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COMPACTNESS PROPERTY OF THE LINEARIZED BOLTZMANN OPERATOR FOR A DIATOMIC SINGLE GAS MODEL

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Abstract. In the following work, we consider the Boltzmann equation that models a diatomic gas by representing the microscopic internal energy by a continuous variable I. Under some convenient assumptions on the collision cross-section $\mathcal{B}$, we prove that the linearized Boltzmann operator $\mathcal{L}$ of this model is a Fredholm operator. For this, we write $\mathcal{L}$ as a perturbation of the collision frequency multiplication operator, and we prove that the perturbation operator $\mathcal{K}$ is compact. The result is established after inspecting the kernel form of $\mathcal{K}$ and proving it to be L^2 integrable over its domain using elementary arguments.This implies that $\mathcal{K}$ is a Hilbert-Schmidt operator.

1. **Introduction.** In the last several decades, the kinetic theory of polyatomic gases witnessed extensive interest due to its vigorous relation with a wide range of practical applications including spacecraft flights, hypersonic flights and aerodynamics [1], plasma physics [20], thermal sciences [13, 23], combustion processes, and chemical reactors. In the context of polyatomic gases, Borgnakke and Larsen proposed a microscopic model [6]. Later on, an entropic kinetic model consistent with [6] has been derived [8]. This model originates from the Boltzmann equation, which was a breakthrough in the kinetic theory, and offered an accurate description of the gas flow.

However, it is usually expensive and cumbersome to solve the Boltzmann equation directly. As an alternative to the Boltzmann equation, kinetic theory provides macroscopic models for not too large Knudsen numbers. These models are derived as approximations to the Boltzmann equation and offer high computational speed and explicit equations for macroscopic variables, which are helpful for understanding and analyzing the flow behavior. Macroscopic models are classically obtained by Chapman-Enskog method [5] and moments method [22, 18]. Using the Chapman-Enskog method, Nagnibeda and Kustova [19] studied the strong vibrational nonequilibrium in diatomic gases and reacting mixture of polyatomic gases, and derived the first-order distribution function and governing equations. Cai and Li [10] extended the NRxx model to polyatomic gases using the ES-BGK model of

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[2] and [9]. In [24], the existence result of the ES-BGK model was achieved in the case where the solution lies close to equilibrium.

Simplified Boltzmann models for mixtures of polyatomic gases have also been proposed in [3, 12]. The authors of [4] developed a generalized macroscopic 14 field theory for the polyatomic gases, based on the methods of extended thermodynamics [18]. In the full non-linear Boltzmann equation, Gamba and Pavić-Čolić [15] established existence and uniqueness theory in the space homogeneous setting.

The relation of the kinetic theory with the spectral theory was initiated by Grad [17], who was behind the history of serious investigation of the spectral properties of the linearized Boltzmann operator for monoatomic gases. With his pioneering work, Grad showed that the linearized collision operator $\mathcal{L}$ for Maxwell and hard potential cases can be decomposed as $\mathcal{L} = \mathcal{K} - \nu$ Id, where ν is called collision frequency, and by using his angular cut-off assumption on the cross-section, he proved that $\mathcal{K}$ is a compact operator in $L^2(\mathbb{R}^3)$. The compactness of $\mathcal{K}$ for a mixture of monoatomic gases was celebrated later by [7], with more explanation in [21]. This result is significant for formally deriving the fluid systems in the Chapman-Enskog expansion which was recently developed in [5] for a mixture of monoatomic and polyatomic gases. One of the focuses of [5] was on the diatomic gases.

In fact, diatomic gases gain a solid importance due to the fact that in the upper atmosphere of the earth, the diatomic molecules Oxygen (O_2) and Nitrogen (N_2) are dominant. We aim in this article to restrict ourselves to diatomic gases, for which the proof is simpler than polyatomic gases. In contrast to monoatomic gases, which have only 3 degrees of freedom coming from the translational motion, diatomic gases have 3 translational and 2 rotational degrees of freedom which sum up to 5 degrees of freedom. We restrict ourselves in this paper for the case where the vibrational degree of freedom is ignored. In this case, the parameter α appearing in the collision operator is equal to zero. Namely, the parameter α is given by $\alpha = \frac{D-5}{2}$, where D is the total number of degrees of freedom of the gas. Therefore, discarding the terms with a power of α in the collision operator simplifies the proof of compactness of $\mathcal{K}$. We remark that using elementary arguments, for the first time we prove that for diatomic gases the operator $\mathcal{K}$ is a Hilbert-Schmidt operator, as an improvement of Grad's result [17] for single monoatomic gases. For a mixture of monoatomic gases, Boudin, Grec, Pavić-Čolić, and Salvarani [7] proved $\mathcal{K}$ to be compact. A generalization of this work for general polyatomic gases is to be carried later.

The plan of the document is the following: In section 2, we give a brief recall on the collision model [8], which describes the microscopic state diatomic gases. In section 3, we define the linearized operator $\mathcal{L}$, which is obtained by approximating the distribution function f around the Maxwellian M. The operator $\mathcal{L}$ is considered as a perturbation of the multiplication operator ν Id by a linear operator $\mathcal{K}$, which we aim to prove to be compact. We write hence $\mathcal{K}$ as $\mathcal{K}_3 + \mathcal{K}_2 - \mathcal{K}_1$, and in section 4 we prove each $\mathcal{K}_i$, with $i = 1, \cdots, 3$, to be a Hilbert-Schmidt operator using classical arguments. The main idea of the proof is to extract the kernel of $\mathcal{K}_i$. For $\mathcal{K}_2$ and $\mathcal{K}_3$, a major step is a change of variable from the post-collisional to the pre-collisional velocity and internal energy. This change of variable is different from the one implemented by [17], as it considers the parameter ω to be fixed, yet takes into consideration the internal energy parameter. In section 5, we give two important properties of the collision frequency: the monotony and coercivity. As a consequence, the linearized Boltzmann operator is a Fredholm operator.

2. **The classical model.** For the sake of clarity, we present the model in [8] on which our work is mainly based. We start with physical conservation equations and proceed as follows.

