The Vicsek-BGK equation for collective dynamics

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SFB PDE Taming Complexity in Partial Differential Systems

Collective motion

- fish schools
- opinion dynamics
- cell dynamics (e.g. sperm)
- pedestrian dynamics

Common feature: individual agents adapt direction to their neighbors.

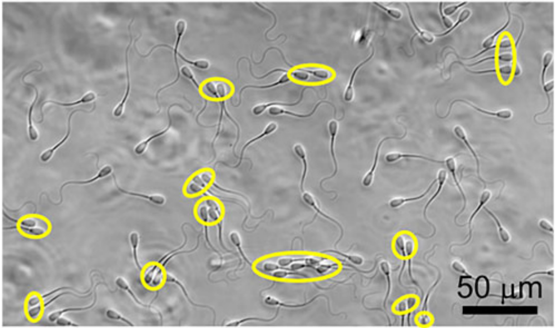


Figure: Collective swimming in sperm, from Phuyal et al. 2022

Qualitative behavior

The observed behavior can be

1. Disordered: the directions of movements are (close to) uniformly distributed, almost no correlation

   Observed if density of agents is (locally) low

2. Ordered: Onset of a collective motion of the group.

   Observed if density of agents is (locally) high

Moreover, we observe phase transition, i.e. critical density ρ where behavior changes from a) to b).

We investigate the Vicsek-BGK model (Degond, Diez, Frouvelle, Merino, 2020):

$$\partial_t F + \omega \cdot \nabla_x F = Q(F),$$
$$Q(F)(x,\omega) = \rho_F(x) M_{J_F}(x,\omega) - F(x,\omega).$$

Kinetic model for the particle density $F(t,x,\omega)$ determining the local density at

1. position $x \in \mathbb{T}^d$
2. direction $\omega \in \mathbb{S}^{d-1}$ (constant speed).

Local relaxation towards von-Mises distribution M_J

$$M_J(w) := \frac{\exp(\omega \cdot J)}{Z(J)}, \quad Z(J) := \int_{\mathbb{S}^{d-1}} \exp(\omega \cdot J)\ \mathrm{d}\omega,$$

given in terms of the local flux $J_f(x)$

$$J_F(x) := \int_{\mathbb{S}^{d-1}} \omega F(x,w)\,\mathrm{d}\omega.$$

Phase transition (Staudner, Merino, 2021)

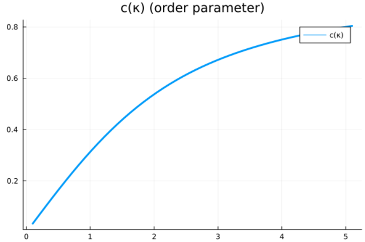
Consider steady state problem for spatially homogeneous equation (for $F(t,\omega)$)

$$0 = \partial_t F = \rho M_{J_F}(\omega) - F(\omega) \quad \Leftrightarrow \quad F = \rho M_{J_F}(\omega).$$

Reduces to closed equation for J_F (consistency relation)

$$J_F = \rho \int_{\mathbb{S}^{d-1}} \omega M_{J_F}(\omega), \quad \text{or} \quad |J| = \rho c(|J|). \tag{1}$$

We always have the *disordered* solution $J_F = 0$.
On the other hand: c is convex, bounded, with $c'(0) = d^{-1}$.



$\rightsquigarrow$ A non-trivial solution exists only for $\rho > d$ (Phase transition)

Key mechanism: coexistence of disordered and ordered behavior for some time, what happens through mixing?

Known properties:

1. Mass conservation
2. No conservation of energy and momentum, neither dissipation of entropy
3. von-Mises steady states (as in homogeneous)

Not known:

1. Well-posedness of initial value problem
2. Stability of von-Mises steady states
3. Existence of more steady states?!

Plan of this talk: Answers to 1. and 2.

Theorem (Merino, Schmeiser, W., 2023+)

Let $d \geq 2$ and assume $F_0 \in L^1(\mathbb{T}^d \times \mathbb{S}^{d-1})$ is a non-negative function with finite entropy:

$$\int_{\mathbb{T}^d \times \mathbb{S}^{d-1}} F_0(x,v) |\log F_0(x,\omega)| \,\mathrm{d}x \,\mathrm{d}\omega = E_0 < \infty. \tag{2}$$

Then there exists a non-negative function $F \in C([0,\infty); L^1(\mathbb{T}^d \times \mathbb{S}^{d-1}))$ which solves the Vicsek-BGK equation in the weak sense.

