GLOBAL WELL-POSEDNESS OF HARTREE TYPE DIRAC EQUATIONS AT CRITICAL REGULARITY

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CONTENTS

HARTREE TYPE DIRAC EQUATION

- **Equations**

$$\text{(D)} \qquad (i\gamma^\mu\partial_\mu - m)\boldsymbol{\psi} = -V_b * (\overline{\boldsymbol{\psi}}\Gamma\boldsymbol{\psi})\Gamma\boldsymbol{\psi}, \quad \boldsymbol{\psi}(0) = \boldsymbol{\psi}_0$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \boldsymbol{x} = (x^\mu)_{\mu=0,1,2,3} = (t, x, y, z)$$

$$\boldsymbol{\psi} : \mathbb{R}^{1+3} \to \mathbb{C}^4, \quad \overline{\boldsymbol{\psi}} = \boldsymbol{\psi}^\dagger\gamma^0, \quad \boldsymbol{\psi}^\dagger = (\boldsymbol{\psi}^*)^T$$

$$m \geq 0$$

$$V_b(x) = \frac{1}{4\pi}\frac{e^{-b|x|}}{|x|}, \quad b \geq 0$$

HARTREE TYPE DIRAC EQUATION

- **Gamma matrices (Dirac-Pauli represention)**

$$\gamma^0 = \begin{bmatrix} 1_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & -1_{2\times 2} \end{bmatrix}, \quad \gamma^j = \begin{bmatrix} 0_{2\times 2} & \sigma^j \\ -\sigma^j & 0_{2\times 2} \end{bmatrix} \quad (j = 1, 2, 3)$$

$$\text{(D)} \qquad (i\gamma^\mu \partial_\mu - m)\psi = -V_b * (\bar{\psi}\Gamma\psi)\Gamma\psi, \quad \psi(0) = \psi_0$$

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0_{2\times 2} & 1_{2\times 2} \\ 1_{2\times 2} & 0_{2\times 2} \end{bmatrix}$$

$$\Gamma = \gamma^0, \quad 1_{4\times 4}, \quad i\gamma^5$$

HARTREE TYPE DIRAC EQUATION

- **Derived from**

$$(\mathrm{MD}) \quad \begin{cases} \Box A_\mu = -\overline{\boldsymbol{\psi}}\gamma_\mu \boldsymbol{\psi}, \\ (i\gamma^\mu \partial_\mu - m)\boldsymbol{\psi} = -A_\mu \gamma^\mu \boldsymbol{\psi} \\ \partial^\mu A_\mu = 0 \end{cases}$$

$(\mathrm{curl}\boldsymbol{A} \equiv 0 \Rightarrow b = 0, \Gamma = \gamma^0$: see Chadam-Glassey (1976))

$$(\mathrm{DKG}) \quad \begin{cases} (i\gamma^\mu \partial_\mu - m)\boldsymbol{\psi} = -\varphi \Gamma \boldsymbol{\psi} \\ (\Box + M^2)\varphi = \overline{\boldsymbol{\psi}}\Gamma \boldsymbol{\psi}, \end{cases} \quad (\Gamma = 1_{4\times 4}, \quad i\gamma^5)$$

scalar, pseudoscalar

(standing wave: $\varphi = e^{i\lambda t}\rho,\ b = \sqrt{M^2 - \lambda^2} > 0$)

(For the role of $i\gamma^5$ see Wick (1958) and Bjorken-Drell(1964))

HARTREE TYPE DIRAC EQUATION

- **In this talk we choose**

$$\Gamma = i\gamma^5 = i \begin{bmatrix} 0_{2\times 2} & 1_{2\times 2} \\ 1_{2\times 2} & 0_{2\times 2} \end{bmatrix} \quad \text{(pseudoscalar)}$$

$$\text{(D)} \qquad (i\gamma^\mu \partial_\mu - m)\boldsymbol{\psi} = V_b * (\overline{\boldsymbol{\psi}}\gamma^5\boldsymbol{\psi})\gamma^5\boldsymbol{\psi}, \quad \boldsymbol{\psi}(0) = \boldsymbol{\psi}_0$$

$$m > 0, \quad V_b(x) = \frac{1}{4\pi}\frac{e^{-b|x|}}{|x|}, \quad b \geq 0$$

$*$ We consider GWP of (D) with **partially small** initial data.

$$\boldsymbol{\psi}_0 = \boldsymbol{\psi}_0' + \boldsymbol{\psi}_0'', \quad \|\boldsymbol{\psi}_0'\|_X \ll 1, \quad \|\boldsymbol{\psi}_0\|_X \gtrsim 1$$

