On the number of Mather measures of Lagrangian systems

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Abstract: In 1996, Ricardo Ricardo Mañé discovered that Mather measures are in fact the minimizers of a ”universal” infinite dimensional linear programming problem. This fundamental result has many applications, one of the most important is to the estimates of the generic number of Mather measures. Mañé obtained the first estimation of that sort by using finite dimensional approximations. Recently, we were able with Gonzalo Contreras to use this method of finite dimensional approximation in order to solve a conjecture of John Mather concerning the generic number of Mather measures for families of Lagrangian systems. In the present paper we obtain finer results in that direction by applying directly some classical tools of convex analysis to the infinite dimensional problem. We use a notion of countably rectifiable sets of finite codimension in Banach (and Frechet) spaces which may deserve independent interest.

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Résumé: En 1996, Ricardo Mañé a découvert que les mesures de Mather peuvent être obtenues comme solutions d’un problème variationnel convexe ”universel” de dimenion infinie. Ce résultat fondamental a de nombreuses applications, il permet par exemple d’estimer le nombre de mesures de Mather des systèmes génériques. Mañé a obtenu la première estimation de ce type en utilisant une approximation par des problèmes variationnels de dimension finie. Nous avons récemment utilisé cette méthode avec Gonzalo Contreras pour résoudre une conjecture de John Mather sur le nombre générique de mesures minimisantes dans les familles de systèmes Lagrangians. Dans le présent article, on obtient des résultat plus fins dans cette direction en appliquant directement au problème de dimension infinie des méthodes classiques de l’analyse convexe. On étudie pour ceci une nouvelle notion d’ensembles rectifiables de codimension finie dans les espaces de Banach (ou de Frechet) qui est peut-être intéressante en elle même.

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1 Introduction

Let M be a compact connected manifold without boundary. We want to study the dynamical system on TM generated by a Tonelli Lagrangian

$$L : TM \longrightarrow \mathbb{R}.$$

By *Tonelli Lagrangian* we mean a C^2 function $L : TM \longrightarrow \mathbb{R}$ such that, for each $x \in M$, the function $v \longmapsto L(x, v)$ is superlinear and convex with positive definite Hessian. Note then that the superlinearity is uniform with respect to x, see [15], section 3.2. To each Tonelli Lagrangian is associated a complete C^1 flow ψ^t on TM, with the property that a curve $(x(t), v(t))$ is a trajectory of ψ^t if and only if (i) $v(t) = \dot{x}(t)$ and (ii) the curve $x(t)$ solves the Euler-Lagrange equation

$$\frac{d}{dt}(\partial_v L(x(t), \dot{x}(t))) = \partial_x L(x(t), \dot{x}(t)).$$

We call this flow the Euler-Lagrange flow associated to L. A standard example is the mechanical case where

$$L(x, v) = \frac{1}{2}\|v\|_x^2 - V(x),$$

the associated Euler Lagrange equation is just the Newton equation

$$\ddot{x}(t) = -\nabla V(x(t)).$$

Variational methods offer interesting tools to investigate the orbits of the Euler-Lagrange flow. We recall the following well-known fact: A C^1 curve $x(t)$ (or more generally an absolutely continuous curve $x(t)$) satisfies the Euler-Lagrange equation on an open interval I if and only if, for each $t_0 \in I$, there exists $\epsilon > 0$ such that $[t_0 - \epsilon, t_0 + \epsilon] \subset I$ and such that

$$\int_{t_0-\epsilon}^{t_0+\epsilon} L(x(t), \dot{x}(t))dt < \int_{t_0-\epsilon}^{t_0+\epsilon} L(\gamma(t), \dot{\gamma}(t))dt$$

for all C^1 curves $\gamma : [t_0 - \epsilon, t_0 + \epsilon] \longrightarrow M$ different from x and satisfying the boundary conditions $\gamma(t_0 - \epsilon) = x(t_0 - \epsilon)$ and $\gamma(t_0 + \epsilon) = x(t_0 + \epsilon)$.

One of the standard applications of variational methods is to the existence of periodic orbits. This can be done as follows. Fix a positive real number T and a homology class $w \in H_1(M, \mathbb{Z})$. Let $W^{1,1}(T, w)$ be the set of absolutely continuous curves $x : \mathbb{R} \longrightarrow M$ which are T-periodic and have homology w (when seen as closed loops on M). It is a classical result that the action functional

$$W^{1,1}(T, w) \ni \gamma \longmapsto \int_0^T L(\gamma(t), \dot{\gamma}(t))dt$$

has a minimum, and that the minimizing curves are C^2 and solve the Euler-Lagrange equation. This is a way to prove the existence of many periodic orbits of the Euler-Lagrange flow.

John Mather had the idea to apply variational methods to measures instead of curves. Let $\mathcal{I}(L)$ be the set of compactly supported Borel probability measures on TM which are invariant under the Euler-Lagrange flow. Note that, if $x(t)$ is a T-periodic solution of the Euler-Lagrange equation, then we associate to it an invariant measure μ characterized by the property that

$$\int_{TM} f(x, v)d\mu(x, v) = \frac{1}{T}\int_0^T f(x(t), \dot{x}(t))dt$$

for each continuous and bounded function f on TM. In this case, we see that the action of the curve x is just $T\int_{TM} Ld\mu$. This suggests to take $\int_{TM} Ld\mu$ as the definition of the action of a

probability measure. A Mather measure is then defined as a minimizer of the action on $\mathcal{I}(L)$. John Mather roved in [20] that Mather measures exist, and moreover that they are supported on a Lipschitz graph. More precisely, there exists a Lipschitz vectorfield $Y(x)$ on M (which depends on the Lagrangian L, but not on the Mather measure) such that all the Mather measures of L are supported on the graph of Y.

In the mechanical case where $L = \|v\|^2/2 - V(x)$, the Mather measures are just the invariant measures associated to the rest points maximizing V (and therefore one can take $Y \equiv 0$ in this case). So we have not gained much insight in the dynamics of the Euler-Lagrange flow of these Lagrangians at that point. A trick due to John Mather yet allows to obtain further information from his theory. Recall first that, if ω is a closed form on M, that we see as a function on TM linear in each fiber, then the Tonelli Lagrangian

$$\tilde{L}(x, v) = L(x, v) + \omega_x \cdot v$$

generates the same Euler-Lagrange flow as L. This can be seen easily by considering the variational characterization of the Euler-Lagrange equation. The remark of Mather is that, although L and $\tilde{L}$ generate the same flow, they do not have the same Mather measures. Actually, the Mather measures of $\tilde{L} = L + \omega$ depend only on the cohomology of ω in $H^1(M, \mathbb{R})$. By definition, they are invariant measures of the flow of L. If the cohomology group $H^1(M, \mathbb{R})$ is not trivial, this construction allows to find non-trivial measures supported on Lipschitz graphs for mechanical Lagrangians. In order to simplify the notations for the sequel, we associate once and for all to each cohomology class $c \in H^1(M, \mathbb{R})$ a smooth closed one-form $c_x \cdot v$ which has cohomology c. We assume that the form depends linearly on the class. We denote by $L + c$ the Lagrangian $L(x, v) + c_x \cdot v$. Finally, we denote by $\mathcal{M}(L)$ the set of Mather measures of L. It is a convex subset of $\mathcal{I}(L)$.