2.1. **Boltzmann equation.** Without loss of generality, we first assume that the particle mass equals unity, and we denote as usual by (v, v_*), (I, I_*) and (v', v'_*), (I', I'_*) the pre-collisional and post-collisional velocity and energy pairs respectively. In this model, the internal energies are assumed to be continuous [9, 12] rather than discrete [14, 16]. The following conservation of momentum and total energy equations hold:

$$v + v_* = v' + v'_* \tag{1}$$

$$\frac{1}{2}v^2 + \frac{1}{2}v_*^2 + I + I_* = \frac{1}{2}v'^2 + \frac{1}{2}v_*'^2 + I' + I'_*. \tag{2}$$

From the above equations, we can deduce the following equation representing the conservation of total energy in the center of mass reference frame:

$$\frac{1}{4}(v - v_*)^2 + I + I_* = \frac{1}{4}(v' - v'_*)^2 + I' + I'_* = E,$$

with E denoting the total energy of the colliding particles. We introduce in addition the parameter $R \in [0, 1]$ which represents the portion allocated to the kinetic energy after collision out of the total energy, and the parameter $r \in [0, 1]$ which represents the distribution of the post internal energy among the two interacting molecules. Namely,

$$\begin{aligned} \frac{1}{4}(v' - v'_*)^2 &= RE \\ I' + I'_* &= (1 - R)E, \end{aligned}$$

and

$$\begin{aligned} I' &= r(1 - R)E \\ I'_* &= (1 - r)(1 - R)E. \end{aligned}$$

Using the above equations, we can express the post-collisional velocities in terms of the other quantities by the following

$$\begin{aligned} v' \equiv v'(v, v_*, I, I_*, \omega, R) &= \frac{v + v_*}{2} + \sqrt{RE}\; T_\omega\left[\frac{v - v_*}{|v - v_*|}\right] \\ v'_* \equiv v'_*(v, v_*, I, I_*, \omega, R) &= \frac{v + v_*}{2} - \sqrt{RE}\; T_\omega\left[\frac{v - v_*}{|v - v_*|}\right], \end{aligned}$$

where $\omega \in S^2$, and $T_\omega(z) = z - 2(z.\omega)\omega$. In addition, we define the parameters $r' \in [0, 1]$ and $R' \in [0, 1]$ for the pre-collisional terms in the same manner as r and R. In particular

$$\begin{aligned} \frac{1}{4}(v - v_*)^2 &= R'E \\ I + I_* &= (1 - R')E, \end{aligned}$$

and

$$\begin{aligned} I &= r'(1 - R')E \\ I_* &= (1 - r')(1 - R')E. \end{aligned}$$

Finally, the post-collisional energies can be given in terms of the pre-collisional energies by the following relation

$$I' = \frac{r(1-R)}{r'(1-R')} I$$
$$I'_* = \frac{(1-r)(1-R)}{(1-r')(1-R')} I_*.$$

The Boltzmann equation for an interacting single polyatomic gas reads

$$\partial_t f + v.\nabla_x f = Q(f,f), \tag{3}$$

where $f = f(t,x,v,I) \geq 0$ is the distribution function, with $t \geq 0, x \in \mathbb{R}^3, v \in \mathbb{R}^3$, and $I \geq 0$. The operator $Q(f,f)$ is the quadratic Boltzmann operator [8] given as

$$Q(f,f)(v,I) = \int_{\mathbb{R}^3\times\mathbb{R}_+\times S^2\times(0,1)^2} \left(\frac{f'f'_*}{(I'I'_*)^\alpha} - \frac{ff_*}{(II_*)^\alpha}\right) \times \mathcal{B} \times (r(1-r))^\alpha (1-R)^{2\alpha} \times I^\alpha I_*^\alpha (1-R) R^{1/2} \, \mathrm{d}R\mathrm{d}r\mathrm{d}\omega\mathrm{d}I_* \, \mathrm{d}v_*, \tag{4}$$

where we use the standard notations $f_* = f(v_*, I_*), f' = f(v', I')$, and $f'_* = f(v'_*, I'_*)$, and $\alpha > -1$. For diatomic molecules, $\alpha = 0$ and the collision operator (4) is relaxed to

$$Q(f,f)(v,I) = \int_{\mathbb{R}^3\times\mathbb{R}_+\times S^2\times(0,1)^2} (f'f'_* - ff_*) \times \mathcal{B} \times (1-R) R^{1/2} \, \mathrm{d}R\mathrm{d}r\mathrm{d}\omega\mathrm{d}I_* \, \mathrm{d}v_*, \tag{5}$$

The function $\mathcal{B}$ is the collision cross-section; a function of $(v, v_*, I, I_*, r, R, \omega)$. In the following, we give some assumptions on $\mathcal{B}$, extended from Grad's assumption for collision kernels of monoatomic gases. In general, $\mathcal{B}$ is assumed to be an almost everywhere positive function satisfying the following microreversibility conditions:

$$\begin{aligned} \mathcal{B}(v, v_*, I, I_*, r, R, \omega) &= \mathcal{B}(v_*, v, I_*, I, 1-r, R, -\omega), \\ \mathcal{B}(v, v_*, I, I_*, r, R, \omega) &= \mathcal{B}(v', v'_*, I', I'_*, r', R', \omega). \end{aligned} \tag{6}$$

Main assumptions on $\mathcal{B}$

Together with the above assumption (6), we assume the following boundedness assumptions on the collision cross section $\mathcal{B}$. In particular, we assume that

$$C_1 \varphi(R)\psi(r) \left|\omega.\frac{(v-v_*)}{|v-v_*|}\right| \left(|v-v_*|^\gamma + I^{\frac{\gamma}{2}} + I_*^{\frac{\gamma}{2}}\right) \leq \mathcal{B}(v, v_*, I, I_*, r, R, \omega), \tag{7}$$

and

$$\mathcal{B}(v, v_*, I, I_*, r, R, \omega) \leq C_2 \varphi_{\tilde{\alpha}}(R)\psi_{\tilde{\beta}}(r) \left(|v-v_*|^\gamma + I^{\frac{\gamma}{2}} + I_*^{\frac{\gamma}{2}}\right), \tag{8}$$

where for any $p > 0$,

$$\psi_p(r) = (r(1-r))^p, \quad \text{and } \varphi_p(R) = (1-R)^p.$$

In addition, $\varphi(R)$, and $\psi(r)$ are positive functions such that

$$\varphi(R) \leq \varphi_{\tilde{\alpha}}(R), \quad \text{and } \psi(r) \leq \psi_{\tilde{\beta}}(r), \tag{9}$$

and $\gamma \geq 0$, $\tilde{\alpha} > \frac{1}{4} + \frac{\gamma}{2}$, $\tilde{\beta} > \frac{3}{4} + \frac{\gamma}{2}$, and $C_2 \geq C_1 > 0$.

We remark that the above assumptions (7) and (8) are compatible with Maxwell molecules, hard spheres and hard potentials in the monoatomic case.

