

Universal norms of p -adic Galois representations

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1. Iwasawa theory of abelian varieties

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$$\text{Sel}_p(A/F) = \text{Ker} \left(H^1(F, A[p^\infty]) \xrightarrow{\text{loc}} \prod_v \frac{H^1(F_v, A[p^\infty])}{\text{Im}(\kappa_{F_v})} \right)$$

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Conjecture (Mazur 1972)

If A has good ordinary reduction at all prime of F dividing p , then the Pontryagin dual of $\text{Sel}_p(A/F_{cyc})$ is a torsion $\mathbb{Z}_p[[\Gamma]]$ -module.

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Theorem (Mazur 1972)

If A has good ordinary reduction at all prime of F dividing p , then the kernel and cokernel of the restriction map

$$\text{Sel}_p(A/F) \rightarrow \text{Sel}_p(A/F_{\text{cyc}})^\Gamma$$

are finite.

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Non-commutative Iwasawa theory

- ▶ Motivated by non-commutative Iwasawa theory, Greenberg (2003) has generalised this last theorem to more general Galois extension F_∞/F with $\text{Gal}(F_\infty/F)$ a p -adic Lie group.
- ▶ In both situation, the study of the structure of $\text{Sel}_p(A/F_\infty)$ essentially reduces to the study of the Kummer map

$$\kappa_{F_v} : A(F_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^1(F_v, A[p^\infty]).$$

at v dividing p .

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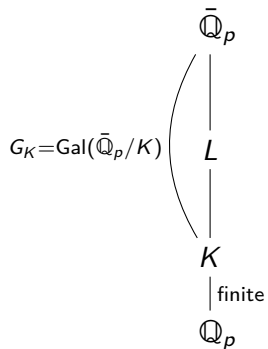
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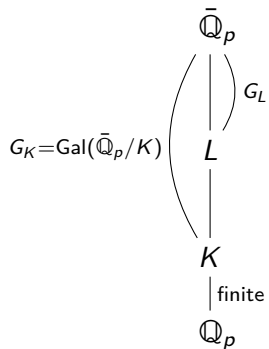
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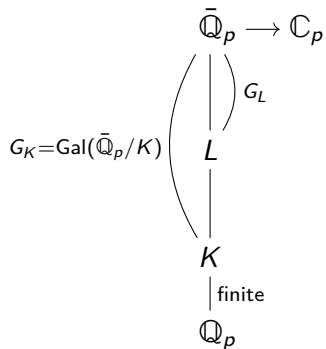
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$$\begin{array}{ccc} & \bar{\mathbb{Q}}_p & \longrightarrow \mathbb{C}_p \\ & \left. \begin{array}{c} | \\ | \\ | \end{array} \right)_{G_L} & \\ G_K = \text{Gal}(\bar{\mathbb{Q}}_p/K) & L & \longrightarrow \hat{L} \\ & | & \\ & K & \\ & | \text{finite} & \\ & \mathbb{Q}_p & \end{array}$$

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Definition (Scholze, 2012)

A complete non-archimedean field F of residue characteristic p is a *perfectoid field* if its valuation group is non-discrete and the p -th power Frobenius map $x \mapsto x^p$ on $\mathcal{O}_F/(p)$ is surjective.

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4. $K(p^{1/p^\infty})^\wedge$ (non Galois over K)

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2. From Abelian varieties to p -adic Galois representations

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Moreover, $V_p(A)_0/T_p(A)_0 = A[p^\infty]_0$.

Theorem (Coates & Greenberg, 1996)

Let A be an abelian variety defined over K . If \hat{L} is perfectoid, then

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Question (Coates & Greenberg)

Does an analogous description of $H_e^1(L, V/T)$ exist when V is a general de Rham p -adic representation?

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Their proofs are specific to the cyclotomic extension.

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Theorem 1 contains Coates and Greenberg's result for abelian varieties (and p -divisible groups).

