsilon\sup_{s\in[r,-k\tau_\varepsilon]}\|f(s)\| = \varepsilon\sup_{s\in[r,-k\tau_\varepsilon]}e^{-\gamma s}e^{\gamma s}\|f(s)\|\le\varepsilon_1 e^{-\gamma r}\sup_{s\in[r,-k\tau_\varepsilon]}e^{\gamma s}\|f(s)\|,$$

while if $\gamma<0$

$$\begin{aligned} \varepsilon\sup_{s\in[r,-k\tau_\varepsilon]}\|f(s)\| &= \varepsilon\sup_{s\in[r,-k\tau_\varepsilon]}e^{-\gamma s}e^{\gamma s}\|f(s)\| \\ &\le \varepsilon e^{\gamma k\tau_\varepsilon}\sup_{s\in[r,-k\tau_\varepsilon]}e^{\gamma s}\|f(s)\| \\ &\le \varepsilon e^{-\gamma r}e^{\gamma(r+k\tau_\varepsilon)}\sup_{s\in[r,-k\tau_\varepsilon]}e^{\gamma s}\|f(s)\| \\ &\le \varepsilon e^{-\gamma r}e^{-\gamma\tau_\varepsilon}\sup_{s\in[r,-k\tau_\varepsilon]}e^{\gamma s}\|f(s)\| \\ &\le \varepsilon_1 e^{-\gamma r}\sup_{s\in[r,-k\tau_\varepsilon]}e^{\gamma s}\|f(s)\|. \end{aligned}$$

Therefore, for each $k\in\mathbb{N}$, each $r\in[-(k+1)\tau_\varepsilon,-k\tau_\varepsilon]$ and $\gamma>-\beta$ we obtain

$$\varepsilon\sup_{s\in[r,-k\tau_\varepsilon]}\|f(s)\|\le\varepsilon_1 e^{-\gamma r}\sup_{s\in[r,-k\tau_\varepsilon]}e^{\gamma s}\|f(s)\|. \tag{5.13}$$

By (5.12) and (5.13) we obtain for each $k\in\mathbb{N}$, each $r\le-(k+1)\tau_\varepsilon$ and $\gamma>-\beta$ that

$$\|U_B^-(r,-k\tau_\varepsilon)v_\lambda(-k\tau_\varepsilon,-(k+1)\tau_\varepsilon)\|\le e^{(\beta+\gamma)(r+(k+1)\tau_\varepsilon)}e^{-\gamma r}\varepsilon_1\sup_{s\in[-(k+1)\tau_\varepsilon,-k\tau_\varepsilon]}e^{\gamma s}\|f(s)\|.$$

Since $-n\tau_\varepsilon-t_0=\theta\in[0,\tau_\varepsilon)$ we obtain from (5.11) and (5.13)

$$\begin{aligned} \|U_B^-(t_0,-n\tau_\varepsilon)v_\lambda(-n\tau_\varepsilon,t_0)\| &\le e^{\beta(t_0+n\tau_\varepsilon)}\varepsilon\sup_{s\in[t_0,-n\tau_\varepsilon]}\|f(s)\| \\ &\le e^{\beta(t_0+n\tau_\varepsilon)}\varepsilon_1 e^{-\gamma t_0}\sup_{s\in[t_0,-n\tau_\varepsilon]}e^{\gamma s}\|f(s)\|, \end{aligned} \tag{5.14}$$

and by using (5.11) and (5.14) it follows that

$$\begin{aligned}\|U_B^-(t_0,0)\Pi^-(0)v_\lambda(0,t_0)\| &\leq \kappa e^{\beta(t_0+n\tau_\varepsilon)}\left[\sum_{k=0}^{n-1} e^{(\beta+\gamma)(-n+k+1)\tau_\varepsilon}e^{\gamma n\tau_\varepsilon}\varepsilon_1 \sup_{s\in[-(k+1)\tau_\varepsilon,-k\tau_\varepsilon]} e^{\gamma s}\|f(s)\|\right] \\ &\quad + e^{\beta(t_0+n\tau_\varepsilon)}\varepsilon_1 e^{-\gamma t_0}\sup_{s\in[t_0,-n\tau_\varepsilon]} e^{\gamma s}\|f(s)\| \\ &\leq \kappa e^{\beta(t_0+n\tau_\varepsilon)}e^{\gamma n\tau_\varepsilon}\left[\sum_{k=0}^{n-1} e^{(\beta+\gamma)(-n+k+1)\tau_\varepsilon}\varepsilon_1 \sup_{s\in[t_0,0]} e^{\gamma s}\|f(s)\|\right] \\ &\quad + e^{\beta(t_0+n\tau_\varepsilon)}\varepsilon_1 e^{-\gamma t_0}\sup_{s\in[t_0,0]} e^{\gamma s}\|f(s)\| \\ &\leq \kappa e^{(\beta+\gamma)(t_0+n\tau_\varepsilon)}e^{-\gamma t_0}\left[\sum_{k=-n+1}^{0} (e^{(\beta+\gamma)\tau_\varepsilon})^k \varepsilon_1 \sup_{s\in[t_0,0]} e^{\gamma s}\|f(s)\|\right] \\ &\quad + e^{\beta(t_0+n\tau_\varepsilon)}\varepsilon_1 e^{-\gamma t_0}\sup_{s\in[t_0,0]} e^{\gamma s}\|f(s)\|.\end{aligned}$$

Finally, since $\gamma+\beta>0$ and $t_0+n\tau_\varepsilon<0$ we get

$$\begin{aligned}\|U_B^-(t_0,0)\Pi^-(0)v_\lambda(0,t_0)\| &\leq \kappa\left[1+\sum_{k=-n+1}^{0}(e^{(\beta+\gamma)\tau_\varepsilon})^k\right]\varepsilon_1 e^{-\gamma t_0}\sup_{s\in[t_0,0]} e^{\gamma s}\|f(s)\| \\ &\leq \kappa\left[1+\sum_{k=-\infty}^{0}(e^{(\beta+\gamma)\tau_\varepsilon})^k\right]\varepsilon_1 e^{-\gamma t_0}\sup_{s\in[t_0,0]} e^{\gamma s}\|f(s)\| \\ &\leq \kappa\left[\frac{2}{1-e^{-(\beta+\gamma)}}\right]\varepsilon_1 e^{-\gamma t_0}\sup_{s\in[t_0,0]} e^{\gamma s}\|f(s)\|.\end{aligned}$$

This completes the proof. ■

Lemma 5.10 *Let Assumptions 1.1, 1.2, 1.3 and 5.6 be satisfied. Assume in addition that*

$$\sup_{t\in\mathbb{R}}\|B(t)\|_{\mathcal{L}(X_0,X)}<+\infty.$$

Let $\eta\in[0,\beta)$ be given. Then for each $\lambda>\omega+1$, each $f\in BC^\eta(\mathbb{R},X)$ and $t\in\mathbb{R}$

$$\mathcal{J}_\lambda^+(f)(t) := \lim_{t_0\to-\infty}\int_{t_0}^t U_B^+(t,s)\lambda R_\lambda(A)f(s)ds := \int_{-\infty}^t U_B^+(t,s)\lambda R_\lambda(A)f(s)ds, \quad (5.15)$$

exists. Moreover, the following properties hold

i) For each $\eta\in[0,\beta)$ and each $\lambda>\omega+1$, $\mathcal{J}_\lambda^+$ is a bounded linear operator from $BC^\eta(\mathbb{R},X)$ into itself. More precisely for any $\nu\in(-\beta,0)$

$$\|\mathcal{J}_\lambda^+(f)\|_\eta \leq \widehat{C}(1,\nu)\|f\|_\eta,\ \forall f\in BC^\eta(\mathbb{R},X) \ \text{ with } \ \eta\in[0,-\nu],$$

where $\widehat{C}(1,\nu)$ is the constant introduced in Proposition 5.8.