The Mather measures are very important objects in themselves, but they also appear as building blocks for more elaborate constructions of orbits of Lagrangian systems as initiated in [21] (see also [6, 7, 10, 11]). It is useful for these constructions to be able to understand the dimension of $\mathcal{M}(L)$. An important result was obtained by Ricardo Mañé in 1996, see [18]:

Theorem 1. *Let L be a Tonelli Lagrangian, and let $\sigma^\infty(L)$ be the set of those potentials $V \in C^\infty(M)$ such that the Tonelli Lagrangian $L - V$ has more than one Mather measure. The set $\sigma^\infty(L)$ is a meager set in the sense of Baire category. It means that it is contained in the union of countably many nowhere dense closed sets.*

When applied in the mechanical case, $L = \|v\|^2/2$ this theorem states that generic smooth functions on M have only one maximum, which is of course not a new result. More interesting situations appear by considering modified kinetic energies $L = \|v\|^2/2 + \omega_x \cdot v$. For applications, however, it is necessary to treat simultaneously all the sets $\mathcal{M}(L - V + c)$, $c \in H^1(M, \mathbb{R})$. We were recently able with Gonzalo Contreras to extend the result of Mañé in that direction, see [4]. These results imply Mañé Theorem as well as the following:

Theorem 2. *Let L be a Tonelli Lagrangian. Let $\Sigma_k^\infty(L)$ be the set of potentials $V \in C^\infty(M)$ such that, for some $c \in H^1(M, \mathbb{R})$, $dim(\mathcal{M}(L - V + c)) \geqslant k$. Then, if $k > b_1$ (where $b_1 = \dim H^1(M, \mathbb{R})$), the set $\Sigma_k^\infty(L)$ is Baire meager in $C^\infty(M)$.*

In the sequel, we shall fix $p \in \{2, 3, \ldots, \infty\}$ and consider potentials in $C^p(M)$ instead of $C^\infty(M)$ (note that the case $p = \infty$ is still included). We call $\Sigma_k^p(L)$ the set of potentials $V \in C^p(M)$ such that

$$\max_{c \in H^1(M, \mathbb{R})} \dim(\mathcal{M}(L - V + c)) \geqslant k.$$

The results we have recalled above remain valid, with the same proof, in this context. They state that it is exceptional in the sense of Baire category to have too many Mather measures.

Now in the separable Frechet space $C^p(M, \mathbb{R})$, there are plenty of other notions of small sets, which are at least as relevant as the Baire category, and are more in the spirit of having measure zero (although there is not a single way to define sets of measure zero on infinite dimensional spaces). Good introductions to these notions and to the literature concerning them are [3], [23] and [13]. In dynamical systems, the most popular notion is prevalence, that we now define:

A subset A of a Frechet space B is said Haar-null if there exists a compactly supported Borel probability measure m on B such that $m(A + x) = 0$ for each $x \in B$. This concept was first introduced by Christensen in the separable case, see [12] or [3]. It was used as a description of the smallness of the sets of non Gâteau differentiability of Lipschitz functions on separable Frechet spaces. It was then rediscovered in the context of dynamical systems, where the name *prevalence* appeared, see [22]. A prevalent set is the complement of a Haar-null set. It is proved in [22] that some versions of the Thom Transversality Theorem still hold in the sense of prevalence.

Another notion is that of Aronzsajn-null sets, or equivalently of Gaussian-null sets, see [13] or [3]. They can be defined as those sets which have zero measure for all Gaussian measure, see [23], [13] and Section 3 for more details. The complement of an Aronzsajn-null set is prevalent.

One can wonder whether the smallness results discussed above concerning the dimension of $\mathcal{M}(L)$ still hold for these notions of small sets. Since these notions have first been introduced to deal with non-differentiability points of Lipschitz or convex functions, and since the proof of the genericity results recalled above boils down to abstract convex analysis, it is not very surprizing that the answer is positive. It is implied by the following stronger statement expressed in terms of a notion of countably rectifiable sets that will be defined in Section 2:

Theorem 3. *Let L be a Tonelli Lagrangian and let $p \in \{2, 3, \ldots, \infty\}$ and $k > b_1$ be given, where b_1 is the first Betti number of M (the dimension of $H^1(M, \mathbb{R})$). The set $\Sigma_k^p(L)$ is countably rectifiable of codimension $k - b_1$ in $C^p(M)$. As a consequence, for each $k > b_1$, the set $\Sigma_k^p(L)$ is meager in the sense of Baire and Aronszajn-null. Its complement is prevalent.*

We will give in Section 5 a more general result which implies Theorem 3 and many similar statements. For example, the set $\sigma_k^p(L)$ of those potentials $V \in C^p(M)$ such that $\dim(\mathcal{M}(L - V)) \geqslant k$ is countably rectifiable of codimension k. This is a refined version of Mañé's result.

Here is an example of a new application: In perturbation theory, one often fixes a Lagrangian L and a potential V and studies the dynamics generated by $L - \epsilon V$, for ϵ small enough. The following result is then useful:

Corollary 1. *Let L be a Tonelli Lagrangian and let $p \in \{2, 3, \ldots, \infty\}$ be given. Let $A_k^p(L)$ be the set of potentials $V \in C^p(M)$ such that*

$$\sup_{\epsilon > 0, c \in H^1(M, \mathbb{R})} \dim(\mathcal{M}(L - \epsilon V + c)) \geqslant k.$$

If $k > 1 + b_1$ (b_1 is the dimension of $H^1(M, \mathbb{R})$), then $A_k^p(L)$ is countably rectifiable of codimension $k - (1 + b_1)$ in $C^p(M)$. As a consequence, the set $A_k^p(L)$ is Baire-meager and Aronszajn-null.

This Corollary is proved in Section 5. Rectifiable sets of finite codimension in Banach spaces are defined and studied in Section 2. To our knowledge, this is the first systematic study of this class of sets, whose definition is inspired by some recent works of Luděk Zajíček in [25]. We extend the definition to Frechet spaces in Section 3 and prove in this more general setting that rectifiable sets of positive codimension are Baire-meager, Haar-null and Aronszajn-null. The proof follows [25]. In section 4, we study the action of differentiable mappings on rectifiable sets in separable Banach spaces. We believe that this study is of independent interest, and hope that it will have other applications. The proof of Theorem 3 (and of a more general statement) is then exposed in section 5. Actually, it consists mainly of stating appropriately some of the ideas developed by Mañé in [18] in order to reduce Theorem 3 to an old result of Zajíček on

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monotone set-valued maps, see [24]. This approach gives a more precise answer with an easier proof than the finite dimensional approximation used in [18] and in [4].

2 Rectifiable sets of finite codimension in Banach spaces

In a finite dimensional Banach space $\mathbb{R}^n$, one can say that a subset A is countably rectifiable of codimension d if there exist countably many Lipschitz maps $F_i : \mathbb{R}^{n-d} \longrightarrow \mathbb{R}^n$ such that A is contained in the union of the ranges of the maps F_i. Many authors also add a set of zero $(n-d)$-dimensional Hausdorff measure, but we do not.

In an infinite dimensional Banach space B, a first attempt might be to define a rectifiable set of codimension d as a set contained in the countable union of ranges of Lipschitz maps $F_i : B_i \longrightarrow B$, where B_i are closed subspaces of B of codimension d. A closer look shows that this definition does not prevent B itself from being rectifiable of positive codimension. For instance, if B is a separable Hilbert space, then $B \times \mathbb{R}^n$ is also a separable Hilbert space. Therefore, it is isomorphic to B, and there exists a Lipschitz (linear) map $B \longrightarrow B \times \mathbb{R}^n$ which is onto. We thus need a finer definition, and the recent work of Luděk Zajíček in [25] opens the way.

A continuous linear map $L : B \longrightarrow B_1$ is called Fredholm if its kernel is finite dimensional and if its range is closed and has finite codimension. We say that L is a Fredholm linear map of type (k, l) if k is the dimension of the kernel of L and l is the codimension of its range. The index of L is the integer $k - l$. Recall that the set of Fredholm linear maps is open in the space of continuous linear maps (for the norm topology), and that the index is locally constant, although the integers k and l are not. They are lower semi-continuous. To better understand the meaning of the index, observe that, when B and B_1 have finite dimension n and n_1, then the index of all linear maps is $i = n - n_1$.

