Topological states in a swarmalator model

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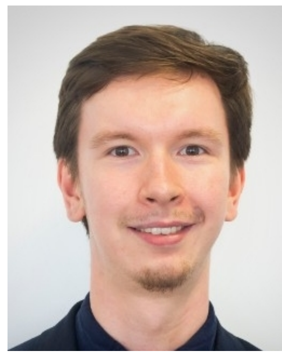


[PD, A. Diez, A. Walczak, Analysis and Applications, 2022]

Summary

1. Presentation
2. Macroscopic model
3. Numerical results
4. Conclusion & perspectives

1. Presentation

Collective dynamics

Micro–scale

Individual agents obey simple rules

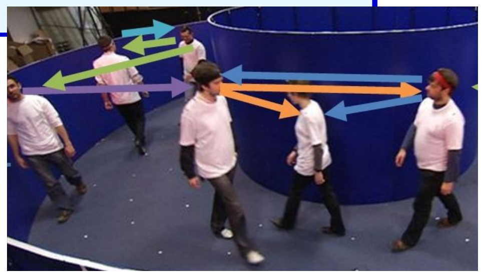
no leader

? →

Macro scale

Emergence of large–scale coherent structures

Not directly encoded in agent's behavior



Questions:

Link between micro-scale geometry and large-scale structures

Topology of collective structures

Object of study: swarmalators

Methodology: dual use of microscopic models and their macroscopic counterparts

Swarmer: Vicsek model [Vicsek et al, PRL 95]

Self-propelled particles ⇒ Speed = constant ($= c_0$)

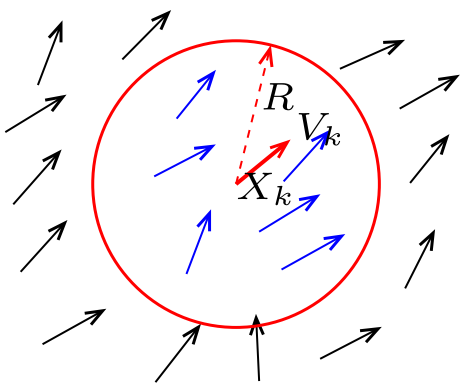
Align with their neighbors up to some noise

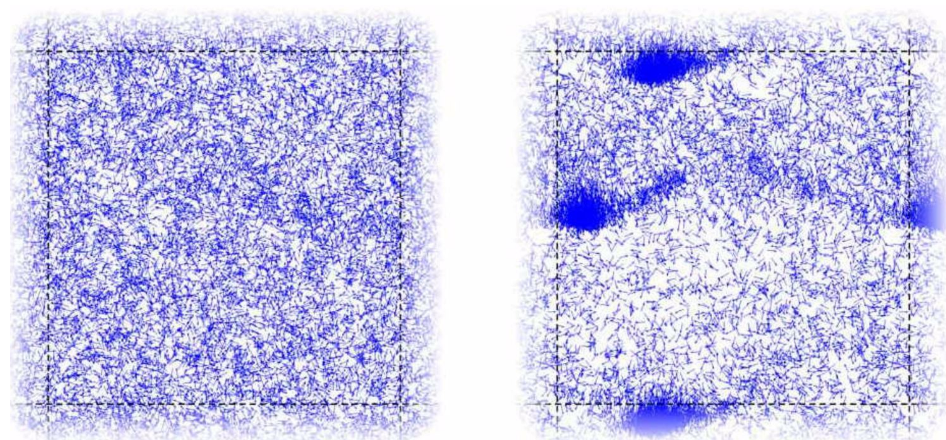
$$\dot{X}_k(t) = c_0 V_k$$

$$dV_k(t) = P_{V_k^\perp} \circ (\nu \bar{V}_k dt + \sqrt{2D}\, dB_t^k)$$

$$\bar{V}_k = \frac{J_k}{|J_k|}, \quad J_k = \sum_{j,\, |X_j - X_k| \le R} V_j$$

$P_{V^\perp} = \mathrm{Id} - V \otimes V$ = orth. proj. on $\{V\}^\perp$ $\circ$ = Stratonovitch





small ν large ν

Simulations by A. Frouvelle

Model describing systems of oscillators which synchronize

Original model (Kuramoto)

$$d\varphi_k(t) = -\frac{\nu}{N}\sum_{j=1}^{N}\sin(\varphi_k - \varphi_j) + \sqrt{2D}\,dB_t^k$$