3. **The linearized boltzmann operator.** We state first the H-theorem for diatomic gases which was initially established for polyatomic gases in [8]. Namely, suppose that the positivity assumption of $\mathcal{B}$ in (8) holds, then the entropy production functional

$$D(f) = \int_{\mathbb{R}^3}\int_{\mathbb{R}_+} Q(f,f)\log f \,\mathrm{d}I\mathrm{d}v \leq 0,$$

and the following are equivalent

1. The collision operator $Q(f,f)$ vanishes, i.e. $Q(f,f)(v,I) = 0$ for every $v \in \mathbb{R}^3$ and $I \geq 0$.
2. The entropy production vanishes, i.e. $D(f) = 0$.
3. There exists $T > 0$, $n > 0$, and $u \in \mathbb{R}^3$ such that

$$f(v,I) = \frac{n}{(2\pi kT)^{\frac{3}{2}}kT} e^{-\frac{1}{kT}(\frac{1}{2}(v-u)^2+I)}, \tag{10}$$

where κ in (10) is the Boltzmann constant. The linearization of the Boltzmann equation of diatomic gases is taken around the local Maxwellian function, which represents the equilibrium state of a diatomic gas and is denoted by $M_{n,u,T}(v,I)$, and given by

$$M_{n,u,T}(v,I) = \frac{n}{(2\pi\kappa T)^{\frac{3}{2}}kT} e^{-\frac{1}{\kappa T}(\frac{1}{2}(v-u)^2+I)}, \tag{11}$$

where n, u, and T in (11) are the number of atoms per unit volume, the hydrodynamic velocity, and the temperature respectively. In particular,

$$n = \int_{\mathbb{R}^3}\int_{\mathbb{R}_+} f\mathrm{d}I\mathrm{d}v, \quad nu = \int_{\mathbb{R}^3}\int_{\mathbb{R}_+} vf\mathrm{d}I\mathrm{d}v, \quad \frac{5}{2}nT = \int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\left(\frac{(v-u)^2}{2}+I\right)f\mathrm{d}I\mathrm{d}v.$$

Without loss of generality, we will consider in the sequel a normalized version $M_{1,0,1}$ of $M_{n,u,T}$, by assuming $\kappa T = n = 1$ and $u = 0$. For the sake of simplicity, the index will be dropped. In particular,

$$M(v,I) = M_{1,0,1}(v,I) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}v^2-I}.$$

We look for a solution f around M having the form

$$f(v,I) = M(v,I) + M^{\frac{1}{2}}(v,I)g(v,I). \tag{12}$$

The linearization of the Boltzmann operator (5) around M (12) leads to introduce the linearized Boltzmann operator $\mathcal{L}$ given as

$$\mathcal{L}g = M^{-\frac{1}{2}}[Q(M, M^{\frac{1}{2}}g) + Q(M^{\frac{1}{2}}g, M)],$$

In particular, $\mathcal{L}$ writes

$$\mathcal{L}g = M^{-\frac{1}{2}}\int_\Delta \left[M'M_*'^{\frac{1}{2}}g_* - MM_*^{\frac{1}{2}}g_* + M'^{\frac{1}{2}}M_*'g' - M^{\frac{1}{2}}M_*g\right] \mathcal{B}(1-R)R^{1/2}\ \mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_*. \tag{13}$$

Thanks to the conservation of total energy (2) we have $MM_* = M'M_*'$, and so $\mathcal{L}$ has the explicit form:

$$\begin{aligned}\mathcal{L}(g) = &-\int_\Delta \mathcal{B} M^{\frac{1}{2}} M_*^{\frac{1}{2}} g_* (1-R) R^{1/2}\,\mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_* \\ &-\int_\Delta \mathcal{B} M_* g (1-R) R^{1/2}\,\mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_* \\ &+\int_\Delta \mathcal{B} M_*^{\frac{1}{2}} M'^{\frac{1}{2}} g'_* (1-R) R^{1/2}\,\mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_* \\ &+\int_\Delta \mathcal{B} M_*^{\frac{1}{2}} M_*'^{\frac{1}{2}} g' (1-R) R^{1/2}\,\mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_*.\end{aligned}$$

Here, Δ refers to the open set $\mathbb{R}^3 \times \mathbb{R}_+ \times S^2 \times (0,1)^2$. In addition, $\mathcal{L}$ can be written in the form

$$\mathcal{L} = \mathcal{K} - \nu\ \mathrm{Id},$$

where

$$\begin{aligned}\mathcal{K}g = &\int_\Delta \mathcal{B} M_*^{\frac{1}{2}} M'^{\frac{1}{2}} g'_* (1-R) R^{1/2}\,\mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_* \\ &+\int_\Delta \mathcal{B} M_*^{\frac{1}{2}} M_*'^{\frac{1}{2}} g' (1-R) R^{1/2}\,\mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_* \\ &-\int_\Delta \mathcal{B} M^{\frac{1}{2}} M_*^{\frac{1}{2}} g_* (1-R) R^{1/2}\,\mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_*,\end{aligned} \tag{14}$$

and

$$\nu(v,I) = \int_\Delta \mathcal{B} M_*\,(1-R) R^{1/2} \mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_*, \tag{15}$$

which represents the collision frequency. We write also $\mathcal{K}$ as $\mathcal{K} = \mathcal{K}_3 + \mathcal{K}_2 - \mathcal{K}_1$ with

$$\mathcal{K}_1 = \int_\Delta \mathcal{B} M^{\frac{1}{2}} M_*^{\frac{1}{2}} g_* (1-R) R^{1/2}\,\mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_*, \tag{16}$$

$$\mathcal{K}_2 = \int_\Delta \mathcal{B} M_*^{\frac{1}{2}} M'^{\frac{1}{2}} g'_* (1-R) R^{1/2}\,\mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_*, \tag{17}$$

and

$$\mathcal{K}_3 = \int_\Delta \mathcal{B} M_*^{\frac{1}{2}} M_*'^{\frac{1}{2}} g' (1-R) R^{1/2}\,\mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_*. \tag{18}$$

The linearized operator $\mathcal{L}$ is a symmetric operator, with kernel

$$\ker \mathcal{L} = M^{1/2} \mathrm{span}\ \{1, v_i, \frac{1}{2}v^2 + I\} \quad i = 1, \cdots, 3.$$

Since $\mathcal{L}$ is symmetric and ν Id is self-adjoint on

$$Dom(\nu\ \mathrm{Id}) = \{g \in L^2(\mathbb{R}^3 \times \mathbb{R}_+) : \nu g \in L^2(\mathbb{R}^3 \times \mathbb{R}_+)\},$$

then $\mathcal{K}$ is symmetric. In the following section, we prove that $\mathcal{K}$ is a bounded compact operator on $L^2(\mathbb{R}^3 \times \mathbb{R}_+)$. Hence, $\mathcal{L}$ is a self adjoint operator on Dom $(\mathcal{L})$ = Dom$(\nu\ \mathrm{Id})$. In section 5 we prove that ν is coercive, and therefore $\mathcal{L}$ is a Fredholm operator on $L^2(\mathbb{R}^3 \times \mathbb{R}_+)$.