ii) For each $\eta \in [0, \beta)$, each $\lambda > \omega + 1$ and each $f \in BC^{\eta}(\mathbb{R}, X)$ we have

$$\mathcal{J}_{\lambda}^{+}(f)(t) = U_B^{+}(t,l)\mathcal{J}_{\lambda}^{+}(f)(l) + \int_l^t U_B^{+}(t,s)\lambda R_{\lambda}(A)f(s)ds, \ \forall (t,l) \in \Delta. \quad (5.16)$$

Proof. Let $\eta \in [0, \beta)$ be given. Let $\lambda > \omega + 1$ be given and fixed. Recall

$$v_{\lambda}(t,t_0) := \int_{t_0}^t U_B(t,s)\lambda R_{\lambda}(A)f(s)ds, \ \forall (t,t_0) \in \Delta,$$

and observe that

$$\Pi^{+}(t)v_{\lambda}(t,t_0) = \int_{t_0}^t U_B^{+}(t,s)\lambda R_{\lambda}(A)f(s)ds, \ \forall (t,t_0) \in \Delta,$$

and

$$\mathcal{J}_{\lambda}^{+}(f)(t) = \lim_{t_0 \to -\infty} \Pi^{+}(t)v_{\lambda}(t,t_0), \ \forall t \in \mathbb{R}.$$

To prove the existence of the limit, we will show that for each fixed $t \in \mathbb{R}$, $\{\Pi^{+}(t)v_{\lambda}(t,t_0)\}_{t_0 \le t}$ is a Cauchy sequence. Fix $t \in \mathbb{R}$. Let $f \in BC^{\eta}(\mathbb{R}, X)$ be given. Let $t_0, r \in \mathbb{R}$ such that $t_0 \le r \le t$. Then we have

$$\begin{aligned} \Pi^{+}(t)v_{\lambda}(t,t_0) &= U_B^{+}(t,r)\int_{t_0}^r U_B^{+}(r,s)\lambda R_{\lambda}(A)f(s)ds + \int_r^t U_B^{+}(t,s)\lambda R_{\lambda}(A)f(s)ds \\ &= U_B^{+}(t,r)v_{\lambda}(r,t_0) + \Pi^{+}(t)v_{\lambda}(t,r) \end{aligned}$$

and

$$\Pi^{+}(t)v_{\lambda}(t,t_0) - \Pi^{+}(t)v_{\lambda}(t,r) = U_B(t,r)\Pi^{+}(r)v_{\lambda}(r,t_0). \quad (5.17)$$

Hence by Proposition 5.8 we can find a constant $\widehat{C}(1,\gamma) > 0$ with $\gamma \in (-\beta, -\eta)$ such that

$$\begin{aligned} \|\Pi^{+}(t)v_{\lambda}(t,t_0) - \Pi^{+}(t)v_{\lambda}(t,r)\| &\le \kappa e^{-\beta(t-r)}\widehat{C}(1,\gamma) \sup_{s\in[t_0,r]} e^{\gamma(r-s)}\|f(s)\| \\ &\le \kappa e^{-\beta(t-r)}\widehat{C}(1,\gamma)\|f\|_{\eta} \sup_{s\in[t_0,r]} e^{\gamma(r-s)}e^{\eta|s|} \\ &\le \kappa e^{-\beta(t-r)}\widehat{C}(1,\gamma)\|f\|_{\eta}e^{-\gamma(t-r)} \sup_{s\in[t_0,r]} e^{\gamma(t-s)}e^{\eta|s|} \\ &\le \kappa e^{-(\beta+\gamma)(t-r)}\widehat{C}(1,\gamma)\|f\|_{\eta} \sup_{s\in[t_0,r]} e^{\gamma(t-s)}e^{\eta(|t-s|+|t|)}. \end{aligned}$$

Then using the fact that $\beta + \gamma > 0$ and $\eta + \gamma < 0$ we obtain

$$\|\Pi^{+}(t)v_{\lambda}(t,t_0) - \Pi^{+}(t)v_{\lambda}(t,r)\| \le \kappa e^{-(\beta+\gamma)(t-r)}\widehat{C}(1,\gamma)\|f\|_{\eta}e^{\eta|t|},$$

that is

$$\lim_{t_0,r \to -\infty} \|\Pi^{+}(t)v_{\lambda}(t,t_0) - \Pi^{+}(t)v_{\lambda}(t,r)\| = 0.$$

This proves the existence of the limit (5.15) for any fixed $t \in \mathbb{R}$.
(i): Let $\eta \in [0,\beta)$ be given. Let $\nu \in (-\beta, 0)$. By Proposition 5.8 we can find a constant $\widehat{C}(1,\nu) > 0$ such that

$$\|\Pi^+(t)v_\lambda(t,t_0)\| \leq \widehat{C}(1,\nu) \sup_{s\in[t_0,t]} e^{\nu(t-s)}\|f(s)\|, \ \forall (t,t_0) \in \Delta.$$

Then for all $(t,t_0) \in \Delta$

$$\begin{aligned}
\|\Pi^+(t)v_\lambda(t,t_0)\| \quad &\leq \widehat{C}(1,\nu) \sup_{s\in[t_0,t]} e^{\nu(t-s)}\|f(s)\| \\
&\leq \widehat{C}(1,\nu)\|f\|_\eta \sup_{s\in[t_0,t]} e^{\nu(t-s)}e^{\eta|s|} \\
&\leq \widehat{C}(1,\nu)\|f\|_\eta \sup_{s\in[t_0,t]} e^{\nu(t-s)}e^{\eta|t-s|+\eta|t|} \\
&\leq \widehat{C}(1,\nu)\|f\|_\eta e^{\eta|t|} \sup_{s\in[t_0,t]} e^{(\nu+\eta)(t-s)},
\end{aligned}$$

and since $\nu + \eta < 0$ we obtain

$$\|\Pi^+(t)v_\lambda(t,t_0)\| \quad \leq \widehat{C}(1,\nu)\|f\|_\eta e^{\eta|t|}. \tag{5.18}$$

The result follows by letting $t_0 \to -\infty$ in (5.18).
(ii): Let $\eta \in [0,\beta)$ and $(t,l) \in \Delta$ be given. Then

$$\begin{aligned}
\mathcal{J}_\lambda^+(f)(t) \quad &= U_B^+(t,l)\int_{-\infty}^l U_B^+(l,s)\lambda R_\lambda(A)f(s)ds + \int_l^t U_B^+(t,s)\lambda R_\lambda(A)f(s)ds \\
&= U_B^+(t,l)\mathcal{J}_\lambda^+(f)(l) + \int_l^t U_B^+(t,s)\lambda R_\lambda(A)f(s)ds.
\end{aligned}$$

This completes the proof. ■

Lemma 5.11 *Let Assumptions 1.1, 1.2, 1.3 and 5.6 be satisfied. Assume in addition that*

$$\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X_0,X)} < +\infty.$$

Let $\eta \in [0,\beta)$ be given. Then for each $\lambda > \omega + 1$, each $f \in BC^\eta(\mathbb{R}, X)$ and $t_0 \in \mathbb{R}$

$$\mathcal{J}_\lambda^-(f)(t_0) := -\lim_{t\to+\infty} \int_{t_0}^t U_B^-(t_0,s)\lambda R_\lambda(A)f(s)ds := -\int_{t_0}^{+\infty} U_B^-(t_0,s)\lambda R_\lambda(A)f(s)ds, \tag{5.19}$$

exists. Moreover the following properties hold

i) For each $\eta \in [0,\beta)$ and each $\lambda > \omega + 1$, $\mathcal{J}_\lambda^-$ is a bounded linear operator from $BC^\eta(\mathbb{R}, X)$ into itself. More precisely for any $\nu \in (-\beta, 0)$

$$\|\mathcal{J}_\lambda^-(f)\|_\eta \leq \widehat{C}(1,\nu)\|f\|_\eta, \ \forall f \in BC^\eta(\mathbb{R}, X) \quad \text{with} \quad \eta \in [0, -\nu],$$

where $\widehat{C}(1,\nu)$ is the constant introduced in Proposition 5.8.