3. Sketch of proof: the Fargues-Fontaine curve

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$$\begin{array}{ccccccc} 1. & 0 & \longrightarrow & \hat{A}(\mathfrak{m}_L) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow & H^1(L, \hat{A}[p^\infty]) & \longrightarrow & H^1(L, \hat{A}) & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow \lambda_L & & \downarrow & & \\ & 0 & \longrightarrow & A(L) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{\kappa_L} & H^1(L, A[p^\infty]) & \longrightarrow & H^1(L, A)[p^\infty] & \longrightarrow & 0 \\ & & & \downarrow & & & & & & \\ & & & 0 & & & & & & \end{array}$$

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$$\begin{array}{ccccccc} 1. & 0 & \longrightarrow & \hat{A}(\mathfrak{m}_L) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow & H^1(L, \hat{A}[p^\infty]) & \longrightarrow & H^1(L, \hat{A}) & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow \lambda_L & & \downarrow & & \\ & 0 & \longrightarrow & A(L) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{\kappa_L} & H^1(L, A[p^\infty]) & \longrightarrow & H^1(L, A)[p^\infty] & \longrightarrow & 0 \\ & & & \downarrow & & & & & & \\ & & & 0 & & & & & & \end{array}$$

2. Theorem (Coates & Greenberg)

Let \mathcal{F} be a commutative formal group defined over \mathcal{O}_K .

Coates and Greenberg's strategy $\text{Im}(\kappa_L) = \text{Im}(\lambda_L)$

Let A be an abelian variety defined over K . May assume A has semi-stable reduction.

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 & & \parallel & & \downarrow \wr & & \downarrow \wr \\
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Lemma/Definition

We have

$$0 \rightarrow H_e^1(L, V/T) \rightarrow H^1(L, V/T) \rightarrow H^1(L, E_{\text{disc}}(V/T)) \rightarrow 0.$$

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\rightsquigarrow We need to compute $H^1(L, E_{\underline{\text{disc}}}(V_0/T_0))$.

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 \text{If } n < 1, E_+(n) \simeq \mathbb{Q}_p(n)/\mathbb{Z}_p(n) \text{ and } \hat{t}_{\mathbb{Q}_p(n)} = 0.
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induces

$$\xi_L : H^1(L, E_{\underline{\text{disc}}}(V/T)) \rightarrow H^1(L, E_+(V/T)).$$

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C & G: compare $H_e^1(L, V/T)$ and $\text{Im}(\lambda_L)$

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+

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Theorem (P.)

Assume V is de Rham. If \hat{L} is perfectoid, then

$$\frac{\text{Im}(\lambda_L)}{H_e^1(L, V/T)} \simeq \text{Ker} \left(H^1(L, E_{\text{disc}}(V/T)) \xrightarrow{\xi_L} H^1(L, E_+(V/T)) \right).$$

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Moreover, if the Hodge-Tate weights of V are ≤ 1 , then

$$H_e^1(L, V/T) = \text{Im}(\lambda_L).$$

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It is a regular, Noetherian, separated and connected 1-dimensional scheme defined over \mathbb{Q}_p with $\Gamma(X^{\text{FF}}, \mathcal{O}_{X^{\text{FF}}}) = \mathbb{Q}_p$. Moreover, X^{FF} is complete, and

$$\pi_1(X^{\text{FF}}) \simeq \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p).$$

Vector bundles over the Fargues-Fontaine curve

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p -adic representations

$\text{Rep}_{\mathbb{Q}_p}(G_K) \rightarrow \text{Bun}_{X^{\text{FF}}}(G_K)$

$$W \mapsto \mathcal{E}(W) = \mathcal{O}_{X^{\text{FF}}} \otimes_{\mathbb{Q}_p} W = (\mathbf{B}_e \otimes_{\mathbb{Q}_p} W, \mathbf{B}_{\text{dR}}^+ \otimes_{\mathbb{Q}_p} W).$$

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► Fontaine (2020):

{ Almost \mathbb{C}_p -representations of G_K }



{ G_K -equivariant coherent sheaves over X^{FF} }.

Le Bras (2018).

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\mathcal{E} is *semi-stable* if it is non-trivial, and $\mu(\mathcal{E}') \leq \mu(\mathcal{E})$ for every non-trivial coherent subsheaf $\mathcal{E}' \subset \mathcal{E}$.