DECOMPOSITION VIA PROJECTION

- **Charge conjugation**

$$\mathsf{C}\boldsymbol{\psi} := i\gamma^2\boldsymbol{\psi}^* \quad \text{(Charge conjugation operator)}$$

$$P^c_\theta\boldsymbol{\psi} := \frac{1}{2}(1_{4\times4} + \theta\mathsf{C})\boldsymbol{\psi} \quad (\theta \in \{+,-\})$$

$$P^c_+ + P^c_- = 1_{4\times4}, \quad (P^c_\theta)^2 = P^c_\theta, \quad P^c_\theta P^c_{-\theta} = 0$$

$$(i\gamma^\mu\partial_\mu - m)\mathsf{C}\boldsymbol{\psi} = -[V_b * (\overline{\mathsf{C}\boldsymbol{\psi}}\gamma^5\mathsf{C}\boldsymbol{\psi})]\gamma^5\mathsf{C}\boldsymbol{\psi}$$

$$(i\gamma^\mu\partial_\mu - m)\boldsymbol{\psi} = V_b * (\overline{\boldsymbol{\psi}}\gamma^5\boldsymbol{\psi})\gamma^5\boldsymbol{\psi}$$

DECOMPOSITION VIA PROJECTION

- **Charge conjugation**

$$\overline{P^c_\theta \boldsymbol{\psi}}\gamma^5 P^c_\theta \boldsymbol{\psi} = 0 \quad (\theta \in \{+,-\})$$

$$(i\gamma^\mu\partial_\mu - m)P^c_+\boldsymbol{\psi} = V_b * (\overline{P^c_+\boldsymbol{\psi}}\gamma^5 P^c_-\boldsymbol{\psi} + \overline{P^c_-\boldsymbol{\psi}}\gamma^5 P^c_+\boldsymbol{\psi})\gamma^5 P^c_+\boldsymbol{\psi},$$

$$(i\gamma^\mu\partial_\mu - m)P^c_-\boldsymbol{\psi} = V_b * (\overline{P^c_+\boldsymbol{\psi}}\gamma^5 P^c_-\boldsymbol{\psi} + \overline{P^c_-\boldsymbol{\psi}}\gamma^5 P^c_+\boldsymbol{\psi})\gamma^5 P^c_-\boldsymbol{\psi}$$

$$P^c_+\boldsymbol{\psi}(0) = P^c_+\boldsymbol{\psi}_0,\ P^c_-\boldsymbol{\psi}(0) = P^c_-\boldsymbol{\psi}_0$$

∗ A good decomposition for **partial smallness**.

$$(i\gamma^\mu\partial_\mu - m)\boldsymbol{\varphi} = V_b * (\overline{\boldsymbol{\varphi}}\gamma^5\boldsymbol{\chi} + \overline{\boldsymbol{\chi}}\gamma^5\boldsymbol{\varphi})\gamma^5\boldsymbol{\varphi},$$

$$(i\gamma^\mu\partial_\mu - m)\boldsymbol{\chi} = V_b * (\overline{\boldsymbol{\varphi}}\gamma^5\boldsymbol{\chi} + \overline{\boldsymbol{\chi}}\gamma^5\boldsymbol{\varphi})\gamma^5\boldsymbol{\chi}$$

DECOMPOSITION VIA PROJECTION

- **Energy projection**

$$\Pi_\theta(\xi) := \frac{1}{2}\left(1_{4\times 4} + \theta\frac{\xi_j\gamma^0\gamma^j + m\gamma^0}{\Lambda(\xi)}\right) \quad (\theta \in \{+,-\})$$

$$\Lambda(\xi) = \sqrt{m^2 + |\xi|^2}$$

$$\Pi_\theta = \Pi_\theta(D) = \mathcal{F}^{-1}\Pi_\theta(\xi), \quad \Lambda(D) = \mathcal{F}^{-1}\Lambda(\xi), \quad D = -i\nabla$$

$$\Pi_\theta(D) + \Pi_{-\theta}(D) = 1_{4\times 4}, \;\; \Pi_\theta(D)\Pi_\theta(D) = \Pi_\theta(D), \;\; \Pi_\theta(D)\Pi_{-\theta}(D) = 0$$

