Universal norms of p-adic Galois representations

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Gautier Ponsinet

Università degli Studi di Genova

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1. Iwasawa theory of abelian varieties

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$$\mathrm{Sel}_p(A/F) = \mathrm{Ker}\left(\mathrm{H}^1(F, A[p^\infty]) \xrightarrow{loc} \prod_v \frac{\mathrm{H}^1(F_v, A[p^\infty])}{\mathrm{Im}(\kappa_{F_v})}\right)$$

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Conjecture (Mazur 1972)

If A has good ordinary reduction at all prime of F dividing p, then the Pontryagin dual of $\mathrm{Sel}_p(A/F_{cyc})$ is a torsion $\mathbb{Z}_p[[\Gamma]]$-module.

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Theorem (Mazur 1972)

If A has good ordinary reduction at all prime of F dividing p, then the kernel and cokernel of the restriction map

$$\mathrm{Sel}_p(A/F) \to \mathrm{Sel}_p(A/F_{cyc})^\Gamma$$

are finite.

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- In both situation, the study of the structure of $\mathrm{Sel}_p(A/F_\infty)$ essentially reduces to the study of the Kummer map

$$\kappa_{F_v} : A(F_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathrm{H}^1(F_v, A[p^\infty]).$$

at v dividing p.

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Definition (Scholze, 2012)

A complete non-archimedean field F of residue characteristic p is a *perfectoid field* if its valuation group is non-discrete and the p-th power Frobenius map $x \mapsto x^p$ on $\mathcal{O}_F/(p)$ is surjective.

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2. From Abelian varieties to p-adic Galois representations

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Moreover, $V_p(A)_0/T_p(A)_0 = A[p^\infty]_0$.

Theorem (Coates & Greenberg, 1996)

Let A be an abelian variety defined over K. If $\hat{L}$ is perfectoid, then

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Question (Coates & Greenberg)

Does an analogous description of $\mathrm{H}^1_e(L, V/T)$ exist when V is a general de Rham p-adic representation?

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Moreover, if the Hodge-Tate weights of V *are* $\leqslant 1$, *then*

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Their proofs are specific to the cyclotomic extension.

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Theorem 1 contains Coates and Greenberg's result for abelian varieties (and p-divisible groups).

3. Sketch of proof: the Fargues-Fontaine curve

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Coates and Greenberg's strategy $\mathrm{Im}(\kappa_L) = \mathrm{Im}(\lambda_L)$

Let A be an abelian variety defined over K. May assume A has semi-stable reduction.
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2. **Theorem (Coates & Greenberg)**

   Let $\mathcal{F}$ be a commutative formal group defined over $\mathcal{O}_K$.

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0 \arrow[r] & A[p^\infty] \arrow[r] & A^{(p)}(\bar{\mathbb{Q}}_p) \arrow[r, "\log_A"] & t_A(\bar{\mathbb{Q}}_p) \arrow[r] & 0.
\end{tikzcd}$$

Almost $\mathbb{C}_p$-representations (Fontaine, 2003)

Lemma/Definition

We have

$$0 \to \mathrm{H}^1_e(L, V/T) \to \mathrm{H}^1(L, V/T) \to \mathrm{H}^1(L, E_{\underline{\mathrm{disc}}}(V/T)) \to 0.$$

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$E_{\underline{\mathrm{disc}}}(V/T)$: "group of points with values in $\bar{\mathbb{Q}}_p$ associated with V/T".

Almost $\mathbb{C}_p$-representations (Fontaine, 2003)

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$\rightsquigarrow$ We need to compute $\mathrm{H}^1(L, E_{\underline{\mathrm{disc}}}(V_0/T_0))$.

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$E_{\underline{\mathrm{disc}}}(A[p^\infty]) \simeq A^{(p)}(\bar{\mathbb{Q}}_p) \to E_{+}(A[p^\infty]) \simeq A^{(p)}(\mathbb{C}_p)$, and $\hat{t}_{V_p(A)}(\bar{\mathbb{Q}}_p) \simeq t_A(\mathbb{C}_p)$.

