Incompressible Navier-Stokes limit of the Boltzmann equation

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Description of a system of particles

Different scales to describe a system composed by a large number of indistinguishable components such as gases:

Microscopic

Individual behavior of each component?

Newton's equations

Mesoscopic

Evolution of the density of particles?

Boltzmann, Landau, Fokker-Planck... equations

Macroscopic

Evolution of observable quantities?

Euler, Navier-Stokes... equations

Kinetic theory

- System described by the evolution of the density of particles $f = f(t,x,v) \geq 0$, $t \in \mathbb{R}^+$ the time, $x \in \Omega = \mathbb{T}^d$ or $\mathbb{R}^d$ the position and $v \in \mathbb{R}^d$ the velocity.

  $f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v$ = quantity of particules in the volume element $\mathrm{d}x\,\mathrm{d}v$ centered in $(x,v) \in \Omega \times \mathbb{R}^d$.

- No external force or interaction: Free transport equation

$$\partial_t f + v \cdot \nabla_x f = 0.$$

- If interaction between particles or with a background medium, equation of kind

$$\partial_t f + v \cdot \nabla_x f = \underbrace{\mathcal{C}(f)}_{\text{collision term}}.$$

- Maxwell (1867), Boltzmann (1872): Boltzmann collision operator for neutral particles (gaz).

The Boltzmann equation

Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q(f,f) \qquad \text{(B)}$$

$$\underbrace{(v', v'_*)}_{\text{before collision}} \rightleftarrows \underbrace{(v, v_*)}_{\text{after collision}}$$

- Conservation of momentum and energy:

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

- Parametrization of (v', v'_*) by an element $\sigma \in \mathbf{S}^{d-1}$.

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma, \quad \sigma \in \mathbf{S}^{d-1}.$$

- Boltzmann collision operator for hard spheres:

$$Q(g,f)(v) = \int_{\mathbf{R}^d \times \mathbf{S}^{d-1}} \underbrace{|v - v_*|}_{\text{collision kernel}} \Big(\underbrace{f(v')\, g(v'_*)}_{\text{“appearing”}} - \underbrace{f(v)\, g(v_*)}_{\text{“disappearing”}} \Big)\, \mathrm{d}\sigma\, \mathrm{d}v_*.$$

Basic properties

For $\phi = \phi(v)$ a test function,

$$\int_{\mathbf{R}^d} Q(f,f)\,\phi\,\mathrm{d}v = \frac{1}{2}\int_{\mathbf{R}^d\times\mathbf{R}^d\times\mathbf{S}^{d-1}} |v-v_*|\,f\,f_*\left(\phi' + \phi'_* - \phi - \phi_*\right)\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v.$$

- Conservation of mass, momentum and energy:

$$\int_{\mathbf{R}^d} Q(f,f)(v)\left(1, v_i, |v|^2\right)\mathrm{d}v = 0.$$

- Entropy inequality (H-Theorem):

$$D(f) := -\int_{\mathbf{R}^d} Q(f,f)(v)\,\log f(v)\,\mathrm{d}v \geq 0,$$

$D(f) = 0 \Leftrightarrow f = \mu =$ Maxwellian (Gaussian in v) (s.t. $Q(\mu,\mu) = 0$).

Rescaling and linearization

- Rescaling in time and space: $(t, x, v) \to \left(t/\varepsilon^2, x/\varepsilon, v\right)$ where ε is the Knudsen number.
- We fix the following centered and normalized Maxwellian:

$$M(v) := (2\pi)^{-d/2} e^{-|v|^2/2}.$$

- Linearization around M of order ε: $f^\varepsilon = M + \varepsilon\sqrt{M} g^\varepsilon$.