Definition 2. *Let B be a Banach space. We say that the subset $A \subset B$ is a Lipschitz graph of codimension d if there exists:*

- *a splitting $B = D \oplus T$ of B, where T is a linear subspace of dimension d and D is a closed linear subspace,*

- *a subset D_1 of D,*

- *a Lipschitz map $g : D_1 \longrightarrow T$,*

such that
$$A = \{(x_1 \oplus g(x_1)) : x_1 \in D_1\}.$$
We then say that A is a Lipschitz graph transverse to T.

We say that the set $A \subset B$ is a rectifiable set of codimension d if there exist an integers k, a Banach space B_1, and a Fredholm linear map $P : B_1 \longrightarrow B$ of type $(k, 0)$ such that
$$A \subset P(A_1),$$
where $A_1 \subset B_1$ is a Lipschitz graph of codimension $d + k$.

Finally, we say that the subset $A \subset B$ is countably rectifiable of codimension d if it is contained in the union of countably many rectifiable sets of codimension d.

This definition is directly inspired by a recent work of Zajíček [25], who proves that the rectifiable sets of positive codimension according to this definition are small sets (see Section 3 for more details). In particular, the space B is not countably rectifiable of positive codimension in itself. This legitimates the systematic study of these sets initiated in the present paper. Denoting by $\mathcal{R}_d(B)$ the collection of all countably rectifiable subsets of codimension d in B, we have $\mathcal{R}_{d+1}(B) \subset \mathcal{R}_d(B)$. This requires a proof:

Lemma 3. *If A is countably rectifiable of codimension $d + 1$, then it is countably rectifiable of codimension d.*

PROOF. We first prove that a Lipschitz graph A of codimension $d+1$ is rectifiable of codimension d. Let T be a transversal to A, and let D be a complement of T in B. Let $g : D \supset D_1 \longrightarrow T$ be a Lipschitz map such that $A = \{x \oplus g(x) : x \in D_1\}$. Let S be a one dimensional subspace of T. The Lipschitz graph $A_2 \subset D \times S \times T$ defined by

$$A_2 = \{(x, \lambda, g(x) - \lambda) : x \in D_1, \lambda \in S\}$$

has codimension $d + 1$. On the other hand, we have $A \subset P(A_2)$, where $P : D \times S \times T \longrightarrow B$ is defined by

$$P : (x, \lambda, t) \longmapsto x + \lambda + t$$

which is of type $(1, 0)$. As a consequence, A is rectifiable of codimension d.

Assume now that A is rectifiable of codimension $d + 1$, and write it $A = L(\tilde{A})$, where $\tilde{A}$ is a Lipschitz graph of codimension $d + i + 1$ and L is linear Fredholm of type $(i, 0)$. Since $\tilde{A}$ is rectifiable of codimension $d + i$, it can be written $\tilde{A} = P(\tilde{A}_1)$, where P is linear Fredholm of type $(l, 0)$ and $\tilde{A}_1$ is a Lipschitz graph of codimension $d + l + i$. Now we have $A = L \circ P(\tilde{A}_1)$, and $L \circ P$ is linear Fredholm of type $(i + l, 0)$. As a consequence, A is rectifiable of codimension d. $\square$

Let us describe the action of Fredholm linear maps on rectifiable sets.

Lemma 4. *Let $L : B \longrightarrow B_1$ be a linear Fredholm map of index i, and let $A \subset B$ be a rectifiable subset of codimension d. Then $L(A)$ is rectifiable of codimension $d - i$ in B_1.*

PROOF. If $A = P(A')$, where $A' \subset B'$ is a Lipschitz graph of codimension $d + k$ and $P : B' \longrightarrow B$ is Fredholm of type $(k, 0)$, then $L(A) = L \circ P(A')$, and $L \circ P$ has index $k + i$. So it is enough to prove the statement when A is a Lipschitz graph.

We now assume that A is a Lipschitz graph of codimension d. Let K be the kernel of L, let $\check{K}$ be a complement of K in B, let R be the range of L and $\check{R}$ be a complement of R in B_1. The set $A \times 0$ is a Lipschitz graph of codimension $d + \dim \check{R}$ in $B \times \check{R}$. On the other hand, the set $L(A)$ can also be written $\tilde{L}(A \times 0)$, where $\tilde{L} : B \times \check{R} \longrightarrow B_1$ is defined by $\tilde{L}(b, r) = L(b) + r$. The linear map $\tilde{L}$ is Fredholm of type $(\dim K, 0)$, hence $L(A) = \tilde{L}(A \times 0)$ is rectifiable of codimension $d + \dim R - \dim K = d - i$. $\square$

Lemma 5. *Let $L : B_1 \longrightarrow B$ be a linear map between two Banach spaces B_1 and B. Assume that $\ker L$ has a closed complement in B_1 and that the range of L is closed and has finite codimension l. If A is a countably rectifiable set of codimension d in B, then $L^{-1}(A)$ is countably rectifiable of codimension $d - l$ in B_1.*

PROOF. Let $R \subset B$ be the range of L. The set $A \cap R$ is countably rectifiable of codimension d in B. Since $A \cap R = \pi(A \cap R)$, where $\pi : B \longrightarrow R$ is a linear projection onto R, and since π is a Fredholm map of type $(l, 0)$, we conclude that $A \cap R$ is countably rectifiable of codimension $d - l$ in R. Let us now consider a splitting $B_1 = R_1 \oplus K_1$, where K_1 is the kernel of L. Let $\tilde{L} : R_1 \longrightarrow R$ be the restriction of L to R_1. Note that $\tilde{L}$ is a linear isomorphism, and therefore $\tilde{L}^{-1}(A \cap R)$ is countably rectifiable of codimension $d - l$ in R_1. The conclusion now follows from the observation that $L^{-1}(A) = \tilde{L}^{-1}(A \cap R) \times K_1$. $\square$

It is obvious from the definition that $\mathcal{R}_d(B)$ is a translation-invariant σ-ideal of subsets of B. More precisely, we have:

$$A \in \mathcal{R}_d(B), A' \subset A \implies A' \in \mathcal{R}_d(B),$$
$$A_n \in \mathcal{R}_d(B) \, \forall n \in \mathbb{N} \implies \cup_{n \in \mathbb{N}} A_n \in \mathcal{R}_d(B),$$
$$A \in \mathcal{R}_d(B), b \in B \implies b + A \in \mathcal{R}_d(B).$$

When $B = \mathbb{R}^n$ a countably rectifiable set of codimension d is what it should be: a set which is contained in the union of the ranges of countably many Lipschitz maps $f_i : \mathbb{R}^{n-d} \longrightarrow \mathbb{R}^n$. Indeed such an range can be written as the projection on the second factor of the graph of f_i in $\mathbb{R}^{n-d} \times \mathbb{R}^n$. Some relations between finite-dimensional rectifiable sets and infinite dimensional rectifiable sets of finite codimension are given in 2.2. They are used in Section 3 to prove, following Zajíček (see [25]), that rectifiable sets of finite codimension are small. A consequence of these results is that a countably rectifiable set of positive codimension has empty interior. As a consequence, we obtain:

Lemma 6. *In a Banach space, a closed linear subspace of codimension d is rectifiable of codimension d, but not countably rectifiable of codimension $d + 1$.*

PROOF. Let B_1 be a closed subspace of codimension d in B. We can see B_1 as the range of a linear Fredholm map $P : B \longrightarrow B_1$ of type $(d, 0)$. Since $B_1 = P(B_1)$, if B_1 was countably rectifiable of codimension $d+1$ in B, it would be countably rectifiable of codimension 1 in itself, which is in contradiction with the fact that countably rectifiable sets of positive codimension have empty interior. $\qquad \square$

2.1 Lipschitz graphs of finite codimension

We collect here some classical useful facts concerning Lipschitz graphs. Given a closed linear subspace T of a Banach space B, we consider the quotient space B/T and the canonical projection $\pi : B \longrightarrow B/T$. We endow B/T with the quotient norm $\|.\|$ defined by $\|y\| := \inf_{x \in \pi^{-1}(y)} \|x\|$. It is well-known that B/T is then itself a Banach space. If B_1 is a closed complement of T, then $\pi_{|B_1}$ is a Banach space isomorphism onto B/T.