ii) For each $\eta \in [0,\beta)$, each $\lambda > \omega + 1$ and each $f \in BC^{\eta}(\mathbb{R}, X)$ we have

$$\mathcal{J}_{\lambda}^{-}(f)(t) = U_B^{-}(t,l)\mathcal{J}_{\lambda}^{-}(f)(l) + \int_l^t U_B^{-}(t,s)\lambda R_{\lambda}(A)f(s)ds, \ \forall (t,l) \in \Delta. \tag{5.20}$$

Proof. Let $\eta \in [0,\beta)$ be given. Let $\lambda > \omega + 1$ be given and fixed. Recall

$$v_{\lambda}(t,t_0) := \int_{t_0}^t U_B(t,s)\lambda R_{\lambda}(A)f(s)ds, \ \forall (t,t_0) \in \Delta.$$

Observe that

$$U_B^{-}(t_0,t)v_{\lambda}(t,t_0) = \int_{t_0}^t U_B^{-}(t_0,s)\lambda R_{\lambda}(A)f(s)ds, \ \forall (t,t_0) \in \Delta,$$

and

$$\mathcal{J}_{\lambda}^{-}(f)(t_0) = -\lim_{t \to +\infty} z_{\lambda}(t,t_0), \ \forall t_0 \in \mathbb{R},$$

with

$$z_{\lambda}(t,t_0) := U_B^{-}(t_0,t)v_{\lambda}(t,t_0), \ \forall (t,t_0) \in \Delta. \tag{5.21}$$

To prove the existence of the limit, we will show that for each $t_0 \in \mathbb{R}$, $\{w_{\lambda}(t,t_0)\}_{t \geq t_0}$ is a Cauchy sequence. Let $f \in BC^{\eta}(\mathbb{R}, X)$ be given. Let $t, r \in \mathbb{R}$ such that $t_0 \leq r \leq t$. Then we have

$$\begin{aligned} z_{\lambda}(t,t_0) &= \int_{t_0}^r U_B^{-}(t_0,s)\lambda R_{\lambda}(A)f(s)ds + U_B^{-}(t_0,r)\int_r^t U_B^{-}(r,s)\lambda R_{\lambda}(A)f(s)ds \\ &= z_{\lambda}(r,t_0) + U_B^{-}(t_0,r)z_{\lambda}(r,t), \end{aligned}$$

and

$$z_{\lambda}(t,t_0) - z_{\lambda}(r,t_0) = U_B^{-}(t_0,r)z_{\lambda}(r,t). \tag{5.22}$$

Then by Proposition 5.9 and the definition of z_{λ} in (5.21) we can find a constant $\widehat{C}(1,\gamma) > 0$ with $\gamma \in (-\beta, -\eta)$ such that

$$\begin{aligned} \|z_{\lambda}(t,t_0) - z_{\lambda}(r,t_0)\| &\leq \kappa e^{-\beta(r-t_0)}\widehat{C}(1,\gamma) \sup_{s \in [t,r]} e^{\gamma(s-t)}\|f(s)\| \\ &\leq \kappa e^{-\beta(r-t_0)}\widehat{C}(1,\gamma)\|f\|_{\eta} e^{-\gamma(t-t_0)} \sup_{s \in [t,r]} e^{\gamma(s-t_0)}e^{\eta|s|} \\ &\leq \kappa e^{-\beta(r-t_0)}\widehat{C}(1,\gamma)\|f\|_{\eta} e^{-\gamma(r-t_0)}e^{-\gamma(t-r)} \sup_{s \in [t,r]} e^{\gamma(s-t_0)}e^{\eta|s|} \\ &\leq \kappa e^{-(\beta+\gamma)(r-t_0)}\widehat{C}(1,\gamma)\|f\|_{\eta} e^{-\gamma(t-r)} \sup_{s \in [t,r]} e^{\gamma(s-t_0)}e^{\eta|s|} \\ &\leq \kappa e^{-(\beta+\gamma)(r-t_0)}\widehat{C}(1,\gamma)\|f\|_{\eta} \sup_{s \in [t,r]} e^{\gamma(s-t_0)}e^{\eta|s|} \\ &\leq \kappa e^{-(\beta+\gamma)(r-t_0)}\widehat{C}(1,\gamma)\|f\|_{\eta} \sup_{s \in [t,r]} e^{\gamma(s-t_0)}e^{\eta(|s-t_0|+|t_0|)} \\ &\leq \kappa e^{-(\beta+\gamma)(r-t_0)}\widehat{C}(1,\gamma)\|f\|_{\eta} \sup_{s \in [t,r]} e^{(\gamma+\eta)(s-t_0)}e^{\eta|t_0|}, \end{aligned}$$

and since $\beta+\gamma>0$, $\gamma+\eta<0$ we obtain

$$\|z_\lambda(t,t_0)-z_\lambda(r,t_0)\|\leq \kappa e^{-(\beta+\gamma)(r-t_0)}\widehat{C}(1,\gamma)\|f\|_\eta e^{\eta|t_0|},$$

which gives

$$\lim_{t,r\to+\infty}\|z_\lambda(t,t_0)-z_\lambda(r,t_0)\|=0,$$

and proves the existence of the limit (5.19).
i) Let $\eta\in[0,\beta)$ be given. Let $\nu\in(-\beta,0)$. Note that by Proposition 5.9 we can find a constant $\widehat{C}(1,\nu)>0$ such that

$$\|w_\lambda(t,t_0)\|\leq \widehat{C}(1,\nu)\sup_{s\in[t_0,t]}e^{\gamma(s-t_0)}\|f(s)\|,\ \forall(t,t_0)\in\Delta.$$

Then for all $(t,t_0)\in\Delta$

$$\begin{aligned}\|w_\lambda(t,t_0)\| &\leq \widehat{C}(1,\nu)\|f\|_\eta\sup_{s\in[t_0,t]}e^{\nu(s-t_0)}e^{\eta|s|}\\ &\leq \widehat{C}(1,\nu)\|f\|_\eta\sup_{s\in[t_0,t]}e^{\nu(s-t_0)}e^{\eta(|s-t_0|+|t_0|)}\\ &\leq \widehat{C}(1,\nu)\|f\|_\eta\sup_{s\in[t_0,t]}e^{(\nu+\eta)(s-t_0)}e^{\eta|t_0|},\end{aligned}$$

and since $\nu+\eta<0$ we obtain

$$\|w_\lambda(t,t_0)\|\leq \widehat{C}(1,\nu)\|f\|_\eta e^{\eta|t_0|}.\tag{5.23}$$