Moreover, the mass of F is constant in time, and the entropy increases at most exponentially, i.e. there exist constants $c, C > 0$ such that

$$\int_{\mathbb{T}^d \times \mathbb{S}^{d-1}} F(t,x,v) |\log F(t,x,v)| \,\mathrm{d}x \,\mathrm{d}v \leq C E_0 e^{ct}. \tag{3}$$

Remarks:

1. Proof similar to Perthame's theory for Boltzmann BGK.
2. Uniqueness and L^∞ solutions unknown.

From the spatially homogeneous case we know

$$F_{\rho,J}(x,\omega) = \rho M_J(\omega)$$

are steady states if

1. $J = 0$.
2. $\rho > d$ and $|J| = L(\rho)$, where $L(\rho) > 0$ unique solution to

$$L = \rho c(L).$$

Observation: In case 2. we have Manifold of steady states $|J| = L(\rho)$.

Problem: The direction of the total flux

$$J_{\text{tot}}(t) = \int_{\mathbb{T}^d \times \mathbb{S}^{d-1}} \omega F(t,x,\omega) dx d\omega$$

does not need to be constant in time.

Theorem (Merino, Schmeiser, W., 2023+)

Let $d = 2, 3$. There exists a constant $\kappa > 0$ such that the following holds: For any $\mu \in (0, \mu_* + \kappa)$ there exists $\varepsilon_0 > 0$ such that for any $\bar{J} \in \mathbb{R}^d$, $|\bar{J}| = L(\mu)$ and any non-negative initial datum $F_0 \in H^2_x(\mathbb{T}^d \times \mathbb{S}^{d-1})$ with $\mu = \int_{\mathbb{T}^d \times \mathbb{S}^{d-1}} F_0(x, \omega) dx d\omega$ and

$$\|F_0 - F_{\mu,\bar{J}}\|_{H^2_x} < \varepsilon_0, \tag{4}$$

there exists a global-in-time solution $F = F(t)$ to the Vicsek-BGK equation with initial datum F_0. Moreover, there exist $c, C > 0$ and $J_\infty \in \mathbb{R}^d$, $|J_\infty| = L(\mu)$ such that

$$\|F(t) - F_{\mu,J_\infty}\|_{H^2_x} \leq Ce^{-ct}. \tag{5}$$

Remarks:

1. H^2_x is the Sobolev space with 2 weak derivatives in x.
2. it looks very hard to compute J_∞ from the initial datum (Manifold of solutions).

Some ingredients of the proof

Let $\mathcal{M}$ be the manifold of von-Mises steady states with density μ.

Sufficient to show:
There are $\varepsilon_0 > 0$ small enough and $T > 0$ large enough such that if

$$\operatorname{dist}(F_0, \mathcal{M}) := \inf_{F \in \mathcal{M}} \|F_0 - F\|_{H^2_x} \leq \varepsilon_0,$$

then

$$\begin{aligned} \operatorname{dist}(F_T, \mathcal{M}) &\leq \tfrac{1}{2} \operatorname{dist}(F_0, \mathcal{M}), \\ \|F_t - F_T\|_{H^2_x} &\leq C \operatorname{dist}(F_0, \mathcal{M}), \quad t \in [0, T]. \end{aligned}$$

Iterating $\rightsquigarrow$ exponential convergence.

Consequence: Reduction to linearized problem

The linearized equation

The linearized equation around $\mu M_{\mathcal{J}}$ reads

$$\partial_t f + \omega \cdot \nabla_x f = \rho_f M_{\mathcal{J}} + \mu J_f \cdot \nabla_J M_{\mathcal{J}} - f.$$