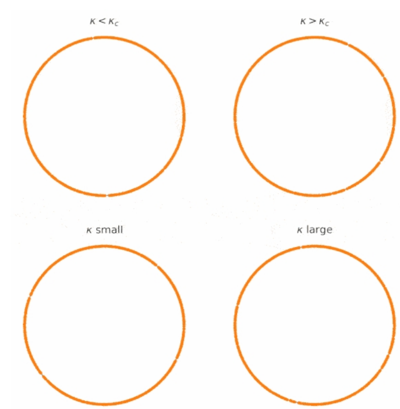
Variant inspired by the Vicsek model

$$d\varphi_k(t) = -\nu\sin(\varphi_k - \bar\varphi(t)) + \sqrt{2D}\,dB_t^k, \qquad \bar\varphi(t) = \mathrm{arg}\Big(\sum_{j=1}^{N} e^{i\varphi_j}\Big)$$

small ν large ν

Original

Variant



[O'Keefe, Hong, Strogartz, Oscillators that sync and swarm, Nature Comm. 2017]

New swarmalator model with original features

- no force reciprocity → pursuit behavior
- second-order model
- noise in velocity and phase

$$\dot{X}_k(t) = c_0 V_k - \gamma \nabla_x W(X_k, \varphi_k)$$
$$dV_k(t) = P_{V_k^\perp} \circ (\nu \bar{V}_k dt + \sqrt{2D}\, dB_t^k)$$
$$d\varphi_k(t) = -\nu' \sin(\varphi_k - \bar{\varphi}(t)) + \sqrt{2D'}\, dB_t^k$$

Position-phase coupling through potential W

$$W(x,\varphi) = \frac{1}{N} \sum_{j=1}^{N} K(|x - X_j|) \sin(\varphi_j - \varphi)$$

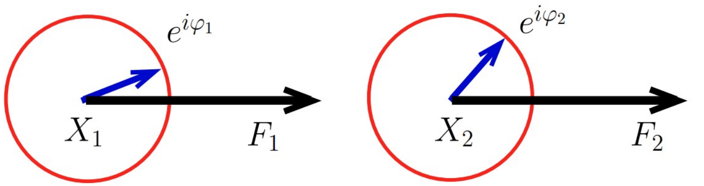
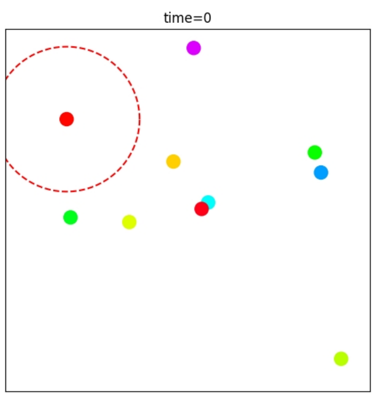
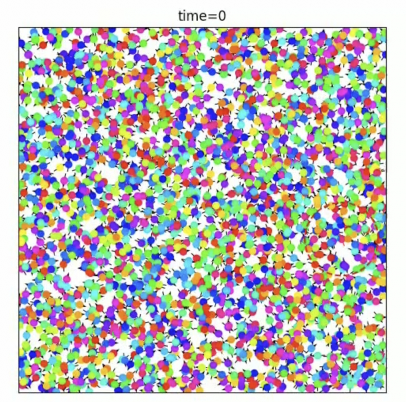


Illustration & applications



$c_0 = 0$



$c_0 \neq 0$

Large literature: cf review [O'Keefe & Bettstetter, 2019]

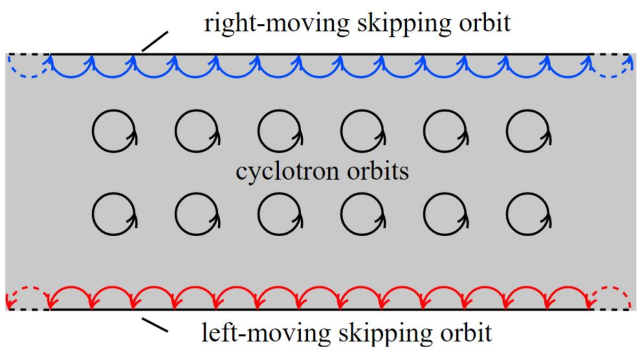
Applications to biology:

- Microswimmers (nematodes or sperm) [Peshkov et al, 2019]
- Cellular interactions with internal state [Japon et al, 2021]

Nematode swarm from [Peshkov et al, 2019]

Topological states of matter

Quantum Hall effect (Klaus von Klitzing, NP 1985)

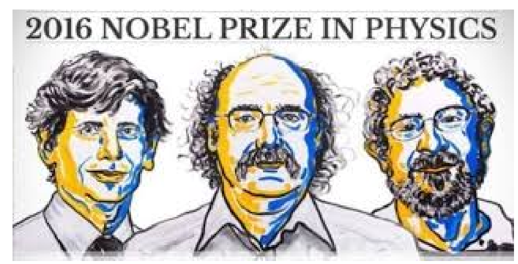


Conducting (chiral) edge states have non-trivial topology

- They are robust against perturbations
- Breaking them requires a "topological phase transition"

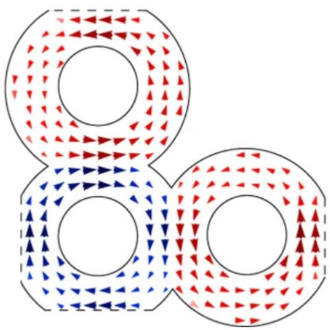
Topological insulators

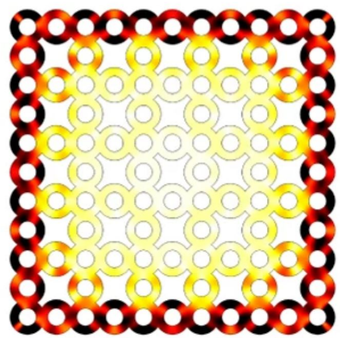
- Thouless, Haldane, Kosterlitz, NP 2016
- Quantum computations, Qubits, . . .



Vicsek on a sphere (Marchetti et al, Phys. Rev. X 2017)

Vicsek in a lattice of rings (Bartolo et al, Nature Phys. 2017, Sone & Ashida, Phys. Rev. Lett. 2019)

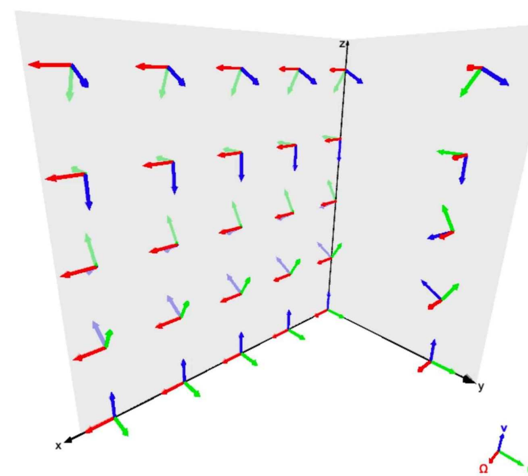




"spatial domain engineering"

Goal: new topological states based on internal degrees of freedom

Previous example: swarming rigid bodies
[D., Diez, Na, SIADS 2022]



Motivation: does topological protection contribute to robustness of living systems?

2. Macroscopic model

Mean-field model ($N \to \infty$)

$f(x, v, \varphi, t)$ distribution function $v \in \mathbb{S}^{n-1}$, $\varphi \in \mathbb{R}/(2\pi\mathbb{Z})$

$f(x, v, \varphi, t)\, dx\, dv\, d\varphi =$ number of particles in $dx\, dv\, d\varphi$ at t

satisfies mean-field kinetic equation

$$\partial_t f^\varepsilon + \nabla_x \cdot \big[(v - \gamma \nabla_x U_{f^\varepsilon}) f^\varepsilon\big] = \frac{1}{\varepsilon} Q(f^\varepsilon)$$

$$Q(f) = D \nabla_v \cdot [-k P_{v^\perp} u_f f + \nabla_v f] + D' \partial_\varphi [-k' \sin(\alpha_f - \varphi) f + \partial_\varphi f]$$

with

$$u_f = \frac{j_f}{|j_f|}, \quad j_f = \int f\, v\, dv\, d\varphi$$

$$\alpha_f = \frac{\ell_f}{|\ell_f|}, \quad \ell_f = \int f\, e^{i\varphi}\, dv\, d\varphi$$

$$U_f = |\ell_f| \sin(\alpha_f - \varphi), \qquad k = \frac{\nu}{D}, \qquad k' = \frac{\nu'}{D'}$$