The result follows by letting $t\to+\infty$ in (5.23).
(ii) Let $\eta\in[0,\beta)$ and $(t,l)\in\Delta$ be given. Then

$$\begin{aligned}\mathcal{J}_\lambda^-(f)(l) &= -\int_l^t U_B^-(l,s)\lambda R_\lambda(A)f(s)ds-\int_t^{+\infty}U_B^-(l,s)\lambda R_\lambda(A)f(s)ds\\ &= -U_B^-(l,t)\int_l^t U_B^-(t,s)\lambda R_\lambda(A)f(s)ds-U_B^-(l,t)\int_t^{+\infty}U_B^-(t,s)\lambda R_\lambda(A)f(s)ds\end{aligned}$$

and because $U_B^-(l,t)$ is invertible from $\Pi^-(t)(X_0)$ into $\Pi^-(l)(X_0)$ with inverse $U_B^-(t,l)$ and $\mathcal{J}_\lambda^-(f)(l)\in\Pi^-(l)(X_0)$ one gets

$$\begin{aligned}U_B^-(t,l)\mathcal{J}_\lambda^-(f)(l) &= -\int_l^t U_B^-(t,s)\lambda R_\lambda(A)f(s)ds-\int_t^{+\infty}U_B^-(t,s)\lambda R_\lambda(A)f(s)ds\\ &= -\int_l^t U_B^-(t,s)\lambda R_\lambda(A)f(s)ds+\mathcal{J}_\lambda^-(f)(t),\end{aligned}$$

and the result follows. ■

Lemma 5.12 *Let Assumptions 1.1, 1.2, 1.3 and 5.6 be satisfied. Assume in addition that*

$$\sup_{t\in\mathbb{R}}\|B(t)\|_{\mathcal{L}(X_0,X)}<+\infty.$$

Let $\eta \in [0, \beta)$ be given. For each $\lambda > \omega + 1$ and each $f \in BC^\eta(\mathbb{R}, X)$ define

$$\mathcal{J}_\lambda(f)(t) := \mathcal{J}_\lambda^+(f)(t) + \mathcal{J}_\lambda^-(f)(t) := \int_{-\infty}^{+\infty} \Gamma_B(t,s)\lambda R_\lambda(A) f(s) ds, \ \forall t \in \mathbb{R}, \tag{5.24}$$

Then the following properties hold

i) For each $\eta \in [0, \beta)$ and each $\lambda > \omega + 1$, $\mathcal{J}_\lambda$ is a bounded linear operator from $BC^\eta(\mathbb{R}, X)$ into itself. More precisely for any $\nu \in (-\beta, 0)$

$$\|\mathcal{J}_\lambda(f)\|_\eta \leq 2\ \widehat{C}(1, \nu)\|f\|_\eta, \ \forall f \in BC^\eta(\mathbb{R}, X) \ \text{ with } \ \eta \in [0, -\nu], \tag{5.25}$$

where $\widehat{C}(1, \nu)$ is the constant introduced in Proposition 5.8.

ii) For each $\eta \in [0, \beta)$, each $\lambda > \omega + 1$ and each $f \in BC^\eta(\mathbb{R}, X)$ we have

$$\mathcal{J}_\lambda(f)(t) = U_B(t, l)\mathcal{J}_\lambda(f)(l) + \int_l^t U_B(t, s)\lambda R_\lambda(A) f(s) ds, \ \forall (t, l) \in \Delta. \tag{5.26}$$

iii) For each $\eta \in [0, \beta)$, each $f \in BC^\eta(\mathbb{R}, X)$, $\mathcal{J}_\lambda(f)$ is uniformly convergent on compact subset of $\mathbb{R}$ as $\lambda \to +\infty$.

iv) For each $f \in BUC(\mathbb{R}, X) \subset BC^0(\mathbb{R}, X)$ with relatively compact range, $\mathcal{J}_\lambda(f)$ is uniformly convergent on $\mathbb{R}$ as $\lambda \to +\infty$.

Proof. The proof of **(i)** follows from Lemmas 5.10 and 5.11.
(ii) Let $\eta \in [0, \beta)$, $f \in BC^\eta(\mathbb{R}, X)$ and $(t, l) \in \Delta$ be given. Since $\mathcal{J}_\lambda^+(f)(l) \in \Pi^+(l)$, $\mathcal{J}_\lambda^-(f)(l) \in \Pi^-(l)$ one gets from (5.16) and (5.20)

$$\mathcal{J}_\lambda^+(f)(t) = U_B(t, l)\Pi^+(l)\mathcal{J}_\lambda^+(f)(l) + \Pi^+(t) \int_l^t U_B(t, s)\lambda R_\lambda(A) f(s) ds. \tag{5.27}$$

and

$$\mathcal{J}_\lambda^-(f)(t) = U_B(t, l)\Pi^-(l)\mathcal{J}_\lambda^-(f)(l) + \Pi^-(t) \int_l^t U_B(t, s)\lambda R_\lambda(A) f(s) ds. \tag{5.28}$$

and the result follows by adding up (5.27) and (5.28) combined with the fact that $\Pi^+(t) + \Pi^-(t) = \Pi^+(l) + \Pi^-(l) = I$.
(iii) To do this we will prove the convergence for $\mathcal{J}_\lambda^+$ and $\mathcal{J}_\lambda^-$ as λ goes to $+\infty$. Let $\eta \in [0, \beta)$ and $f \in BC^\eta(\mathbb{R}, X)$ be given. Let $\varepsilon > 0$ be given given and fixed. Let $r > 0$ be large enough such that

$$2\kappa e^{-\beta r}\widehat{C}(1, \nu)\|f\|_\eta \leq \varepsilon. \tag{5.29}$$

We first prove the convergence for $\mathcal{J}_\lambda^+$ as λ goes to $+\infty$. Indeed by using (5.27) combined with the estimate in (i) we obtain for each $\lambda, \mu > \omega + 1$, each $t \in \mathbb{R}$

$$\begin{aligned} \|\mathcal{J}_\lambda^+(f)(t) - \mathcal{J}_\mu^+(f)(t)\| &\leq \kappa e^{-\beta r}\|\mathcal{J}_\lambda^+(f)(t-r) - \mathcal{J}_\mu^+(f)(t-r)\| \\ &\qquad + \kappa \|\int_{t-r}^t U_B(t, s)[\lambda R_\lambda(A) - \mu R_\mu(A) f(s) ds\| \\ &\leq 2\kappa e^{-\beta r}\widehat{C}(1, \nu)\|f\|_\eta \\ &\qquad + \kappa \|\int_{t-r}^t U_B(t, s)[\lambda R_\lambda(A) - \mu R_\mu(A) f(s) ds\| \end{aligned}$$

and by using (5.29) we obtain the estimate

$$\|\mathcal{J}_\lambda^+(f)(t) - \mathcal{J}_\mu^+(f)(t)\| \leq \varepsilon + \kappa \| \int_{t-r}^{t} U_B(t,s)[\lambda R_\lambda(A) - \mu R_\mu(A) f(s) ds\|, \ \forall t \in \mathbb{R}. \tag{5.30}$$

Now infer from Theorem 1.6 that

$$\lim_{\lambda,\mu \to +\infty} \int_{t-r}^{t} U_B(t,s)[\lambda R_\lambda(A) - \mu R_\mu(A) f(s) ds = 0,$$

uniformly for t in a compact subset of $\mathbb{R}$ and (5.30) yields

$$\lim_{\lambda,\mu \to +\infty} \|\mathcal{J}_\lambda^+(f)(t) - \mathcal{J}_\mu^+(f)(t)\| \leq \varepsilon,$$

uniformly for t in a compact subset of $\mathbb{R}$. Since $\varepsilon > 0$ is arbitrarily fixed we conclude by a Cauchy sequence argument that $\lim_{\lambda \to +\infty} \mathcal{J}_\lambda^+(f)(t)$ exists uniformly for t in a compact subset of $\mathbb{R}$.