Taking Fourier Laplace transform: $\tilde{f}_k(z)$, $k \in \mathbb{Z}^d$, $\Re(z) \geq 0$:

$$\tilde{f}_k(z, \omega) := \int_0^\infty \int_{\mathbb{T}^d} e^{-zt} e^{-ik\cdot x} f(t, x, \omega) dx dt,$$

the equation becomes

$$\tilde{J} = \tilde{\rho} \int_{\mathbb{S}^{d-1}} \frac{\omega M_{\mathcal{J}}}{1 + z + ik \cdot \omega} d\omega + \mu \int_{\mathbb{S}^{d-1}} \frac{\omega \otimes \nabla_J M_{\mathcal{J}}}{1 + z + ik \cdot \omega} d\omega \cdot \tilde{J} + R_1$$
$$\tilde{\rho} = \tilde{\rho} \int_{\mathbb{S}^{d-1}} \frac{M_{\mathcal{J}}}{1 + z + ik \cdot \omega} d\omega + \mu \int_{\mathbb{S}^{d-1}} \frac{\nabla_J M_{\mathcal{J}}}{1 + z + ik \cdot \omega} d\omega \cdot \tilde{J} + R_2.$$

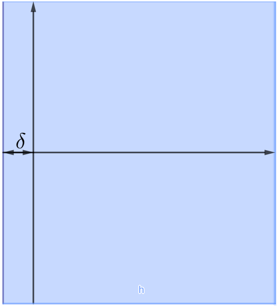
Linearized equation

The F-L transforms of density and flux $\tilde{\rho}$ and $\tilde{J}$ solve

$$\mathcal{K}(k,z,\mu)\cdot\begin{pmatrix}\tilde{J}_k(z)\\ \tilde{\rho}_k(z)\end{pmatrix} = R_k(z).$$

Extension of Fourier-Laplace Kernel

Exponential convergence in time of the k-th Fourier component holds if $K^{-1}(k,z,\mu)$ can be analytically extended to $\Re(z)\geq -\delta$, $\delta>0$ in the left-hand side of the complex plane.



Strategy: Show this property for $\mu \leq \mu_*$, $\mathcal{J} = 0$ and extend by continuity.

Problem reduces to:

$$1 \neq b + \mu \frac{\lambda^2}{1 - \mu a},$$

in some uniform region $\Re(z) \geq -\delta$, $0 \neq k \in \mathbb{Z}^d$, $\mu \in [0, \mu_*]$

$$\begin{aligned} b(z,k) &= \int_{\mathbb{S}^{d-1}} \frac{M_0}{1 + z + ik \cdot \omega} d\omega, \\ \lambda(z,k) &= \frac{1}{|k|} \int_{\mathbb{S}^{d-1}} \frac{\omega \cdot k M_0}{1 + z + ik \cdot \omega} d\omega, \\ a(z,k) &= \frac{1}{|k|^2} \int_{\mathbb{S}^{d-1}} \frac{(k \cdot \omega)^2 M_0}{1 + z + ik \cdot \omega} d\omega. \end{aligned}$$

$d = 2$: integration and some tricks $\rightsquigarrow$ roots of 4-th order polynomials in z ✓

$d = 3$: no such luck, need to try harder. Actually false for $k \in \mathbb{R}^3$.

In dimension $d = 3$, use argument principle: We need to show

$$1 \neq b + \mu \frac{\lambda^2}{1 - \mu a} =: h(z),$$

for $\Re(z) \geq 0$, $0 \neq k \in \mathbb{Z}^d$, $\mu \in [0, \mu_*]$.

Since h(z) analytic on $\Re(z) > -1$, sufficient to show

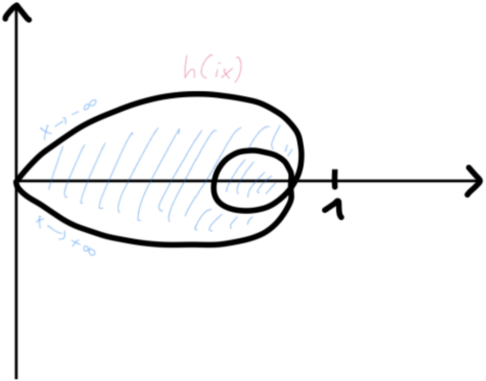
$$\begin{aligned} \gamma : \mathbb{R} &\to \mathbb{C} \\ x &\mapsto b(ix) + \mu \frac{\lambda^2(ix)}{1 - \mu a(ix)} \end{aligned}$$

does not cross the half-line

$$H_+ = \{y \in \mathbb{R} : y \geq 1\}.$$

In order to show

$$h_{\mu,k}(z) \neq 1, \quad \Re(z) \geq 0,$$



Use rather accurate estimates to show $\Re(h(ix)) < 1$.

Thank you!