4. **Main result.** We give now the main result on the linearized Boltzmann operator based on the assumptions of the collision cross section (8) and (7). In particular, using (7) we prove that the multiplication operator by ν Id is coercive and using (8) we prove that $\mathcal{K}$ is compact. This leads to the Fredholm property of $\mathcal{L}$ on $L^2(\mathbb{R}^3 \times \mathbb{R}_+)$.

We state the following theorem, which is the main result of the paper.

Theorem 4.1. *The operator $\mathcal{K}$ of diatomic gases defined in (14) is a compact operator from $L^2(\mathbb{R}^3 \times \mathbb{R}_+)$ to $L^2(\mathbb{R}^3 \times \mathbb{R}_+)$, and the multiplication operator by ν is coercive. As a result, the linearized Boltzmann operator $\mathcal{L}$ is an unbounded self adjoint Fredholm operator from $Dom(\mathcal{L}) = Dom(\nu\ Id) \subset L^2(\mathbb{R}^3 \times \mathbb{R}_+)$ to $L^2(\mathbb{R}^3 \times \mathbb{R}_+)$.*

We carry out the proof of the coercivity of ν Id in section 5, and we dedicate the following proof for the compactness of $\mathcal{K}$.

Proof. Throughout the proof, we prove the compactness of each $\mathcal{K}_i$ with $i = 1, \cdots, 3$ separately.

Compactness of $\mathcal{K}_1$. The compactness of $\mathcal{K}_1$ is straightforward as $\mathcal{K}_1$ already possesses a kernel form. Thus, we can inspect the operator kernel of $\mathcal{K}_1$ (16) to be

$$k_1(v, I, v_*, I_*) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{S^2 \times (0,1)^2} \mathcal{B} e^{-\frac{1}{4}v_*^2 - \frac{1}{4}v^2 - \frac{1}{2}I_* - \frac{1}{2}I} (1 - R) R^{1/2} \, \mathrm{d}r \mathrm{d}R \mathrm{d}\omega,$$

and therefore

$$\mathcal{K}_1 g(v, I) = \int_{\mathbb{R}^3 \times \mathbb{R}_+} g(v_*, I_*) \, k_1(v, I, v_*, I_*) \, \mathrm{d}I_* \mathrm{d}v_* \quad \forall (v, I) \in \mathbb{R}^3 \times \mathbb{R}_+.$$

If $\mathcal{B}$ is constant in $|v - v_*|$, I, and I_*, then $\mathcal{K}_1$ is a rank one operator and thus compact. However, in general, we give the following lemma that yields to the compactness of $\mathcal{K}_1$.

Lemma 4.2. *With the assumption (8) on $\mathcal{B}$, the function k_1 belongs to $L^2(\mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+)$.*

Proof. Applying Cauchy-Schwarz we get

$$\begin{aligned}
||k_1||_{L^2}^2 &\leq c \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} (I^\gamma + I_*^\gamma + |v - v_*|^{2\gamma}) e^{-\frac{1}{2}v_*^2 - \frac{1}{2}v^2 - I_* - I} \mathrm{d}I \mathrm{d}v \mathrm{d}I_* \mathrm{d}v_* \\
&\leq c \int_{\mathbb{R}^3} e^{-\frac{1}{2}v_*^2} \left[\int_{|v - v_*| \leq 1} e^{-\frac{1}{2}v^2} \mathrm{d}v + \int_{|v - v_*| \geq 1} |v - v_*|^{\lceil 2\gamma \rceil} e^{-\frac{1}{2}v^2} \mathrm{d}v \right] \mathrm{d}v_* \\
&\leq c \int_{\mathbb{R}^3} e^{-\frac{1}{2}v_*^2} \left[\int_{|v - v_*| \geq 1} \sum_{k=0}^{\lceil 2\gamma \rceil} |v|^k |v_*|^{\lceil 2\gamma \rceil - k} e^{-\frac{1}{2}v^2} \mathrm{d}v \right] \mathrm{d}v_* \\
&\leq c \sum_{k=0}^{\lceil 2\gamma \rceil} \int_{\mathbb{R}^3} |v_*|^{\lceil 2\gamma \rceil - k} e^{-\frac{1}{2}v_*^2} \left[\int_{\mathbb{R}^3} |v|^k e^{-\frac{1}{2}v^2} \mathrm{d}v \right] \mathrm{d}v_* < \infty,
\end{aligned}$$

where $\lceil 2\gamma \rceil$ is the ceiling of 2γ, and $c > 0$ is a generic constant. □

This implies that $\mathcal{K}_1$ is a Hilbert-Schmidt operator, and thus compact. We prove now the compactness of $\mathcal{K}_2$, similarly by proving it to be a Hilbert-Schmidt operator.

Compactness of $\mathcal{K}_2$. Additional work is required to inspect the kernel form of $\mathcal{K}_2$, since the kernel is not obvious. As a first step, we simplify the expression of $\mathcal{K}_2$ by writing it in the $\sigma-$notation through a change of variable on ω explained in the following lemma.