Solutions of $Q(f) = 0$ given by von Mises distribution

$$f_{\text{eq}} = \rho M_u(v) N_\alpha(\varphi)$$

$$M_u(v) \sim e^{k v \cdot u}, \quad N_\alpha(\varphi) \sim e^{k' \cos(\varphi - \alpha)}$$

$$(\rho, u, \alpha) \text{ arbitrary in } [0, \infty) \times \mathbb{S}^{n-1} \times \mathbb{R}/(2\pi\mathbb{Z})$$

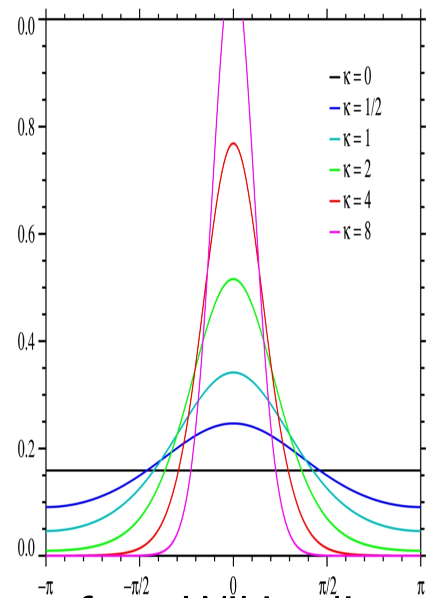
When $\varepsilon \to 0$, $f^\varepsilon \to f_{\text{eq}}$ with $(\rho, u, \alpha)(x,t)$:

$\rho(x,t) \geq 0$: mean density

$u(x,t) \in \mathbb{S}^{n-1}$: mean direction of motion

$\alpha(x,t) \in \mathbb{R}/(2\pi\mathbb{Z})$: mean phase

Eq. satisfied by $(\rho, u, \alpha) \equiv$ macroscopic eqs.



from Wikipedia

Macroscopic model

Swarmalator hydrodynamics:

$$\partial_t \rho + \nabla_x \cdot \big[\rho(c_1 u + b\rho\nabla_x \alpha)\big] = 0$$

$$\partial_t u + \big[(c_2 u + b\rho\nabla_x \alpha) \cdot \nabla_x\big] u + \Theta P_{u^\perp} \nabla_x \log \rho = 0$$

$$\rho\Big(\partial_t \alpha + \big[(c_1 u + b'\rho\nabla_x \alpha) \cdot \nabla_x\big]\alpha\Big) - \Theta' \nabla_x \cdot (\rho \nabla_x \rho) = 0$$

Coefficients given explicitly in terms of those of kinetic model

Derivation not straightforward due to lack of conservations

Generalized Collision Invariant [D. Motsch, M3AS 2008]

Travelling-wave solution

No noise in phase eq. $k' \to \infty$: $\Rightarrow$ phase eq. simplifies

$$\partial_t \alpha + \big[(c_1 u + b\rho \nabla_x \alpha) \cdot \nabla_x\big] \alpha = 0$$

Introducing $z = \nabla_x \alpha$, System equivalent to

$$\partial_t \rho + \nabla_x \cdot \big[\rho(c_1 u + b\rho z)\big] = 0$$

$$\partial_t u + \big[(c_2 u + b\rho z) \cdot \nabla_x\big] u + \Theta P_{u^\perp} \nabla_x \log \rho = 0$$