Proposition 7. *The following statements are equivalent for a subset $A \subset B$ and a finite dimensional (or more generally closed with a closed complement) subspace T of B:*

1. *There exists a closed complement D of T in B, a subset $D_1 \subset D$ and a Lipschitz map $g : D_1 \longrightarrow T$ such that $A = \{x \oplus g(x) : x \in D_1\}$.*

2. *For each closed complement D of T in B, there exists a subset $D_1 \subset D$ and a Lipschitz map $g : D_1 \longrightarrow T$ such that $A = \{x \oplus g(x) : x \in D_1\}$.*

3. *The restriction to A of the natural projection $\pi : B \longrightarrow B/T$ is a bi-Lipschitz homeomorphism onto its image.*

In this case, we say that A is a Lipschitz graph transverse to T.

PROOF. If 1. holds, then $\pi(A) = \pi(D_1)$, and the restriction $\pi_{|A}$ can be inverted by the Lipschitz map

$$x \longmapsto \pi_{|D}^{-1}(x) \oplus g(\pi_{|D}^{-1}(x))$$

so we have 3.

Let us now assume 3. If D is a complement of T, then we have 2. with $D_1 = \pi_{|D}^{-1}(\pi(A))$ and

$$g = P \circ \pi_{|A}^{-1} \circ \pi_{|D},$$

where $P : B \longrightarrow T$ is the projection associated to the splitting $B = D \oplus T$. This map is Lipschitz since we have assumed that $\pi_{|A}^{-1}$ is Lipschitz. $\qquad \square$

Proposition 8. *Let A be a Lipschitz graph transverse to T in B. Then there exists $\delta > 0$ such that, if $F : B \longrightarrow B$ is Lipschitz with $\mathrm{Lip}(F) < \delta$, then $(Id + F)(A)$ is a Lipschitz graph transverse to T.*

PROOF. We consider a complement D of T in B, a subset D_1 of D and a Lipschitz map $g : D_1 \longrightarrow T$ such that $A = \{x \oplus g(x) : x \in D_1\}$. Let us set $G(x) = x \oplus g(x)$. Let $\pi : B \longrightarrow B/T$ be the canonical projection. We want to prove that $\pi \circ (Id + F)$ restricted to A is a bi-Lipschitz homeomorphism. It is equivalent to prove that

$$\pi \circ (Id + F) \circ G = Id + \pi \circ F \circ G$$

is a bi-Lipschitz homeomorphism of D_1 onto its image. This holds if $\mathrm{Lip}(\pi \circ F \circ G) < 1$ by the classical inversion theorem for Lipschitz maps. But $\mathrm{Lip}(\pi \circ F \circ G) \leqslant \mathrm{Lip}(F)\mathrm{Lip}(G)$. $\qquad \square$

We have two useful corollaries:

Lemma 9. *Let $A \subset B$ be a Lipschitz graph of codimension d, and let $F : U \longrightarrow B_1$ be a C^1 diffeomorphism onto its range, where U is an open subset of B. For each $a \in A \cap U$, there exists an open neighborhood $V \subset U$ of a such that $F(A \cap V)$ is a Lipschitz graph of codimension d.*

Lemma 10. *Given a Lipschitz graph $A \subset B$ of codimension n, the set of tranversals to A is open for the natural topology on n-dimensional linear subspaces of B.*

PROOF OF LEMMA 10: Let $\mathcal{G}^d(B)$ be the set of d-dimensional linear subspaces of B. Let $\mathcal{L}(B)$ be the set of bounded linear selfmaps of B. Let A be a Lipschitz graph transverse to $T \in \mathcal{G}^d(B)$. By Proposition 8, there exists $\delta > 0$ such that $(Id + L)(A)$ is a Lipschitz graph transverse to T when $L \in \mathcal{L}(B)$ satisfies $\|L\| \leqslant \delta$. We can assume that $\delta < 1$, which implies that the linear map $(Id + L)$ is a Banach space isomorphism. As a consequence, the set $A = (Id + L)^{-1}\big[(Id + L)(A)\big]$ is a Lipschitz graph transverse to $(Id + L)^{-1}(T)$. In other words, for each $L \in \mathcal{L}(B)$ such that $\|L\| \leqslant \delta$, the space $(Id + L)^{-1}(T)$ is a transversal to A. These spaces, when L vary, form a neighborhood of T in $\mathcal{G}^d(B)$ because the linear maps $\{(Id + L)^{-1}, \|L\| \leqslant \delta\}$ form a neighborhood of the identity in $\mathcal{L}(B)$. $\qquad \square$

2.2 Pre-transversals of Rectifiable sets

Let us now return to rectifiable sets which are not necessarily Lipschitz graphs.

Definition 11. *Let A be a rectifiable subset of codimension d in the Banach space B. We say that the subspace $Q \subset B$ is a pre-transversal of A if there exists:*

- *A Banach space B_1 and a Fredholm map $P : B_1 \longrightarrow B$ of type $(k, 0)$.*

- *A Lipschitz graph A_1 such that $P(A_1) = A$.*

- *A transversal T of A_1 in B_1 such that $P_{|T}$ is an isomorphism onto $Q = P(T)$.*

The dimension of T and Q is necessarily $d + k$.

Lemma 12. *Let A be a rectifiable subset of codimension $d \geqslant 1$ in the Banach space B. Then there exists an integer $n \geqslant d$ such that the set of pre-transversals of A contains a non-empty open set of $\mathcal{G}^n(B)$ (the set of all n-dimensional linear subspaces of B).*

PROOF. Let us write $A = P(A_1)$, where $P : B_1 \longrightarrow B$ is a Fredholm map of type $(k, 0)$ and $A_1 \subset B_1$ is a Lipschitz graph of codimension $d + k$. Let K be the kernel of P, and let T_0 be a transversal of A_1 such that $T_0 \cap K = 0$, such a transversal exists by Lemma 10. Then, by definition, $P(T_0) = Q_0$ is a pre-transversal of A. Let B_2 be a complement of K in B_1 containing T_0, and let $U \subset \mathcal{G}^{d+k}(B_2)$ be a neighborhood of T_0 in the space of $(d+k)$-dimensional subspaces of B_2. If U is small enough, then each $T \in U$ (seen as a subspace of B_1) is a transversal of A_1 such that $T \cap K = 0$, so that $Q = P(T)$ is a pre-transversal of A. Since $P_{|B_2}$ is an isomorphism, the spaces $P(T), T \in U$ form a neighborhood of Q_0 in $\mathcal{G}^{d+k}(B)$. $\qquad\square$

Lemma 13. *Let A be a rectifiable subset of codimension d in the Banach space B. If Q is a pre-transversal to A, then $(A + x) \cap Q$ is rectifiable of codimension d (or equivalently it is rectifiable of dimension $(\dim Q) - d$) in the finite dimensional space Q for each $x \in B$.*