Lemma 4.3. *Let*

$$\sigma = T_\omega\Big(\frac{v-v_*}{|v-v_*|}\Big) = \frac{v-v_*}{|v-v_*|} - 2\frac{v-v_*}{|v-v_*|}.\omega\omega, \tag{19}$$

then the Jacobian of the $\omega - \sigma$ transformation is given in [11] *as*

$$d\omega = \frac{d\sigma}{2\Big|\sigma - \frac{v-v_*}{|v-v_*|}\Big|}.$$

Proof. It's enough to assume that ω is not collinear to $\frac{v-v_*}{|v-v_*|}$. The differential map for (19) is

$$\begin{aligned} d\sigma_\omega : \mathbb{R}^3 &\longmapsto \mathbb{R}^3 \\ \vec{\omega} &\longrightarrow \vec{\sigma} = -2\left\langle \frac{v-v_*}{|v-v_*|}, \vec{\omega}\right\rangle \omega - 2\left\langle \frac{v-v_*}{|v-v_*|}, \omega\right\rangle \vec{\omega}. \end{aligned} \tag{20}$$

Let T_1 be the tangent plane to ω, and T_2 be the plane determined by ω and $\frac{v-v_*}{|v-v_*|}$, i.e. $T_2 = \text{span}\Big\{\omega, \frac{v-v_*}{|v-v_*|}\Big\}$. Choose $\{\vec{\omega}_1, \vec{\omega}_2\} \subset T_1$ orthonormal basis such that $\vec{\omega}_1 \in T_2$ and $\vec{\omega}_2 \perp T_2$, and let $(\vec{\sigma_1}, \vec{\sigma_2}) = (d\sigma_\omega(\vec{\omega}_1), d\sigma_\omega(\vec{\omega}_2))$. Then, $\vec{\sigma}_1 \in T_2$ and $\vec{\sigma}_2 \perp T_2$. The Gram determinant is given by

$$\text{Gram} = |\vec{\sigma}_1|^2|\vec{\sigma}_2|^2 - \langle \vec{\sigma}_1, \vec{\sigma}_2\rangle^2,$$

where

$$\begin{aligned} |\vec{\sigma}_1|^2 &= 4\left(\left\langle \frac{v-v_*}{|v-v_*|}, \vec{\omega}_1\right\rangle^2 + \left\langle \frac{v-v_*}{|v-v_*|}, \omega\right\rangle^2\right) = 4\left|\frac{v-v_*}{|v-v_*|}\right|^2 = 4, \\ |\vec{\sigma}_2|^2 &= 4\left(\left\langle \frac{v-v_*}{|v-v_*|}, \vec{\omega}_2\right\rangle^2 + \left\langle \frac{v-v_*}{|v-v_*|}, \omega\right\rangle^2\right) = 4\left\langle \frac{v-v_*}{|v-v_*|}, \omega\right\rangle^2, \end{aligned}$$

and

$$\langle \sigma_1, \sigma_2\rangle = 0.$$

As a result,

$$\text{Gram} = 16\left\langle \frac{v-v_*}{|v-v_*|}, \omega\right\rangle^2 = 4\left|\sigma - \frac{v-v_*}{|v-v_*|}\right|^2.$$

□

We thus write $\mathcal{K}_2$ as

$$\begin{aligned} \mathcal{K}_2 g(v,I) = \int_\Delta e^{-\frac{I_*}{2} - \frac{1}{2}r(1-R)\left(\frac{(v-v_*)^2}{4} + I + I_*\right) - \frac{1}{4}v_*^2 - \frac{1}{4}\left(\frac{v+v_*}{2} + \sqrt{R\left(\frac{1}{4}(v-v_*)^2 + I + I_*\right)}\sigma\right)^2} \times \\ g\Big(\frac{v+v_*}{2} - \sqrt{R(\frac{1}{4}(v-v_*)^2 + I + I_*)}\sigma, (1-R)(1-r)\Big[\frac{1}{4}(v-v_*)^2 + I + I_*\Big]\Big) \\ \frac{1}{(2\pi)^{\frac{3}{2}}}(1-R)R^{\frac{1}{2}}\mathcal{B}\left|\sigma - \frac{v-v_*}{|v-v_*|}\right|^{-1} \mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}I_*\mathrm{d}v_*. \end{aligned} \tag{21}$$

We seek first to write $\mathcal{K}_2$ in its kernel form. For this, we define $h_{v,I,r,R,\sigma}$; where for simplicity the index will be omitted; as

$$\begin{aligned} h : \mathbb{R}^3 \times \mathbb{R}_+ &\longmapsto h(\mathbb{R}^3 \times \mathbb{R}_+) \subset \mathbb{R}^3 \times \mathbb{R}_+ \\ (v_*, I_*) \quad &\longmapsto (x, y) = \Big(\frac{v + v_*}{2} - \sqrt{R(\frac{1}{4}(v - v_*)^2 + I + I_*)}\sigma, \\ &\qquad\qquad (1 - R)(1 - r)\Big[\frac{1}{4}(v - v_*)^2 + I + I_*\Big]\Big), \end{aligned}$$

for fixed v,I,r,R, and σ. The function h is invertible, and (v_*, I_*, v', I') can be expressed in terms of (x, y) as

$$v_* = 2x + 2\sqrt{Ray}\sigma - v, \quad I_* \quad = ay - I - (x - v + \sqrt{Ray}\sigma)^2,$$

and

$$v' = x + 2\sqrt{Ray}\sigma, \quad I' \quad = \frac{r}{1 - r}y,$$

where $a = \frac{1}{(1-r)(1-R)}$. The Jacobian of h^{-1} is computed as

$$J = \left| \frac{\partial v_* \partial I_*}{\partial x \partial y} \right| = \frac{8}{(1 - r)(1 - R)},$$

and the positivity of I_* restricts the variation of the variables (x, y) in integral (21) over the space

$$H^{v,I}_{R,r,\sigma} = h(\mathbb{R}^3 \times \mathbb{R}_+) = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}_+ \colon ay - I - (x - v + \sqrt{Ray}\sigma)^2 > 0\}. \quad (22)$$

In fact, $H^{v,I}_{R,r,\sigma}$ can be explicitly expressed as

$$H^{v,I}_{R,r,\sigma} = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}_+ : x \in B_{v - \sqrt{Ray}\sigma}(\sqrt{ay - I}) \text{ and } y \in ((1-r)(1-R)I, +\infty)\}.$$

Therefore, equation (21) becomes

$$\begin{aligned} \mathcal{K}_2 g = &\frac{1}{(2\pi)^{\frac{3}{2}}} \int_{(0,1)^2 \times S^2} \int_{H^{v,I}_{R,r,\sigma}} (1 - R) R^{\frac{1}{2}} J \, \mathcal{B}\Big|\sigma - \frac{v - x - \sqrt{Ray}\sigma}{|v - x - \sqrt{Ray}\sigma|}\Big|^{-1} g(x, y) \times \\ &e^{-\frac{ay - I - (x - v + \sqrt{Ray}\sigma)^2}{2} - \frac{r}{2(1-r)}y - \frac{1}{4}(2x + 2\sqrt{Ray}\sigma - v)^2 - \frac{1}{4}(x + 2\sqrt{Ray}\sigma)^2} \\ &\mathrm{d}y\mathrm{d}x\mathrm{d}\sigma\mathrm{d}r\mathrm{d}R. \end{aligned} \quad (23)$$