Rescaled Boltzmann equation

$$\partial_t g^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x g^\varepsilon = \frac{1}{\varepsilon^2} L g^\varepsilon + \frac{1}{\varepsilon}\Gamma(g^\varepsilon, g^\varepsilon) \qquad (\mathrm{B}_\varepsilon)$$

with

$$\Gamma(f_1, f_2) := \frac{1}{2\sqrt{M}}\left(Q(\sqrt{M} f_1, \sqrt{M} f_2) + Q(\sqrt{M} f_2, \sqrt{M} f_1)\right)$$

and

$$Lf := \Gamma(\sqrt{M}, f).$$

Formal convergence

- x ε^2 and $\varepsilon \to 0$: $Lg = 0$.
- We deduce that

$$g \in \mathrm{Ker}\, L = \mathrm{Span}\left\{\sqrt{M}, v_1\sqrt{M}, \dots, v_d\sqrt{M}, |v|^2\sqrt{M}\right\}$$

$$\iff \quad g(x,v) = \sqrt{M(v)}\Big(\rho_g(x) + u_g(x)\cdot v + \frac{1}{2}(|v|^2 - d)\theta_g(x)\Big)$$

with

$$\rho_g(x) := \int_{\mathbf{R}^d} g(x,v)\sqrt{M(v)}\,\mathrm{d}v\,, \quad u_g(x) := \int_{\mathbf{R}^d} v\, g(x,v)\sqrt{M(v)}\,\mathrm{d}v\,,$$

$$\theta_g(x) := \frac{1}{d}\int_{\mathbf{R}^d}(|v|^2 - d)g(x,v)\sqrt{M(v)}\,\mathrm{d}v\,.$$

- Local conservation laws (equations satisfied by ρ_{g^ε}, u_{g^ε} and θ_{g^ε}) and then $\varepsilon \to 0$.
- For example, the first local conservation law gives:

$$\partial_t \rho_{g^\varepsilon} + \frac{1}{\varepsilon}\nabla_x \cdot u_{g^\varepsilon} = 0 \rightsquigarrow \nabla_x \cdot u_g = 0.$$

Fluid system - I

Incompressible Navier-Stokes-Fourier system

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu_1 \Delta u = -\nabla p \\ \partial_t \theta + u \cdot \nabla \theta - \nu_2 \Delta \theta = 0 \\ \nabla \cdot u = 0 \\ \nabla(\rho + \theta) = 0 \end{cases} \qquad \text{(NSF)}$$

with (ρ, u, θ, p) = (mass, velocity, temperature, pressure) and ν_i the viscosity coefficients fully determined by L.

Fluid system - II

Theorem

For $(\rho_{\text{in}}, u_{\text{in}}, \theta_{\text{in}}) \in H^{\frac{d}{2}-1}(\Omega)$, $\exists!$ *maximal time* $T^* > 0$,

$$\exists!\, (\rho, u, \theta) \in L^\infty\Big([0,T], H^{\frac{d}{2}-1}(\Omega)\Big) \cap L^2\Big([0,T], H^{\frac{d}{2}}(\Omega)\Big)$$

solution to (NSF) *for all times* $T < T^*$. *It satisfies*

$$\|(\rho, u, \theta)\|_{\tilde{L}^\infty\left([0,T], H^{\frac{d}{2}-1}(\Omega)\right)} + \|(\nabla\rho, \nabla u, \nabla\theta)\|_{L^2\left([0,T], H^{\frac{d}{2}-1}(\Omega)\right)} \lesssim C\Big(\|(\rho_{\text{in}}, u_{\text{in}}, \theta_{\text{in}})\|_{H^{\frac{d}{2}-1}(\Omega)}\Big).$$

Leray, Fujita-Kato, Chemin, Chemin-Lerner, Bahouri-Chemin-Danchin etc...

Globally well-posed in 2D, and in 3D for small data for example.