PROOF. We have $A = P(A_1)$ and $Q = P(T)$, where $P : B_1 \longrightarrow B$ is Fredholm of type $(k, 0)$ and where A_1 is a Lipschitz graph transverse to T. Let $K \subset B_1$ be the kernel of P, we have $K \cap T = 0$. The relation

$$(A + x) \cap Q = P\Big((A_1 + y) \cap (T \oplus K)\Big)$$

holds for each point $y \in P^{-1}(x)$. The set $(A_1 + y) \cap (T \oplus K)$ is a Lipschitz graph of dimension at most $k = \dim K$ in $T \oplus K$. As a consequence, the set $P((A_1 + y) \cap (T \oplus K))$ is rectifiable of dimension at most k in Q. In other words, it is rectifiable of codimension d (recall that $\dim Q = d + k$). $\qquad\square$

The closure of a Lipschitz graph of codimension d is obviously a Lipschitz graph of codimension d. This property does not hold for rectifiable sets, but we have:

Lemma 14. *Each rectifiable set of codimension d is contained in a rectifiable set of codimension d which is a countable union of closed sets.*

PROOF. Using the notations of Definition 2, we have $A = P(A_1)$, where A_1 is a Lipschitz graph of codimension d in B_1. The closure $\bar{A}_1$ of A_1 is a Lipschitz graph of codimension d, hence $P(\bar{A}_1)$ is rectifiable of codimension d. Let us write $\bar{A}_1 = \cap_{i \in \mathbb{N}} A_1^i$ where the sets $A_1^i, i \in \mathbb{N}$ are closed bounded subsets of $\bar{A}_1$. Note that the sets A_1^i are Lipschitz graphs. We claim that $P(A_1^i)$ is closed for each n, which implies the thesis. Consider a sequence a_n of points of $P(A_1^i)$ converging to a in B. We want to prove that $a \in P(A_1^i)$. Let $\tilde{B}_1$ be a closed complement of $\ker P$ in B_1, and let x_n be the sequence of preimages of a_n in $\tilde{B}_1$. Since $P_{|\tilde{B}_1}$ is an isomorphism, the sequence x_n has a limit x in $\tilde{B}_1$ with $P(x) = a$. The point x does not necessarily belong to A_1^i, but there exists a sequence k_n in $\ker P$ such that $x_n \oplus k_n \in A_1$ and such that $P(x_n \oplus k_n) = a_n$. The sequence k_n is bounded because A_1 is bounded, and therefore it has a subsequence converging to a limit k in the finite dimensional space $\ker P$. We have $x \oplus k \in A_1^i$ because A_1^i is closed, and thus $a = P(x \oplus k)$ belongs to $P(A_1^i)$. $\qquad\square$

3   Rectifiable sets in Frechet spaces

In this section, we work in the more general setting of Frechet spaces. A Frechet space F is a complete topological vector space whose topology is generated by a countable family of semi-norms $\|.\|_k, k \in \mathbb{N}$. Equivalently, the topology on F is generated by a translation invariant metric which turns F into a complete metric space. The main example here is $C^\infty(M)$. Let us first define Lipschitz graphs and rectifiable sets in the context of Frechet spaces.

Definition 15. ?? *We say that the set $A \subset F$ is a Lipschitz graph of codimension d if there exists a continuous linear map $P : F \longrightarrow B$ with dense range in the Banach space B such that $P(A)$ is a Lipschitz graph of codimension d in B.*

   This definition is coherent with the already existing one in the case where F is a Banach space in view of the following:

Proposition 16. *Let F and B be Banach spaces, and let $P : F \longrightarrow B$ be a continuous linear map with dense range. If $A \subset B$ is a Lipschitz graph of codimension d in B, then $P^{-1}(A)$ is a Lipschitz graph of codimension d in F.*

PROOF. Let T be a transversal of A which belongs to the range of P (such a transversal exists by Lemma 10), and let $\tilde{B}$ be a complement of T in B. Let $T' \subset F$ be a subspace such that $P_{|T'}$ is an isomorphism onto T, and let $\tilde{F}$ be the preimage of $\tilde{B}$. Note that $F = \tilde{F} \oplus T'$. This can be proved as follows: let $\pi : B \longrightarrow T$ and $\tilde{\pi} : B \longrightarrow \tilde{B}$ be the projections corresponding to the splitting $B = T \oplus \tilde{B}$. Each point $f \in F$ can be written $f = t + (f - t)$ with $t = P_{|T'}^{-1} \circ \pi \circ P(f) \in T'$. It is enough to see that $P(f - t) \in \tilde{B}$. This inclusion holds because $P(f - t) = P(f) - \pi(P(f)) = \tilde{\pi}(P(f)) \in \tilde{B}$. Let $g : \tilde{B} \supset D \longrightarrow T$ be the Lipschitz map such that $A = \{x \oplus g(x), x \in D\}$. Setting $D' = P^{-1}(D)$ and $g' = P_{|T'}^{-1} \circ g \circ P$, we get that

$$P^{-1}(A) = \{f \oplus g'(f), f \in D'\}$$

and $g' : D' \longrightarrow T'$ is Lipschitz. $\qquad \square$

Proposition 17. *The following statements are equivalent for a subset A of the Frechet space F:*

- *There exists a Banach space B and a continuous linear map $P : F \longrightarrow B$ with dense range such that $P(A)$ is rectifiable of codimension d in B.*

- *There exists a Frechet space F_1, a continuous linear map π_1 which is Fredholm of type $(k, 0)$, and a Lipschitz graph A_1 of codimension $d + k$ in F_1 such that $A = \pi_1(A_1)$.*

We say that A is rectifiable of codimension d if these properties are satisfied.

PROOF. Assume that there exists a linear Fredholm map $\pi_1 : F_1 \longrightarrow F$ as in the statement. Then, there exists a Banach space B_1 and a continuous linear map $P_1 : F_1 \longrightarrow B_1$ with dense range such that $P_1(A_1)$ is a Lipschitz graph of codimension $d + k$ in B_1. The space $K := P_1(\ker \pi_1) \subset B_1$ has dimension at most k. Let us set $B := B_1/K$, and let $\pi : B_1 \longrightarrow B$ be the standard projection. There is a unique map $P : F \longrightarrow B$ such that $P \circ \pi_1 = \pi \circ P_1$. In other words, the following diagram commutes.

$$
\begin{array}{ccc}
F_1 & \xrightarrow{\;P_1\;} & B_1 \\
{\scriptstyle \pi_1}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\
F & \xrightarrow{\;P\;} & B
\end{array}
$$

Let us check that P has dense range and that $P(A)$ is rectifiable of codimension d. The range of P is the range of $\pi \circ P_1$, which is dense because π is onto and P_1 has dense range. We have $P(A) = P(\pi_1(A_1)) = \pi(P_1(A_1))$ which implies that $P(A)$ is rectifiable of codimension d because $P_1(A_1)$ is a Lipschitz graph of codimension $d + k$ and π has type $(k', 0)$ with $k' \leqslant k$.