$$\Lambda(D)(\Pi_+(D) - \Pi_-(D)) = \gamma^0\gamma^j(-i\partial_j) + m\gamma^0$$

DECOMPOSITION VIA PROJECTION

- **Energy projection**

$$\Lambda(D)(\Pi_+(D) - \Pi_-(D)) = \gamma^0\gamma^j(-i\partial_j) + m\gamma^0$$

$$(i\partial_t - \theta\Lambda(D))\Pi_\theta\boldsymbol{\varphi} = \sum_{\substack{\theta_j\in\{+,-\}\\ j=1,\cdots,5}} \Pi_\theta[V_b * (\overline{\Pi_{\theta_1}\boldsymbol{\varphi}}\gamma^5\Pi_{\theta_2}\boldsymbol{\chi} + \overline{\Pi_{\theta_3}\boldsymbol{\chi}}\gamma^5\Pi_{\theta_4}\boldsymbol{\varphi})\gamma^0\gamma^5\Pi_{\theta_5}\boldsymbol{\varphi}]$$

$$(i\partial_t - \theta'\Lambda(D))\Pi_{\theta'}\boldsymbol{\chi} = \sum_{\substack{\theta'_j\in\{+,-\}\\ j=1,\cdots,5}} \Pi_{\theta'}[V_b * (\overline{\Pi_{\theta'_1}\boldsymbol{\varphi}}\gamma^5\Pi_{\theta'_2}\boldsymbol{\chi} + \overline{\Pi_{\theta'_3}\boldsymbol{\chi}}\gamma^5\Pi_{\theta'_4}\boldsymbol{\varphi})\gamma^0\gamma^5\Pi_{\theta'_5}\boldsymbol{\chi}]$$

$$\theta, \theta', \theta_j \in \{+,-\}$$

DECOMPOSITION VIA PROJECTION

- **Energy projection**

$$(D) \text{ is equivalent to find } (\boldsymbol{\varphi}_+, \boldsymbol{\varphi}_-, \boldsymbol{\chi}_+, \boldsymbol{\chi}_-) :$$

$$\boldsymbol{\varphi}_\theta = \Pi_\theta \boldsymbol{\varphi}, \boldsymbol{\chi}_{\theta'} = \Pi_{\theta'} \boldsymbol{\chi}, \quad \theta, \theta' \in \{+, -\}$$

$$(i\partial_t - \theta\Lambda(D))\boldsymbol{\varphi}_\theta = \sum_{\substack{\theta_j \in \{+,-\} \\ j=1,\cdots,5}} \Pi_\theta [V_b * (\overline{\boldsymbol{\varphi}_{\theta_1}} \gamma^5 \boldsymbol{\chi}_{\theta_2} + \overline{\boldsymbol{\chi}_{\theta_3}} \gamma^5 \boldsymbol{\varphi}_{\theta_4}) \gamma^0 \gamma^5 \boldsymbol{\varphi}_{\theta_5}]$$

$$(i\partial_t - \theta'\Lambda(D))\boldsymbol{\chi}_{\theta'} = \sum_{\substack{\theta'_j \in \{+,-\} \\ j=1,\cdots,5}} \Pi_{\theta'} [V_b * (\overline{\boldsymbol{\varphi}_{\theta'_1}} \gamma^5 \boldsymbol{\chi}_{\theta'_2} + \overline{\boldsymbol{\chi}_{\theta'_3}} \gamma^5 \boldsymbol{\varphi}_{\theta'_4}) \gamma^0 \gamma^5 \boldsymbol{\chi}_{\theta'_5}]$$

$$\boldsymbol{\varphi}_\theta(0) = \Pi_\theta P_+^c \boldsymbol{\psi}_0, \quad \boldsymbol{\chi}_{\theta'}(0) = \Pi_{\theta'} P_-^c \boldsymbol{\psi}_0$$

PROBLEM AND RESULTS

$$(D) \qquad (i\gamma^\mu \partial_\mu - m)\boldsymbol{\psi} = V_b * (\overline{\boldsymbol{\psi}}\gamma^5 \boldsymbol{\psi})\gamma^5 \boldsymbol{\psi}, \quad \boldsymbol{\psi}(0) = \boldsymbol{\psi}_0$$

$$m > 0, \quad V_b(x) = \frac{1}{4\pi} \frac{e^{-b|x|}}{|x|}, \quad b \geq 0$$

- L^2-conservation

$\|\boldsymbol{\psi}(t)\|_{L^2} = \|\boldsymbol{\psi}_0\|_{L^2}$ if the solution exists.