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If $n < 1$, $E_{+}(n) \simeq \mathbb{Q}_p(n)/\mathbb{Z}_p(n)$ and $\hat{t}_{\mathbb{Q}_p(n)} = 0$.

Almost $\mathbb{C}_p$-representations (Fontaine, 2003)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V/T & \longrightarrow & E_+(V/T) & \longrightarrow & \hat{t}_V(\bar{\mathbb{Q}}_p) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & V/T & \longrightarrow & E_{\underline{\mathrm{disc}}}(V/T) & \longrightarrow & t_V(\bar{\mathbb{Q}}_p) & \longrightarrow & 0 \end{array}$$

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\end{array}
$$

induces

$$\xi_L : \mathrm{H}^1(L, E_{\underline{\mathrm{disc}}}(V/T)) \to \mathrm{H}^1(L, E_+(V/T)).$$

Situation

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$$+$$

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Assume V is de Rham. If $\hat{L}$ is perfectoid, then

$$\frac{\mathrm{Im}(\lambda_L)}{\mathrm{H}^1_e(L, V/T)} \simeq \mathrm{Ker}\left(\mathrm{H}^1(L, E_{\underline{\mathrm{disc}}}(V/T)) \xrightarrow{\xi_L} \mathrm{H}^1(L, E_+(V/T))\right).$$

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Moreover, if the Hodge-Tate weights of V are $\leqslant 1$, then

$$\mathrm{H}^1_e(L, V/T) = \mathrm{Im}(\lambda_L).$$

The Fargues-Fontaine curve ($\sim$2009, publ. 2018)

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It is a regular, Noetherian, separated and connected 1-dimensional scheme defined over $\mathbb{Q}_p$ with $\Gamma(\mathsf{X}^{\mathsf{FF}}, \mathcal{O}_{\mathsf{X}^{\mathsf{FF}}}) = \mathbb{Q}_p$. Morevover, X^{FF} is complete, and

$$\pi_1(\mathsf{X}^{\mathsf{FF}}) \simeq \mathsf{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p).$$

Vector bundles over the Fargues-Fontaine curve

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\mathsf{Bun}_{\mathsf{X}^{\mathsf{FF}}} \simeq \left\{ \begin{array}{l} (\mathcal{E}_{\mathsf{e}}, \mathcal{E}_{\mathsf{dR}}^{+}, \iota_{\mathcal{E}}), \\ \mathcal{E}_{\mathsf{e}} \text{ f.g. torsion free } \mathbf{B}_{\mathsf{e}}\text{-mod}, \\ \mathcal{E}_{\mathsf{dR}}^{+} \text{ f.g. torsion free } \mathbf{B}_{\mathsf{dR}}^{+}\text{-mod}, \\ \iota_{\mathcal{E}} : \mathbf{B}_{\mathsf{dR}} \otimes_{\mathbf{B}_{\mathsf{e}}} \mathcal{E}_{\mathsf{e}} \xrightarrow{\sim} \mathbf{B}_{\mathsf{dR}} \otimes_{\mathbf{B}_{\mathsf{dR}}^{+}} \mathcal{E}_{\mathsf{dR}}^{+} \end{array} \right\}
$$

$$
\mathcal{E} \mapsto \left(\begin{array}{l} \mathcal{E}_{\mathsf{e}} = \Gamma(\mathsf{X}^{\mathsf{FF}} \setminus \{\infty\}, \mathcal{E}) \\ \mathcal{E}_{\mathsf{dR}}^{+} = \mathbf{B}_{\mathsf{dR}}^{+} \otimes_{\mathcal{O}_{\mathsf{X}^{\mathsf{FF}},\infty}} \mathcal{E}_{\infty} \\ \iota_{\mathcal{E}} \text{ glueing data} \end{array} \right)
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$\mathrm{Bun}_{\mathsf{X}^{\mathsf{FF}}}(G_K)$: G_K-equivariant vector bundles