Conversely, assume that $P : F \longrightarrow B$ exists as in the statement. Then there exists a Lipschitz graph A'_1 of codimension $d + k$ in some Banach space B_1 and a linear Fredholm map $\pi : B_1 \longrightarrow B$ of type $(k, 0)$ such that $P(A) = \pi(A'_1)$. Let K be the kernel of π, which has dimension k, and let us set $F_1 := F \times K$. Let $\pi_1 : F_1 \longrightarrow F$ be the projection on the first factor. In order to complete the diagram with a map $P_1 : F_1 \longrightarrow B_1$, we consider a right inverse L of π, (which is a Fredholm linear map of type $(0, k)$) and set $P_1(f, k) = L(P(f)) + k$. Note then that $P \circ \pi_1 = \pi \circ P_1$. The range of P_1 is $\pi^{-1}(P(F))$, it is dense because the range of P is dense. As a consequence, the set $A_1 := P_1^{-1}(A'_1)$ is a Lipschitz graph of codimension $d + k$, by definition. On the other hand, the map π_1 is Fredholm of type $(k, 0)$ hence the thesis follows from the inclusion $A \subset \pi_1(A_1)$. In order to prove this inclusion, let us consider a point $a \in A$. There exist $a'_1 \in A'_1$ such that $\pi(a'_1) = P(a)$. Then there exists $a_1 \in A_1$ such that $\pi \circ P_1(a_1) = P(a)$, which implies that $P(\pi_1(a_1)) = P(a)$. As a consequence, the difference $f := a - \pi_1(a_1)$ belongs to $\ker P$. Let us consider the point $b_1 = a_1 + (f, 0) \in F_1$. We have $P_1(b_1) = P_1(a_1) \in A'_1$, thus $b_1 \in A_1$. On the other hand, $\pi_1(b_1) = \pi_1(a_1) + f = a$ hence $a \in \pi_1(A_1)$. $\qquad\square$

A subset $A \in F$ is said countably rectifiable of codimension d if it is a countable union of rectifiable sets of codimension d. The special case $F = C^\infty(M)$ may help to understand the definitions.

Lemma 18. *A subset $A \in C^\infty(M)$ is rectifiable of codimension d if and only if there exists $p \in \mathbb{N}$ and a set $A' \subset C^p(M)$ which is rectifiable of codimension d and such that $A = C^\infty(M) \cap A'$.*

PROOF. Assume that A is rectifiable of codimension d. Then there exists a Banach space B and a continuous linear map $P : C^\infty(M) \longrightarrow B$ with dense range such that $A' = P(A)$ is rectifiable of codimension d. Then, the map P is continuous for some C^p norm and extends to a continuous linear map $P_p : C^p(M) \longrightarrow B$ for some p. Since the map P_p has dense range, the set $A^p = P_p^{-1}(A')$ is rectifiable of codimension d, and $A \subset A^p \cap C^\infty(M)$.

Conversely, if $A = C^\infty(M) \cap A'$ for some rectifiable set $A' \subset C^p(M)$, then we have $A = P^{-1}(A')$, where $P : C^\infty(M) \longrightarrow C^p(M)$ is the standard inclusion. This inclusion is continuous with dense range hence A is rectifiable. $\qquad\square$

We shall now explain, following Zajíček (see [25]) that rectifiable sets of positive codimension in Frechet spaces are small in various meanings of that term. We first recall definitions.

A subset $A \subset F$ is called **Baire-meager** if it is contained in a countable union of closed sets with empty interior. Baire Theorem states that a Baire-meager subset of a Frechet space has empty interior.

A subset $A \subset F$ is called **Haar-null** if there exists a compactly supported Borel probability measure μ on F such that $\mu(A + f) = 0$ for all $f \in F$. The equality $\mu(A + f) = 0$ means that the set $A + f$ is contained in a Borel set $\tilde{A}_f$ such that $\mu(\tilde{A}_f) = 0$. A countable union of Haar-null sets is Haar-null, see [12, 3] and [22] for the non-separable case.

A subset $A \subset F$ of a separable Frechet space F is called **Aronszajn-null** if, for each sequence f_n generating a dense subset of F, there exists a sequence A_n of Borel subsets of F such that $A \subset \cup_n A_n$ and such that, for each $f \in F$ and for each n, the set

$$\{x \in \mathbb{R} : f + x f_n \in A_n\} \subset \mathbb{R}$$

has zero Lebesgue measure. A countable union of Aronszajn-null sets is Aronszajn-null, and each Aronszajn-null set is Haar null, see [1, 3].

Theorem 4. *Let F be a Frechet space, and let $A \subset F$ be countably rectifiable of positive codimension. Then A is Baire-meager, Haar null, and (if F is separable) Aronszajn-null.*

This Theorem is due to Luděk Zajíček (see [25]) in the case of Banach spaces. The extension to Frechet spaces that we now expose is not very different.

PROOF OF THEOREM 3: Let A be a rectifiable set of positive codimension in the Frechet space F, and let $P : F \longrightarrow B$ be a continuous linear map with dense image in the Banach space B such that $P(A)$ is rectifiable of positive codimension. We can assume without loss of generality that $A = P^{-1}(P(A))$.

Let us prove that A is Baire meager. By Lemma 14, we can assume without loss of generality that $P(A)$ is a countable union of closed sets in B, which implies that $A = P^{-1}(P(A))$ is a countable union of closed sets in F. It is thus enough to prove that A has empty interior. Let $\tilde{Q}$ be a pre-transversal of $P(A)$ in B (see Definition 11) contained in $P(F)$. Such a space $\tilde{Q}$ exists by Lemma 12 because P has dense range. Let $Q \subset F$ be a linear subspace such that $P_{|Q}$ is an isomorphism onto $\tilde{Q}$. Given $f \in F$, the set $(A + f) \cap Q$ has empty interior in Q, and therefore there exists a sequence $q_n \in Q$ such that $q_n \longrightarrow 0$ (in Q thus in F) and $(f + q_n) \notin A$. We conclude that the complement of A is dense in F.

Let us prove that A is Haar-null. Let $\tilde{Q}$ be a pre-transversal of A contained in $P(F)$ (see Definition 11 and Lemma 12) and let $Q \subset F$ be a linear space such that $P_{|Q}$ is an isomorphism onto $\tilde{Q}$. Let μ be the normalized Lebesgue measure on a bounded open subset of Q, that we also see as a compactly supported Borel probability measure on F. Since all rectifiable sets of positive codimension in the finite-dimensional space Q have zero Lebesgue measure, we conclude that $\mu(A + f) = 0$ for each $f \in F$. Therefore, A is Haar-null in F.

Finally, we assume that F is separable and prove that A is Aronszajn-null. Let $f_n \in F$ be a sequence generating a dense subspace of F. Since P has a dense range, the sequence $P(f_n)$ generates a dense subset in B. As a consequence, there exists an integer N such that the space

$$\text{vect}\{P(f_n), n \leqslant N\} \subset B$$

contains a pre-transversal $\tilde{Q}$ of $P(A)$. Then, the space

$$F_N := \text{vect}\{f_n, n \leqslant N\} \subset F$$

contains a subspace Q such that $P_{|Q}$ is an isomorphism onto $\tilde{Q}$. Since $(A + f) \cap Q$ has zero Lebesgue measure in Q for each $f \in F$, we conclude from Fubini Theorem that $A \cap (F_n - f)$ has zero Lebesque measure in F_N for each $f \in F$. By standard arguments, (see for example [3], Proposition 6.29, p 144 or [1], Proposition 1, p 151) this implies that A can be written as the union

$$A = \cup_{n \leqslant N} A_n$$

of Borel sets A_n which are such that the set

$$\{x \in \mathbb{R} : f + x f_n \in A_n\}$$

has zero measure in $\mathbb{R}$ for each $f \in F$ and each $n \leqslant N$. □

4 Countably rectifiable sets and differential calculus

In separable Banach spaces, the concepts of countably rectifiable sets can be localized and behaves well with differential calculus. This Section is not used in the proof of our results in Lagrangian dynamics, except a small part of it for Corollary 1.

We say that a set $A \subset B$ is locally countably rectifiable of codimension d if, for each $a \in A$, there exists a neighborhood U of a in B such that $U \cap A$ is countably rectifiable of codimension d. Every countably rectifiable set is obviously locally countably rectifiable, and conversely we have:

Lemma 19. *Let B be a separable Banach space. If the subset $A \subset B$ is locally countably rectifiable of codimension d, then it is countably rectifiable of codimension d.*

PROOF. The space A is a separable metric space for the metric induced from the norm of B, and therefore it has the Lindelöf property: each open cover of A admits a countable subcover. Now the hypothesis of local countably rectifiability implies that A can be covered by a union of open subsets of A each of which is countably rectifiable of codimension d in B. Therefore, by the Lindelöf property, A is the union of countably many sets each of which is countably rectifiable of codimension d in B. $\qquad\square$

Let B and B_1 be two Banach spaces, $U \subset B$ be an open subset of B, and $F : U \longrightarrow B_1$ be a C^1 map. We say that F is Fredholm if the Frechet differential dF_x is Fredholm at each $x \in U$. If U is connected, then the index of dF_x does not depend on x, we say that this is the index of F. The following result shows that rectifiable sets of codimension d in separable Banach spaces could have been equivalently defined as the image by a C^1 Fredholm map of index i of a Lipschitz graph of codimension $d + i$.