$$\partial_t z + \nabla_x \big[(c_1 u + b\rho z) \cdot z\big] = 0$$

$$\nabla_x \wedge z = 0$$

Uniform state (ρ_0, u_0, z_0) is a solution

Corresponds to a travelling-wave in phase

$$\alpha(x,t) = z_0 \cdot x - (c_1 u_0 + b\rho_0 z_0) \cdot z_0 \, t$$

Hyperbolicity

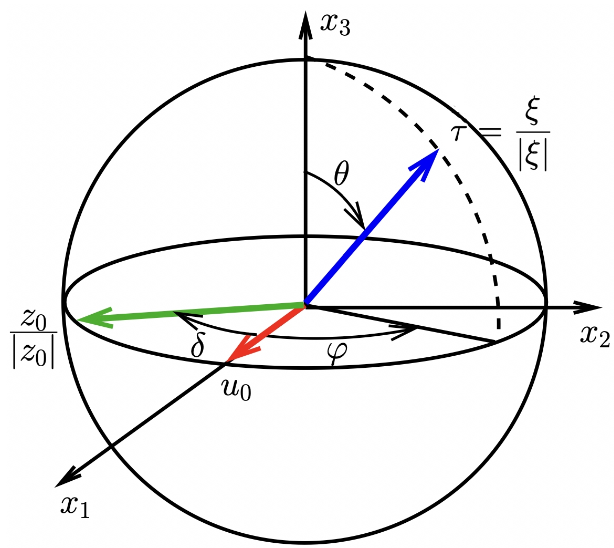
Linearize about (ρ_0, u_0, z_0)

Hyperbolicity $\approx$ stability in Fourier variable ξ

Theorem:

(i) if $z_0 = 0$ or $z_0 \parallel u_0$ then hyperbolic

(ii) For $\rho_0|b||z_0|$ either small or large and for some values of $\delta = \measuredangle(u_0, z_0)$ then not hyperbolic



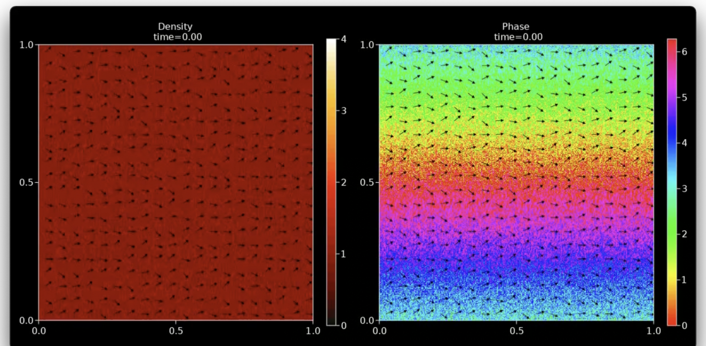
Original system (with noise in phase eq.) on 2-d unit torus $\mathbb{T}^2$

Proposition: given $(p,m) \in \mathbb{Z}^2$, $\alpha_0 \in \mathbb{R}$, $U = (U_1, U_2) \in \mathbb{S}^1$. Then:

$$\rho = 1 \qquad u = U \qquad \alpha = 2\pi(px_1 + mx_2) - \lambda t + \alpha_0$$

is a travelling-wave solution: with travelling-wave speed

$$\lambda = 2\pi c_1(pU_1 + mU_2) + 4\pi^2 b'(p^2 + m^2)$$



Example with
$(p,m) = (0,1)$
$U = (1,0)$

Topological state:

$e^{i\alpha}$ makes a complete turn when x_2 goes from 0 to 1

(p,m) is the topological index of the solution

3. Numerical results

1. Validate hydro model against particle model
2. Check stability of topological states in particle model
3. Check stability of topological states in hydro model

Numerical method for particle model:

GPU simulations in Python using the SiSyPHE library developed by A. Diez [Diez, J Open Source Software 2021]

Numerical method for hydro model:

relaxation approximation by conservative hyperbolic system [Motsch & Navoret, Mult. Model. Simul. 2011]

Dimensional splitting and HLLE scheme; code in Julia

Both codes available at: https://github.com/antoinediez

Validation of hydro model

Use particle simulation in the hydro regime

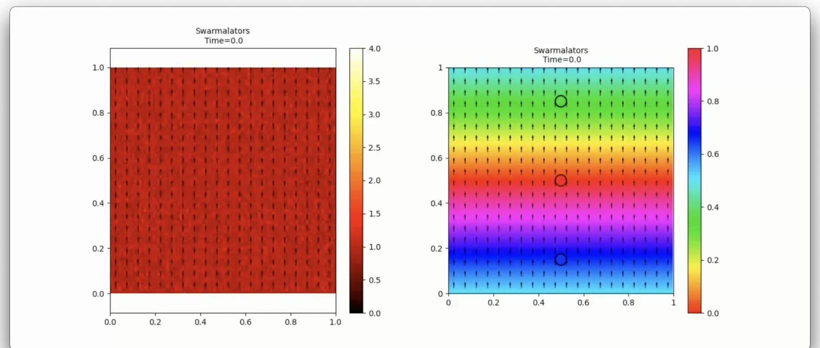
$$R \ll 1, \qquad \nu,\, \nu',\, \sigma,\, \sigma',\, \gamma,\, N \gg 1, \qquad k,\, k' \sim 1$$