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p-adic representations

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{Bun}_{\mathsf{X}^{\mathsf{FF}}}(G_K)$$

$$W \mapsto \mathcal{E}(W) = \mathcal{O}_{\mathsf{X}^{\mathsf{FF}}} \otimes_{\mathbb{Q}_p} W = (\mathbf{B}_{\mathrm{e}} \otimes_{\mathbb{Q}_p} W, \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} W).$$

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- Fontaine (2020):

$$\{\text{ Almost } \mathbb{C}_p\text{-representations of } G_K \}$$
$$\updownarrow$$
$$\{\ G_K\text{-equivariant coherent sheaves over } \mathrm{X}^{\mathrm{FF}}\ \}.$$

Le Bras (2018).

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Theorem

There exists an unique increasing filtration of $\mathcal{E}$ by coherent subsheaves

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{n-1} \subset \mathcal{E}_n = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semi-stable for each $i \in \{1, \ldots, n\}$, and

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$(\mathcal{E}_i)_{0 \leqslant i \leqslant n}$: the *Harder-Narasimhan filtration* of $\mathcal{E}$,
$(\mu(\mathcal{E}_i/\mathcal{E}_{i-1}))_{1 \leqslant i \leqslant n}$: the *Harder-Narasimhan slopes* of $\mathcal{E}$.

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By-product:

Proposition (Fargues & Fontaine)

Assume $\hat{L}$ is perfectoid. If $\mathcal{E}$ is a G_L-equivariant coherent sheaf over X^{FF} whose Harder-Narasimhan slopes are > 0, then

$$\mathrm{H}^1(L, \Gamma(\mathsf{X}^{\mathsf{FF}}, \mathcal{E})) \simeq \mathrm{Ext}^1_{G_L}(\mathcal{O}_{\mathsf{X}^{\mathsf{FF}}}, \mathcal{E}) = 0.$$

Application to our problem

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The Harder-Narasimhan slopes of $\mathcal{E}_+(V)$ *are* > 0 *if and only if* $V = V_0$.

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If $\hat{L}$ is perfectoid, then

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- Define a modification of (de Rham) G_K-equivariant vector bundles,

$$\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}} \underset{\text{forget}}{\overset{\tau_{\mathrm{HT}}^{\leqslant 0}}{\rightleftarrows}} \mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}^{\leqslant 0}$$

$$\mathcal{E} = (\mathcal{E}_{\mathrm{e}}, \mathcal{E}_{\mathrm{dR}}^+) \mapsto (\mathcal{E}_{\mathrm{e}}, \mathcal{E}_{\mathrm{dR}}^+ + \mathbf{B}_{\mathrm{dR}}^+ \otimes \mathbf{D}_{\mathrm{dR}}(\mathcal{E})),$$

of which $\mathcal{E}_+(V)$ is a particular case, that is $\mathcal{E}(V) \mapsto \mathcal{E}_+(V)$.

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0 \arrow[r] & V/T \arrow[r] & E_+(V/T) \arrow[r] & \hat{t}_V(\bar{\mathbb{Q}}_p) \arrow[r] & 0 \\
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In particular,

$$\frac{\mathrm{Im}(\lambda_L)}{\mathrm{H}^1_e(L, V/T)} \simeq \mathrm{Ker}\left(\mathrm{H}^1(L, E_{\underline{\mathrm{disc}}}(V/T)) \xrightarrow{\xi_L} \mathrm{H}^1(L, E_+(V/T))\right).$$

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General case

While $\bar{\mathbb{Q}}_p$ is dense in $\mathbf{B}^+_{\mathrm{dR}}$; if $\hat{L}$ perfectoid, then L is in general not dense in $(\mathbf{B}^+_{\mathrm{dR}})^{G_L}$ or $(\mathbf{B}^+_{\mathrm{dR}}/\operatorname{Fil}^n \mathbf{B}_{\mathrm{dR}})^{G_L}$ for $n > 1$ (Iovita & Zaharescu, 1999).

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Assume V is de Rham. If $\hat{L}$ is perfectoid, then

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- If L/K is p-adic Lie extension, can we obtain a more precise result?

Thank You!