Well prepared data (WP)

For $f = f(x,v)$, we define

$$\rho_f(x) = \int_{\mathbf{R}^d} f(x,v)\sqrt{M(v)}\,\mathrm{d}v\,, \quad u_f(x) = \int_{\mathbf{R}^d} v f(x,v)\sqrt{M(v)}\,\mathrm{d}v\,,$$
$$\theta_f(x) = \frac{1}{d}\int_{\mathbf{R}^d} (|v|^2 - d) f(x,v)\sqrt{M(v)}\,\mathrm{d}v\,.$$

(WP_1): $f \in \operatorname{Ker} L$ i.e.

$$f(x,v) = \sqrt{M(v)}\Big(\rho_f(x) + u_f(x)\cdot v + \frac{1}{2}(|v|^2 - d)\theta_f(x)\Big).$$

(WP_2): $\nabla_x \cdot u_f = 0$ and $\rho_f + \theta_f = 0$.

Main result - I

- $\rho_{\rm in}, u_{\rm in}, \theta_{\rm in} \in H^\ell(\Omega)$, $\ell > d/2$ satisfying (WP_2)
  - $\rho_{\rm in}, u_{\rm in}, \theta_{\rm in} \in L^1(\Omega)$ if $\Omega = \mathbf{R}^2$,
  - $\rho_{\rm in}, u_{\rm in}, \theta_{\rm in}$ are mean free if $\Omega = \mathbf{T}^d$.
- Consider
$$(\rho, u, \theta) \in L^\infty\Big([0,T], H^\ell(\Omega)\Big) \cap L^2\Big([0,T], H^{\ell+1}(\Omega)\Big)$$
  the unique solution to (NSF) associated with initial data $(\rho_{\rm in}, u_{\rm in}, \theta_{\rm in})$ on a time interval $[0,T]$.
- Set
$$g_{\rm in}(x,v) := \sqrt{M(v)}\Big(\rho_{\rm in}(x) + u_{\rm in}(x)\cdot v + \frac{1}{2}(|v|^2 - d)\theta_{\rm in}(x)\Big),$$
  and define on $[0,T]\times\Omega\times\mathbf{R}^d$
$$g(t,x,v) := \sqrt{M(v)}\Big(\rho(t,x) + u(t,x)\cdot v + \frac{1}{2}(|v|^2 - d)\theta(t,x)\Big).$$

Main result - II

Theorem (WP data in the whole space or the torus)

$\exists \varepsilon_0 > 0$ *s.t.* $\forall \varepsilon \leq \varepsilon_0$, $\exists !\, g^{\varepsilon} \in L^{\infty}([0,T], X)$ *solution to* $(\mathrm{B}_{\varepsilon})$ *with initial data* g_{in} *and it satisfies*

$$\lim_{\varepsilon \to 0} \|g^{\varepsilon} - g\|_{L^{\infty}([0,T],X)} = 0.$$

Moreover, if the solution (ρ, u, θ) *to* (NSF) *is defined on* $\mathbf{R}^{+}$, *then* ε_0 *depends only on the initial data and not on* T *and there holds*

$$\lim_{\varepsilon \to 0} \|g^{\varepsilon} - g\|_{L^{\infty}(\mathbf{R}^{+},X)} = 0.$$

$X := L^{\infty}_v H^{\ell}_x(\langle v \rangle^k)$, $k > d/2 + 1$ defined by

$$\|f\|_X := \sup_{v \in \mathbf{R}^d} \langle v \rangle^k \|f(\cdot, v)\|_{H^{\ell}_x}.$$

Known results

- Framework of weak solutions (DiPerna-Lions for Boltzmann equation and Leray for Navier-Stokes): Bardos-Golse-Levermore (90s), Lions, Masmoudi, Saint-Raymond etc...

  "Obtain a theorem that only requires a priori estimates given by the physics: Mass, energy and entropy."

- Framework of strong solutions :
  - De Masi-Esposito-Lebowitz (90'), Bardos-Ukai (91'),
  - Guo (06'), Briant (15'), Briant-Merino-Mouhot (18') etc...