- L^2-scaling invariance

$\lambda^{\frac{3}{2}}\boldsymbol{\psi}(\lambda t, \lambda x)$ is a solution to (D) with m, b replaced by $\frac{m}{\lambda}, \frac{b}{\lambda}$.

(L^2 is the critial space for well-posedness)

PROBLEM AND RESULTS

- Problem 1: Gobal well-posedness in $L^2_x(\mathbb{R}^3)$
  - For every $\psi_0 \in L^2_x \;\; \exists! \; \psi \in C(\mathbb{R}; L^2_x)$

- Problem 2: Linear scattering

$$\boldsymbol{\psi} \text{ scatters if } \; \exists \boldsymbol{\psi}^\ell :$$

$$(i\gamma^\mu \partial_\mu - m)\boldsymbol{\psi}^\ell = 0$$

$$\|\boldsymbol{\psi}(t) - \boldsymbol{\psi}^\ell(t)\|_{L^2_x} \xrightarrow{t \to +\infty} 0.$$

(L^2 problems are completely open in 3d)

PROBLEM AND RESULTS

- Known results for $\Gamma = 1_{4\times 4}$ (scalar source)

$$\text{(DKG)} \quad \begin{cases} (i\gamma^\mu\partial_\mu - m)\boldsymbol{\psi} = -\varphi\boldsymbol{\psi} \\ (\Box + M^2)\varphi = \overline{\boldsymbol{\psi}}\boldsymbol{\psi}, \end{cases}$$

- Candy-Herr (2018): scattering in

$$L^{2,\sigma} \times H^{\frac{1}{2},\sigma} \times H^{-\frac{1}{2},\sigma} (\sigma > 0)$$

with one of $\|P_+^c\boldsymbol{\psi}_0\|_{L^{2,\sigma}}$ and $\|P_-^c\boldsymbol{\psi}_0\|_{L^{2,\sigma}}$ small

($H^{s,\sigma} = (1-\Delta_{\mathbb{S}^2})^{-\frac{\sigma}{2}} H^s$ and $L^{2,\sigma} = H^{0,\sigma}$)

* $L^{2,\sigma}$ is scaling critical subspace of L^2.

PROBLEM AND RESULTS

- Known results for $\Gamma = 1_{4\times 4}$ or $1_{2\times 2}$ (scalr source)

$$(D) \qquad (i\gamma^\mu \partial_\mu - m)\boldsymbol{\psi} = V_b * (\overline{\boldsymbol{\psi}}\boldsymbol{\psi})\boldsymbol{\psi}, \quad \boldsymbol{\psi}(0) = \boldsymbol{\psi}_0$$

  - GWP and Scattering $(m > 0, b > 0)$
    - Yang (2019): $H^s(\mathbb{R}^3)(s > 0)$ (small data)
    - Tesfahun (2020): $H^s(\mathbb{R}^d)(d = 2, 3, s > 0)$ (small data)
    - Georgiev-Shakarov (2021): $H^s(\mathbb{R}^2)(s > 0)$ (large data GWP)
    - C-Hong-Lee (in preprint): $L^{2,\sigma}(\mathbb{R}^3)(\sigma > 0)$ (partial smallness)
    - C-Hong-Lee (in preprint): $L^2(\mathbb{R}^2)$ (partial smallness)

PROBLEM AND RESULTS

- Known results for $\Gamma = \gamma^0 = \begin{bmatrix} 1_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & -1_{2\times 2} \end{bmatrix}$

$$\text{(D)} \qquad (i\gamma^\mu \partial_\mu - m)\boldsymbol{\psi} = (V_b * |\boldsymbol{\psi}|^2)\gamma^0\boldsymbol{\psi}, \quad \boldsymbol{\psi}(0) = \boldsymbol{\psi}_0$$

  - GWP and Scattering $(m > 0, b = 0)$
    - C-Lee-Ozawa (2022): (2d) No linear scattering
    - C-Hong-Lee (in preprint): (3d) No linear scattering
    - C-Kwon-Lee-Yang (in preprint): (3d) modified scattering

PROBLEM AND RESULTS

$$\text{(D)} \qquad (i\gamma^\mu \partial_\mu - m)\boldsymbol{\psi} = V_b * (\overline{\boldsymbol{\psi}}\gamma^5\boldsymbol{\psi})\gamma^5\boldsymbol{\psi}, \quad \boldsymbol{\psi}(0) = \boldsymbol{\psi}_0$$

$$m > 0, \quad V_b(x) = \frac{1}{4\pi}\frac{e^{-b|x|}}{|x|}, \quad b \geq 0$$

Theorem 1 (C-Hong-Ozawa)

Let $b > 0$ and $\sigma > 0$. Assume that

one of $\|P_+^c\boldsymbol{\psi}_0\|_{L^{2,\sigma}}$ and $\|P_-^c\boldsymbol{\psi}_0\|_{L^{2,\sigma}}$ is sufficiently small.