We now point out the kernel form of $\mathcal{K}_2$ and prove after by the help of assumption (8) that the kernel of $\mathcal{K}_2$ is in $L^2(\mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+)$. Indeed, we recall the definition of Δ, with $\Delta := \mathbb{R}^3 \times \mathbb{R}_+ \times S^2 \times (0, 1) \times (0, 1)$, and we define $H^{v,I}$ to be

$$\begin{aligned} H^{v,I} := \{(y, x, \sigma, r, R) \in \Delta \ : \ &R \in (0, 1), \ r \in (0, 1), \ \sigma \in S^2, \\ &x \in B_{v - \sqrt{Ray}\sigma}(\sqrt{ay - I}), \text{ and } y \in ((1 - r)(1 - R)I, +\infty)\}. \end{aligned}$$

We remark that $H^{v,I}_{R,r,\sigma}$ is a slice of $H^{v,I}$, and we define the slice $H^{v,I}_{x,y} \subset (0, 1) \times (0, 1) \times S^2$ such that

$H^{v,I} = H^{v,I}_{x,y} \times \mathbb{R}^3 \times \mathbb{R}_+$ which is equivalent to $H^{v,I} = (0, 1) \times (0, 1) \times S^2 \times H^{v,I}_{R,r,\sigma}$.

In other words,

$$H^{v,I}_{x,y} = \{(r, R, \sigma) \in (0, 1) \times (0, 1) \times S^2 : (y, x, \sigma, r, R) \in H^{v,I}\}. \quad (24)$$

Then by Fubini theorem, it holds that

$$\begin{aligned}\mathcal{K}_2 g(v,I) =& \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{H^{v,I}}(1-R)R^{\frac{1}{2}}J\,\mathcal{B}\Big|\sigma-\frac{v-x-\sqrt{Ray}\sigma}{|v-x-\sqrt{Ray}\sigma|}\Big|^{-1}g(x,y)\times\\ & e^{-\frac{ay-I-(x-v+\sqrt{Ray}\sigma)^2}{2}-\frac{r}{2(1-r)}y-\frac{1}{4}(2x+2\sqrt{Ray}\sigma-v)^2-\frac{1}{4}(x+2\sqrt{Ray}\sigma)^2}\mathrm{d}y\mathrm{d}x\mathrm{d}\sigma\mathrm{d}r\mathrm{d}R\\ =& \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3\times\mathbb{R}_+}\int_{H^{v,I}_{x,y}}(1-R)R^{\frac{1}{2}}J\,\mathcal{B}\Big|\sigma-\frac{v-x-\sqrt{Ray}\sigma}{|v-x-\sqrt{Ray}\sigma|}\Big|^{-1}g(x,y)\times\\ & e^{-\frac{ay-I-(x-v+\sqrt{Ray}\sigma)^2}{2}-\frac{r}{2(1-r)}y-\frac{1}{4}(2x+2\sqrt{Ray}\sigma-v)^2-\frac{1}{4}(x+2\sqrt{Ray}\sigma)^2}\mathrm{d}\sigma\mathrm{d}r\mathrm{d}R\mathrm{d}y\mathrm{d}x.\end{aligned}\tag{25}$$

The kernel of $\mathcal{K}_2$ is thus inspected and written explicitly in the following lemma.

Lemma 4.4. *With the assumption (8) on $\mathcal{B}$, the kernel of $\mathcal{K}_2$ given by*

$$\begin{aligned}k_2(v,I,x,y) =& \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{H^{v,I}_{x,y}}(1-R)R^{\frac{1}{2}}J\,\mathcal{B}\Big|\sigma-\frac{v-x-\sqrt{Ray}\sigma}{|v-x-\sqrt{Ray}\sigma|}\Big|^{-1}\times\\ & e^{-\frac{ay-I-(x-v+\sqrt{Ray}\sigma)^2}{2}-\frac{r}{2(1-r)}y-\frac{1}{4}(2x+2\sqrt{Ray}\sigma-v)^2-\frac{1}{4}(x+2\sqrt{Ray}\sigma)^2}\mathrm{d}\sigma\mathrm{d}r\mathrm{d}R\end{aligned}$$

is in $L^2(\,\mathbb{R}^3\times\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}_+)$.

Proof. Rewriting k_2 in the $\omega-$notation and applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}\|k_2\|^2_{L^2} \le c\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{(0,1)^2\times S^2}&(1-R)^2RJ^2\mathcal{B}^2\times\\ & e^{-\left[ay-I-\left(x-v+\sqrt{Ray}T_\omega\left(\frac{v-v_*}{|v-v_*|}\right)\right)^2\right]-\frac{r}{(1-r)}y-\frac{1}{2}(2x+2\sqrt{Ray}T_\omega\left(\frac{v-v_*}{|v-v_*|}\right)-v)^2}\\ & e^{-\frac{1}{2}(x+2\sqrt{Ray}T_\omega\left(\frac{v-v_*}{|v-v_*|}\right))^2}\mathrm{d}\omega\mathrm{d}r\mathrm{d}R\mathrm{d}y\mathrm{d}x\mathrm{d}I\mathrm{d}v.\end{aligned}$$

Writing back in σ notation, then by means of h^{-1}, and back to the ω notation with omitting the term $e^{-\frac{1}{2}v'^2-r(1-R)I_*}$ in the last integral, we get

$$\begin{aligned}\|k_2\|^2_{L^2} \le c\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{(0,1)^2\times S^2}&e^{-I_*-\frac{1}{2}v_*^2-r(1-R)\left(\frac{(v-v_*)^2}{4}+I\right)}\\ &(1-R)^2RJ\mathcal{B}^2(v,v_*,I,I_*,r,R,\omega)\mathrm{d}\omega\mathrm{d}r\mathrm{d}R\mathrm{d}I_*\mathrm{d}v_*\mathrm{d}I\mathrm{d}v.\end{aligned}$$

Assumption (8) on $\mathcal{B}$ yields

$$\begin{aligned}\|k_2\|^2_{L^2} &\le c\int_{(0,1)^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}(1-R)^2RJ\Big(|v-v_*|^{2\gamma}+I^\gamma+I_*^\gamma\Big)(r(1-r))^{2\tilde\beta}\\ &\qquad (1-R)^{2\tilde\alpha}\times e^{-I_*-\frac{1}{2}v_*^2-r(1-R)\left(\frac{(v-v_*)^2}{4}+I\right)}\mathrm{d}I\mathrm{d}v\mathrm{d}I_*\mathrm{d}v_*\mathrm{d}r\mathrm{d}R\\ &\le c\int_{(0,1)^2} r^{2\tilde\beta-\frac{5}{2}-\gamma}(1-r)^{2\tilde\beta-1}R(1-R)^{2\tilde\alpha-\frac{3}{2}-\gamma}\mathrm{d}r\mathrm{d}R<\infty.\end{aligned}$$

with $c>0$. We give the following remark for better understanding of the above computations.