Rewriting the problem - Duhamel formula

- $U^\varepsilon(t)$ semigroup associated with $-\varepsilon^{-1}v\cdot\nabla_x + \varepsilon^{-2}L$.
- We rewrite the rescaled Boltzmann equation

$$\partial_t g^\varepsilon + \frac{1}{\varepsilon} v\cdot\nabla_x g^\varepsilon = \frac{1}{\varepsilon^2} L g^\varepsilon + \frac{1}{\varepsilon}\Gamma(g^\varepsilon, g^\varepsilon) \qquad (\mathrm{B}_\varepsilon)$$

with Duhamel formula:

$$g^\varepsilon(t) = U^\varepsilon(t) g_{\text{in}} + \underbrace{\frac{1}{\varepsilon}\int_0^t U^\varepsilon(t-s)\Gamma(g^\varepsilon, g^\varepsilon)(s)\,\mathrm{d}s}_{\Psi^\varepsilon(t)(g^\varepsilon,g^\varepsilon)}.$$

- In some sense, $U^\varepsilon(t) \to U(t)$ and $\Psi^\varepsilon(t) \to \Psi(t)$ so that the limit g writes

$$g(t) = U(t) g_{\text{in}} + \Psi(t)(g,g).$$

Rewriting the problem - Fixed point argument

- Introduction of $h^\varepsilon := g^\varepsilon - g$.
- h^ε satisfies:

$$h^\varepsilon(t) = \underbrace{(U^\varepsilon(t) - U(t))g_{\text{in}} + (\Psi^\varepsilon(t) - \Psi(t))(g,g)}_{D^\varepsilon(t) = \text{source terms}}$$

$$+ \underbrace{2\Psi^\varepsilon(t)(g, h^\varepsilon)}_{\mathcal{L}^\varepsilon(t)h^\varepsilon = \text{linear part}} + \underbrace{\Psi^\varepsilon(t)(h^\varepsilon, h^\varepsilon)}_{\text{quadratic part}}$$

↪ Fixed point argument?

E Banach space, $\mathcal{L} \in \mathscr{L}(E,E)$ and $\mathcal{B} \in \mathscr{B}(E^2, E)$.
If $||\mathcal{L}|| < 1$, for any $x_0 \in E$ small enough, the equation

$$x = x_0 + \mathcal{L}x + \mathcal{B}(x,x)$$

has a unique solution in the ball $B\left(0, \frac{1-||\mathcal{L}||}{2||\mathcal{B}||}\right)$ and there exists a constant $C_0 > 0$ such that $||x|| \le C_0||x_0||$.

Ellis and Pinsky decomposition - estimates in $H^\ell_x L^2_v$

- Fourier transform in x of $-v \cdot \nabla_x + L$: $L_\xi := L - iv \cdot \xi$.
- Decomposition of the semigroup $U^1(t)$:

$$U^1(t,\xi) = \sum_{j=1}^{d+2} e^{t\lambda_j(\xi)} P_j(\xi) + U^{1\sharp}(t,\xi)$$

  with Taylor expansion of the eigenvalues $\lambda_j(\xi)$.
- $U^\varepsilon(t,\xi) = U^1\left(\varepsilon^{-2} t, \varepsilon\xi\right) \rightsquigarrow$ decomposition of $U^\varepsilon(t)$.
- Decay estimates on $U^\varepsilon(t)$:

$$\left\| \frac{1}{\varepsilon} U^\varepsilon(t)(\mathrm{I} - \Pi_{L,0}) \right\|_{H^\ell_x L^2_v \to H^\ell_x L^2_v} \leq \chi_\Omega(t).$$

Estimate on the linear part - contraction?

Depending on the norm of g, there is no reason for $\mathcal{L}^{\varepsilon}(t)$ to be a contraction!