Now we prove the convergence for $\mathcal{J}_\lambda^-$. First recall that for each $t \in \mathbb{R}$, $U_B^-(t+r,t)$ is invertible from $\Pi^-(t)(X_0)$ into $\Pi^-(t+r)(X_0)$ with inverse $U_B^-(t,t+r)$. Then by applying $U_B^-(t,t+r)$ to the left side of (5.28) one gets for all $t \in \mathbb{R}$

$$\begin{aligned} U_B^-(t,t+r)\mathcal{J}_\lambda^-(f)(t+r) \quad &= U_B^-(t,t+r)U_B(t+r,t)\Pi^-(t)\mathcal{J}_\lambda^-(f)(t) \\ &+U_B^-(t,t+r)\Pi^-(t+r)\int_t^{t+r} U_B(t+r,s)\lambda R_\lambda(A) f(s) ds, \ \forall t \in \mathbb{R}, \end{aligned}$$

that is

$$U_B^-(t,t+r)\mathcal{J}_\lambda^-(f)(t+r) \quad = \mathcal{J}_\lambda^-(f)(t) + U_B^-(t,t+r)\int_t^{t+r} U_B(t+r,s)\lambda R_\lambda(A) f(s) ds, \ \forall t \in \mathbb{R},$$

so that

$$\mathcal{J}_\lambda^-(f)(t) = U_B^-(t,t+r)\mathcal{J}_\lambda^-(f)(t+r) - U_B^-(t,t+r)\int_t^{t+r} U_B(t+r,s)\lambda R_\lambda(A) f(s) ds, \ \forall t \in \mathbb{R}.$$

Then for each $\lambda, \mu > \omega + 1$, each $t \in \mathbb{R}$

$$\begin{aligned} \|\mathcal{J}_\lambda^-(f)(t) - \mathcal{J}_\mu^-(f)(t)\| \quad &\leq \kappa e^{-\beta r}\|\mathcal{J}_\lambda^-(f)(t+r) - \mathcal{J}_\mu^-(f)(t+r)\| \\ &+\| \int_t^{t+r} U_B^-(t,s)[\lambda R_\lambda(A) - \mu R_\mu(A)] f(s) ds\| \\ &\leq 2\kappa e^{-\beta r}\widehat{C}(1,\nu)\|f\|_\eta \\ &+\kappa \| \int_t^{t+r} U_B(t+r,s)[\lambda R_\lambda(A) - \mu R_\mu(A)] f(s) ds\|, \end{aligned}$$

and by using (5.29) we obtain the estimate

$$\|\mathcal{J}_\lambda^-(f)(t) - \mathcal{J}_\mu^-(f)(t)\| \leq \varepsilon + \kappa \| \int_t^{t+r} U_B(t+r,s)[\lambda R_\lambda(A) - \mu R_\mu(A)] f(s) ds\|. \tag{5.31}$$

Now we infer from Theorem 1.6 that

$$\lim_{\lambda,\mu\to+\infty}\int_t^{t+r}U_B(t+r,s)[\lambda R_\lambda(A)-\mu R_\mu(A)]f(s)ds=0,$$

uniformly for t in a compact subset of $\mathbb{R}$ and (5.31) yields

$$\lim_{\lambda,\mu\to+\infty}\|\mathcal{J}_\lambda^-(f)(t)-\mathcal{J}_\mu^-(f)(t)\|\leq\varepsilon.$$

uniformly for t in a compact subset of $\mathbb{R}$. Since $\varepsilon>0$ is arbitrarily fixed we conclude by a Cauchy sequence argument that $\lim_{\lambda\to+\infty}\mathcal{J}_\lambda^-(f)(t)$ exists uniformly for t in a compact subset of $\mathbb{R}$.

Finally we obtain that

$$\lim_{\lambda\to+\infty}\mathcal{J}_\lambda(f)(t)=\lim_{\lambda\to+\infty}\mathcal{J}_\lambda^+(f)(t)+\lim_{\lambda\to+\infty}\mathcal{J}_\lambda^-(f)(t),$$

exists uniformly for t in a compact subset of $\mathbb{R}$.

(iv) The proof use the same argument as in the proof of *iii)*. The uniform convergence on $\mathbb{R}$ is obtained by using Proposition 4.1 which ensures that the limits

$$\lim_{\lambda,\mu\to+\infty}\int_{t-r}^{t}U_B(t,s)[\lambda R_\lambda(A)-\mu R_\mu(A)]f(s)ds=0,$$

and

$$\lim_{\lambda,\mu\to+\infty}\int_t^{t+r}U_B(t+r,s)[\lambda R_\lambda(A)-\mu R_\mu(A)]f(s)ds=0,$$

are uniform for $t\in\mathbb{R}$. ■

Now we are ready to prove an analogue of Theorem 5.4 for our purpose.

Theorem 5.13 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Assume in addition that*

$$\sup_{t\in\mathbb{R}}\|B(t)\|_{\mathcal{L}(X_0,X)}<+\infty.$$

Then the following assertions are equivalent

- *i) The evolution family $\{U_B(t,s)\}_{(t,s)\in\Delta}$ has an exponential dichotomy.*
- *ii) For each $f\in BC(\mathbb{R},X)$, there exists a unique integrated solution $u\in BC(\mathbb{R},X_0)$ of (1.1).*

Moreover if U_B has an exponential dichotomy with exponent $\beta>0$, then for each $\eta\in[0,\beta)$ and each $f\in BC^\eta(\mathbb{R},X)$ there exists a unique integrated solution $u\in BC^\eta(\mathbb{R},X_0)$ of (1.1) which is given by

$$u(t)=\lim_{\lambda\to+\infty}\mathcal{J}_\lambda(f)(t)=\lim_{\lambda\to+\infty}\int_{-\infty}^{+\infty}\Gamma_B(t,s)\lambda R_\lambda(A)f(s)ds,\ \forall t\in\mathbb{R},$$

where $\{\Gamma_B(t,s)\}_{(t,s)\in\mathbb{R}^2}$ is the Green's operator function associated to $\{U_B(t,s)\}_{(t,s)\in\Delta}$.

Proof. **(i)** $\Rightarrow$**(ii)** This is a direct consequence of Lemma 5.12 by taking the limit when λ goes to $+\infty$ in (5.26).

(ii) $\Rightarrow$**(i)** First of all note that since $BC(\mathbb{R}, X_0) \subset BC(\mathbb{R}, X)$, the property *ii)* ensures that for each $f \in BC(\mathbb{R}, X_0)$ there exists a unique integrated solution $u \in BC(\mathbb{R}, X_0)$ of (1.1). Furthermore note that if $u_f \in BC(\mathbb{R}, X_0)$ is a solution of (1.1) for $f \in BC(\mathbb{R}, X_0)$ then by Corollary 3.1 we know that it satisfies the integral equation

$$u_f(t) = U_B(t,t_0)x_0 + \int_{t_0}^{t} U_B(t,s)f(s)ds, \ \forall t \geq t_0,$$

and **(i)** follows by Theorem 5.4. The proof is complete. ■

As a consequence of the foregoing theorem we can obtain the following persistence result for exponential dichotomy

Theorem 5.14 *Let Assumptions 1.1, 1.2, 1.3 and 5.6 be satisfied and assume in addition that*

$$\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X_0,X)} < +\infty.$$

Then there exists $\varepsilon > 0$ *such that for each strongly continuous family* $\{C(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(X_0, X)$ *satisfying*

$$\sup_{t\in\mathbb{R}} \|B(t) - C(t)\|_{\mathcal{L}(X_0,X)} \leq \varepsilon,$$

the evolution family generated by

$$\frac{du(t)}{dt} = (A + C(t))u(t), \ \text{for } t \in \mathbb{R}. \tag{5.32}$$

has an exponential dichotomy.