Remark 1. For any $a, b, c \in \{0, \gamma, 2\gamma\}$, by using the spherical coordinates of $(v - v_*)$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+} I^a I_*^b |v-v_*|^c e^{-I_*-\frac{1}{2}v_*^2 - r(1-R)\frac{(v-v_*)^2}{4} - r(1-R)I} \mathrm{d}I\mathrm{d}v\mathrm{d}I_*\mathrm{d}v_* \\
&\quad \le C\Big(\int_{\mathbb{R}_+} I^a e^{-r(1-R)I}\mathrm{d}I\Big)\Big(\int_{\mathbb{R}^3}\Big[\int_{\mathbb{R}^3}|v-v_*|^c e^{-r(1-R)\frac{(v-v_*)^2}{4}}\mathrm{d}v\Big]e^{-\frac{1}{2}v_*^2}\mathrm{d}v_*\Big) \\
&\quad \le C[r(1-R)]^{-a-1}[r(1-R)]^{-\frac{c+3}{2}},
\end{aligned}$$

for some constant $C > 0$.

The lemma is thus proved, which implies that $\mathcal{K}_2$ is a Hilbert-Schmidt operator. □

Compactness of $\mathcal{K}_3$. The proof of the compactness of $\mathcal{K}_3$ (18) is very similar to that of $\mathcal{K}_2$. The operator $\mathcal{K}_3$ which has the explicit form

$$\begin{aligned}
\mathcal{K}_3 g(v,I) = \int_\Delta & e^{-\frac{I_*}{2}-\frac{1}{2}(1-r)(1-R)\left(\frac{(v-v_*)^2}{4}+I+I_*\right)} e^{-\frac{1}{4}v_*^2-\frac{1}{4}\left(\frac{v+v_*}{2}-\sqrt{R(\frac{1}{4}(v-v_*)^2+I+I_*)}\,\sigma\right)^2} \\
& g\Big(\frac{v+v_*}{2}+\sqrt{R(\frac{1}{4}(v-v_*)^2+I+I_*)}\sigma, r(1-R)\Big[\frac{1}{4}(v-v_*)^2+I+I_*\Big]\Big) \\
& \frac{1}{(2\pi)^{\frac{3}{2}}} R^{\frac{1}{2}}(1-R)\mathcal{B}\,\Big|\sigma-\frac{v-v_*}{|v-v_*|}\Big|^{-1}\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}I_*\mathrm{d}v_*,
\end{aligned}$$

inherits the same form as $\mathcal{K}_2$, with a remark that the Jacobian of the transformation $\tilde{h} : \mathbb{R}^3 \times \mathbb{R}_+ \longmapsto \mathbb{R}^3 \times \mathbb{R}_+$

$$\begin{aligned}
(v_*, I_*) \longmapsto (x, y) = \Big(&\frac{v+v_*}{2}+\sqrt{R(\frac{1}{4}(v-v_*)^2+I+I_*)}\sigma, \\
& r(1-R)\Big[\frac{1}{4}(v-v_*)^2+I+I_*\Big]\Big),
\end{aligned}$$

is calculated to be

$$\tilde{J} = \frac{8}{r(1-R)}.$$

The final requirement for the kernel of $\mathcal{K}_3$ to be L^2 integrable is

$$\int_{(0,1)^2}(1-r)^{2\tilde{\beta}-\frac{5}{2}-\gamma} r^{2\tilde{\beta}-1} R(1-R)^{2\tilde{\alpha}-\frac{3}{2}-\gamma}\mathrm{d}r\mathrm{d}R < \infty,$$

which holds by the change of variable $r \mapsto 1 - r$. □

To this extent, the perturbation operator $\mathcal{K}$ is proved to be Hilbert-Schmidt, and thus $\mathcal{K}$ is a bounded compact operator. As a result, the linearized operator $\mathcal{L}$ is a self adjoint operator.

5. **Properties of the collision frequency.** We give in this section some properties of ν. The first is the coercivity property, which implies that $\mathcal{L}$ is a Fredholm operator, and we prove the monotony of ν which depends on the choice of the collision cross section $\mathcal{B}$. The latter property is used for locating the essential spectrum of $\mathcal{L}$.

Proposition 1 (Coercivity of ν Id)**.** *With the assumption* (7)*, there exists* $c > 0$ *such that*

$$\nu(v, I) \ge c(|v|^\gamma + I^{\gamma/2} + 1),$$

for any $\gamma \ge 0$*. As a result, the multiplication operator* ν Id *is coercive.*

Proof. The collision frequency (15) is

$$\nu(v,I) = \int_\Delta \mathcal{B} e^{-I_* - \frac{1}{2}v_*^2} \mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_*,$$

where by (7) we get

$$\begin{aligned}\nu(v,I) &\geq c\int_{S^2}\int_{\mathbb{R}^3}\left(|v-v_*|^\gamma + I^{\gamma/2}\right)e^{-\frac{1}{2}v_*^2}\,\mathrm{d}\omega\mathrm{d}v_* \\ &\geq c\Big(I^{\gamma/2} + \int_{\mathbb{R}^3}||v|-|v_*||^\gamma e^{-\frac{1}{2}v_*^2}\,\mathrm{d}v_*\Big),\end{aligned}$$

where c is a generic constant. We consider the two cases, $|v| \geq 1$ and $|v| \leq 1$. If $|v| \geq 1$ we have

$$\begin{aligned}\nu(v,I) &\geq c\Big(I^{\gamma/2} + \int_{|v_*|\leq\frac{1}{2}|v|}(|v|-|v_*|)^\gamma e^{-\frac{1}{2}v_*^2}\,\mathrm{d}v_*\Big) \\ &\geq c\Big(I^{\gamma/2} + |v|^\gamma\int_{|v_*|\leq\frac{1}{2}} e^{-\frac{1}{2}v_*^2}\mathrm{d}v_*\Big) \\ &\geq c(|v|^\gamma + I^{\gamma/2} + 1).\end{aligned}$$

For $|v| \leq 1$,

$$\begin{aligned}\nu(v,I) &\geq c\Big(I^{\gamma/2} + \int_{|v_*|\geq 2}(|v_*|-|v|)^\gamma e^{-\frac{1}{2}v_*^2}\,\mathrm{d}v_*\Big) \\ &\geq c\Big(I^{\gamma/2} + \int_{|v_*|\geq 2} e^{-\frac{1}{2}v_*^2}\,\mathrm{d}v_*\Big) \\ &\geq c(1 + I^{\gamma/2} + |v|^\gamma).\end{aligned}$$

□

The result is thus proved. We give now the following proposition, which is a generalization of the work of Grad [17], in which he proved that the collision frequency of monoatomic single gases is monotonic based on the choice of the collision cross section $\mathcal{B}$.