Use doubly-periodic travelling-wave

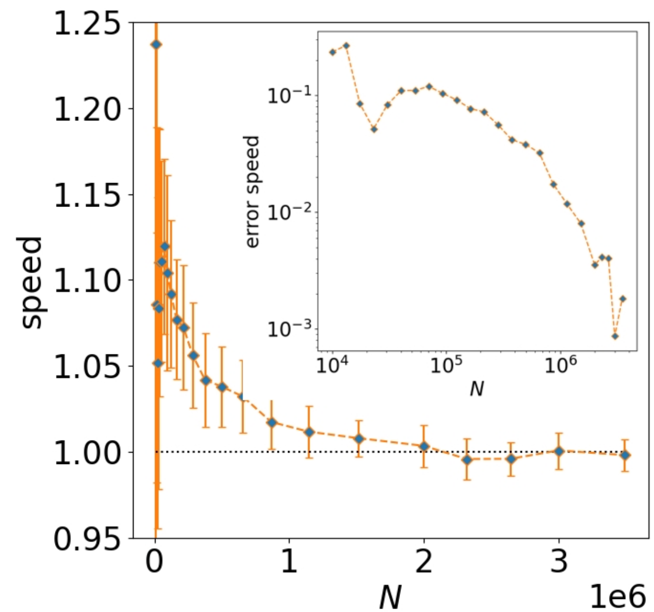
anti-aligned: phase-force and self-propulsion velocity opposite

$$(p, m) = (0, 1)$$

$$U = (0, 1)$$

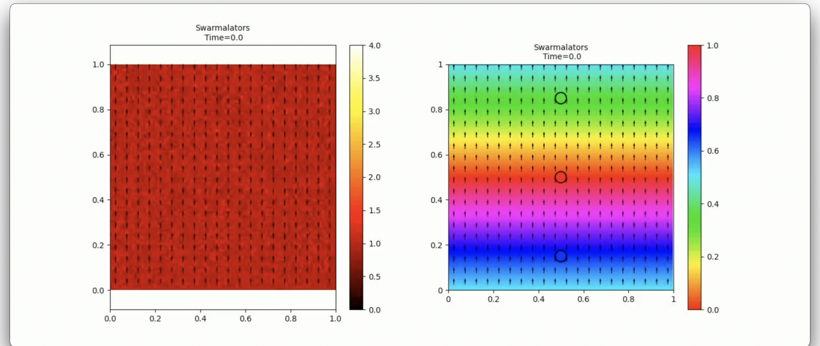


Compare travelling-wave speeds



Stability of topological states (particles)