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Theorem

There exists an unique increasing filtration of \mathcal{E} by coherent subsheaves

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{n-1} \subset \mathcal{E}_n = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semi-stable for each $i \in \{1, \dots, n\}$, and

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Harder-Narasimhan filtration

Let \mathcal{E} be a vector bundle over X^{FF} .

degree: $\deg(\mathcal{E}) \in \mathbb{Z}$ rank: $\text{rk}(\mathcal{E}) \in \mathbb{N}$,

slope: $\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\text{rk}(\mathcal{E})} \in \mathbb{Q}$.

\mathcal{E} is *semi-stable* if it is non-trivial, and $\mu(\mathcal{E}') \leq \mu(\mathcal{E})$ for every non-trivial coherent subsheaf $\mathcal{E}' \subset \mathcal{E}$.

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$(\mathcal{E}_i)_{0 \leq i \leq n}$: the *Harder-Narasimhan filtration* of \mathcal{E} ,

$(\mu(\mathcal{E}_i/\mathcal{E}_{i-1}))_{1 \leq i \leq n}$: the *Harder-Narasimhan slopes* of \mathcal{E} .

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Theorem (Fargues & Fontaine)

$$\text{Rep}_{\mathbb{Q}_p}(G_K) \xrightarrow{\simeq} \text{Bun}_{X^{\text{FF}}}^0(G_K) \quad (:H\text{-}N \text{ slopes } 0).$$

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By-product:

Proposition (Fargues & Fontaine)

Assume \hat{L} is perfectoid. If \mathcal{E} is a G_L -equivariant coherent sheaf over X^{FF} whose Harder-Narasimhan slopes are > 0 , then

$$H^1(L, \Gamma(X^{\text{FF}}, \mathcal{E})) \simeq \text{Ext}_{G_L}^1(\mathcal{O}_{X^{\text{FF}}}, \mathcal{E}) = 0.$$

Application to our problem

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The Harder-Narasimhan slopes of $\mathcal{E}_+(V)$ are > 0 if and only if $V = V_0$.

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- ▶ Define a modification of (de Rham) G_K -equivariant vector bundles,

$$\mathrm{Bun}_{\mathrm{XFF}}(G_K)_{\mathrm{dR}} \begin{array}{c} \xrightarrow{\tau_{\mathrm{HT}}^{\leq 0}} \\ \xleftrightarrow{\text{forget}} \\ \xrightarrow{\tau_{\mathrm{HT}}^{\leq 0}} \end{array} \mathrm{Bun}_{\mathrm{XFF}}(G_K)_{\mathrm{dR}}^{\leq 0}$$

$$\mathcal{E} = (\mathcal{E}_e, \mathcal{E}_{\mathrm{dR}}^+) \mapsto (\mathcal{E}_e, \mathcal{E}_{\mathrm{dR}}^+ + \mathbf{B}_{\mathrm{dR}}^+ \otimes \mathbf{D}_{\mathrm{dR}}(\mathcal{E})),$$

of which $\mathcal{E}_+(V)$ is a particular case, that is $\mathcal{E}(V) \mapsto \mathcal{E}_+(V)$.

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In particular,

$$\frac{\text{Im}(\lambda_L)}{H_e^1(L, V/T)} \simeq \text{Ker} \left(H^1(L, E_{\underline{\text{disc}}}(V/T)) \xrightarrow{\xi_L} H^1(L, E_+(V/T)) \right).$$

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General case



While $\bar{\mathbb{Q}}_p$ is dense in \mathbf{B}_{dR}^+ ; if \hat{L} perfectoid, then L is in general not dense in $(\mathbf{B}_{\text{dR}}^+)^{G_L}$ or $(\mathbf{B}_{\text{dR}}^+ / \text{Fil}^n \mathbf{B}_{\text{dR}})^{G_L}$ for $n > 1$ (Iovita & Zaharescu, 1999).

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Assume V is de Rham. If \hat{L} is perfectoid, then

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- ▶ If L/K is p -adic Lie extension, can we obtain a more precise result?

Thank You!