Proposition 2 (monotony of ν)**.** *Under the assumption that*

$$\int_{(0,1)^2\times S^2}(1-R)R^{\frac{1}{2}}\mathcal{B}(|V|,I,I_*,r,R,\omega)\mathrm{d}r\mathrm{d}R\mathrm{d}\omega \tag{26}$$

is increasing (respectively decreasing) in $|V|$ and I for every I_*, the collision frequency ν is increasing (respectively decreasing), where $|V| = |v - v_*|$.

In particular, for Maxwell molecules, where $\mathcal{B}$ is constant in $|V|$ and I, ν is constant. On the other hand, for collision cross-sections of the form

$$\mathcal{B}(v,v_*,I,I_*,r,R,\omega) = C\varphi(r)\psi(R)\Big(|v-v_*|^\gamma + I^{\gamma/2} + I_*^{\gamma/2}\Big),$$

the integral (26) is increasing, and thus ν is increasing, where $C > 0$, $\gamma \geq 0$, and φ and ψ are positive functions that belong to $L^1((0,1))$.

In fact, if φ and ψ satisfy in addition (9), then this collision cross section satisfies our main assumptions (7) and (8).

Proof. We remark first that ν is a radial function in $|v|$ and I. In fact, we perform the change of variable $V = v - v_*$ in the integral (15), where the expression of ν becomes

$$\nu(|v|, I) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_\Delta (1-R) R^{\frac{1}{2}} \mathcal{B}(|V|, I, I_*, r, R, \omega) e^{-\frac{1}{2}(v-V)^2 - I_*} \mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}V, \tag{27}$$

where $\Delta = \mathbb{R}^3 \times \mathbb{R}_+ \times S^2 \times (0,1)^2$. The integration in V in the above integral (27) is carried in the spherical coordinates of V, with fixing one of the axes of the reference frame along v, and therefore, the above integral will be a function of $|v|$ and I.

The partial derivative of ν in the v_i direction is

$$\begin{aligned} \frac{\partial \nu}{\partial v_i} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int & (1-R) R^{\frac{1}{2}} \frac{v_i - v_{*i}}{|v - v_*|} \frac{\partial \mathcal{B}}{\partial |v - v_*|} (|v - v_*|, I, I_*, r, R, \omega) \\ & e^{-\frac{1}{2}v_*^2 - I_*} \mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}v_*. \end{aligned} \tag{28}$$

Perform the change of variable $V = v - v_*$ in (28), then

$$\frac{\partial \nu}{\partial v_i} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int (1-R) R^{\frac{1}{2}} \frac{V_i}{|V|} \frac{\partial \mathcal{B}}{\partial |V|} (|V|, I, I_*, r, R, \omega) e^{-\frac{1}{2}(v-V)^2 - I_*} \mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}V,$$

and thus,

$$\sum_{i=1}^{3} v_i \frac{\partial \nu}{\partial v_i} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int (1-R) R^{\frac{1}{2}} \frac{v.V}{|V|} \frac{\partial \mathcal{B}}{\partial |V|} (|V|, I, I_*, r, R, \omega) \tag{29}$$

$$e^{-\frac{1}{2}(v-V)^2 - I_*} \mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}V. \tag{30}$$

Applying Fubini theorem, we write (29) as

$$\sum_{i=1}^{3} v_i \frac{\partial \nu}{\partial v_i} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left[\int (1-R) R^{\frac{1}{2}} \frac{\partial \mathcal{B}}{\partial |V|} (|V|, I, I_*, r, R, \omega) \mathrm{d}r\mathrm{d}R\mathrm{d}\omega \right] \frac{v.V}{|V|} \tag{31}$$

$$e^{-\frac{1}{2}(v-V)^2 - I_*} \mathrm{d}I_*\mathrm{d}V. \tag{32}$$

The partial derivative of ν along I is

$$\begin{aligned} I \frac{\partial \nu}{\partial I} &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int (1-R) R^{\frac{1}{2}} I \frac{\partial \mathcal{B}}{\partial I} (|V|, I, I_*, r, R, \omega) e^{-\frac{1}{2}(v-V)^2 - I_*} \mathrm{d}r\mathrm{d}R\mathrm{d}\omega\mathrm{d}I_*\mathrm{d}V \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} I \int \left[\int (1-R) R^{\frac{1}{2}} \frac{\partial \mathcal{B}}{\partial I} (|V|, I, I_*, r, R, \omega) \mathrm{d}r\mathrm{d}R\mathrm{d}\omega \right] e^{-\frac{1}{2}(v-V)^2 - I_*} \mathrm{d}I_*\mathrm{d}V. \end{aligned} \tag{33}$$

When $v.V > 0$, the exponential in the integral (29) is greater than when $v.V < 0$, and so the term $v.V$ doesn't affect the sign of the partial derivatives of ν. Therefore, the sign of the partial derivative of ν along $|v|$ has the same sign as

$$\int (1-R) R^{\frac{1}{2}} \frac{\partial \mathcal{B}}{\partial |V|} (|V|, I, I_*, r, R, \omega) \mathrm{d}r\mathrm{d}R\mathrm{d}\omega.$$

It's clear as well that the partial derivative of ν with respect to I (33) has the same sign as

$$\int (1-R) R^{\frac{1}{2}} \frac{\partial \mathcal{B}}{\partial I} (|V|, I, I_*, r, R, \omega) \mathrm{d}r\mathrm{d}R\mathrm{d}\omega.$$

As a result, for a collision cross-section $\mathcal{B}$ satisfying the condition that the integral

$$\int_{(0,1)^2\times S^2} (1-R)R^{\frac{1}{2}}\mathcal{B}(|V|,I,I_*,r,R,\omega)\mathrm{d}r\mathrm{d}R\mathrm{d}\omega$$

is increasing (respectively decreasing) in $|V|$ and I, the collision frequency is increasing (respectively decreasing). □

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