Then (D) is globally well-posed in $L^{2,\sigma}$ and $\boldsymbol{\psi}$ scatters in $L^{2,\sigma}$.

PROBLEM AND RESULTS

$$\text{(D)} \qquad (i\gamma^\mu \partial_\mu - m)\boldsymbol{\psi} = V_b * (\overline{\boldsymbol{\psi}}\gamma^5\boldsymbol{\psi})\gamma^5\boldsymbol{\psi}, \;\; \boldsymbol{\psi}(0) = \boldsymbol{\psi}_0$$

$$m > 0, \;\; V_b(x) = \frac{1}{4\pi}\frac{e^{-b|x|}}{|x|}, \;\; b \geq 0$$

Theorem 2 (C-Hong-Ozawa)

Let $b = 0$. Suppose that there exists a linear solution $\boldsymbol{\psi}^\ell$:

$$\int [V_0 * (\overline{\boldsymbol{\psi}^\ell}\gamma^5\boldsymbol{\psi}^\ell)]\overline{\boldsymbol{\psi}^\ell}\gamma^5\boldsymbol{\psi}^\ell \, dx \geq \alpha \int [V_0 * |\boldsymbol{\psi}^\ell|^2]|\boldsymbol{\psi}^\ell|^2 \, dx \;\; (\alpha > 0)$$

$$\|\boldsymbol{\psi}(t) - \boldsymbol{\psi}^\ell(t)\|_{L^2} \xrightarrow{t \to +\infty} 0.$$

Then $\boldsymbol{\psi} = \mathbf{0} = \boldsymbol{\psi}^\ell$ in L^2.

PROOF OF

Theorem 1 (C-Hong-Ozawa)

Let $b > 0$ and $\sigma > 0$. Assume that

one of $\|P_+^c \boldsymbol{\psi}_0\|_{L^{2,\sigma}}$ and $\|P_-^c \boldsymbol{\psi}_0\|_{L^{2,\sigma}}$ is sufficiently small.

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$$\text{(D)} \qquad (i\gamma^\mu \partial_\mu - m)\boldsymbol{\psi} = V_b * (\overline{\boldsymbol{\psi}}\gamma^5 \boldsymbol{\psi})\gamma^5 \boldsymbol{\psi}, \quad \boldsymbol{\psi}(0) = \boldsymbol{\psi}_0$$

$$(i\partial_t - \theta\Lambda(D))\boldsymbol{\varphi}_\theta = \sum_{\substack{\theta_j \in \{+,-\} \\ j=1,\cdots,5}} \Pi_\theta [V_b * (\overline{\boldsymbol{\varphi}_{\theta_1}}\gamma^5 \boldsymbol{\chi}_{\theta_2} + \overline{\boldsymbol{\chi}_{\theta_3}}\gamma^5 \boldsymbol{\varphi}_{\theta_4})\gamma^0\gamma^5 \boldsymbol{\varphi}_{\theta_5}]$$

$$(i\partial_t - \theta'\Lambda(D))\boldsymbol{\chi}_{\theta'} = \sum_{\substack{\theta'_j \in \{+,-\} \\ j=1,\cdots,5}} \Pi_{\theta'} [V_b * (\overline{\boldsymbol{\varphi}_{\theta'_1}}\gamma^5 \boldsymbol{\chi}_{\theta'_2} + \overline{\boldsymbol{\chi}_{\theta'_3}}\gamma^5 \boldsymbol{\varphi}_{\theta'_4})\gamma^0\gamma^5 \boldsymbol{\chi}_{\theta'_5}]$$

$$\boldsymbol{\varphi}_\theta(0) = \Pi_\theta P_+^c \boldsymbol{\psi}_0, \quad \boldsymbol{\chi}_{\theta'}(0) = \Pi_{\theta'} P_-^c \boldsymbol{\psi}_0$$

$$\boldsymbol{\varphi}_\theta = \Pi_\theta \boldsymbol{\varphi}, \ \boldsymbol{\chi}_{\theta'} = \Pi_{\theta'} \boldsymbol{\chi}, \ \ \theta, \theta' \in \{+,-\}$$