Proof. The proof of this theorem is classical. Then we will only sketch the proof. Note that the the evolution family generated by (5.32) has an exponential dichotomy if and only if for each $f \in BC(\mathbb{R}, X)$ there exists a unique $u \in BC(\mathbb{R}, X_0)$ satisfying

$$\frac{du(t)}{dt} = (A + C(t))u(t) + f(t), \ \text{for } t \in \mathbb{R}.$$

or equivalently

$$\frac{du(t)}{dt} = (A + B(t))u(t) + [C(t) - B(t)]u(t) + f(t), \ \text{for } t \in \mathbb{R}.$$

This is equivalent to solve for each $f \in BC(\mathbb{R}, X)$ the fixed point problem to find $u \in BC(\mathbb{R}, X_0)$ such that

$$u(t) = \mathcal{J}([C(.) - B(.)]\, u(.) + f(.))(t)$$

where

$$\mathcal{J}(g)(t) = \lim_{\lambda\to+\infty} \int_{-\infty}^{+\infty} \Gamma_B(t,s)\lambda R_\lambda(A)g(s)ds, \ \forall t \in \mathbb{R}$$

which can be performed by using the uniform estimates (5.25) (for $\eta = 0$) obtained in Lemma 5.12. ■

6 Example 1

In order to illustrate our results we will apply some of the results to a parabolic equation . Let $p \in [1, +\infty)$ and $I := (0,1)$. Consider the following parabolic equation with non-local boundary condition for each initial time $t_0 \in \mathbb{R}$

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + \alpha u(t,x) + g(t,x), \text{ for } t \geq t_0 \text{ and } x \in (0,1), \\ -\dfrac{\partial u(t,0)}{\partial x} = \displaystyle\int_I \beta_0(t,x)\varphi(x)dx + h_0(t) \\ +\dfrac{\partial u(t,1)}{\partial x} = \displaystyle\int_I \beta_1(t,x)\varphi(x)dx + h_1(t) \\ u(t_0,.) = \varphi \in L^p(I,\mathbb{R}), \end{cases} \tag{6.1}$$

with $\alpha > 0$, $g \in C(\mathbb{R}, L^p(I,\mathbb{R}))$, $h_0, h_1 \in C(\mathbb{R},\mathbb{R})$ and $\beta_0, \beta_1 \in C(\mathbb{R}, L^q(I,\mathbb{R}))$ (with $\frac{1}{p} + \frac{1}{q} = 1$).

Abstract reformulation: In order to incorporate the boundary condition into the state variable, we consider

$$X := \mathbb{R}^2 \times L^p(I,\mathbb{R})$$

which is a Banach space endowed with the usual product norm

$$\left\| \begin{pmatrix} x_0 \\ x_1 \\ \varphi \end{pmatrix} \right\| = |x_0| + |x_1| + \|\varphi\|_{L^p}$$

and we set

$$X_0 := \{0_{\mathbb{R}^2}\} \times L^p(I,\mathbb{R}).$$

We consider $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ the linear operator defined by

$$\mathcal{A} \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} := \begin{pmatrix} \varphi'(0) \\ -\varphi'(1) \\ \varphi'' \end{pmatrix}$$

with

$$D(\mathcal{A}) := \{0_{\mathbb{R}^2}\} \times W^{2,p}(I,\mathbb{R}).$$

By construction $\mathcal{A}_0$ the part of $\mathcal{A}$ in X_0 coincides with the usual formulation for the parabolic equation (6.1) with homogeneous boundary conditions. Indeed $\mathcal{A}_0 : D(\mathcal{A}_0) \subset X_0 \to X_0$ is a linear operator on X_0 defined by

$$\mathcal{A}_0 \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi'' \end{pmatrix}$$

with

$$D(\mathcal{A}_0) = \{0_{\mathbb{R}^2}\} \times \left\{ \varphi \in W^{2,p}\left((0,1), \mathbb{R}\right) : \varphi'(0) = \varphi'(1) = 0 \right\}.$$

In the following lemma we will first summarize some classical properties for the linear operator $\mathcal{A}_0$.

Lemma 6.1 *The linear operator $\mathcal{A}_0$ is the infinitesimal generator of $\{T_{\mathcal{A}_0}(t)\}_{t\geq 0}$, an analytic semigroup of bounded linear operators on X_0. Moreover, $T_{\mathcal{A}_0}(t)$ is compact for each $t > 0$ and $(0, +\infty) \subset \rho(\mathcal{A}_0)$. The spectrum of $\mathcal{A}_0$ is given by*

$$\sigma(\mathcal{A}_0) = \left\{-(\pi k)^2 : k \in \mathbb{N}\right\}$$

and each eigenvalue $\lambda_k := -(\pi k)^2$ is associated to the eigenfunction

$$\psi_k(x) := \cos(\pi k x).$$

Furthermore each eigenvalue λ_k is simple and the projector on the generalized eigenspace associated to this eigenvalue is given by

$$\Pi_{k,0}\begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} := \begin{pmatrix} 0_{\mathbb{R}^2} \\ \frac{\int_0^1 \psi_k(r)\varphi(r)dr}{\int_0^1 \psi_k(r)^2 dr}\psi_k \end{pmatrix}.$$

Set

$$\Omega_\omega = \{\lambda \in \mathbb{C} : \mathcal{R}e\,(\lambda) > \omega\}\,,\ \forall \omega \in \mathbb{R},$$

define for $\lambda \in \mathbb{C}$,

$$\Delta\,(\lambda) := \mu^2(e^{\mu} - e^{-\mu}),$$

where

$$\mu := \sqrt{\lambda}.$$

Next we compute explicitly the resolvent of $\mathcal{A}$.

Lemma 6.2 *For each $\omega_{\mathcal{A}} \geq 0$ such that*

$$\Omega_{\omega_{\mathcal{A}}} \subset \{\lambda \in \mathbb{C} : \Delta\,(\lambda) \neq 0\} \subset \rho\,(\mathcal{A})\,,$$

and for each $\lambda := \mu^2 \in \Omega_{\omega_{\mathcal{A}}}$ we have

$$\begin{aligned}
&(\lambda I - \mathcal{A})\begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ f \end{pmatrix} \Leftrightarrow \\
&\begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} = (\lambda I - \mathcal{A})^{-1}\begin{pmatrix} y_0 \\ y_1 \\ f \end{pmatrix} \Leftrightarrow \\
&\varphi(x) = \frac{\Delta_1(x)}{\Delta\,(\lambda)}\frac{1}{\mu}y_0 + \frac{\Delta_2(x)}{\Delta\,(\lambda)}\frac{1}{\mu}y_1 + \frac{\Delta_1(x)}{\Delta\,(\lambda)}\frac{1}{2\mu}\int_0^1 e^{-\mu s}f(s)ds \\
&\qquad + \frac{\Delta_2(x)}{\Delta\,(\lambda)}\frac{1}{2\mu}\int_0^1 e^{-\mu(1-s)}f(s)ds + \frac{1}{2\mu}\int_0^1 e^{-\mu|x-s|}f(s)ds
\end{aligned}$$

where

$$\Delta_1(x) = \mu^2\left[e^{\mu(1-x)} + e^{-\mu(1-x)}\right] \text{ and } \Delta_2(x) = \mu^2\left[e^{-\mu x} + e^{\mu x}\right].$$

Proof. In order to compute the resolvent we set

$$u(x) := \frac{1}{2\mu}\int_0^1 e^{-\mu|x-s|} f(s)ds = \frac{1}{2\mu}\int_{-\infty}^{+\infty} e^{-\mu|x-s|}\bar{f}(s)ds$$

where $\bar{f}$ extends f by 0 on $\mathbb{R}\setminus[0,1]$. We have

$$u(x) = \frac{1}{2\mu}\left[\int_{-\infty}^{x} e^{-\mu(x-s)}\bar{f}(s)ds + \int_{x}^{+\infty} e^{\mu(x-s)}\bar{f}(s)ds\right]$$

so

$$u'(x) = -\frac{1}{2}\int_{-\infty}^{x} e^{-\mu(x-s)}\bar{f}(s)ds + \frac{1}{2}\int_{x}^{+\infty} e^{\mu(x-s)}\bar{f}(s)ds.$$

We set

$$u(0) = \gamma_0 := \frac{1}{2\mu}\int_0^1 e^{-\mu s} f(s)ds \text{ and } u(1) = \gamma_1 := \frac{1}{2\mu}\int_0^1 e^{-\mu(1-s)} f(s)ds$$

and we observe that

$$u'(0) = \mu\gamma_0 \text{ and } u'(1) = -\mu\gamma_1$$

We set

$$u_1(x) := e^{-\mu x} \text{ and } u_2(x) := e^{\mu x}.$$

In order to solve the problem

$$(\lambda I - \mathcal{A})\begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ f \end{pmatrix}$$

we look for φ under the form

$$\varphi(x) = u(x) + z_1 u_1(x) + z_2 u_2(x),$$

where $z_1, z_2 \in \mathbb{R}$.