When noise ↘ doubly-periodic traveling wave destabilizes

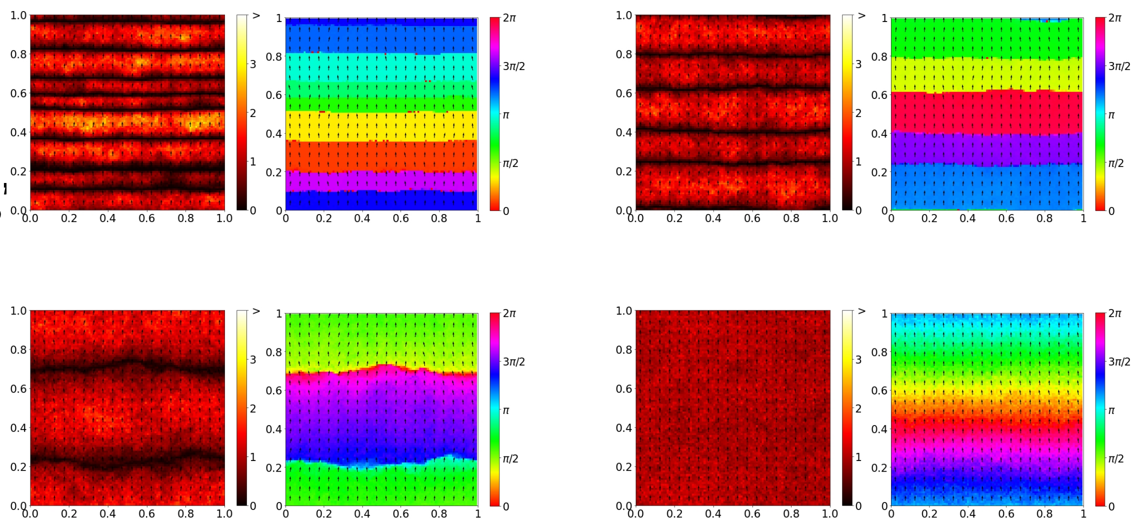


same as before
but small noise

Emergence of segregated constant-phase regions

converge at large times to band-like structure

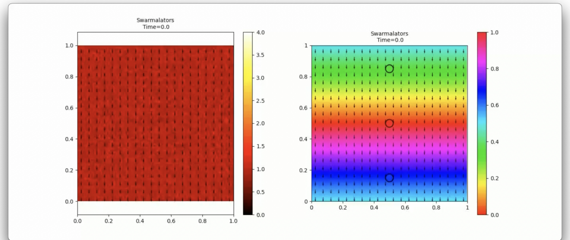
“Final states”
by ↗ noise



Other initial conditions

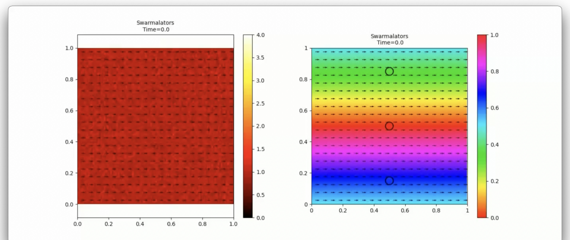
Doubly-periodic travelling wave, large noise case

Positively aligned: phase-force and self-propulsion velocity equal



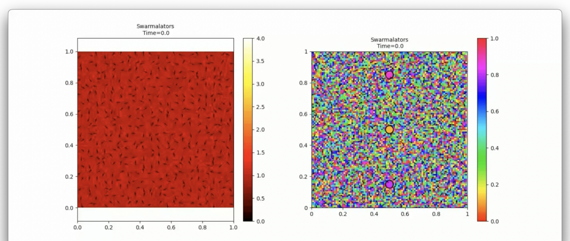
less stable

Orthogonal: phase-force and self-propulsion velocity perpendicular



unstable

Random initial condition, large noise case



complex patterns

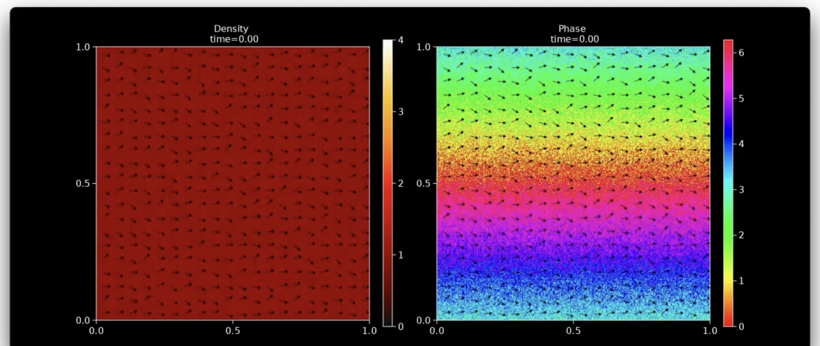
Stability of topological states (hydro)

No or very small noise: shock formation and blow-up

Small noise:

Anti-aligned: stable
Aligned: stable
Orthogonal: unstable

Consistent with hyperbolicity theorem



orthogonal, low noise

Larger noises: all stable

4. Conclusion & perspectives

Conclusion & perspectives

New swarmalator model

no force reciprocity, second order, with noise in velocity and phase

Derivation of a macroscopic model

Hyperbolicity analysis, travelling-wave topological states

Numerical simulations

validation of macro model, stability of topological states

Perspectives (theory)

existence / uniqueness of solutions to kinetic / macro models

particle $\to$ kinetic & kinetic $\to$ macro convergence proofs

segregated solutions supported by macro model ?

Perspectives (modelling)

other geometrical configurations: strip and ring

diffusive corrections, higher-dimensional phase-vector space