Proposition 20. *Let B and B_1 be separable Banach spaces, let U be an open subset of B, and let $A \subset U$ be a countably rectifiable set of codimension d. If $F : U \longrightarrow B_1$ is a C^1 Fredholm map of index i then $F(A)$ is countably rectifiable of codimension $d - i$ in B_1.*

PROOF. It is enough to prove that, for each i and d, the image of a Lipschitz graph of codimension d by a Fredholm map of index i is locally countably rectifiable of codimension $d - i$. In fact, if $A = \cup_n P_n(A_n)$ is a countably rectifiable set of codimension d, where A_n are Lipschitz graphs of codimension $d + i_n$ and P_n are linear Fredholm maps of type $(i_n, 0)$, then $F(A) = \cup_n F \circ P_n(A_n)$. The map $F \circ P_n$ is Fredholm of index $i + i_n$, and therefore, if our claim is proved, then $F \circ P_n(A_n)$ is countably rectifiable of codimension $(d + i_n) - (i + i_n) = d - i$.

So we now assume that A is a Lipschitz graph. In order to prove that the image $F(A)$ is countably rectifiable of codimension $d - i$, it is enough to prove that each point $a \in A$ has a neighborhood U in A such that $F(U)$ is countably rectifiable of codimension $d - i$. Let $a \in A$ be given. Assume that the linear Fredholm map dF_a is of type (k, l). There is a local C^1 diffeomorphism ϕ of B around a and a splitting $B = B_1 \oplus C$ of B, with $\dim C = l$, such that $F = \tilde{F} \circ \phi$ in a neighborhood of a, where $\tilde{F}$ is of the form

$$x \longmapsto \pi(x) \oplus f(x),$$

where $\pi : B \longrightarrow B_1$ is a linear Fredholm map of type $(k, 0)$ and $f : B \longrightarrow C$ is a C^1 map satisfying $df_a = 0$, see [9], Theorem 1.1. By Lemma 9, $\phi(A)$ is locally a Lipschitz graph of codimension d. In other words, there exists a neighborhood U of a in A such that $\phi(U)$ is a Lipschitz graph of codimension d. Hence there exists a splitting $B = \tilde{B} \oplus T$, a subset $\tilde{D} \subset \tilde{B}$, and a Lipschitz map $g : \tilde{D} \longrightarrow T$ such that $\phi(U) = \{x \oplus g(x) : x \in \tilde{D}\}$. Let us define $A_2 \subset B \times C$ as

$$A_2 := \left\{ \big(x, f(x)\big) : x \in \phi(U) \right\} = \left\{ \big(\tilde{x} \oplus g(\tilde{x}), f(\tilde{x} \oplus g(\tilde{x}))\big) : \tilde{x} \in \tilde{D} \right\}.$$

Since the map

$$\tilde{D} \ni \tilde{x} \longmapsto (g(\tilde{x}), f(\tilde{x} \oplus g(\tilde{x}))) \in T \times C$$

is Lipschitz, the set A_2 is a Lipschitz graph transverse to $T \times C$ in $B \times C$, it is thus of codimension $d + l$. Now the set $\tilde{F}(A)$ is just the image of A_2 by the linear map

$$B \times C \ni (x, z) \longmapsto \pi(x) \oplus z \in B.$$

This linear map is Fredholm of type $(k, 0)$, so by definition $\tilde{F}(A)$ is rectifiable of codimension $d + l - k = d - i$. $\qquad\square$

The following direct corollary is especially important:

Corollary 21. *Let B be a separable Banach space, $U \subset B$ and open set, and $\phi : U \longrightarrow B$ a C^1 diffeomorphism onto its image V. Then the set $A \subset U$ is countably rectifiable of codimension d if and only if its image $\phi(A)$ is countably rectifiable of codimension d.*

A consequence of all these observations is that the notion of countably rectifiable subsets of codimension d is well-defined in separable manifolds modeled on separable Banach spaces:

Definition 22. *Let W be a separable C^1 manifold modeled on the separable Banach space B. The set $A \subset B$ is said countably rectifiable of codimension d if, for each point $a \in A$, there exists a neighborhood $U \subset W$ of a in W and a chart $\phi : U \longrightarrow V \subset B$ such that $\phi(U \cap A)$ is countably rectifiable of codimension d in B.*

In view of Corollary 21, if A is a countably rectifiable set of codimension d in W and $\phi : U \subset W \longrightarrow V \subset B$ is any chart of W, then $\phi(A \cap U)$ is countably rectifiable of codimension d in B. Obviously, when $W = B$, this definition of countably rectifiable sets of codimension d coincides with the former one. Proposition 20 has a straightforward generalization:

Theorem 5. *Let W and W_1 be separable manifolds modeled on separable Banach spaces, and let $A \subset W$ be a countably rectifiable set of codimension d. If $F : W \longrightarrow W_1$ is a C^1 Fredholm map of index i then $F(A)$ is countably rectifiable of codimension $d - i$ in W_1.*

If W and W_1 are two separable manifolds modeled on separable Banach spaces B and B_1, the C^1 map $F : W \longrightarrow W_1$ is called a submersion at x if the differential $dF_x : T_x W \longrightarrow T_{F(x)} W_1$ is onto, and if its kernel splits (if these conditions are satisfied, we say that dF_x is a linear submersion). It means that there exists a closed linear subspace $\tilde{B}$ in $T_x M$ such that $T_x M = \ker(dF_x) \oplus \tilde{B}$. It is known that the map F is a submersion at x if and only if there exist local charts at x and $F(x)$ such that the expression of F in these charts is a linear submersion. The following statement then follows from Lemma 5:

Proposition 23. *Let W and W_1 be two separable manifolds modeled on separable Banach spaces. Let $A \subset W_1$ be a countably rectifiable subset of codimension d, and let $F : W \longrightarrow W_1$ be a C^1 map which is a submersion at each point of $F^{-1}(A)$. Then $F^{-1}(A)$ is countably rectifiable of codimension d.*

5 Application to Lagrangian systems

We now return to the study of Minimizing measures of Lagrangian systems. Let us begin with a general abstract result:

Theorem 6. *Let F be a Frechet space of C^2 functions on TM (but not necessarily with the C^2 topology) and U be an open subset of F. Assume that*

- *The topology on F is stronger than the compact-open topology. In other words, for each compact set $K \subset TM$ the natural map $F \longrightarrow C(K)$ is continuous.*

- *The space F contains a dense subset of $C(M)$ (where the functions of $C(M)$ are seen as functions on TM).*

- *For each $f \in U$, the function $L - f$ is a Tonelli Lagrangian.*

Then, for each $k \in \mathbb{N}$, the set

$$\{f \in U : \dim(\mathcal{M}(L - f)) \geqslant k\}$$

is a countable union of Lipschitz graphs of codimension k in F (and therefore it is countably rectifiable of codimension k).

Before we turn to the proof, let us see how to derive the statements of the introduction from this result.

By taking $U = F = C^p(M), p \in \{2, 3, \ldots, \infty\}$, we obtain that the set $\sigma^p(L)$ of functions $f \in C^p(M)$ such that $L - f$ has more than one Mather mesure is countably rectifiable of codimension 1, which is stronger than the Theorem of Mañé (Theorem 1).