↪ Introduction of a "filter": For some fixed and well-chosen r,

$$h_{\lambda}^{\varepsilon}(t) := h^{\varepsilon}(t) \exp\left(-\lambda \int_0^t \|g(\tau)\|^r \, \mathrm{d}\tau\right), \quad \lambda > 0$$

so that

$$h_{\lambda}^{\varepsilon}(t) = D_{\lambda}^{\varepsilon}(t) + \mathcal{L}_{\lambda}^{\varepsilon}(t) h_{\lambda}^{\varepsilon} + \Psi_{\lambda}^{\varepsilon}(t)(h_{\lambda}^{\varepsilon}, h_{\lambda}^{\varepsilon})$$

with

$$\Psi_{\lambda}^{\varepsilon}(t)(f_1, f_2) = \frac{1}{\varepsilon} \int_0^t U^{\varepsilon}(t-s) \; \exp\left(-\lambda \int_s^t \|g\|^r\right) \Gamma(f_1, f_2)(s) \, \mathrm{d}s$$

Estimate on the linear part - stability?

- The nonlinear collision operator Γ induces a loss of weight:

$$\|\Gamma(f_1, f_2)\|_{L^\infty_v(\langle v\rangle^k)} \lesssim \|f_1\|_{L^\infty_v(\langle v\rangle^{k+1})}\|f_2\|_{L^\infty_v(\langle v\rangle^{k+1})}.$$

- Splitting of $L = \Gamma(\sqrt{M}, \cdot)$:

$$Lh = Kh - \nu(v)h \quad \text{with} \quad \nu(v) := \int_{\mathbf{R}^d \times \mathbf{S}^{d-1}} |v - v_*| M(v_*) \, \mathrm{d}v_*$$

$$K : L^2_v \to L^\infty_v \quad \text{and} \quad K : L^\infty_v\left(\langle v\rangle^j\right) \to L^\infty_v\left(\langle v\rangle^{j+1}\right), \quad j \geq 0.$$

- Duhamel formula $\rightsquigarrow$ the problem boils down to perform estimates in $H^\ell_x L^2_v$.

Source terms - I

$$D_1^\varepsilon(t) := (U^\varepsilon(t) - U(t))g_{\text{in}}.$$

Ellis and Pinsky decomposition gives:

$$U^\varepsilon(t)g_{\text{in}} = \underbrace{U(t)g_{\text{in}}}_{\text{independent of } \varepsilon} + \underbrace{V^\varepsilon(t)g_{\text{in}}}_{\text{"nice terms"}} + \underbrace{U^\varepsilon_{\text{disp}}(t)g_{\text{in}}}_{0 \text{ if } (WP_1)} + \underbrace{U^{\varepsilon\sharp}(t)g_{\text{in}}}_{\text{small if } (WP_2)}.$$

$$\|(U^\varepsilon(t) - U(t))g_{\text{in}}\|_{L^\infty_t(X)} \xrightarrow[\varepsilon\to 0]{} 0 \quad \text{if } g_{\text{in}} \text{ is WP.}$$

For ill-prepared data, we introduce $\tilde{g}^\varepsilon(t) := (U^\varepsilon_{\text{disp}}(t) + U^{\varepsilon\sharp}(t))g_{\text{in}}$ and write the equation satisfied by $\tilde{h}^\varepsilon := g^\varepsilon - g - \tilde{g}^\varepsilon$.

Source terms - II

$$D_2^{\varepsilon}(t) := (\Psi^{\varepsilon}(t) - \Psi(t))(g, g).$$

Requires estimates on (ρ, u, θ) and on their derivatives of type $\tilde{L}_t^{\infty} H_x^{\ell}$ (Chemin-Lerner spaces), $L_t^2 H_x^{\ell+1}$ etc... as well as pointwise decay estimates (Wiegner and Schonbek).

$$\|D_2^{\varepsilon}(t)\|_{L_t^{\infty}(X)} \xrightarrow[\varepsilon \to 0]{} 0.$$

Thank you for your attention!