We observe that to verify the boundary conditions

$$-\varphi'(0) = y_0 \text{ and } \varphi'(1) = y_1$$

is equivalent to verify

$$\begin{cases} u'(0) + z_1 u_1'(0) + z_2 u_2'(0) = -y_0 \\ u'(0) + z_1 u_1'(0) + z_2 u_2'(0) = y_1 \end{cases}$$

so we must solve the system

$$\begin{aligned} z_1 u_1'(0) + z_2 u_2'(0) &= -y_0 - u'(0) \\ z_1 u_1'(1) + z_2 u_2'(1) &= y_1 - u'(1) \end{aligned}$$

which is equivalent to

$$\begin{array}{lll} -\mu z_1 & +\mu z_2 & = -y_0 - \mu\gamma_0 \\ -\mu e^{-\mu} z_1 & +\mu e^{\mu} z_2 & = y_1 + \mu\gamma_1 \end{array}$$

hence

$$\begin{array}{l} z_1 = \frac{1}{\Delta(\lambda)} \left[-\mu e^{\mu} \left(-y_0 - \mu\gamma_0 \right) + \mu \left(y_1 + \mu\gamma_1 \right) \right] \\ z_2 = \frac{1}{\Delta(\lambda)} \left[-\mu e^{-\mu} \left(-y_0 - \mu\gamma_0 \right) + \mu \left(y_1 + \mu\gamma_1 \right) \right] \end{array}$$

and the result follows. ■

The following estimation shows that $\mathcal{A}$ is not a Hille-Yosida operator.

Lemma 6.3 *We have the following estimations*

$$0 < \liminf_{\lambda(\in\mathbb{R})\to+\infty} \lambda^{\frac{1}{p^*}} \left\| (\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(X)} \leq \limsup_{\lambda(\in\mathbb{R})\to+\infty} \lambda^{\frac{1}{p^*}} \left\| (\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(X)} < +\infty,$$

with

$$p^* = \frac{2p}{1+p}.$$

Proof. Let $\lambda > 0$ be large enough. We have

$$\left\| (\lambda I - \mathcal{A})^{-1} \begin{pmatrix} 0 \\ y_1 \\ 0_{L^p} \end{pmatrix} \right\| = |y_1| \frac{\sqrt{\lambda}}{\Delta(\lambda)} \left\| e^{\sqrt{\lambda}\cdot} + e^{-\sqrt{\lambda}\cdot} \right\|_{L^p}.$$

Set

$$\gamma_\lambda := \frac{\sqrt{\lambda}}{\Delta(\lambda)} \left\| e^{\sqrt{\lambda}\cdot} + e^{-\sqrt{\lambda}\cdot} \right\|_{L^p},$$

we have

$$\frac{\sqrt{\lambda}}{\Delta(\lambda)} \left[\|e^{\sqrt{\lambda}\cdot}\|_{L^p} - \|e^{-\sqrt{\lambda}\cdot}\|_{L^p} \right] \leq \gamma_\lambda \leq \frac{\sqrt{\lambda}}{\Delta(\lambda)} \left[\|e^{\sqrt{\lambda}\cdot}\|_{L^p} + \|e^{-\sqrt{\lambda}\cdot}\|_{L^p} \right]$$

and

$$\frac{\sqrt{\lambda}}{\Delta(\lambda)} \|e^{\sqrt{\lambda}\cdot}\|_{L^p} = \frac{\sqrt{\lambda}}{\lambda(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}})} \left(\int_0^1 e^{p\sqrt{\lambda}x} dx \right)^{1/p}$$

and

$$\lim_{\lambda\to+\infty} \gamma_\lambda \lambda^{\frac{p+1}{2p}} = (1/p)^{1/p} > 0,$$

and the result follows. ■

By using Lemmas 6.1-6.3, we deduce that Assumption 3.4 in Ducrot, Magal and Prevost [17] is satisfied. Therefore by applying Theorem 3.11 in [17] we obtain the following lemma.

Lemma 6.4 *The linear operator $\mathcal{A}$ satisfies Assumption 1.1 and Assumption 1.2.*

Remark 6.5 *Since $\rho(\mathcal{A}) \neq \emptyset$, one can prove that $\sigma(\mathcal{A}_0) = \sigma(\mathcal{A})$ (see [33]).*

Abstract Cauchy problem: By identifying $u(t,.)$ and $v(t) := \begin{pmatrix} 0_{\mathbb{R}^2} \\ u(t,.) \end{pmatrix}$ we can rewrite equation (6.1) as the following abstract Cauchy problem for each initial time $t_0 \in \mathbb{R}$

$$\frac{dv(t)}{dt} = \mathcal{A}v(t) + \alpha v(t) + \mathcal{B}(t)v(t) + f(t), \text{ for } t \geq t_0 \text{ and } v(t_0) = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix}, \tag{6.2}$$

where

$$\mathcal{B}(t)\begin{pmatrix} 0_{\mathbb{R}} \\ \varphi \end{pmatrix} := \begin{pmatrix} \int_I \beta_0(t,x)\varphi(x)dx \\ \int_I \beta_1(t,x)\varphi(x)dx \\ 0_{L^p} \end{pmatrix} \text{ and } f(t) := \begin{pmatrix} h_0(t) \\ h_1(t) \\ g(t,.) \end{pmatrix}.$$

By using Lemma 6.1 we know that $(\mathcal{A}+\alpha I)_0$ the part of $(\mathcal{A}+\alpha I)$ in $X = \mathbb{R}^2 \times L^p((0,+\infty),\mathbb{R})$, is the infinitesimal generator of $\left\{T_{(\mathcal{A}+\alpha I)_0}(t)\right\}_{t\geq 0}$ an analytic semigroup of bounded linear operators on X_0. By using Lemma 6.4 we deduce that $(\mathcal{A}+\alpha I)$ generates an integrated semigroup $\left\{S_{(\mathcal{A}+\alpha I)}(t)\right\}_{t\geq 0}$. Consider for each initial time $t_0 \in \mathbb{R}$ the parabolic equation

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + \alpha u(t,x), \text{ for } t \geq t_0 \text{ and } x \in (0,1), \\ -\dfrac{\partial u(t,0)}{\partial x} = \int_I \beta_0(t,x)\varphi(x)dx \\ +\dfrac{\partial u(t,1)}{\partial x} = \int_I \beta_1(t,x)\varphi(x)dx \\ u(t_0,.) = \varphi \in L^p(I,\mathbb{R}), \end{cases} \tag{6.3}$$

this equation corresponds to the abstract Cauchy problem for each initial time $t_0 \in \mathbb{R}$

$$\frac{dv(t)}{dt} = (\mathcal{A}+\alpha I)v(t) + \mathcal{B}(t)v(t), \text{ for } t \geq t_0 \text{ and } v(t_0) = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix}. \tag{6.4}$$

Variation of constants formula : By using Proposition 1.5 we obtain the following result.

Proposition 6.6 *The Cauchy problem (6.4) generates a unique evolution family $\{U_{\mathcal{B}}(t,s)\}_{(t,s)\in\Delta} \subset \mathcal{L}(X_0)$. Moreover $U_{\mathcal{B}}(\cdot,t_0)x_0 \in C([t_0,+\infty),X_0)$ is the unique solution of the fixed point problem*

$$U_{\mathcal{B}}(t,t_0)x_0 = T_{(\mathcal{A}+\alpha I)_0}(t-t_0)x_0 + \frac{d}{dt}\int_{t_0}^t S_{(\mathcal{A}+\alpha I)}(t-s)\mathcal{B}(s)U_{\mathcal{B}}(s,t_0)x_0 ds, \ t \geq t_0. \tag{6.5}$$

If we assume in addition that

$$\sup_{t\in\mathbb{R}} \|\beta_0(t,.)\|_{L^q} + \|\beta_1(t,.)\|_{L^q} < +\infty$$

then the evolution family $\{U_{\mathcal{B}}(t,s)\}_{(t,s)\in\Delta}$ is exponentially bounded.