PROOF OF THEOREM 3: As earlier, let us identify the space $H^1(M, \mathbb{R})$ with a b_1-dimensional space of smooth forms, and therefore with a b_1-dimensional space of funtions on TM. Let us take $U = F = H^1(M, \mathbb{R}) \times C^p(M)$, and apply Theorem 6. We obtain that the set of pairs $(c, f) \in H^1(M, \mathbb{R}) \times C^p(M)$ such that $\dim(\mathcal{M}(L - c - f)) \geqslant k$ is countably rectifiable of codimension k. Since Σ_k^p is the projection of this set on the second factor $C^p(M)$, we conclude by Proposition 17 that Σ_k^p is countably rectifiable of codimension $k - b_1$. □

PROOF OF COROLLARY 1: Let $P_p : C^p(M) \longrightarrow C^2(M)$ be the standard inclusion, which is a continuous linear map with dense range. We observe that

$$A_k^p(M) = A_k^2(M) \cap C^p(M) = P_p^{-1}(M)$$

so, by Proposition 17, it is enough to prove the result for $p = 2$. The map

$$(0, \infty) \times C^2(M, \mathbb{R}) \ni (\epsilon, V) \longmapsto \epsilon V \in C^2(M, \mathbb{R})$$

is a smooth submersion. By theorem 3, the set

$$\Sigma_k^2 := \{V \in C^2(M, \mathbb{R}) : \max_{c \in H^1(M, \mathbb{R})} \dim \mathcal{M}(L - V + c) \geqslant k\}$$

is countably rectifiable of codimension $k - b_1$, hence Proposition 23 implies that the set of pairs $(\epsilon, V) \in (0, \infty) \times C^2(M, \mathbb{R})$ such that $\epsilon V \in \Sigma_k^2$ is countably rectifiable of codimension $k - b_1$ in $(0, \infty) \times C^2(M, \mathbb{R})$. The conclusion now follows by Lemma 4 since A_k^2 is the projection of this set on the second factor. □

The proof of Theorem 6 will occupy the rest of the section. It is useful first to recall the notion of a **closed probability measure** on TM. The Borel probability measure μ is said closed if it is compactly supported and if, for each function $f \in C^\infty(M)$, we have

$$\int_{TM} df_x \cdot v \, d\mu(x, v) = 0.$$

If μ is compactly supported and invariant under the Euler-Lagrange flow of a Tonelli Lagangian L, then μ is closed. We recall the proof first given by Mather, see [20].

We can express the invariance of the measure μ by saying that, for each smooth function $g : TM \longrightarrow \mathbb{R}$, we have

$$\int_{TM} g d\mu = \int_{TM} g \circ \psi^t d\mu.$$

Differentiating at $t = 0$, we get

$$\int_{TM} dg_{(x,v)} \cdot \mathcal{X}(x,v) d\mu(x,v) = 0,$$

where $\mathcal{X}$ is the Euler-Lagrange vector-field (the generator of ψ^t). This is true in particular if $g = f \circ \pi$ is a function which depends only on the position x, and writes

$$\int_{TM} df_x \cdot \Pi(\mathcal{X}(x,v)) d\mu(x,v) = 0$$

for each smooth function f on M, where $\Pi : T_{(x,v)}(TM) \longrightarrow T_x M$ is the differential of the standard projection. The proof now follows from the observations that $\Pi(\mathcal{X}(x,v)) = v$.

Let us now fix a Riemannian metric on M, and define, for each $n \in \mathbb{N}$, the compact subset K_n of TM as follows:

$$K_n = \{(q,v) \in TM : \|v\|_q \leqslant n\}.$$

We denote by $\mathcal{C}_n$ the set of closed probability measures supported on K_n. We also define the Banach space B_n as the closure, in $C(K_n)$, of the restrictions to K_n of the functions of F. Since $C(K_n)$ is separable, so is B_n. Moreover, it follows from our assumptions on F that B_n contains $C(M)$. Each probability measure μ on K_n gives rise to a linear form on $C(K_n)$ and, by restriction, to a linear form $l_n(\mu) \in B_n^*$, which is just defined by

$$l_n(\mu) \cdot f = \int f d\mu.$$

Because B_n is not necessarily dense in $C(K_n)$, the map l_n is not necessarily one-to-one.

Let us now define the functional $A_n : B_n \longrightarrow \mathbb{R}$ by

$$A_n(f) = \sup_{\mu \in \mathcal{C}_n} \int (f - L) d\mu.$$

This functional is convex, and it is bounded from below (because any Dirac supported on a point of the zero section of TM belongs to $\mathcal{C}_n$). By standard results of convex analysis, the set

$$\{f \in B_n : \dim(\partial A_n(f)) \geqslant k\}$$

is a countable union of Lipschitz graphs of codimension k in the separable Banach space B_n for each $k \in \mathbb{N}$ (see [24] or [3], Theorem 4.20, p. 93), where $\partial A_n(f)$ is the sub-differential in the sense of convex analysis of the function A_n at point f. We will now be able to conclude if we can relate the set $\mathcal{M}(L - f)$ of Mather measures and the set $\partial A_n(f)$. This is the content of the following, where we denote by $P_n : F \longrightarrow B_n$ the natural map :

Lemma 24. *If $L - f$ is a Tonelli Lagrangian, then there exists $n \in \mathbb{N}$ such that*

$$\dim(\mathcal{M}(L - f)) \leqslant \dim \partial A_n(P_n(f)).$$

Assuming Lemma 24, let us finish the proof of Theorem 6. The map $P_n : F \longrightarrow B_n$ is continuous by our assumptions on F, and it has a dense range by the definition of B_n. In view of the Lemma we have

$$\{f \in U : \dim \mathcal{M}(L - f) \geqslant k\} \subset \bigcup_n P_n^{-1}\big(\{f \in B_n : \dim \partial A_n(f) \geqslant k\}\big). \tag{U}$$

Each of the sets

$$\{f \in B_n : \dim \partial A_n(f) \geqslant k\}$$

is a countable union of Lipschitz graphs of codimension k in B_n, hence the preimage

$$P_n^{-1}\big(\{f \in B_n : \dim \partial A_n(f) \geqslant k\}\big)$$

is a countable union of Lipschitz graphs of codimension k in F, by Definition **??**. Therefore Theorem 6 follows from (U). □

The last step is to prove Lemma 24. The main tool is the following beautiful variational principle which has been established by Bangert [2] and Fathi and Siconolfi [16] following fundamental ideas of Mañé, [18] (see also [5, 8]):

Theorem 7. *The Mather measures of the Tonelli Lagrangian L are those which minimize the action $\int L d\mu$ in the class of all compactly supported closed measures.*

The key point here is that the invariance of the measure is obtained as a consequence of its minimization property, and not imposed as a constraint as in Mather's work. By applying this general result to the Lagrangian $L - f$, we obtain that there exists $n \in \mathbb{N}$ such that the set $\mathcal{M}(L - f)$ of Mather measures is the set of measures $\mu \in \mathcal{C}_n$ which minimize the action $\int (L - f) d\mu$ in $\mathcal{C}_n$. Now if μ is such a measure, then the associated $l_n(\mu)$ belongs to $\partial A_n(P_n(f))$, as can easily be seen from the definition of A_n. In other words, we have proved that

$$l_n(\mathcal{M}(L - f)) \subset \partial A_n(P_n(f))$$

when n is large enough. In order to prove Lemma 24, it is enough to observe that l_n is one to one on $\mathcal{M}(L - f)$. This property holds because B_n contains $C(M)$ and because the elements of $\mathcal{M}(L - f)$ are all supported on a Lipschitz graph. □

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