PROOF OF THEOREM 1

$$\boldsymbol{\varphi}_\theta = e^{-\theta i\Lambda(D)}\boldsymbol{\varphi}_{0,\theta} - i\sum_{\theta_j\in\{+,-\}}\int_0^t e^{-\theta i(t-t')\Lambda(D)}\Pi_\theta[V_b * (\overline{\boldsymbol{\varphi}_{\theta_1}}\gamma^5\boldsymbol{\chi}_{\theta_2} + \overline{\boldsymbol{\chi}_{\theta_3}}\gamma^5\boldsymbol{\varphi}_{\theta_4})\gamma^0\gamma^5\boldsymbol{\varphi}_{\theta_5}]\,dt'$$

$$\boldsymbol{\chi}_{\theta'} = e^{-\theta' i\Lambda(D)}\boldsymbol{\chi}_{0,\theta'} - i\sum_{\theta_j\in\{+,-\}}\int_0^t e^{-\theta' i(t-t')\Lambda(D)}\Pi_{\theta'}[V_b * (\overline{\boldsymbol{\varphi}_{\theta_1}}\gamma^5\boldsymbol{\chi}_{\theta_2} + \overline{\boldsymbol{\chi}_{\theta_3}}\gamma^5\boldsymbol{\varphi}_{\theta_4})\gamma^0\gamma^5\boldsymbol{\chi}_{\theta_5}]\,dt'$$

PROOF OF THEOREM 1

- **Trilinear estimate:** For all $\boldsymbol{f}_j \in X^\sigma_{\theta_j}$ $(\theta_j \in \{+,-\})$

$$\left\| \int_0^t e^{-\theta i(t-s)\Lambda(D)} \Pi_\theta [V_b * (\overline{\boldsymbol{f}_1} \gamma^5 \boldsymbol{f}_2) \gamma^0 \gamma^5 \boldsymbol{f}_3] \, ds \right\|_{X^\sigma_\theta} \leq C \|\boldsymbol{f}_1\|_{X^\sigma_{\theta_1}} \|\boldsymbol{f}_2\|_{X^\sigma_{\theta_2}} \|\boldsymbol{f}_3\|_{X^\sigma_{\theta_3}}$$

$$X^\sigma_\theta := \{\boldsymbol{f} \in C(\mathbb{R}; L^{2,\sigma}) : \Pi_{-\theta} \boldsymbol{f} = \boldsymbol{0}, \ \ \|\boldsymbol{f}\|_{X^\sigma_\theta} < \infty\}$$

$$\|\boldsymbol{f}\|_{X^\sigma_\theta} := \left(\sum_{\lambda, N \in 2^{\mathbb{N} \cup \{0\}}} N^{2\sigma} \|P_\lambda H_N \boldsymbol{f}\|^2_{V^2_\theta} \right)^{\frac{1}{2}}$$

H_N is the spherical harmonic projection of degree $\ell \sim N \in 2^{\mathbb{N} \cup \{0\}}$

PROOF OF THEOREM 1

- **Trilinear estimate follows from**

  - Null structure

$$|\Pi_{\theta_1}(\xi)\gamma^0\gamma^5\,\Pi_{\theta_2}(\eta)| \lesssim \angle(\theta_1\xi, \theta_2\eta) + \frac{|\theta_1|\xi| + \theta_2|\eta||}{(1+|\xi|)(1+|\eta|)}$$

  - Bilinear estimates

Let $\epsilon > 0$. Then there exists $\delta > 0$ such that

$$\|P_{\lambda_0}H_{N_0}(\overline{\Pi_{\theta_1}P_{\lambda_1}H_{N_1}\boldsymbol{f}}\gamma^5\,\Pi_{\theta_2}P_{\lambda_2}H_{N_2}\boldsymbol{g})\|_{L^2_{t,x}}$$

$$\lesssim \lambda_0\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^{\delta} N_{\min}^{\epsilon}\|P_{\lambda_1}H_{N_1}\boldsymbol{f}\|_{V^2_{\theta_1}}\|P_{\lambda_2}H_{N_2}\boldsymbol{g}\|_{V^2_{\theta_2}}$$

$$\theta_j \in \{+,-\},\quad \lambda_j, N_j \in 2^{\mathbb{N}\cup\{0\}}$$

PROOF OF THEOREM 1

$$\text{Solution map}: \boldsymbol{\Phi}(\boldsymbol{\varphi}_+, \boldsymbol{\varphi}_-, \boldsymbol{\chi}_+, \boldsymbol{\chi}_-) = (\Phi^1_+, \Phi^1_-, \Phi^2_+, \Phi^2_-)$$