By using Theorem 1.6 we obtain the following result.

Theorem 6.7 *For each $t_0 \in \mathbb{R}$, each $x_0 \in X_0$ and each $f \in C([t_0, +\infty], X)$ the unique integrated solution $v_f \in C([t_0, +\infty], X_0)$ of (6.2) is given for each $t \geq t_0$ by*

$$v_f(t) = U_{\mathcal{B}}(t, t_0)x_0 + \lim_{\lambda \to +\infty} \int_{t_0}^{t} U_{\mathcal{B}}(t, s)\lambda R_\lambda(\mathcal{A} + \alpha I)f(s)ds \tag{6.6}$$

where the limit exists in X_0. Moreover the convergence in (6.6) is uniform with respect to $t, t_0 \in I$ for each compact interval $I \subset \mathbb{R}$.

Exponential dichotomy result : By using Theorem 5.13 we obtain the following result

Theorem 6.8 *Assume that*

$$\sup_{t \in \mathbb{R}} \|\beta_0(t, .)\|_{L^q} + \|\beta_1(t, .)\|_{L^q} < +\infty$$

Then the following assertions are equivalent

i) The evolution family $\{U_{\mathcal{B}}(t, s)\}_{(t,s)\in\Delta}$ has an exponential dichotomy.

ii) For each $f \in BC(\mathbb{R}, X)$, there exists a unique integrated solution $u \in BC(\mathbb{R}, X_0)$ of (6.2).

Assumption 6.9 *Assume that $\alpha > 0$ and $\alpha \neq -(\pi k)^2, \forall k \in \mathbb{N}$.*

Then the spectrum of $\mathcal{A} + \alpha I$ does not contain any purely imaginary eigenvalue, and by using Lemma 6.1 and Remark 6.5 we deduce that

$$\sigma(\mathcal{A} + \alpha I) = \sigma(\mathcal{A}_0 + \alpha I) = \left\{-(\pi k)^2 + \alpha : k \in \mathbb{N}\right\}.$$

Therefore

$$0 \notin \sigma(\mathcal{A}_0 + \alpha I).$$

Then $U(t, s) := T_{\mathcal{A}+\alpha I}(t - s)$ has an exponential dichotomy and we can apply Theorem 5.14 with $A + B(t) := \mathcal{A} + \alpha I$ and $C(t) := \mathcal{B}(t)$.

Theorem 6.10 *Let Assumption 6.9 be satisfied. There exists $\varepsilon > 0$ such that*

$$\sup_{t \in \mathbb{R}} \|\beta_0(t, .)\|_{L^q} + \|\beta_1(t, .)\|_{L^q} < \varepsilon$$

implies that the evolution family $\{U_{\mathcal{B}}(t, s)\}_{(t,s)\in\Delta} \subset \mathcal{L}(X_0)$ has an exponential dichotomy.

7 Example 2

In this section we briefly illustrate our results with an application to an age-structured model which is a hyperbolic partial differential equation. We consider

$$\begin{cases} \partial_t u(t, a) + \partial_a u(t, a) = -\mu u(t, a), \text{ for } t \geq t_0 \text{ and } a \geq 0, \\ u(t, 0) = \displaystyle\int_0^{+\infty} \beta(t, a)u(t, a)da \\ u(t_0, .) = \varphi \in L^p_+((0, +\infty), \mathbb{R}), \end{cases} \tag{7.1}$$

with $\mu > 0$, $\beta \in C(\mathbb{R}, L^q((0,+\infty),\mathbb{R}))$ (with $\frac{1}{p}+\frac{1}{q}=1$).
As in the preceding example we define the Banach space

$$X := \mathbb{R}\times L^p((0,+\infty),\mathbb{R})$$

in order to incorporate the boundary condition. The Banach space X is endowed with the usual product norm

$$\left\| \begin{pmatrix} r \\ \varphi \end{pmatrix} \right\| = |r| + \|\varphi\|_{L^p}.$$

Set

$$X_0 := \{0\} \times L^p((0,+\infty),\mathbb{R})).$$

We consider $A : D(A) \subset X \to X$ the linear operator defined by

$$A \begin{pmatrix} 0 \\ \varphi \end{pmatrix} := \begin{pmatrix} -\varphi(0) \\ -\varphi' - \mu\varphi \end{pmatrix}$$

with

$$D(A) := \{0\} \times W^{1,p}((0,+\infty),\mathbb{R}).$$

The following lemma is proved in [32, Lemma 8.1 and Lemma 8.3].

Lemma 7.1 *The linear operator A satisfies Assumptions 1.1 and 1.2. Moreover we have*

$$0 < \lim_{\lambda\to+\infty} \lambda^{\frac{1}{p}} \|R_\lambda(A)\|_{\mathcal{L}(X)} < +\infty.$$

Lemma 7.1 implies that the part A_0 of A on $X_0 := \overline{D(A)}$ generates a C_0-semigroup $\{T_{A_0}(t)\}_{t\geq 0}$.

Now we consider the family of bounded linear operators $\{B(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ given by

$$B(t) \begin{pmatrix} r \\ \varphi \end{pmatrix} := \begin{pmatrix} \int_0^{+\infty} \beta(t,a)\varphi(a)da \\ \\ 0 \end{pmatrix}, \ \forall \begin{pmatrix} r \\ \varphi \end{pmatrix} \in X.$$

Hence by identifying $u(t,.)$ and $v(t) := \begin{pmatrix} 0_{\mathbb{R}^2} \\ u(t,.) \end{pmatrix}$, system (7.1) rewrites as the following abstract Cauchy problem for each initial time $t_0 \in \mathbb{R}$

$$\frac{dv(t)}{dt} = Av(t) + B(t)v(t), \text{ for } t \geq t_0 \text{ and } v(t_0) = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix}. \tag{7.2}$$

Proposition 7.2 *The Cauchy problem (7.1) generates a unique evolution family $\{U_B(t,s)\}_{(t,s)\in\Delta} \subset \mathcal{L}(X_0)$. Moreover $U_B(\cdot,t_0)x_0 \in C([t_0,+\infty),X_0)$ is the unique solution of the fixed point problem*

$$U_B(t,t_0)x_0 = T_{A_0}(t-t_0)x_0 + \frac{d}{dt}\int_{t_0}^t S_A(t-s)B(s)U_B(s,t_0)x_0 ds, \ t \geq t_0.$$

If we assume in addition that

$$\sup_{t\in\mathbb{R}} \|\beta(t,.)\|_{L^q} < +\infty$$

then the evolution family $\{U_B(t,s)\}_{(t,s)\in\Delta}$ *is exponentially bounded.*

By using Theorem 1.6 we also obtain the following result concerning the existence of exponential dichotomy.

Theorem 7.3 *Assume that*

$$\sup_{t\in\mathbb{R}} \|\beta(t,.)\|_{L^q} < +\infty$$

Then the following assertions are equivalent

i) The evolution family $\{U_B(t,s)\}_{(t,s)\in\Delta}$ *has an exponential dichotomy.*

ii) For each $f \in BC(\mathbb{R}, X)$*, there exists a unique integrated solution* $v \in BC(\mathbb{R}, X_0)$ *of*

$$\frac{dv(t)}{dt} = Av(t) + B(t)v(t) + f(t), \text{ for } t \in \mathbb{R}.$$

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