$$\Phi^1_\theta = e^{-\theta i \Lambda(D)} \boldsymbol{\varphi}_\theta(0) - i \sum_{\theta_j \in \{+,-\}} \int_0^t e^{-\theta i (t-t')\Lambda(D)} \Pi_\theta [V_b * (\overline{\boldsymbol{\varphi}_{\theta_1}} \gamma^5 \boldsymbol{\chi}_{\theta_2} + \overline{\boldsymbol{\chi}_{\theta_3}} \gamma^5 \boldsymbol{\varphi}_{\theta_4}) \gamma^0 \gamma^5 \boldsymbol{\varphi}_{\theta_5}]\, dt'$$

$$\Phi^2_{\theta'} = e^{-\theta' i \Lambda(D)} \boldsymbol{\varphi}_{\theta'}(0) - i \sum_{\theta_j \in \{+,-\}} \int_0^t e^{-\theta' i (t-t')\Lambda(D)} \Pi_{\theta'} [V_b * (\overline{\boldsymbol{\varphi}_{\theta_1}} \gamma^5 \boldsymbol{\chi}_{\theta_2} + \overline{\boldsymbol{\chi}_{\theta_3}} \gamma^5 \boldsymbol{\varphi}_{\theta_4}) \gamma^0 \gamma^5 \boldsymbol{\chi}_{\theta_5}]\, dt'$$

PROOF OF

Theorem 1 (C-Hong-Ozawa)
Let $b > 0$ and $\sigma > 0$. Assume that
one of $\|P_+^c \psi_0\|_{L^{2,\sigma}}$ and $\|P_-^c \psi_0\|_{L^{2,\sigma}}$ is sufficiently small.
Then (D) is globally well-posed in $L^{2,\sigma}$ and ψ scatters in $L^{2,\sigma}$.

$$\text{Solution map}: \mathbf{\Phi}(\varphi_+, \varphi_-, \chi_+, \chi_-) = (\Phi^1_+, \Phi^1_-, \Phi^2_+, \Phi^2_-)$$

$$\mathbf{X} = \{(\varphi_+, \varphi_-, \chi_+, \chi_-) \in X^\sigma_+ \times X^\sigma_- \times X^\sigma_+ \times X^\sigma_- : \|\varphi_\theta\|_{X^\sigma_\theta} \le 2\|\varphi_\theta(0)\|_{L^{2,\sigma}}, \ \|\chi_{\theta'}\|_{X^\sigma_{\theta'}} \le 2\|\chi_{\theta'}(0)\|_{L^{2,\sigma}}\}$$

$$\|(\varphi_+, \varphi_-, \chi_+, \chi_-)\|_{\mathbf{X}} := \mathsf{a}^{-1}\sum_\theta \|\varphi_\theta\|_{X^\sigma_\theta} + \mathsf{A}^{-1}\sum_{\theta'} \|\chi_{\theta'}\|_{X^\sigma_{\theta'}}$$

$$\mathsf{A} = \|P^c_+\psi_0\|_{L^{2,\sigma}}, \quad \mathsf{a} = \|P^c_-\psi_0\|_{L^{2,\sigma}}$$

$$\|\mathbf{\Phi}(\varphi_1, \chi_1) - \mathbf{\Phi}(\varphi_2, \chi_2)\|_{\mathbf{X}} \le 24C\mathsf{A}\mathsf{a}\|(\varphi_1, \chi_1) - (\varphi_2, \chi_2)\|_{\mathbf{X}}$$

If $\mathsf{A}\mathsf{a} \le \frac{1}{48C}$, then $\mathbf{\Phi}$ is a contraction on $\mathbf{X}$.

The linear scattering follows from the definition of V^2 space.

PROOF OF

Theorem 2 (C-Hong-Ozawa)

Let $b = 0$. Suppose that there exists a linear solution $\boldsymbol{\psi}^\ell$:

$$\int [V_0 * (\overline{\boldsymbol{\psi}^\ell}\gamma^5\boldsymbol{\psi}^\ell)]\overline{\boldsymbol{\psi}^\ell}\gamma^5\boldsymbol{\psi}^\ell \, dx \geq \alpha \int [V_0 * |\boldsymbol{\psi}^\ell|^2]|\boldsymbol{\psi}^\ell|^2 \, dx \quad (\alpha > 0)$$

$$\|\boldsymbol{\psi}(t) - \boldsymbol{\psi}^\ell(t)\|_{L^2} \xrightarrow{t\to+\infty} 0.$$

Then $\boldsymbol{\psi} = \mathbf{0} = \boldsymbol{\psi}^\ell$ in L^2.

Suppose that $\|\boldsymbol{\psi}_0\|_{L^2} \neq 0$.

$$\mathfrak{H}(t) := \mathrm{Im}\left\langle \boldsymbol{\psi}, \boldsymbol{\psi}^\ell \right\rangle_{L^2_x}, \quad |\mathfrak{H}(t)| \leq \|\boldsymbol{\psi}_0\|^2_{L^2}$$

$$\frac{d}{dt}\mathfrak{H}(t) = \frac{1}{4\pi}\mathrm{Re}\left\langle [|\cdot|^{-1} * (\overline{\boldsymbol{\psi}}\gamma^5\boldsymbol{\psi})]\gamma^0\gamma^5\boldsymbol{\psi}, \psi^\ell \right\rangle_{L^2_x}$$

$$\geq \frac{\alpha}{4\pi}\int [|\cdot|^{-1} * (\overline{\boldsymbol{\psi}^\ell}\gamma^5\boldsymbol{\psi}^\ell)]\overline{\boldsymbol{\psi}^\ell}\gamma^5\boldsymbol{\psi}^\ell \, dx + o_{\|\psi_0\|_{L^2}}(t^{-1})$$

$$\geq \frac{A}{4\pi}t^{-1} + o(t^{-1}) \quad (t \gg 1)$$

Thank you for your attention

DECOMPOSITION VIA PROJECTION

- **Chirality**

$$(\text{Chiral operator})\quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0_{2\times 2} & 1_{2\times 2} \\ 1_{2\times 2} & 0_{2\times 2} \end{bmatrix}$$

$$\boldsymbol{\psi}_R := \frac{1}{2}(1_{4\times 4} + \gamma^5)\boldsymbol{\psi}, \qquad \boldsymbol{\psi}_L := \frac{1}{2}(1_{4\times 4} - \gamma^5)\boldsymbol{\psi}$$

(Right-handed spinor) (Left-handed spinor)

$$\boldsymbol{\psi} = \boldsymbol{\psi}_R + \boldsymbol{\psi}_L,\ \ (\boldsymbol{\psi}_R)_R = \boldsymbol{\psi}_R,\ \ (\boldsymbol{\psi}_L)_L = \boldsymbol{\psi}_L,\ \ (\boldsymbol{\psi}_R)_L = (\boldsymbol{\psi}_L)_R = \mathbf{0}$$

$$(\gamma^5\boldsymbol{\psi}_R = \boldsymbol{\psi}_R, \quad \gamma^5\boldsymbol{\psi}_L = -\boldsymbol{\psi}_L)$$

DECOMPOSITION VIA PROJECTION

- **Chirality**

$$\gamma^5(i\gamma^\mu\partial_\mu\boldsymbol{\psi}) = -i\gamma^\mu\partial_\mu\gamma^5\boldsymbol{\psi}$$

$$\overline{\boldsymbol{\psi}_R}\boldsymbol{\psi}_R = \overline{\boldsymbol{\psi}_L}\boldsymbol{\psi}_L = \mathbf{0}$$

$$\begin{aligned} i\gamma^\mu\partial_\mu\boldsymbol{\psi}_R &= m\boldsymbol{\psi}_L - [V_b * (\overline{\boldsymbol{\psi}_L}\boldsymbol{\psi}_R - \overline{\boldsymbol{\psi}_R}\boldsymbol{\psi}_L)]\boldsymbol{\psi}_L, \\ i\gamma^\mu\partial_\mu\boldsymbol{\psi}_L &= m\boldsymbol{\psi}_R - [V_b * (\overline{\boldsymbol{\psi}_L}\boldsymbol{\psi}_R - \overline{\boldsymbol{\psi}_R}\boldsymbol{\psi}_L)]\boldsymbol{\psi}_R, \\ \boldsymbol{\psi}_R(0) = \boldsymbol{\psi}_{0,R}, &\quad \boldsymbol{\psi}_L(0) = \boldsymbol{\psi}_{0,L} \\ \boldsymbol{\psi}_0 &= \boldsymbol{\psi}_{0,R} + \boldsymbol{\psi}_{0,L} \end{aligned}$$

$*$ Not a good decomposition for **partial smallness**.